EXAMINATION

14 September 2005 (am)

Subject CT4 (103) — Models (103 Part) Core Technical

Time allowed: One and a half hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all 7 questions, beginning your answer to each question on a separate sheet.*
5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

1 An insurance company has a block of in-force business under which policyholders have been given options and investment-related guarantees. A stochastic model has been developed which projects option and guarantee costs. You have used the model to estimate, for the Company Board, the probability of the insurance company having insufficient assets to honour the payouts under the policies. A Board member has asked whether there are any factors which could cause this probability to be inaccurate.

Outline the items you would mention in your response. [5]

2 (i) In the context of a stochastic process denoted by $\{X_t : t \in J\}$, define:

(a) state space
(b) time set
(c) sample path

[2]

(ii) Stochastic process models can be placed in one of four categories according to whether the state space is continuous or discrete, and whether the time set is continuous or discrete. For each of the four categories:

(a) State a stochastic process model of that type.

(b) Give an example of a problem an actuary may wish to study using a model from that category.

[4]
[Total 6]

3 A die is rolled repeatedly. Consider the following two sequences:

I B_n is the largest number rolled in the first n outcomes.
II C_n is the number of sixes rolled in the first n outcomes.

For each of these two sequences:

(a) Explain why it is a Markov chain.
(b) Determine the state space of the chain.
(c) Derive the transition probabilities.
(d) Explain whether the chain is irreducible and/or aperiodic.
(e) Describe the equilibrium distribution of the chain. [7]

4 A life insurance company prices its long-term sickness policies using a three-state Markov model in continuous time. The states are healthy (*H*), ill (*I*) and dead (*D*). The forces of transition in the model are $\sigma_{HI} = \sigma$, $\sigma_{IH} = \rho$, $\sigma_{HD} = \mu$, $\sigma_{ID} = v$ and they are assumed to be constant over time.

For a group of policyholders observed over a 1-year period, there are:

23 transitions from State H to State I;
15 transitions from State I to State H;
3 deaths from State H;
5 deaths from State I.

The total time spent in State H is 652 years and the total time spent in State I is 44 years.

(i) Write down the likelihood function for these data. [3]

(ii) Derive the maximum likelihood estimate of σ. [2]

(iii) Estimate the standard deviation of $\tilde{\sigma}$, the maximum likelihood estimator of σ. [2]

[Total 7]

5 Claims arrive at an insurance company according to a Poisson process with rate λ per week.

Assume time is expressed in weeks.

(i) Show that, given that there is exactly one claim in the time interval $[t, t + s]$, the time of the claim arrival is uniformly distributed on $[t, t + s]$. [3]

(ii) State the joint density of the holding times $T_0, T_1, \ldots, T_n$ between successive claims. [1]

(iii) Show that, given that there are n claims in the time interval $[0, t]$, the number of claims in the interval $[0, s]$ for $s < t$ is binomial with parameters n and s/t. [3]

[Total 7]

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6 A Markov jump process X_t with state space $S = \{0, 1, 2, \ldots, N\}$ has the following transition rates:

$\sigma_{ii} = -\lambda$ for $0 \le i \le N-1$

$\sigma_{i,i+1} = \lambda$ for $0 \le i \le N-1$

$\sigma_{ij} = 0$ otherwise

(i) Write down the generator matrix and the Kolmogorov forward equations (in component form) associated with this process. [3]

(ii) Verify that for $0 \le i \le N-1$ and for all $j \ge i$, the function

$$p_{ij}(t) = e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!}$$

is a solution to the forward equations in (i). [2]

(iii) Identify the distribution of the holding times associated with the jump process. [2]

[Total 7]

7 A time-inhomogeneous Markov jump process has state space {A, B} and the transition rate for switching between states equals $2t$, regardless of the state currently occupied, where t is time.

The process starts in state A at $t = 0$.

(i) Calculate the probability that the process remains in state A until at least time s. [2]

(ii) Show that the probability that the process is in state B at time T, and that it is in the first visit to state B, is given by $T^2 \times \exp^{-T^2}$. [3]

(iii) (a) Sketch the probability function given in (ii).

(b) Give an explanation of the shape of the probability function.

(c) Calculate the time at which it is most likely that the process is in its first visit to state B.

[6]

[Total 11]

END OF PAPER