Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

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Certificate in Derivatives: Mathematics and Basic Principles

EXAMINERS' REPORT

1 (i) To construct a straddle, an investor buys a put and a call option on the same underlying asset with the same strike price and expiration date.

The maximum loss is the sum of the premiums paid.

The maximum profit if the stock price moves up is unlimited; if the stock price moves down there is an upper limit when the stock price falls to zero.

The cost to profit ratio is driven by the choice of strike price. Far out of the money options are cheaper than at the money options but a bigger move in the stock price is required to make a profit.

To construct a strangle, an investor buys a put and a call option on the same underlying asset with the same expiration date but with different strike prices.

The maximum loss is the sum of the premiums paid.

The maximum profit if the stock price moves up is unlimited; if the stock price moves down there is an upper limit on the profit — stock price cannot go below zero.

The cost to profit ratio is driven by how far apart the strike prices are. The further they are apart the lower the cost but the bigger the movement in the stock price required to profit from the strategy.

(ii) A strangle will be less expensive than a straddle.

However a much bigger move in the stock price will be needed to profit from a strangle compared with a straddle.

Option prices for stocks expected to exhibit extreme volatility tend to be higher than for options where the stock price is not expected to exhibit such volatility. So it is difficult enough to make a profit from such strategies.

2 (i) Let S be the current spot price in Australian dollars of the US dollar.
Let K be the delivery price agreed to in the forward contract.
Let r_f be the risk free rate in US dollars
Let r be the risk free rate in Australian dollars
Let f be the value of the forward contract

Consider portfolio A consisting of one long forward contract plus an amount of cash equal to $Ke^{-r(T-t)}$; and portfolio B consisting of an amount $e^{-r_f(T-t)}$ of the US dollar.

Both portfolios will become worth the same as one unit of the foreign currency at time T. To avoid arbitrage opportunities, they must therefore be equally valuable at time t. Hence

$$f + Ke^{-r(T-t)} = Se^{-r_f(T-t)} \qquad \text{(Equation 1)}$$

The forward price, F, (or forward exchange rate) is that value of K which makes $f = 0$.

Hence

$$F = K = Se^{(r-r_f)(T-t)}. \qquad \text{(Equation 2)}$$

(ii) When the foreign interest rate is greater than the domestic interest rate ($r_f > r$), then equation (2) shows that F is always less than S and that F decreases as T increases.

When the foreign interest rate is less than the domestic rate, then F is always greater than S and F increases with T.

When $r_f = r$, $F = S$ for all T.

3 (i) (a) The hedge ratio is defined as the ratio of the size of the position taken in a futures contract relative to the size of the underlying exposure.

(b) Let σ_S be the standard deviation of ΔS.

Let σ_F be the standard deviation of ΔF.

Let ρ be the coefficient of correlation between ΔS and ΔF.

Let h be the hedge ratio.

The variance of the hedged position, v, is given by $\mathrm{VAR}(\Delta S - h\Delta F)$

$$= \sigma_S^2 + h^2\sigma_F^2 - 2h\rho\sigma_S\sigma_F$$

$$\frac{\partial v}{\partial h} = 2h\sigma_F^2 - 2\rho\sigma_S\sigma_F$$

Setting $\dfrac{\partial v}{\partial h} = 0$ we have

$$h = \rho \frac{\sigma_S}{\sigma_F}$$

(ii) Since the company knows that it due to sell gold in the future, it can protect itself from fluctuations in the price at which it will be able to sell its gold by entering into a short futures position. This is known as a short hedge.

If the price of gold goes down, the company does not fare well from the sale of its gold but makes a gain on the short futures position.

If the price of gold goes up, the company makes a gain from the sale of its gold but makes a loss on its short futures position.

It is important to realise that futures hedging does not necessarily improve the financial outcome. In fact under many scenarios, the impact of the futures hedging can be to make the financial outcome worse. What futures hedging can do is make the financial outcome more certain.

It can be said that the assertion made by the colleague depends on how risk is defined. If risk is defined in terms of variability or certainty of outcome, then his assertion is theoretically possible.

For gold, the basis risk, defined as the difference between the spot price of gold being hedged and the futures price of the contract used, is relatively small. This is because efficiencies in the market eliminate arbitrage opportunities so that there is a well defined relationship between spot and futures prices. The main source of basis risk for gold will be due to uncertainties in risk free interest rates.

There are however a number of practical issues to address in order to determine whether the effective and whether the relative gains/ losses in the spot price of gold will be over or under compensated for by the relative gains/ losses in the futures position. Some of these issues include:

- The company may be uncertain as to the exact volumes of gold it expects to sell on exact dates in the future.

- The hedge may require the futures contract to be closed out well before its expiration date. The company's overall liquidity must be sufficient to cover margining payments during the life of the hedge.

- The company will need to ensure that it can find any margin cashflows during the hedging period.

The volume of futures contract that minimises risk is not straight forward to determine. The optimal size of the futures position relative to the size of the underlying gold portfolio (hedge ratio) depends on the correlation between the change in the price of the futures contract over the life of the hedge with the change in the price of the underlying gold, and the variabilities in these prices.

It is unlikely that the company in question will need to roll over the hedge, since it is primarily concerned with its results to the end of the current financial year. Longer term hedges may however need to be rolled over and the basis risk then increases.

The counter party risk for an exchange traded futures contract should not be significant.

In summary, hedging using futures can in theory dramatically reduce risk (when measured in terms of certainty of outcome), but it is difficult to completely eliminate risk. The administrative risks of managing a futures hedging portfolio should not be discounted.

4 (i)

Delta = –1,500(0.5) – 750(0.8) – 3,000(–0.4) – 750(0.7) = –675
Gamma = –1,500(2.2) – 750(0.6) – 3,000(1.3) – 750(1.8) = –9,000
Vega = –1,500(1.8) – 750(0.2) – 3,000(0.7) – 750(1.4) = –6,000

(ii) The gamma of the traded option is 1.5 so a long position in (+9,000 / 1.5) traded options would be required.

The delta of the portfolio will then be – 675 + 6,000(0.6) = 2,925.

So a short position in dollars of $2,925 will be needed to neutralise this.

(iii) The vega of the traded option is 0.8 so a long position in 7,500 traded options would be required.

The delta of the portfolio will then be – 675 + 7,500(0.6) = 3,825.

So a short position in dollars of $3,825 will be needed to neutralise this.

5 (i) (a) The writer of the option has no credit exposure to the purchaser of the option as she usually receives the option premium up-front.

If the premium is paid in instalments over time then the writer has a credit exposure to the purchaser.

By contrast, the purchaser of an option always has a credit exposure to the writer as she is dependent on him for receipt of any profit on the option.

(b) The party with the long position always has a potential, if not current, credit exposure to the party with the short position and vice versa as forwards can quickly move from being an asset on one's balance sheet to being a liability.

(ii) (a) Third party guarantees and letters of credit could be used to reduce the probability of default.

Limiting one's dealings to counterparties with a minimum credit rating, like A+, is another way of reducing the probability of default.

This however reduces the number of counterparties one can deal with.

(b) The underlying exposure can be reduced by daily marking to market (margining).

The party with credit risk exposure receives collateral (usually cash and/or government securities) from the other party.

(iii) In essence, the investment bank is holding a zero coupon bond as security for cash lent.

The investment bank has two credit exposures: one to the pension scheme (it may default on its obligations to buy the zero coupon bond back for cash plus interest) and one to the issuer of the zero coupon bond.

The credit risk of the investment bank will increase if the value of the zero coupon bond falls below the cash + interest to be collected from the pension scheme.

If the zero coupon bond defaults the pension scheme is still obliged to buy it back at the price agreed.

The exposure on the zero coupon bond becomes an issue if the pension scheme defaults at the same time as the zero coupon bond defaults.

The credit risk could be reduced by:

- Insisting on AAA rated government issuers for the zero coupon bond.
- Insist that that the value of the zero coupon bond always exceeds the value of the cash + interest by a minimum margin say, 2%.
- Insist on marking credit exposure to market when the value of the zero coupon bond falls below the value of the cash + interest + the margin.

6 (i)

(a) The settlement price for any given day is the average the prices at which the contract traded immediately before the end of trading.

(b) For a given contract, open interest at the end of a trading day is the sum of all the short positions open at an exchange. For each short position there is a long position so it could also be defined in terms of the number of long positions open in the contract at the end of a trading day.

(c) The volume of trading for any given day in a given contract is the number of futures contracts bought (and equally the number sold as for every contract bought there is one sold).

(ii) (a) The change in settlement price between one day and the next determines the movement in the investment bank's margin account with the exchange. A study of settlement price changes from one day to the next could help the bank in its assessment of the volatility of the futures price and hence in its risk management.

Open interest allows the investment bank to assess the size of its open position relative to the market. It ought to be concerned if it has speculative open positions approximately equal to that of the whole market.

Volume history is an indication of the level of trading activity and hence liquidity in the contract. This can be useful in deciding whether there is sufficient liquidity in the market to close out a large open position.

As part of its market risk and liquidity risk management program, the investment bank would probably want to monitor the number of contracts it has open at the exchange in the light of movements in open interest and volume for the contract in question.

(b) The volume of trading in a particular contract for a given day may exceed the change in open interest over the trading day if there were a very large number of "day trades" (new positions opened and closed out within the trading day).

(iii) (a) Volume for the day is two contracts. Open interest at the end of the trading day is zero as there are no contracts open at the end of the trading day. The key issue is that the same counterparty is used to open the contract and close it out.

(b) Volume for the day is two contracts. Open interest at the end of the trading day is one as there is one contract open at the exchange — the one between the exchange and A matched up with the one between the exchange and C.

(iv) The trader agreed to buy 1,000 (10 contracts × 100 ounces) ounces of gold for \$300 an ounce — \$300,000. Up to the point of delivery, the trader will have paid over \$50,000 (\$50 × 1,000) in margin payments to the party with the short position. So on delivery of the gold the trader just pays:

<u>The final futures price × number of contracts × 100</u>

which in this case is \$250,000.

7 (i) The Black-Scholes assumptions are:

1. The stock price follows a Brownian Motion process with drift μ, such that

$$dS = \mu S dt + \sigma S dZ$$

2. The short selling of securities with full use of proceeds is permitted

3. There are no transaction costs or taxes

4. All securities are perfectly divisible

5. There are no dividends during the life of the derivative

6. There are no riskless arbitrage opportunities

7. Security trading is continuous

8. The risk free rate of return is constant and the same for all maturities

(ii) (a) From Ito's lemma

$$df = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz \qquad (1)$$

(b) The equation

$$dS = \mu S dt + \sigma S dZ$$

$$\lim_{\Delta t \to 0} \Delta S = (\mu S \Delta t + \sigma S \Delta z) \qquad (2)$$

while the equation (1)

$$\lim_{\Delta t \to 0} \Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right)\Delta t + \frac{\partial f}{\partial S}\sigma S dz \qquad (3)$$

Note that $\Delta f = \lim_{\Delta t \to 0} f; \quad \Delta S = \lim_{\Delta t \to 0} S; \quad \Delta Z = \lim_{\Delta t \to 0} Z$

and the Brownian motion processes ΔZ in equations (2) and (3) are the same.

Now considering the portfolio Π consisting of:

-1 units of derivative

$\frac{\partial f}{\partial S}$ units of stock

The value of this portfolio is

$$V_\Pi = -f + \frac{\partial f}{\partial S} S \qquad (4)$$

and hence

$$\lim_{\Delta t \to 0} \Delta V_\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S\,. \qquad (5)$$

Substituting (2) and (3) into (5) gives

$$\lim_{\Delta t \to 0} \Delta V_\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right)\Delta t \qquad (6)$$

The equation (6) does not involve Z or ΔZ. This means that it is non-random and riskless over the short time interval Δt. Under the Black-Sholes assumptions (6,7,8), it follows that the portfolio will earn the risk free rate of return over the short time interval Δt (otherwise riskless arbitrage opportunities would exist). This means that

$$\lim_{\Delta t \to 0} \Delta V_\Pi = r V_\Pi \Delta t \qquad (7)$$

Substituting (4) into (7) gives

$$\lim_{\Delta t \to 0} \Delta V_\Pi = r\left(-f + \frac{\partial f}{\partial S} S \right)\Delta t \qquad (8)$$

Equating (8) and (6) and collecting like terms leads to the result:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{9}$$

(iii) Equation (9) above is highly significant.

Solving this differential equation gives a method for pricing derivatives. There are often many solutions to the equation. However the solution which is dependant on boundary conditions governed by the value of the derivative at a particular point in time (e.g. at maturity) leads to a unique valuation formula for all t.

The equation does not involve any variables that depend on the risk preferences of the investor. Since risk preferences do not enter into the equation, they also do not affect into the solution.

Therefore, any set of risk preferences can be used when valuing f. In particular, we can assume a risk neutral world where the expected return on all securities is the risk free rate.

This considerably simplifies the valuation of derivatives.

8 (i) Definition: **S** is a martingale with respect to measure P and filtration F if and only if

$$E_{\mathcal{P}}[S_j \mid \mathcal{F}_i] = S_i \text{ for all } i \le j$$

Explanation: This means that if we know the past history of the stock price until any time t (F_t), then the process is a martingale if (and only if) the expected future stock value is equal to its current stock price at time t (S_t).

The probabilities from the set P are used to calculate expected values for this purpose.

It also means that with this probability measure used for calculating expected values, the process has no drift and no up or down bias in its expected future value relative to its current value.

(ii) Definition: The process ϕ is previsible if (and only if) its value ϕ_t at any time t is dependent only on the history (or filtration) upto time $t - 1$.

Explanation: Relative to the stock price S_t at time t, it means that the previsible process is known one time unit in advance (since its value ϕ_t can be derived at time $t - 1$ once the filtration F_{t-1} has been determined).

(iii) The Binomial Representation Theorem says that if we are given:

- a process **S**
- a measure **Q** such that **S** is **Q** martingale
- any other **Q**-martingale process **N**

Then there exists a previsible process ϕ such that

$$N_i = N_0 + \sum_{k=1}^{i} \phi_k \Delta S_k$$

where:

- $\Delta S_i = S_i - S_{i-1}$ is the change in S from tick time $i-1$ to tick time i
- ϕ_i is the value of ϕ at the appropriate node at tick time i

The idea behind the application of the Binomial Representation Theorem involves establishing whether a probability measure **Q** can be found makes the stock price process **S** a martingale. If we can do this, then we can:

- Find some other process **E** which can be represented in terms of the stock prices S_t.

- Buy the appropriate quantities ϕ_i of the stock and follow the gains and losses of the martingale E_i.

To apply this we first calculate the conditional expectation of the claim X at each tick time t along the binomial tree. That is we compute

$$E_t = \mathbf{E}_{\mathbf{Q}}(X \mid F_t)$$

If **Q** is such that S_t is martingale, then E_t is martingale also.

The Binomial Representation Theorem tells us that there exists a previsible process ϕ such that

$$E_t = E_0 + \sum_{k=1}^{t} \phi_k \Delta S_k$$

This is extremely helpful since it tells us how much stock and how many units of cash to hold at each tick time t.

At each time t, we need to buy a portfolio π_t consisting of:

ϕ_{t+1} units of stock S

$\psi_{t+1} = E_t - \phi_{t+1} S_t$ units of cash

(We can do this because ϕ is previsible)

At time zero, this is worth

$$\phi_1 S_0 + \psi_1 = E_0 = E_{\mathbf{Q}}(X)$$

Moreover, if we held the portfolio π_0 at time 0, then one tick time later the value of the constituents would have changed to

$$\begin{aligned} &\phi_1 S_1 + \psi_1 \\ &= E_0 + \phi_1 (S_1 - S_0) \end{aligned}$$

The Binomial Representation Theorem also tells us that this is equal to E_1.

Thus our portfolio π_1 at time 1 is equal to the expected claim value at time 1, *whatever* happened to the stock value between time 0 and time 1.

More generally, the portfolio π_t at time t will be equal to the expected claim value at time t, and will eventually equal the value of the claim amount at the node of the tree, whatever path the stock values take to reach the end of the tree.