EXAMINATIONS

September 1997

Subject C — Statistics

Paper Two

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Write your surname in full, the initials of your other names and your Candidate's Number on the front of the answer booklet.*

2. *Begin your answers to Parts One, Two and Three on a separate sheet.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 14 questions.*

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet and this question paper.

In addition to this paper you should have available Actuarial Tables and an electronic calculator.

PART ONE

For questions 1–6 indicate in your answer booklet which one of the answers A, B, C or D is correct.

1 The annual claim numbers associated with a particular risk are assumed to have independent Poisson distributions, each with unknown parameter λ. The parameter λ has a gamma prior distribution with mean 1 and standard deviation $1/\sqrt{2}$. A total of 5 claims have been made in the past 8 years. The Bayesian estimate of λ with respect to the quadratic loss function is:

A 0.6
B 0.625
C 0.7
D 0.725 [3]

2 Let X have a Poisson distribution with mean 10, and let Y be a related variable with conditional mean and variance given by

$$E(Y|X=x) = \frac{x}{2} + 5, \quad V(Y|X=x) = \frac{x}{4}.$$

The value of the unconditional variance of Y is:

A 2.5
B 5
C 7.5
D 10 [3]

3 A random sample of size 5 is taken from a normal distribution with mean μ (unknown) and variance $s^2 = 25$. The value of the sample mean is $\bar{x} = 36$.

A prior distribution for μ is available — it is normal with mean 40 and variance 4.

The Bayes estimate of μ is:

A 36.1
B 38.0
C 38.2
D 39.2 [3]

4 The random variable Y is defined as:

$$Y = \sum_{i=1}^{N} X_i$$

where the X_is are independent random variables, each with an exponential distribution with mean ϕ, and N is a non-negative integer-valued random variable, independent of the X_is, with probability generating function:

$$G(t) = p/(1 - qt)$$

where $0 < p < 1$ and $q = 1 - p$.

The moment generating function of Y is:

A $p(\phi - t)/(p\phi - t)$
B $(1 - qt)/(1 - qt - p\phi)$
C $\phi/(\phi - p - \phi qt)$
D $p(1 - \phi t)/(p - \phi t)$ [3]

5 A coin is selected at random from a pair of coins and tossed. Coin 1 is a double headed coin (i.e. a head on both sides). Coin 2 is a standard unbiased coin. The result of the toss is a Head. What is the probability that it was coin 1 which was tossed?

A 1/3
B 1/2
C 2/3
D 3/4 [2]

6 Claims occur on a portfolio of general insurance policies according to a Poisson process at rate λ per annum. The probability density function of individual claim amounts, X, is $f(x)$. The insurer arranges excess loss reinsurance with retention level M.

Let $l^* = \lambda P[X > M]$.

The variance of the aggregate annual claims paid by the reinsurer is:

A $l^* \int_M^\infty (x - M)^2 f(x)dx$

B $l^* \int_M^\infty (x - M)^2 \dfrac{f(x)}{P[X > M]} dx$

C $l \int_M^\infty (x - M)^2 \dfrac{f(x)}{P[X > M]} dx$

D $l \int_M^\infty (x^2 - M^2) f(x)dx$ [3]

PART TWO

7 A single observation, x, is drawn from a distribution with the probability density function:

$$f(x \mid \theta) = q^{-1}, \qquad 0 < x < \theta$$

$$f(x \mid \theta) = 0, \qquad \text{otherwise.}$$

The prior distribution of θ is given by:

$$g(\theta) = \theta \exp\{-\theta\}, \qquad \theta > 0$$

Derive an expression in terms of x for the Bayes estimator of θ with respect to the absolute error loss function. [4]

8 Consider the time series process $\{Y_t, t = 0, 1, 2, ...\}$ defined by:

$$Y_t - \tfrac{1}{3}Y_{t-1} = Z_t + \tfrac{2}{3}Z_{t-1}$$

where $Z_t \sim N(0, s^2)$ denotes white noise.

(i) Classify this process. [1]

(ii) Determine the autocorrelation function for the process. [4]

(iii) Now suppose Y_t denotes the first differences of another process W_t, so that $Y_t = W_t - W_{t-1}$. Write down the expression in terms of W_t and Z_t which defines the new process $\{W_t\}$ and classify this process.
[2]
[Total 7]

9 The prior distribution of the number of children, N, in a family is given by:

$$P[N = n] = 2^n e^{-2} / n!, \qquad n = 0, 1, 2, ...$$

The probability that a child is male is a known constant p and it is assumed that the sex of any child is independent of the sex of any other child in the family.

A family is known to contain exactly r male children.

(i) Show that the posterior distribution of $(N - r)$ is Poisson. [6]

(ii) Determine the posterior mean of N in terms of p and r. [1]
[Total 7]

10 A life insurer has a portfolio of life policies for a group of 30 independent lives. The probability of death next year for each life is 0.01. Ten of the lives have just one policy each, 15 of the lives have two policies each and five of the lives have three policies each.

Let N denote the total number of policies on which a claim is made next year.

(i) Derive the moment generating function of N. [4]

(ii) Calculate the mean and variance of N. [4]
[Total 8]

11 An autoregressive process of order 3 is given by

$$(1 - 0.3B)(1 - 0.4B)(1 - 0.5B)\,Y_t = Z_t$$

where

Z_t is white noise with variance s^2, and
B is the backward shift operator, so that $BY_s = Y_{s-1}$.

Calculate the first two autocorrelations, r_1 and r_2. [4]

PART THREE

12 The random variable S_i represents the aggregate claims in year i from an insurance portfolio. These claims are paid at the end of the year. $\{S_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables, each with a $N(1, 1)$ distribution. The annual premium, payable at the start of the year, is 1.2 and the insurer's initial surplus for this portfolio is 1. Interest at 5% per annum is earned on the surplus at the start of the year plus the premium for the year. The random variable U_n represents the insurer's surplus at time n, i.e. at the start of year $n + 1$, so that:

$$U_n = (U_{n-1} + 1.2) \times 1.05 - S_n \qquad n = 1, 2, \ldots$$

with $U_0 = 1$.

Let $y_1(U_0, n)$ denote the probability that U_m is negative for some m, where $m = 1, 2, \ldots, n$, given initial surplus U_0.

(i) Calculate $y_1(U_0, 1)$. [2]

(ii) Calculate $P[U_2 < 0]$. [3]

(iii) Explain why:

$$P[U_2 < 0] < y_1(U_0, 2).$$ [3]

(iv) Explain why:

$$y_1(U_0, 2) - y_1(U_0, 1) < y_1(U_0, 1)$$ [3]

(v) Explain carefully how to derive the following formula:

$$y_1(U_0, n+1) = y_1(U_0, 1) + \int_{-\infty}^{(U_0+1.2)\times 1.05} y_1((U_0 + 1.2) \times 1.05 - x, n) f(x) dx$$

for $n = 1, 2, \ldots$, where $f(x)$ denotes the probability density function of S_i.

[6]

[Total 17]

13 A motor insurer operates a no claims discount system which has four levels of discount: 0%, 20%, 40% and 60%. The full premium is £400. For each policyholder the distribution of the number of accidents in any year is:

P[no accidents] = 0.7
P[1 accident] = 0.2
P[2 accidents] = 0.1

The rules for moving between the discount levels are as follows:

(a) If no claim is made in a particular year, the policyholder moves in the next year to the next higher level of discount (or remains at the highest discount level).

(b) If one claim is made in a year, the policyholder remains at the same discount level next year.

(c) If two claims are made in a year, the policyholder moves in the next year to the 0% discount level.

It is assumed that the cost of an accident has an exponential distribution with mean £500.

When deciding whether or not to make a claim following an accident, a policyholder considers the consequent increase in premiums over the next two years and assumes he will have no more accidents. For example, following the second accident in a year the policyholder will consider the difference in future premiums between making one claim and making two claims, assuming a claim was made following the first accident, or between making no claim and making one claim, assuming a claim was not made following the first accident.

(i) For each discount level, calculate the probability that a policyholder will make a claim following the first accident in a year. [6]

(ii) For each discount level, calculate the probability that a policyholder will make a claim following the second accident in a year, given that a claim was made following the first accident. [4]

(iii) Derive the transition matrix for this no claims discount system. [8]
[Total 18]

14 (i) The random variable X has a lognormal distribution with parameters μ and s^2. Show that, for L, $d > 0$:

$$\int_d^{L+d} xf(x)dx = \exp\{m + \tfrac{1}{2}s^2\}\left[\Phi\left(\frac{\log(L+d) - m - s^2}{s}\right) - \Phi\left(\frac{\log(d) - m - s^2}{s}\right)\right]$$

[6]

(ii) Claims on a certain class of insurance policy are lognormally distributed with mean £2,000 and standard deviation £2,500. The insurance company arranges a special excess loss reinsurance treaty with a retention level of £5,000. Under this treaty, the maximum amount that the reinsurer will pay on any individual claim is £10,000.

(a) Calculate the mean claim amount paid by the reinsurer on claims which involve the reinsurer.

(b) Next year the claim amounts on these policies are expected to increase by 10% but the reinsurance treaty will remain unchanged. Calculate the mean claim amount to be paid next year by the reinsurer on claims which involve the reinsurer. [12]

[Total 18]