Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

April 1998

Subject A — Fundamentals of Actuarial Mathematics

Paper One

EXAMINERS' REPORT

PART ONE

1 D

2 A

3 B

4 D

5 D

6 B

7 A

PART TWO

8 $90 \times 12 \times \ a_{\overline{5}|\,i}^{(12)} = 4000$

$\Rightarrow \ a_{\overline{5}|\,i}^{(12)} = 3.7037$

At 12% $\ a_{\overline{5}|\,i}^{(12)} = 3.7990$
15% $= 3.5769$

By interpolation $i \triangleq 13.3\%$

At 13.3% $\ a_{\overline{5}|\,i}^{(12)} = 3.6997$
13.2% $= 3.7072$

$\Rightarrow \quad 13.2\% < i < 13.3\% \qquad \Rightarrow \text{APR} = 13.2\%$

For full marks, candidates must show that the APR lies between 13.2% and 13.3%, and state that the APR is rounded down to the lower 0.1%.

9 (i) Annual income $= \dfrac{10000}{a_{\overline{5}|}}$ at 6¼% $= \dfrac{10000}{4.1839} = £2{,}390.13$

(ii) Real yield p.a. on investment A $= \left[(1.035)(1.045)\ldots(1.075) \times \dfrac{240}{275.6}\right]^{1/5} - 1$

$= 2.61\%$

Real yield p.a. on investment B $= \left(\dfrac{274.0}{237.6}\dfrac{240.0}{275.6}1.0275^5\right)^{1/5} - 1$

$= 2.84\%$

(iii) Real yield on investment C is given by

$$10000 = \left(\frac{240}{250}v + \frac{240}{264.4}v^2 + \ldots + \frac{240}{275.6}v^5\right)2390.13$$

At 2.84%, RHS = £9,967, so real yield is less than 2.84%.

At 2.83%, RHS = £9,970

Hence investment B gives greatest real yield.

Many candidates calculated the nominal rates of return on investments A and B rather than the real rate of return. Few candidates calculated the real rate of return on investment C correctly.

10 (i) $E[i_t] = j = 0.08 \times 0.625 + 0.04 \times 0.25 + 0.02 \times 0.125$

$= 0.0625$

$E[S_3] = (1+j)^3 = 1.0625^3 = 1.1995$

$s^2 = V[i_t] = 0.08^2 \times 0.625 + 0.04^2 \times 0.25 + 0.02^2 \times 0.125 - 0.0625^2$

$= 0.000544$

Now $V[S_n] = ((1+j)^2 + s^2)^n - (1+j)^{2n}\,;\, j = E[i_t],\ s^2 = V[i_t]$

so $V[S_3] = ((1.0625)^2 + 0.000544)^3 - 1.0625^6$

$= 1.440792 - 1.438711$

$= 0.002081$

so $\text{sd}[S_3] = 0.04562$

Generally very well answered, although some candidates forgot to calculate the standard deviation asked for in the question, having calculated the variance.

11 (i) The n-year spot rate of interest is the yield to maturity on a zero coupon bond with remaining time to maturity of term n years.

(ii) Let $f_{t,n}$ = n-year forward rate at time t

Let $y_{0,k}$ = k-year spot rate of interest at time $t = 0$

then $f_{3,2}$ is such that $(1+y_{0,3})^3\,(1+f_{3,2})^2 = (1+y_{0,5})^5$

so $(1+f_{3,2})^2 = 1.075^5 / 1.06^3 = 1.205382$

so $f_{3,2} = 1.205382^{½} - 1 = 0.09790 = 9.79\%$

(iii) Let y_n = n-year spot yield

Let y_{c_n} = n-year par yield

then 6-year par yield is given by

$$1 = y_{c_6}(v_{y_1} + v_{y_2}^2 + \ldots + v_{y_6}^6) + 1v_{y_6}^6$$

$$\text{so} \quad 1 = y_{c_6}\left(\frac{1}{1.04} + \frac{1}{1.05^2} + \frac{1}{1.06^3} + \frac{1}{1.07^4} + \frac{1}{1.075^5} + \frac{1}{1.08^6}\right) + \frac{1}{1.08^6}$$

$$= y_{c_6}\,4.797811 + 0.63017$$

$$\text{so} \quad y_{c_6} = (1 - 0.63017) / 4.797811$$

$$= 0.07708$$

$$= 7.71\%$$

12 Present value of outgoings $= 2{,}000{,}000 + 450{,}000v^{½}$ at 12%

$$= 2{,}425{,}210$$

Present value of income $= 180{,}000v\ddot{a}_{\overline{1}|}^{(4)} + 180{,}000(1+k)\,v^2\ddot{a}_{\overline{1}|}^{(4)}$

$$+ \ldots + 180{,}000v^5(1+k)^4\,\ddot{a}_{\overline{1}|}^{(4)} + 3{,}400{,}000v^6$$

$$= 180{,}000\ddot{a}_{\overline{1}|}^{(4)}\,v(1 + v_j + v_j^2 + v_j^3 + v_j^4) + 3{,}400{,}000v^6$$

$$\text{where } j = \frac{1.12}{1+k} - 1$$

$\ddot{a}^{(4)}_{\overline{1}|} = 0.95887$ at 12%

$$\text{so PV of income } = 180{,}000 \times 0.95887 \times \frac{1}{1.12} \times \ddot{a}^{j}_{\overline{5}|} + 3{,}400{,}000v^6$$

$$= 154{,}104.6\, \ddot{a}^{j}_{\overline{5}|} + 1{,}722{,}546$$

Hence, for IRR = 12%, $2{,}425{,}210 = 154{,}104.6\ddot{a}^{j}_{\overline{5}|} + 1{,}722{,}546$

so $\ddot{a}^{j}_{\overline{5}|} = 4.55961$

At 4% $\ddot{a}_{\overline{5}|} = 4.6299$

5% $\ddot{a}_{\overline{5}|} = 4.5460$

$$\text{so} \quad j \simeq 4 + \frac{4.6299 - 4.5596}{4.6299 - 4.5460}$$

$$= 4.838$$

$\ddot{a}^{.04838}_{\overline{5}|} = 4.5593$

$\ddot{a}^{.0483}_{\overline{5}|} = 4.5600$

so $j \simeq 0.04834$

and $1 + k = \dfrac{1.12}{1+j} = 6.84\%$

Not all candidates appreciated that whilst rent is payable quarterly in advance, the amount of the annual rent only increases annually, not quarterly. Those who obtained the correct expression for valuing the rental income did not always realise that this could be expressed in terms of an annuity at a rate of interest equal to 12 per cent net of the rental income increases per annum.

PART THREE

13 (i) Let $A(t)$ = accumulation at time t of 1 at time $t = 0$

$$A(t) = \exp(\int_0^t \delta(r)dr)$$

For $0 \le t < 4$, $\int_0^t \delta(r)dr = \int_0^t (0.06 + 0.005r)\,dr$

$$= \left[0.06r + 0.005\frac{r^2}{2}\right]_0^t$$

$$= 0.06t + 0.0025t^2$$

so for $0 \le t < 4$, $A(t) = \exp(0.06t + 0.0025t^2)$

For $4 \le t < 6$, $A(t) = A(4)\exp\left(\int_4^t (0.12 - 0.01r)\,dr\right)$

$$= A(4)\exp\left(\left[0.12r - 0.01\frac{r^2}{2}\right]_4^t\right)$$

$$= \exp(0.24 + 0.04) \times \exp\left(0.12t - 0.01\frac{t^2}{2} - 0.48 + 0.08\right)$$

$$= \exp(0.12t - 0.005t^2 - 0.12)$$

For $t \ge 6$, $A(t) = A(6)\exp\left(\int_6^t 0.06\,dr\right)$

$$= \exp(0.72 - 0.18 - 0.12)\exp([0.06r]_6^t)$$

$$= \exp(0.42)\exp(0.06t - 0.36)$$

$$= \exp(0.06 + 0.06t)$$

(ii) Present value = 100 / $A(5)$ = 100 / exp(0.6 – 0.125 – 0.12)

= 100 / exp(0.355)

= £70.12

(iii) The accumulation up to time 6 is

$$S_6 = \int_4^6 r(t)\, e^{\int_t^6 d(r)\,dr} dt$$

$$= \int_4^6 \rho(t)\, e^{\int_t^6 (0.12-0.01r)dr} dt$$

$$= \int_4^6 \rho(t) \exp\left[0.12r - 0.01\frac{r^2}{2}\right]_t^6 dt$$

$$= \int_4^6 \rho(t) \exp\left[0.72 - 0.18 - 0.12t + 0.01\frac{t^2}{2}\right] dt$$

$$= e^{0.54} \int_4^6 \rho(t)\, e^{-0.12t+0.005t^2}\, dt$$

$$= e^{0.54} \int_4^6 (12-t)\, e^{-0.12t+0.005t^2}\, dt$$

Let $u = -0.12t + 0.005t^2$, then $du = (-0.12 + 0.01t)\, dt$

so $S_6 = e^{0.54} \int_{-0.40}^{-0.54} -100\, e^u\, du$

$$= e^{0.54}\, 100(e^{-0.4} - e^{-0.54})$$

$$= 100e^{0.54}\; 0.087572$$

$$= 1.7160068 \times 8.7572$$

$$= 15.0274$$

Accumulated to time $t = 12 = 15.0274 \times e^{6\times 0.06}$

$$= 15.0274e^{0.36}$$

$$= £21.54$$

Many candidates failed to answer part (i) correctly, giving only the accumulations at times t=4 and t=6, or the accumulations equivalent to A(0,4), A(4,6) and A(6,t) rather than A(0,t).

Part (ii) required the amount at time t=0 which would accumulate to £100 at time t=5, not the accumulation at t=5 of £100 from t=0, which was often the answer given.

Part (iii) was generally poorly answered.

14 (i) Redemption = 110% nominal

$t_1 = 0.25$

If P = price at 1 March 1990,

$$P = 1{,}000\,(1-0.25)\ a_{\overline{10}|}^{(2)} + 11{,}000v^{10} \text{ at 8\% p.a.}$$

$$= 750 \times 1.019615 \times 6.7101 + 11{,}000 / 1.08^{10}$$

$$= 5{,}131.29 + 5{,}095.13$$

$$= £10{,}226.42$$

so price per £100 nominal = £102.26

(ii) If the price to yield 6% net for the second investor is P_2 at 1/3/98, given $t_1 = t_2 = 0.4$, then

$$P_2 = 1{,}000(1-0.4)\ a_{\overline{2}|}^{(2)} + 11{,}000v^2 - \text{CGT} \text{ at 6\%}$$

$$= 600 \times 1.014782 \times 1.8334 + 11{,}000 / 1.06^2 - \text{CGT}$$

$$= 1{,}116.30 + 9.789.96 - \text{CGT}$$

$$= 10{,}906.26 - 0.4(11{,}000 - P_2)\,v^2$$

so $P_2 = (10{,}906.26 - 3{,}915.98) / (1 - 0.356)$

$$= £10{,}854.47$$

Price per £100 nominal = £108.54

(iii) The net effective yield after tax earned by the first investor is given by i, where

$$10{,}226.42 = 750a_{\overline{8}|}^{(2)} + 10{,}854.47v_i^8$$

For i = 8%, RHS = 10,258.60
i = 9%, RHS = 9,689.76

By interpolation, $i = 8 + \dfrac{10{,}258.60 - 10{,}226.42}{10{,}258.60 - 9{,}689.76}$

$= 8.057$

$= 8.06\%$

Generally very well answered, although when calculating the yield in part (iii) some candidates forgot that the first investor paid £102.26 per £100 nominal initially for the stock, not £100. In part (ii), several candidates calculated, without comment, a price per £100 nominal after allowing for capital gains tax which was greater than £110, the redemption proceeds.

15 (i) For immunisation, let L_{t_k} = liability at time t_k and A_{t_k} = asset at time t_k, then we require

$$\sum_k L_{t_k} v^{t_k} = \sum_k A_{t_k} v^{t_k}$$

and $$\frac{\Sigma\, t_k\, L_{t_k} v^{t_k+1}}{\Sigma\, L_{t_k}\, v^{t_k}} = \frac{\Sigma\, t_k\, A_{t_k}\, v^{t_k+1}}{\Sigma\, A_{t_k}\, v^{t_k}}$$

$\Sigma\, L_{t_k}\, v^{t_k} = 20{,}000v^{15} + 5{,}000\ {}_{10|}\ddot{a}^{(2)}_{\overline{25}|}$ at 7%

$= 7{,}248.92 + 5{,}000v^{10}\dfrac{i}{d^{(2)}}a_{\overline{25}|}$

$= 7{,}248.92 + 31{,}166.80$

$= 38{,}415.72$

$\Sigma\, A_{t_k}\, v^{t_k} = 25{,}000v^{10} + Yv^t$

$= 12{,}708.73 + Yv^t$

$\Rightarrow Yv^t = 38{,}415.72 - 12{,}708.73$

$= 25{,}706.99$

(ii) $\Sigma\, t_k\, L_{t_k}\, v^{t_k+1} = 15 \times 20{,}000v^{16} + \dfrac{5{,}000}{2}[10v^{11} + 10\tfrac{1}{2}v^{11\frac{1}{2}} + \ldots + 34\tfrac{1}{2}v^{35\frac{1}{2}}]$

$= 101{,}620.37 + \dfrac{5{,}000}{4}[20v^{11} + 21v^{11\frac{1}{2}} + 22v^{12} + \ldots + 69v^{35\frac{1}{2}}]$

$= 101{,}620.37 + 1{,}250\ [38\ {}_{11|}\ddot{a}^{(2)}_{\overline{25}|} + v_i^{11}(I\ddot{a})^{j}_{\overline{50}|}]$

where $j = 1.07^{½} - 1$

$${}_{11|}\ddot{a}^{(2)}_{\overline{25|}} = v^{11} \times \frac{i}{d^{(2)}} \times a_{\overline{25|}} = 0.475093 \times 1.052204 \times 11.6536$$

$$= 5.82557$$

$$v_i^{11}(I\ddot{a})^j_{50} = v_i^{11}\left[\frac{\ddot{a}_{50} - 50v^{50}}{d}\right]^j$$

$$= v_i^{11}\ \frac{24.5239 - 9.2125}{0.033264}$$

$$= 218.68493$$

$$\Rightarrow \quad \Sigma\, t_k\, L_{t_k}\ v^{t_k+1} = 101{,}620.37 + 1{,}250\ (38 \times 5.82557 + 218.68493)$$

$$= 651{,}691.10$$

$$\Rightarrow \quad t_y\, Yv^{t_y+1} = 651{,}691.10 - t_x\ Xv^{t_x+1}$$

$$= 651{,}691.10 - 10 \times 25{,}000v^{11}$$

$$= 532{,}917.90$$

so $\quad t_y = 532{,}917.90 / Yv^{t_y+1}$

$$= (532{,}917.90 / 25{,}706.99) \times 1.07$$

$$= 22.182 \text{ years}$$

Redemption proceeds for bond $Y = 25{,}706.99 \times 1.07^{22.182}$

$$= £115{,}303.41$$

(iii) Check that the convexity of the assets is greater than that of the liabilities so that any small change in the interest rate (either up or down) will lead to a positive surplus.

That is $\dfrac{\Sigma\, t_k(t_k+1)\, L_{t_k}\ v^{t_k+2}}{\Sigma\, L_{t_k}\ v^{t_k}} < \dfrac{\Sigma\, t_k(t_k+1)\, A_{t_k}\ v^{t_k+2}}{\Sigma\, A_{t_k}\ v^{t_k}}$

Part (i) was generally well answered. In part (ii) many candidates were unable to calculate the volatility or discounted mean term for the liabilities correctly. For both marks in part (iii) it was necessary to define what is meant by convexity.