Institute of Actuaries **Faculty of Actuaries**

EXAMINATIONS

April 1997

Subject A — Fundamentals of Actuarial Mathematics

Paper One

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

J S R Ritchie
Chairman of the Board of Examiners

10 June 1997

PART ONE

1 C

2 C

3 B

4 C

5 D

6 B

7 B

8 D

PART TWO

9 $130 \times 12 \times a^{(12)}_{\overline{4}|i} = 5000$

$\Rightarrow$	$a^{(12)}_{\overline{4}|i} = 3.2051$	at	10%	$a^{(12)}_{\overline{4}|} = 3.3126$
			12%	3.2010
	$i \approx 11.9\%$	at	11.9%	$a^{(12)}_{\overline{4}|} = 3.2064$
$\Rightarrow$	$11.9\% < i < 12\%$	$\Rightarrow$	APR = 11.9%	

Comments: For full marks candidates must show that the APR lies between 11.9% and 12%, and state that the APR is rounded down to the lower 0.1%.

10 DPP is t s.t. $t \le 20$, $4t$ is an integer and t is the smallest such number satisfying $1800\, a^{(4)}_{\overline{t}|} \ge 10{,}000$

$\Rightarrow \quad a^{(4)}_{\overline{t}|} \ge 5.5556 \quad$ at $\quad 15\%$

$\Rightarrow \quad 1 - v^t \ge 0.79019 \quad \Rightarrow \quad v^t \le .209814$

$\Rightarrow \quad t \log v \ge \log .209814 \quad \Rightarrow \quad t \ge 11.17$

$\Rightarrow \quad t = 11.25$

11 Convert to amounts at time 0 values:

Loan = 20,000

Interest at $t = 1$: 1826.99
$t = 2$: 1736.36
$t = 3$: 1603.93

Capital repaid at $t = 3$ = 16039.28

$\Rightarrow$ Equation of Value is

$$20 = 1.827v + 1.736v^2 + 17.643v^3$$

Real rate of return ≈ 10%–7.6% as 10% is the money rate of return, and 7.6% is average inflation rate.

At 2% RHS is 20.085 $\Rightarrow$ $i \approx 2.16\%$ p.a.
2½% 19.818

12 (i) In 000s

MWRR is i s.t.

$$100(1+i)^3 + 20(1+i)^2 + 35(1+i) = 180$$

At 6.32% LHS is $100 \times 1.2018 + 20 \times 1.13039 + 35 \times 1.0632 = 180.00$ as required.

(ii) TWRR is i s.t.

$$(1+i)^3 = \frac{115}{100} \times \frac{155}{135} \times \frac{180}{190} \quad \Rightarrow \quad i = 7.75\%$$

(iii) MWRR is lower than TWRR because of the large cash flow on 1/7/95; the overall return in the final year is much lower than the first 2 years, and the payment at 1/7/95 gives this final year more weight in the MWRR, but does not affect the TWRR.

Having derived the requisite equation, candidates were required to SHOW that 1.0632 is a solution, not merely state that it is.

13 (i) Annual rate is X s.t.

$$4Xa^{(4)}_{\overline{20|}} = 20000 \quad \Rightarrow \quad X = 566.48$$

(ii) (a) Cap o/s at $t = 6$ is $4Xa^{(4)}_{\overline{14|}} = 2265.91 \times 7.637 = 17305.78$

(b) Interest part of 25th instalment is $\frac{i^{(4)}}{4} \times 17305.78 = 417.31$

Capital part is $X - 417.31 = 149.17$

14 Let g = rate of rental increase p.a. (vesting every 5 years)

The equation of value is (000s)

$$1700 + 270v^{\frac{1}{2}} = 100\bar{a}_{\overline{3|}} + 100(1+g)^5\, v^3\, \bar{a}_{\overline{5|}} + \ldots + 100(1+g)^{40}\, v^{38}\, \bar{a}_{\overline{5|}}$$

and irr = 10.0% p.a. $\Rightarrow$ where $v_{i'} = v_i^5(1+g)^5$

$$1700 + 270v^{\frac{1}{2}} = 100(2.6092) + 100\bar{a}_{\overline{5|}}\,(1+i)^2 \times (v_{i'} + v_{i'}^2 + \ldots + v_{i'}^8)$$

$\Rightarrow$ $1957.435 = 260.92 + 481.255 a_{\overline{8}|i'} \Rightarrow a_{\overline{8}|i'} = 3.5252$

$\Rightarrow$ $i' = .2293$

$\Rightarrow$ $g = 5.55\%$ as reqd.

Many candidates did not realise that the first rent increase would include 5 years' increases rather than 3 (as it occurs five years after the previous increase).

15 (i) Let S_n = accumulated value; let i_t = interest rate $t-1$ to t.

$$S_n = (1+i_1)(1+i_2)\ldots(1+i_n)$$

$$E[S_n] = E[(1+i_1)(1+i_2)\ldots(1+i_n)]$$

$$= E[(1+i_i)]\,E[(1+i_2)]\ldots E[(1+i_n)]$$ as yields i_t are independent

$$\therefore \quad E[S_n] = (1+j)^n \qquad \text{as } E[(1+i_t)] = (1+j) \text{ for all } t$$

$$E[S_n^2] = E[((1+i_1)(1+i_2)\ldots(1+i_n))^2]$$

$$= E[(1+i_1)^2]\,E[(1+i_2)^2]\ldots E[(1+i_n)^2]$$ as i_t independent

$$E[(1+i_t)^2] = E[1+2i_t+i_t^2]$$

$$= 1+2j+(j^2+s^2) \qquad \text{for all } t$$

$$\Rightarrow \quad E[S_n^2] = (1+2j+(j^2+s^2))^n$$

$$\Rightarrow \quad V[S_n] = (1+2j+(j^2+s^2))^n - (1+j)^{2n}$$

$$= ((1+j)^2+s^2)^n - (1+j)^{2n}$$

(ii) $S_n = (1+i_1)\ldots(1+i_n)$

$\log S_n = \sum_{k=1}^{n} \log(1+i_k)$ each variable $\log(1+i_k) \sim N(m, s^2)$

$$\Rightarrow \quad \sum_{1}^{n} \log(1+i_k) \sim N(n\mu, n\sigma^2)$$

(as i_k independent)

$$\Rightarrow \qquad S_{16} \sim \text{lognormal } (m' = 16 \times .08, \sigma = 4 \times .04)$$

$$\sim \text{lognormal } (1.28, .16)$$

$$\Pr[S_{16} > 4.25] = 1 - \Phi\left(\frac{1.4469 - 1.28}{0.16}\right) = 0.1486$$

In part (ii) candidates had to DERIVE the distribution of S_{16}*, not merely state what the distribution is.*

16 (i) $i^{(p)} = .06^{(4)} = .058695$

$$g(1 - t_1) = \frac{.08}{1.05} \times 0.6 = .045714$$

So $i^{(P)} > g(1 - t_1) \Rightarrow$ there is a capital gain on the contract.

The minimum value of the loan arises if the repayment is at the latest possible date. Hence we assume redemption occurs 20 years after issue.

If A is the price per £100 of loan:

$$A = 100 \times .08 \times 0.6 \times a_{\overline{20}|}^{(4)} + (105 - 0.3(105 - A))v^{20} \text{ at } 6\%$$

$$= 56.280 + 22.917 + 0.09354A$$

$$= 87.370 \quad \text{or} \quad £87,370 \text{ for the whole loan}$$

(ii) (a) The price is A' where

$$A' = 100 \times 0.08 \times 0.8 \times \ddot{a}_{\overline{14\frac{1}{4}}|}^{(4)} \times v^{\frac{2}{12}} + 105v^{14\frac{2}{12}}$$
$$- \text{PV of Capital Gains Tax (if any)}$$

$$= 61.607 + 45.993 - \text{PV of Capital Gains Tax}$$

$$= 107.80$$

Since $A' > 105$ there is no capital gains tax liability.

(b) $$A'' = 100 \times 0.08 \times .8 \times \ddot{a}_{\overline{19\frac{1}{4}}|}^{(4)} \times v^{\frac{2}{12}} + 105v^{19\frac{2}{12}} - \text{PV(CGT)} = 108.25$$

(iii) Investor should pay no more than A'. If investor pays A'' and bond is redeemed early, yield achieved will be less than 6% p.a.

Some candidates assumed that redemption was at the option of their purchaser, rather than the borrower.

17 (i) (a) Volatility $= -\dfrac{1}{P(i)}\dfrac{d}{di}P(i)$

where $P(i)$ is the PV of the cash flow

$$= +\frac{1}{\sum_{k=1}^{n} C_{t_k} v^{t_k}} \times \left\{ \sum_{k=1}^{n} C_{t_k}\ t_k\ v^{t_k+1} \right\}$$

(b) Convexity $= \dfrac{1}{P(i)}\dfrac{d^2}{di^2}P(i)$

$$= \frac{1}{\sum_{k=1}^{n} C_{t_k}\ v^{t_k}} \times \left\{ \sum_{k=1}^{n} C_{t_k}\ t_k\ (t_k+1)\ v^{t_k+2} \right\}$$

(ii) (a) Volatility is $\dfrac{.08 \times v \times (Ia)_{\overline{10|}} + 10v^{11}}{.08a_{\overline{10|}} + v^{10}} = \dfrac{v\ddot{a}_{\overline{10|}}}{1}$

$$= a_{\overline{10|}} = 6.7101$$

(b) Volatility is $= \dfrac{100{,}000 \times 7.247 \times v^{8.247}}{100{,}000 \times v^{7.247}} = 7.247v = 6.7102$

Convexity is $100{,}000 \times 7.247 \times 8.247 \times v^{9.247} / 100{,}000v^{7.247} = 51.240$

(c) Conditions:

(1) PV of assets = PV of liabilities: $V_L = 57{,}250$

Price of loan stock at 8% = 100%

Invest 57,250 in assets so that PV of assets = PV of liabilities.

(2) Volatility of assets = volatility of liabilities — shown in (ii)(a) and (b).

(3) Require convexity of asset > convexity of liabilities.

Convexity of assets = 60.53 > 51.24 = convexity of liabilities.

Hence liability is immunised.

(iii) PV of assets at 8½% = $57250 \times (.08a_{\overline{10}|} + v^{10})_{8\frac{1}{2}\%}$ = 55371

PV of liabilities at 8½% = 55366

PV of profit after ½% rise in interest rates = 5

Candidates who stated formulae which they were asked to derive could not be given full credit. Part (iii) was rarely answered correctly.