Faculty of Actuaries **Institute of Actuaries**

REPORT OF THE BOARD OF EXAMINERS ON THE EXAMINATIONS HELD IN

April 2002

Subject — Certificate In Derivatives: Mathematics and Basic Principles

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The examiners are mindful that a number of interpretations maybe drawn from the syllabus and Core Reading. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

The report does not attempt to offer a specimen solution for each question - that is, a solution that a well prepared candidate might have produced in the time allowed. For most questions substantially more detail is given than would normally be necessary to obtain a clear pass. There can also be valid alternatives which would gain equal marks.

K Forman

Chairman of the Board of Examiners

25 June 2002

QUESTION 1

(i)

The Cameron Martin Girsanov Theorem (CMG)

If W_t is a ***P***-Brownian motion and γ_t is an F-previsible process satisfying the boundedness condition $E_P\left[\exp\left(\frac{1}{2}\int_0^T \gamma_t^2 dt\right)\right] < \infty$,

then there exists a measure ***Q*** such that:

Q is equivalent to ***P*** (ie, events are impossible under ***Q*** if and only if they are impossible under ***P***)

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \gamma_t W_t - \frac{1}{2}\int_0^T \gamma_t^2 dt\right)$$

$\widetilde{W}_t = W_t + \int_0^t \gamma_s ds$ is ***Q***-Brownian motion

In other words, W_t is a drifting ***Q***-Brownian motion with drift $-\gamma_t$ at time t.

$\frac{dQ}{dP}$ is the Radon-Nikodym derivative which can be defined as the ratio that the joint probability density function of W over measure ***Q*** bears to the probability function of W over measure ***P***.

(ii)

Martingales

This means that all expected future values of M, given its history up to and value at time t, are equal to its current value at time t.

Martingale Representation Theorem

Suppose M_t is a ***Q***-Martingale process, whose volatility σ_t satisfies the additional condition that is (with probability one) always non-zero. Then if N_t is any other ***Q***-Martingale, there exists an F-previsible process ϕ such that $\int_0^T \phi^2\sigma^2 dt < \infty$ with probability one, and N_t can be written as

$$N_t = N_0 + \int_0^t \phi_s M_s$$

(iii)

Significance of Theorems to the Martingale Approach to Valuing Derivatives

The CMG and MRT are very important to the theory of valuing derivatives.

This is because derivatives can be valued by finding a portfolio of cash and stock which replicates the derivative. The value of the derivative must be equal to the value of the replicating portfolio otherwise arbitrage opportunities would exist.

The approach for finding the replicating portfolio involves finding a measure ***Q*** which makes the underlying stock price process S_t a Martingale. This means that all expected future values of S, given its history up to and value at time t, are equal to its current value at time t.

Converting a stochastic process into a Martingale involves finding the probability measure ***Q*** under which it is driftless. The CMG theorem tells us that this can be done. It gives us a very powerful tool for controlling the drift of most stochastic processes that are encountered in practice, and in particular, for making the process driftless.

Now if the process S_t is a Martingale, then the process $E_t = E_Q[X \mid F_t]$ which represents the expected value of the derivative X on S (conditional on the history up to time t), is also a Martingale. The MRT then leads us to the construction of the replicating portfolio, ie the appropriate volumes ϕ_t of stock to hold. Since E_t and S_t are both Martingales, the MRT tells us that there is a previsible process ϕ_t such that $dE_t = \phi_t dS_t$ under the probability measure ***Q***. This is important because if we are holding a volume ϕ_t of stock, then changes in the value of our stock portfolio will match changes in the derivative's expected value.

To complete the replication, the volume of cash needed is then $\psi_t = E_t - \phi_t S_t$. The portfolio consisting of ϕ_t units of stock and ψ_t units of cash will always have value equal to the value of the derivative since its current value is equal to the value of the derivative and changes in the value of the components of the portfolio exactly match changes in the value of the derivative.

(iv)

Risk Neutral Valuations

Risk neutral valuations are valuations in which the risk preferences of investors do not affect the valuation result.

It follows that any set of risk preferences can be used, including the assumption that investors are indifferent about risk so that the expected rate of return on all securities is the risk free rate, regardless of their volatility.

Thus, the valuation of a derivative under a risk neutral valuation typically involves:

- assuming that the expected return from the underlying asset is the risk free rate (i.e. replace the risk averse investor's expected return of μ with the risk free rate r)
- calculating the expected payoff from the derivative at maturity using the risk neutral assumptions
- discounting the expected payoff at the risk free rate.

Significance of CMG Theorem

By changing the probability measure such that the stock price process becomes driftless, one is effectively moving from a risk averse world (under which the expected return for risk is the drift of the process) to a risk neutral world, under which the process is driftless.

The CMG theorem is significant because it says that such translations are possible and moreover, the CMG change of measure involves changing only the drift of W. Volatility is unchanged.

QUESTION 2

(i)

Let F_k being the futures price at the end of day k. ($0<k<30$).

Consider the following strategy (*due to Cox, Ingersoll and Ross*):

Take a long futures position of e^{δ} at the end of day 0 (i.e. at the beginning of the contract)

Increase the position to $e^{2\delta}$ at the end of day 1

Increase the position to $e^{3\delta}$ at the end of day 2 … and so on.

By the beginning of day k, the investor has a long position of $e^{\delta k}$.

The profit (or loss) from the position on day k is $(F_k - F_{k-1})e^{\delta k}$.

Compounding this at the risk free rate until the end of day n leads to a value at the end of day n of $(F_k - F_{k-1})e^{\delta k}e^{(n-k)\delta} = (F_k - F_{k-1})e^{n\delta}$.

The value at the end of day n = 30 of the entire investment strategy is therefore

$$\sum_{k=1}^{n}(F_k - F_{k-1})e^{n\delta}$$
$$= [(F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1})]e^{\delta n}$$
$$= (F_n - F_0)e^{\delta n}$$

Because F_n is the same as the terminal price S_{30}, the terminal value of the investment strategy equals $(S_T - F_0)e^{n\delta}$. (T = day 30)

Thus an investment of F_0 in a risk free bond combined with the strategy given above yields

$$F_0e^{n\delta} + (S_T - F_0)e^{n\delta} = S_Te^{n\delta}$$

at time T.

No investment is needed for all the long futures positions described. In other words, an amount F_0 can be invested to give an amount $S_Te^{n\delta}$ at time T.

Now suppose that the forward price at the end of day 0 is G_0.

By investing G_0 in a riskless bond and taking a long forward position of $e^{n\delta}$ forward contracts, and amount $S_Te^{n\delta}$ is also guaranteed at time T.

Thus there are two investment strategies, one requiring an initial outlay of F_0, the other requiring an initial outlay of G_0, both of which yield $S_T e^{n\delta}$ at time T.

It follows that $F_0 = G_0$, otherwise arbitrage opportunities would exist.

(ii)

Positive correlation between interest rates and the asset would increase the value of the futures price …

because if S rises, a long futures position would generate settlement gain (margin payment), which can be invested at higher rates. Similar, if S decreases the opposite occurs.

… but forward price not affected.

Three other factors from this list:

- Transaction costs
- Liquidity
- Futures basis risk (supply/demand differences)
- Margin payment method on futures
- Taxes.

QUESTION 3

(i)

Bank A	240 x (1.06) = 254.4	
Bank B	240 x (1.065) = 255.6	
Bank C	purchase	240 x (1.0625) = 255
	short sale	238 x (1.0550) = 251.09

[*Marks were also awarded if the candidate assumes continuously compounded rates – with the solution adjusted accordingly.*]

(ii)

<u>Riskless rate</u>: There is no true riskless rate, it depends on the financing distribution of each market participant.

<u>Frictionless markets</u>: Frictionless markets require an ability to short stock. This is only available to a limited number of market participants and has significant cost.

<u>Zero-cost hedging</u>: Hedging is not zero cost since the stock appears to have a bid offer spread.

(iii)

There is a "smile" in the volatility for different strikes i.e. the volatility is not constant for every strike, generally trending either up or down from the ATM volatility …

… caused possibly by fear of sharp downward movements ("crash" effect), increasing the put volatilities at the expense of the calls

… or possibly by the convexity of the share, being riskier (hence more volatile) at lower prices than higher.

Generally, it is reckoned that put volatility rises as the strike moves further away from the current share price, and call volatility falls.

If ATM volatility is 25%, expect calls to be priced from 20% to 15%, and puts to be priced from 30% to 40%, the difference increasing as they go out of the money.

(iv) (a)

Try to make a wide bid-offer spread in the option prices.

Try to combine call and put options to create natural hedges, without using large positions of Quist.

Balance rehedging frequency against the transaction cost situation.

Don't go short of Quist - use sold Calls or bought Puts instead, where possible.

(b)

Use the model as a guide, not a rule.

The delta is a function of the various assumptions and we don't really know what these are.

Model will perform relatively poorly where a short position is taken since few (only a third) of the participants in the market can do this.

Preferably use B&S to give a result generally equivalent to that required.

QUESTION 4

(i)

A mathematical definition of process $W_n(t)$ is given by:

$W_n(0) = 0$ (defines the origin of the process)

$$W_n\left(\frac{i+1}{n}\right) = W_n\left(\frac{i}{n}\right) + \frac{X_i}{\sqrt{n}}$$

where

$i = 1, 2, 3 \ldots n$ is a positive integer

X_1, X_2, … X_n are independent identically distributed Bernouilli random variables given by X_i = +1 or -1 with equal probability (i.e. ½).

(ii)

The process W_t is ***P***-Brownian if and only if:

- W_t is continuous
- the value of W_t is distributed, under ***P***, as a normal random variable with mean zero and variance t
- the increment $W_{t+s} - W_s$ is distributed, under ***P***, as a normal random variable with mean zero and variance t
- the increment $W_{t+s} - W_s$ is independent of the filtration (path or history) F_s up to time s

[*The process is said to be "unshifted" if $W_0 = 0$.*]

(iii)

Let X_i be i.i.d. Bernouilli random variables as defined in (i)

Then:

$$W_n(0) = 0$$

$$W_n\left(\tfrac{1}{n}\right) = \frac{X_1}{\sqrt{n}}$$

$$W_n\left(\tfrac{2}{n}\right) = \frac{X_1}{\sqrt{n}} + \frac{X_2}{\sqrt{n}}$$

$$W_n\left(\tfrac{3}{n}\right) = \frac{X_1 + X_2 + X_3}{\sqrt{n}}$$

$$\vdots$$

$$W_n\left(\tfrac{nt}{n}\right) = \frac{X_1 + X_2 + X_3 + \cdots X_{nt}}{\sqrt{n}}$$

$$= \sqrt{t}\left(\frac{\sum_{i=1}^{nt} X_i}{\sqrt{nt}}\right)$$

As $n \to \infty$ this becomes a continuous function of t.

Furthermore, under the probability measure ***P***, the central limit theorem tells us that $\lim_{n \to \infty}\left(\frac{\sum_{i=1}^{nt} X_i}{\sqrt{nt}}\right)$ is distributed as an N(0,1) random variable.

It follows that under the same probability measure, the distribution of:

$$\lim_{n \to \infty} W_n(t) = \lim_{n \to \infty} W_n\left(\tfrac{nt}{n}\right) = \sqrt{t}\left(\frac{\sum_{i=1}^{nt} X_i}{\sqrt{nt}}\right) = \sqrt{\mathrm{t}}\ \mathrm{N}(0,1) = \mathrm{N}(0,t)\,.$$

Next consider the displacement $W_\mathrm{n}(t + s) - W_\mathrm{n}(s)$. This is equal to

$$\left(\frac{\sum_{i=1}^{n(t+s)} X_i}{\sqrt{n}}\right) - \left(\frac{\sum_{i=1}^{ns} X_i}{\sqrt{n}}\right)$$

$$= \left(\frac{\sum_{i=ns+1}^{nt+ns} X_i}{\sqrt{n}}\right)$$

which is independent of what the process did up to time s.

Furthermore, under probability measure $\boldsymbol{P}$, the increment follows a binomial distribution with mean zero and variance t.

By the Central Limit Theorem , $\lim_{n \to \infty}(W_n(t+s) - W_n(s))$ is N(0,t).

(iv)

Let W_t be a Brownian Motion process and let S_t be the stock price process.

Brownian motion wanders over time with an element of randomness which is desirable for modelling the uncertain behaviour of stock price changes.

The randomness underlying the Brownian Motion may however be too volatile (or not volatile enough) compared with the volatility of the relevant stock.

We can adjust for this by adding a volatility parameter σ such that

$$S_t = \sigma W_t$$

However Brownian motion wanders with zero mean, whereas the stock price of a company normally grows (or falls) at some rate, depending on the company's performance and the rate of inflation in the relevant industries.

We can adjust for this by adding a drift parameter μ such that

$$S_t = \sigma W_t + \mu t$$

If the stock price process followed a Brownian Motion with volatility σ and drift μ, then the stock price process would have a normal distribution with mean μt and variance $\sigma^2 t$ ...

... but this means that there is a non zero probability that the stock price can take on negative values, which in reality is impossible.

We can adjust our model and eliminate this problem by using exponentials; i.e. setting

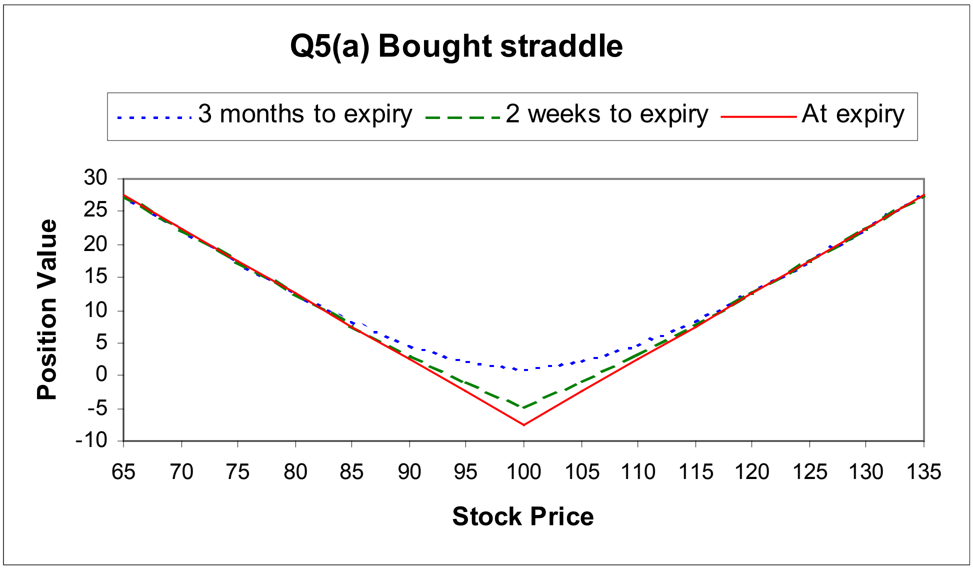
$$S_t = \exp(\sigma W_t + \mu t)$$

QUESTION 5

For each of (a), (b) and (c), two marks for correct expiry diagram and one for each correct intermediate line. Half marks for nearly accurate lines.

The intention of adding the intermediate lines is to see that the candidate can understand the concept of time premium

(a)



<u>Expiry</u>

- significant initial receipt (in example, shown as 7.5 but could be more or less)
- 45º lines, symmetric about strike
- equiv to bought 100 put + bought 100 call

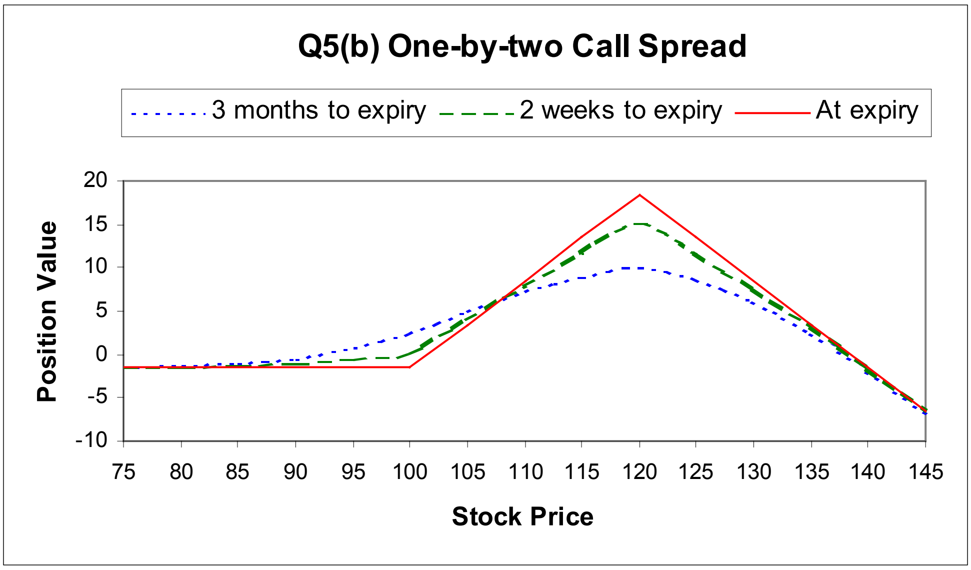
<u>2 weeks prior</u>

- dotted line is close to the expiry diagram except at the apex
- at 100 value should be positive less premium

<u>3 months prior</u>

- dotted line is further away from the expiry diagram
- at 100 value should be positive less premium

(b)



Expiry

- net premium can be positive or negative but is small
- symmetric 45º lines about 120
- unlimited loss as price increases
- flat line below 100 at the level of the net cost/receipt

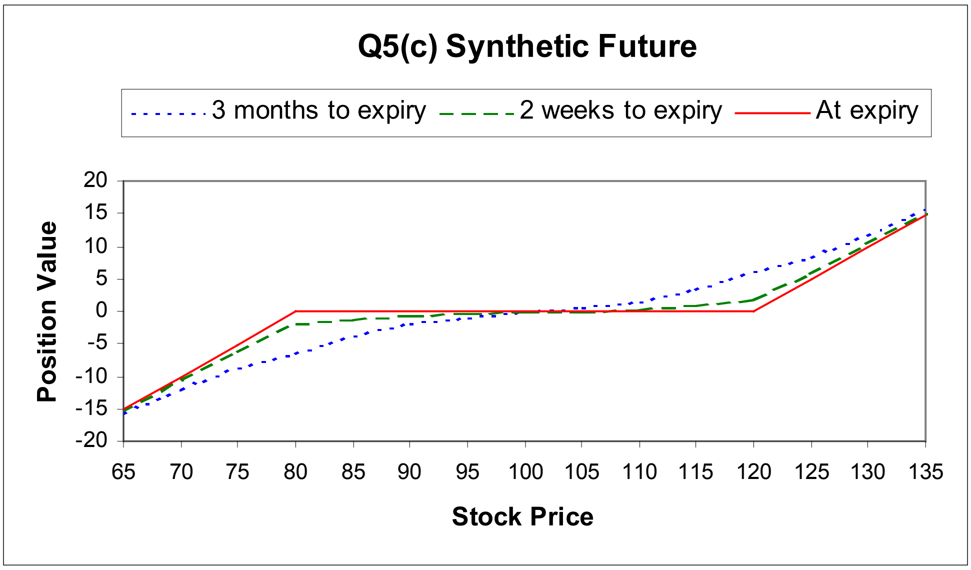
2 weeks prior

- similar dotted line will rise from 100 to @ 110 and fall above
- line is close to the expiry diagram below 100 and above 130

3 months prior

- similar dotted line will rise from 90 to @ 110 and fall above, as before
- always remain further away from the expiry line than the 2 week line

(c)



Expiry

- either slight net cost or net profit to undertake strategy (depends on volatilities)
- 45^o lines above 120 and below 80
- flat line between

2 weeks prior

- dotted line asymmetric to expiry line above 120 and below 80
- should pass very close to the expiry value at 100 which will be a turning point

3 months prior

- dotted line asymmetric to expiry line above 120 and below 80
- should pass further from the expiry value than the 2 week line, but 100 will still be a turning point

QUESTION 6

(i)

Calculate up probability p given by:

$$S = e^{-r\Delta t}\left[p \,.\, S_u + (1 - p) \,.\, S_d\right] \qquad (*)$$

so:

$$p = (e^{r\Delta t} - d) / (u - d) = 0.80067.$$

Then get values of the put at each node by discounting back through tree in the same method as equation (*) above, using the value of p just calculated, as follows:

		$V_{uu} = 0$
	$V_u = 0.4144$	
$V = 0.9608$		$V_{ud} = 2.1$
	$V_d = 3.204$	
		$V_{dd} = 7.8$

For example, $V_u = e^{-r\Delta t}\left[p \,.\, V_{uu} + (1 - p) \,.\, V_{ud}\right]$ etc.

Actually, it's quite simple to re-arrange the algebra to prove this. Also, there maybe slight differences due to rounding.

For the delta, look at the nodes for t = 1:

$$\Delta \approx (V_u - V_d) / (S_u - S_d) = (0.4144 - 3.204) / (82 - 76) = -0.4649, \text{ as before.}$$

(ii)

This takes much longer to write down!

First, we are given risk-free rate $r = 0.04$ and quarterly time step $\Delta t = 0.25$, and we will need the discount factor:

$$e^{-r\Delta t} = e^{-0.01} = 0.990050$$

Strike $K = 80$, and stock price S follows binomial tree with up factor $u = 1.025$ and down factor $d = 0.95$.

Next, prepare the stock price tree for $t = 0$, 1 and 2:

		$S_{uu} = 84.05$
	$S_u = 82$	
$S = 80$		$S_{ud} = 77.9$
	$S_d = 76$	
		$S_{dd} = 72.2$

and the equivalent put price tree:

		$V_{uu} = 0$
	V_u	
V		$V_{ud} = 2.1$
	V_d	
		$V_{dd} = 7.8$

where, for example, $V_{dd} = \max(0, K - S_{dd}) = 80 - 72.2 = 7.8$ etc.

"No arbitrage" implies setting up a portfolio (which will change from $t = 1$ to $t = 2$) of:

short one put option
long Δ (delta) amount of cash equity.

Start at $t = 1$, since we know the outcomes for $t = 2$, then work back to $t = 0$. All three stages here use the value of a riskless portfolio over a complete time step.

<u>Up step at $t = 1$</u>

$\Pi_u(2) = S_{uu} \,.\, \Delta_u - V_{uu} = S_{ud} \,.\, \Delta_u - V_{ud}$

⇨ $84.05\ \Delta_u - 0 = 77.9\ \Delta_u - 2.1$

⇨ $\Delta_u = -0.341463$

⇨ $\Pi_u(2) = -28.7$

But $\Pi_u(2)$ has a value independent of the path up or down, so must have earned the risk-free rate from $t = 1$ to $t = 2$.

Hence $\Pi_u(1) = \Pi_u(2)\, e^{-r\Delta t} = -28.7 \times 0.990050 = -28.4144$.

Thus $V_u = S_u \,.\, \Delta_u - \Pi_u(1) = 82 \times -0.341463 + 28.4144 = 0.4144$.

<u>Down step at $t = 1$</u>

Similarly, $\Delta_d = -1$ (100% delta because option is exercised at both nodes),

so $\Pi_d(2) = -80$ and $\Pi_d(1) = \Pi_d(2)\, e^{-r\Delta t} = -79.204$.

Hence $V_d = 76 \times -1 + 79.204 = 3.204$.

<u>Initial step at $t = 0$</u>

$\Pi(1) = S_u \,.\, \Delta - V_u = S_d \,.\, \Delta - V_d$

$\Rightarrow$ $82\,\Delta - 0.4144 = 76\,\Delta - 3.204$

$\Rightarrow$ $\Delta = -0.4649$

$\Rightarrow$ $\Pi(1) = -38.5362$

But $\Pi(1)$ has a value independent of the path up or down, so must have earned the risk-free rate from $t = 0$ to $t = 1$.

Hence $\Pi(0) = \Pi(1)\, e^{-r\Delta t} = -38.5362 \times 0.99005 = -38.1528$.

Thus $V = S \,.\, \Delta - \Pi(0) = 80 \times -0.4649 + 38.1528 = 0.9608$.

These values agree with those in part (i), subject to rounding differences.

The algebra above is for clarity – it is not required in the answer.

(iii)

To value American options, need to check whether exercise occurs at each intermediate node, as well as at the end, i.e. at $t = 0$ and $t = 1$ as well as $t = 2$.

Compare option value at that node with intrinsic value, and take higher.

For S = 80,

		$V_{uu} = 0$
	$V_u = 0.4144$	
$V = 1.1179$		$V_{ud} = 2.1$
	$V_d = 4$	
		$V_{dd} = 7.8$

since V_d = higher of $\{\max(0, 80 - 76), 3.204\} = 4$

so $V = e^{-r\Delta t}[p \,.\, V_u + (1 - p) \,.\, V_d]$

$= 0.99005\,(0.800067 \times 0.4144 + 0.19933 \times 4) = 1.1179$.

This is not exerciseable immediately, as it is above intrinsic (= 0).

QUESTION 7

(i)

(a) Locals

Locals are members of LIFFE who deal on behalf of themselves and other LIFFE members but not on behalf of clients. (*The new name for them is Non Public Order members.*)

They exist to provide speculative activity, which adds liquidity to that provided by market-makers, end-user client orders and hedgers.

(b) Options "Autoquote"

Options have many strikes and expiry dates, so trading prices do not always exist for every series. It is useful to "fill in the blanks" to try to generate more business by showing a price for each series.

Autoquote automatically calculates option prices based on underlying volatility quotes, updating as the underlying equity price moves.

The quotes are indicative only, but reflect business that the option market-makers wish to do.

(ii)

LCH was originally founded to clear trades for commodity trading, now clears for a variety of products, including LIFFE contracts.

It is owned by its (currently 114) members, by means of 'A' shares.

Members are mostly investment banks, brokerages houses and commodity producers. (*The LIFFE 1996 booklet says LCH is owned by the six clearing banks, but this changed in 1996.*)

Clearing means acting as a central counterparty for settlement, so that there is minimal systemic risk if one side defaults on their payments.

Only LCH members can settle with LCH as principals.

LCH handles all the margin payments for LIFFE contracts, and operates SPAN (see below).

Each member must maintain adequate reserves with LCH to cover their SPAN liabilities ...

... which is in the form of acceptable collateral, such as cash or Treasury bonds.

Other aspects of LCH risk management include counterparty risk assessment, price and position monitoring ...

... central banking, delivery and default management.

Other points of interest (not covered in the reading):

LCH is a not-for-profit organisation, not tied to any particular electronic trading system. Its capital is made up of member contributions to the Default Fund, and its income is volume related.

LCH also allows standardisation across a range of products - such as Swaps, Repo, Equities etc - which is more efficient for members involved in several diverse markets.

(iii)

Description of SPAN

The principle of SPAN is to set an ***initial margin*** (the returnable deposit in cash or collateral required by the LCH from its members when opening certain futures and options positions) which reflects the risk of the entire portfolio of derivatives held by a particular market participant or account.

SPAN generates the ***scanning loss***, which is the worst-case potential risk in a portfolio of derivatives (as calculated by the SPAN system) for the next day …

… by constructing a series of scenarios of changing underlying prices and volatilities for each derivative instrument in the portfolio, based on current market conditions.

SPAN then adds up all the scanning losses for the portfolio to give the initial margin.

(a) inter-commodity spread adjustment

Price movements tend to correlate fairly well between related underlying instruments.

As a result, gains from positions in one derivatives instrument will sometimes offset losses in another related instrument …

… SPAN will reduces the initial margin for these spread position according to their contribution to the reduction in risk.

(b) equity option adjustment

Purchase of option - the premium is paid upfront (different from an interest-rate option).

Sale of option - short option minimum charge created to allow for gamma movements in short option positions …

… SPAN margin is maximum of initial margin and short option charge.