Institute of Actuaries **Faculty of Actuaries**

EXAMINATIONS

September 1997

Subject A — Fundamentals of Actuarial Mathematics

Paper One

EXAMINERS' REPORT

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PART ONE

1 B

2 C

3 B

4 D

5 B

6 C

7 B

PART TWO

8 (i) Let A be the accumulated proceeds:

$$A = 100e^{\int_0^{10} d(t)dt} + 100e^{\int_5^{10} d(t)dt}$$

$$= 100\left\{ \exp\left[.05t + \frac{.001t^2}{2} + \frac{.0001t^3}{3} \right]_0^{10} + \exp\left[.05t + \frac{.001t^2}{2} + \frac{.0001t^3}{3} \right]_5^{10} \right\}$$

$$= 100\left\{ e^{.58333} + e^{0.58333-0.26667} \right\}$$

$$= 316.45$$

(ii) The equivalent constant force is δ where

$$316.45 = 100(e^{10d} + e^{5d})$$

$$\Rightarrow \quad e^{5d} = \frac{-100 \pm \sqrt{100^2 + 4 \times 316.45 \times 100}}{2 \times 100} \text{ and } e^{5d} > 0$$

so $e^{5d} = 1.34784 \quad \Rightarrow \quad \delta = .05970 \quad$ or 5.97%

9 Let d be the annual simple rate of discount required.

For 100 due in 3 months the deposit account requires an investment of

$$100v_{2\%}^{\frac{1}{2}} = 99.01475$$

The TB requires $100(1 - \frac{1}{4}d)$

If both provide the same effective rate of return, then

$$d = 4 \times (100 - 99.01475)\% = 3.941\%$$

10 TWRR is i where

$$1 + i = \frac{1.4}{1.2} \times \frac{1.8}{1.6} \quad \Rightarrow \quad i = 31.25\%$$

11 (i) Capital outstanding at $t = 5$ is $10{,}000 a^{(12)}_{\overline{15}|10\%} = 79{,}487$

$\Rightarrow$ interest due at $t = 5\tfrac{1}{12}$ is $79487 \times \dfrac{i^{(12)}}{12} = 633.84$

(ii) Total capital cost at $t = 0$ is $10{,}000 a^{(12)}_{\overline{20}|10\%} = 88{,}971$

$\Rightarrow$ total capital repaid during first 5 years is $88971 - 79487 = 9{,}484$

Total capital + interest paid is $5 \times 10000 = 50000$

$\Rightarrow$ total interest paid is 40,516

12 (i) $(1 + i_t) \sim \log N(m, s)$

$E[1 + i_t] = e^{m + s^2/2} = 1.05 \qquad \Rightarrow \qquad m + s^2/2 = 0.048790$

$V[1 + i_t] = (e^{m + s^2/2})^2 (e^{s^2} - 1) = (1.05)^2 (e^{s^2} - 1) = 0.1 1^2$

$\Rightarrow \quad e^{s^2} = 1.010975$

$\Rightarrow \quad s^2 = 0.010915$

$\Rightarrow \quad \mu = 0.04333, \quad \sigma = 0.104476$

(ii) $\Pr[1.04 < (1 + i_t) \le 1.07]$

$= \Phi\left[\dfrac{\log(1.07) - m}{s}\right] - \Phi\left[\dfrac{\log(1.04) - m}{s}\right]$

$= \Phi(0.233) - \Phi(-0.039)$

$= 0.59211 - (1 - 0.51555) = 0.10766$

PART THREE

13 *Many candidates tried to evaluate the part of the revenue component that was increasing at 3% pa using an $\bar{a}$ function at an adjusted rate of interest. This is incorrect as it assumes the increases are continuous rather than annual.*

(i) PV of outgo (£000s)

$$105(1 + v^{\frac{1}{2}} + v) + 200v^{15} = 366.31 \quad \text{at } 8\%$$

PV of income

$$\bar{a}_{\overline{1}|}\{20v + 23v^2 + 26v^3 + 29v^4$$

$$+ 29v^5 \,.\, 1.03(1 + (1.03v) + (1.03v)^2 + \ldots + (1.03v)^{24})\}$$

$$= \bar{a}_{\overline{1}|}\{80.193 + 20.329 \times 14.996\} = 370.61$$

So NPV is 4.30

(ii) Required to show there is no DPP in first 15 years, i.e. $\nexists\ t$ such that

$$105\left(1 + v^{\frac{1}{2}} + v\right) \le 80.193\bar{a}_{\overline{1}|} + 20.329 \times \left(\bar{a}_{\overline{1}|} + v(1.03)\bar{a}_{\overline{1}|} + \ldots + (1.03v)^k\, \bar{a}_{\overline{t-k}|}\right)$$

where $k = [t]$, and $t \le 10$

(Clearly the payback period cannot be within the first 5 years, as $80.193\bar{a}_{\overline{1}|} < 100(1 + v^{\frac{1}{2}} + v)$)

$$\Leftrightarrow \qquad 303.26 \le 77.185 + 20.329 \times \left(\bar{a}_{\overline{1}|} + \ldots + (1.03v)^k\, \bar{a}_{\overline{t-k}|}\right)$$

RHS is maximised when $t = 10$, when RHS

$$= 77.185 + 20.329 \times \bar{a}_{\overline{1}|} \times \left(\frac{1 - (1.03)^{10}v^{10}}{1 - (1.03)v}\right)$$

$$= 236.73$$

$\therefore$ the initial borrowing cannot be paid back from income in the first 15 years.

(iii) Since the NPV is very small, considerably less than the PV of the final year's income $\left(29 \times (1.03)^{25} \times \bar{a}_{\overline{1}|} \times v^{29} = 6.272\right)$, the DPP must fall in the final year. We know the DPP exists as the NPV > 0.

So DPP is 29 + r where

$$366.31 = \bar{a}_{\overline{1}|} \times \left\{ 80.193 + 20.329 \times \left(\frac{1-(1.03)^{24}v^{24}}{1-1.03v} \right) \right\}$$

$$+ 29 \times 1.03^{25} \times v^{29} \times \bar{a}_{\overline{r}|} \quad \text{at } 8\%$$

$\Rightarrow$ $366.31 = 364.335 + 6.5169\bar{a}_{\overline{r}|}$

$\Rightarrow$ $\bar{a}_{\overline{r}|} = 0.3031$

$\Rightarrow$ $v^r = 0.97668$ $\Rightarrow$ $r = 0.307$

So the DPP is 29.31.

14 (i) (a) Price is A where

$$A = 10.0 \times 0.75 \times a^{(2)}_{\overline{20}|} + 110v^{20} \quad \text{at } 10\%$$

$$= 7.5 \times 8.7214 + 110 \times 0.148644$$

$$= 81.761 \quad \text{or} \quad £81.76\%$$

(b) Volatility is ν where

$$\nu = \frac{\frac{7.5}{2}\left(\frac{1}{2}v^{1\frac{1}{2}} + v^2 + 1\frac{1}{2}v^{2\frac{1}{2}} + \ldots + 20v^{21}\right) + 20 \times 110 \times v^{21}}{81.761}$$

Let $X = \frac{1}{2}v^{1\frac{1}{2}} + v^2 + 1\frac{1}{2}v^{2\frac{1}{2}} + \ldots + 20v^{21}$

$$(1+i)^{\frac{1}{2}}X = \frac{1}{2}v + v^{1\frac{1}{2}} + 1\frac{1}{2}v^2 + 2v^{2\frac{1}{2}} + \ldots + 20v^{20\frac{1}{2}}$$

$$\left((1+i)^{\frac{1}{2}} - 1\right)X = \frac{1}{2}\left(v + v^{1\frac{1}{2}} + v^2 + \ldots + v^{20\frac{1}{2}}\right) - 20v^{21}$$

$\Rightarrow$ $$X = 2v.\left(\frac{\ddot{a}^{(2)}_{\overline{20}|} - 20v^{20}}{i^{(2)}}\right) = 114.998$$

$\Rightarrow$ $\nu = 8.910$

(ii) (a) $Y(v^{10} + v^t) = 81.761$ Assume w.l.o.g. assets of 1 bond

$t = 9.61 \Rightarrow Y = 104.063$

$$n_L = \frac{10Yv^{11} + tYv^{t+1}}{81.761}$$

$t = 9.61 \Rightarrow n_L = 8.910$ as required

(b) $$C_L = \frac{1}{81.761}\left\{Y \times 9.61 \times 10.61 \times v^{11.61} + Y10 \times 11 \times v^{12}\right\}$$

$= 87.526$

(c) Convexity measures the rate of change of duration with the rate of interest. In general payments which are more spread out will have higher convexity than single payments or (as here) two payments made very close together.

In this fund the two liability payments are made within ½ year of each other.

The assets are spread over the time period $t = ½$ to $t = 20$.
∴ expect assets to have higher convexity.

The conditions for immunisation are:

equal expected value;
equal duration or volatility;
convexity of assets > convexity of liabilities.

All are satisfied here, so the fund is immunised.

15 *Candidates who did well on this question approached the valuation methodically, showing all stages in their working. In particular, the nominal payments should be identified; the indexed payments should be found by applying the appropriate lagged index, they should then be devalued, allowing for the effects of inflation. The best marks were scored by candidates who clearly understood the principles of indexation and devaluation and their relationship.*

(i) (a) Let $Q(t)$ be RPI at t, where t is measured in half-years from the issue date.

Let $D(t)$ be the dividend due at t, and let R be the redemption money due at $t = 30$ half-years. Consider a bond of £100 nominal amount.

$D(1) = 2 \times 206/200 = 2.06$

$D(2) \quad = D(1) \times 1.07^{\frac{1}{2}}$

$\vdots$

$D(t) \quad = D(1) \times (1.07)^{\frac{t}{2}-\frac{1}{2}}$

$R \quad = 100 \times 1.03 \times (1.07)^{14\frac{1}{2}}$

So the real yield equation of value is, where A is the price

$$A = D(1) . \frac{Q(0)}{Q(1)} . v + D(2) . \frac{Q(0)}{Q(2)} v^2 + \ldots + D(30) \frac{Q(0)}{Q(30)} v^{30} + R \frac{Q(0)}{Q(3)} v^{30}$$

Now $Q(0) = 2.06 \times 1.07^{\frac{2}{12}}, \quad Q(t) = Q(0) \times 1.07^{\frac{t}{2}}$

so $\dfrac{Q(0)}{Q(t)} = (1.07)^{-\frac{t}{2}}$

so $D(t)\dfrac{Q(0)}{Q(t)} = D(1) \times (1.07)^{-\frac{1}{2}}$

and $R . \dfrac{Q(0)}{Q(30)} = 103 \times (1.07)^{-\frac{1}{2}}$

So the real yield equation of value is

$$A = D(1) \times (1.07)^{-\frac{1}{2}} \left(v + v^2 + \ldots + v^{30} \right) + 103 (1.07)^{-\frac{1}{2}} v^{30} \quad \text{at } i = 1½\%$$

$$= (1.07)^{-\frac{1}{2}} \left\{ D(1) a_{\overline{30|}} + 103 v^{30} \right\}$$

(b) $A = 0.96674 \times \{2.06 \times 24.0158 + 103 \times 0.63976\} = 111.53$

(ii) Let i = real yield. The $\text{eq}^{\underline{n}}$ of real values is now

$$A = D(1) \times (1.05)^{-\frac{1}{2}} \times a_{\overline{30|}i} + R \frac{Q(0)}{Q(30)} v_i^{30} - \text{Real PV of CGT}$$

CGT is due if $R > A \dfrac{Q(30)}{Q(0)}$ $\qquad R = 103 \times (1.05)^{14\frac{1}{2}}$

$$A \frac{Q(30)}{Q(0)} = 111.53 \times (1.05)^{15} > R$$

Hence real pv of capital gains tax = 0

so $\quad A = D(1) \times (1.05)^{-\frac{1}{2}} a_{\overline{30}|i} + 103 \times (1.05)^{-\frac{1}{2}} v_i^{30}$

The real yield must be greater than 1½%, as RHS at 1½%

$$= A \times \frac{1.07^{\frac{1}{2}}}{1.05^{\frac{1}{2}}} > A$$

At 2% RHS is 100.52 $= 112.59$

$A \;\Rightarrow\; i \;\approx\;$ 1.544% per ½ year

$=$ 3.09% p.a. convertible ½ yearly

(Actually 1.54115% per ½ year.)