# EXAMINATIONS

April 2003

## Certificate in Derivatives: Mathematics and Basic Principles

EXAMINERS' REPORT

**Introduction**

**The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The examiners are mindful that a number of interpretations may be drawn from the syllabus and Core Reading. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.**

**The report does not attempt to offer a specimen solution for each question — that is, a solution that a well prepared candidate might have produced in the time allowed. For most questions substantially more detail is given than would normally be necessary to obtain a clear pass. There can also be valid alternatives which would gain equal marks.**

**Mrs J Curtis**

**Chairman of the Board of Examiners**

**17 June 2003**

**QUESTION 1**
***Syllabus: 8.1***
***Reading: Hull Ch 4.5***

(i)

The forward interest rate for year the period ($m$, $n$) is the future reinvestment rate implied by the current zero coupon rates covering the start and end of the period.

Let $f_{m,n}$ be the forward rate for the period ($m$, $n$).

If 1 unit of cash is invested for a term of $m$ years in a zero coupon bond, then at time $m$ at the start of the time interval ($m$, $n$), this would have accumulated to $(1+z_m)^m$.

If the proceeds of this $m$ year zero coupon bond were reinvested over the time interval ($m$, $n$), then the proceeds would accumulate further by a factor $(1+f_{m,n})^{(n-m)}$.

Instead of reinvesting the proceeds of the $m$-year zero coupon bond for the interim period $n - m$, one could simply invest in an $n$-year bond, where 1 unit would accumulate to $(1+z_n)^n$.

The two procedures invest the same amount of money for the same period, so (in the absence of arbitrage) they must produce the same proceeds.

Therefore $f_{m,n}$ is the solution of:

$$(1+z_m)^m(1+f_{m,n})^{(n-m)} = (1+z_n)^n$$

(ii)

If the $n$-year zero-coupon rate is $z_n$ and the equivalent ($n$-year continuously compounded) force of interest is $\delta_n$, then:

$$e^{n\delta_n} = (1+z_n)^n$$

i.e. $\delta_n = \log(1+z_n)$ by taking logarithms and dividing by $n$.

Using the result in (i):

$$\begin{aligned}(1+f_{m,n})^{(n-m)} &= \frac{(1+z_n)^n}{(1+z_m)^m} \\ &= e^{n\log(1+z_n)} / e^{m\log(1+z_m)} \\ &= e^{n\log(1+z_n)-m\log(1+z_m)}\end{aligned}$$

Taking logarithms of both sides gives

$$(n-m)\log(1+f_{m,n}) = n\log(1+z_n) - m\log(1+z_m)$$
$$\Leftrightarrow (n-m)\delta_{f\,m,n} = n\delta_n - m\delta_m$$
$$\Leftrightarrow \delta_{f\,m,n} = (n\delta_n - m\delta_m)\big/(n-m)$$

where $\delta_{f\,m,n}$ is the force of interest in respect of the (continuously compounded) forward rate.

(iii)

[*Candidates may calculate either the annually compounded or continuously compounded rate]*

The forward interest rate for year 15 is given by the solution *f* of:

$$1.049^{14}(1+f) = 1.05^{15}$$
$$\Rightarrow f = 1.05^{15}\big/1.049^{14} - 1$$
$$= 6.41\%$$

[*Thus the continuously compounded forward interest rate is* $\log_e(1.0641) = 6.21\%$.]

<u>*Alternative solution*</u>

Use $\delta_{f\,m,n} = (n\delta_n - m\delta_m)\big/(n-m)$ =>

$$\delta_{f\,m,n} = (15\log(1.05) - 14\log(1.049))\big/(15-14) = 0.0621.$$

[*Thus the annually compounded forward rate is* $e^{0.0621} - 1 = 6.41\%.$]

<u>Comment</u>

The rate of interest in year 15 is considerably higher than the zero coupon rates for 14 or 15 years. That is because the rise from a 14-year average of 4.9% to a 15-year average of 5%, although small, has to be accounted for entirely by the increase in the final year.

Approximately, in annual terms the forward rate will be need to be higher than the 14-year zero rate by 15 × (5.0% – 4.9%) = 1.5%, which it is.

**QUESTION 2**

***Syllabus: 6.2, 6.4, 7.1.2***

***Reading: Baxter and Rennie Ch 3.3, Hull Ch 10.6, 11.5***

(i)

Ito's formula (or lemma) forms the basic extension of differential calculus to variables which are stochastic in nature.

It is used to derive stochastic differential equations (SDEs) for derivatives whose payoffs depend on the evolution of a stochastic process. These SDEs can then be solved analytically or numerically to give derivative prices and sensitivities.

(ii)

Firstly, note that the PDE's for the derivatives in (a) and (b) are identical, because they follow the same underlying process. Only the boundary conditions differ.

Let $\mathrm{d}s = \mu s + \sigma s \mathrm{d}z$

Define derivative as "$f(s,t)$". From Ito's lemma the stochastic process is:

$$\mathrm{d}f = \left(\frac{\partial f}{\partial s}\mu s + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2\right)\mathrm{d}t + \frac{\partial f}{\partial s}\sigma s \mathrm{d}z$$

Construct a portfolio $\pi$ consisting of one unit of derivative and $\alpha$ units of stock:

$$\pi = f + \alpha s$$

over a small time interval,

$$\Delta\pi = \Delta f + \alpha\,\Delta s$$

or, using the discrete versions of the stochastic process above

$$\Delta\pi = \Delta t\left\{\mu s\frac{df}{ds} + \alpha\mu s + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{d^2 f}{ds^2}\sigma^2 s^2\right\} + \Delta z\left\{\frac{\partial f}{\partial s}\sigma s + \alpha\sigma s\right\}$$

Thus, if $\alpha$ is chosen to be $\alpha = \dfrac{-\partial f}{\partial s}$

$$\Delta\pi = \Delta t\left\{\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2\right\}$$

Since this portfolio is riskless, it will earn the riskless rate of return, i.e. $\Delta\pi = r\pi\Delta t$

Thus,

$$r[f - \frac{\partial f}{\partial s} s] = \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial s^2} \sigma^2 s^2$$

i.e.

$$rf = \frac{\partial f}{\partial t} + rs \frac{\partial f}{\partial s} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 f}{\partial s^2}$$

<u>Boundary conditions for (a)</u>

$f \geq 0$ for $0 \leq t \leq T$
$f = \max(s - k\ 0)$ at $t = T$

<u>Boundary conditions for (b)</u>

$f \geq 0$ for $0 \leq t \leq T$
$f = \max(s^2 - k, 0)$ at $t = T$
$f \geq s^2 - k$ for $0 \leq t \leq T$ (American feature)

(iii)

In both cases the instantaneous stock hedge is $\partial f / \partial s$.

However, the solutions to the P.D.E. will be different in each case and therefore the amounts are different.

Whilst neither hedge will be constant with respect to stock price, the hedge in the squared pay out option will change more rapidly.

(iv)

Even though the volatility element of the process is now more complicated and depends on time $t$, it is still deterministic (i.e. not stochastic).

Therefore the same differential equation of value is valid, and so the answers in (ii) will be identical, except that $\sigma$ will be replaced by $\sigma_0 e^{-\beta t}$.

For the hedges, these depend on the solution to the P.D.E. which will be different because of the time dependent volatility factor.

**QUESTION 3**

***Syllabus: 6.1 to 6.3***

***Reading: Baxter & Rennie Ch 3 – for part (ii) see Exercise 3.9***

(i)

(a) The continuous process $W_t$ is **P**-Brownian Motion if and only if

1. $W_0 = 0$
2. Under probability measure **P**, $W_t \sim N(0,t)$.
3. The increment $W_{t+s} - W_s$ is distributed as a normal $N(0, t)$, under **P**, and is independent of the $\mathbf{F}_s$, the history of what the process did up to time s.

(b) The two measures P and Q are equivalent if and only if

1. They operate on the same sample space; and
2. They agree on what is possible (and what is impossible)

<u>Alternative answer</u>: If A is any event in the common sample space on which P and Q operate, then P and Q are equivalent if and only if $P(A) > 0 \Leftrightarrow Q(A) > 0$.

(ii)

$\tilde{W}_t$ is clearly continuous. We need to show that it satisfies the three necessary and sufficient conditions for a continuous process to be Q-Brownian Motion.

[i.e.
1. $\tilde{W}_0 = 0$.
2. Under probability measure **Q**, $\tilde{W}_t \sim N(0,t)$.
3. The increment $\tilde{W}_{t+s} - \tilde{W}_s$ is distributed as a normal $N(0, t)$, under **Q**, and is independent of the $\mathbf{F}_s$, the history of what the process did up to time $s$. ]

<u>First condition:</u>

$$\begin{aligned}\tilde{W}_0 &= W_0 + \gamma \times 0 \\ &= W_0 \\ &= 0\end{aligned}$$

<u>Second condition:</u>

The second condition will be met if the third condition is met in the special case with $s = 0$. Therefore we only need to show that the third condition is met for all $s$ (including $s = 0$).

<u>Third condition:</u>

$$(\tilde{W}_{t+s} - \tilde{W}_s) \sim N(0,t) \text{ under } \mathbf{Q} \text{ and independent of } F_s$$
$$\Leftrightarrow \text{ the moment generating function } \mathrm{E}_{\mathbf{Q}}\left[\exp\left(\theta(\tilde{W}_{t+s} - \tilde{W}_s)\right) | F_s\right] = \exp\left(\tfrac{1}{2}\sigma^2 \mathrm{t}\right)$$
$$\Leftrightarrow \mathrm{E}_{\mathbf{Q}}[\exp\left(\theta(W_{t+s} - W_s + \gamma(t+s) - \gamma t)\right) | F_s] = \exp\left(\tfrac{1}{2}\sigma^2 \mathrm{t}\right)$$
$$\Leftrightarrow \frac{1}{\varsigma_s}\mathrm{E}_{\mathbf{P}}\left[\varsigma_{t+s}\exp\left(\theta(W_{t+s} - W_s + \gamma t)\right) | F_s\right] = \exp\left(\tfrac{1}{2}\sigma^2 \mathrm{t}\right) \text{ ......................} \quad (*1)$$

where the last line follows from the Radon-Nikodym result for equivalent measures **P** and **Q**.

Now, since $(W_T - W_t) \sim N(0, T-t)$ under **P**, and is independent of $F_t$
it follows that

$$\begin{aligned}
\varsigma_t &= E_P\left[\frac{dQ}{dP} | F_t\right] \\
&= E_P\left[\exp\left(-\gamma W_T - \tfrac{1}{2}\gamma^2 T\right) | F_t\right] \\
&= E_P\left[\exp\left(-\gamma(W_T - W_t)\right)\exp\left(-\gamma W_t - \tfrac{1}{2}\gamma^2 T\right) | F_t\right] \\
&= \exp\left(-\gamma W_t - \tfrac{1}{2}\gamma^2 T\right)\ \exp\left(\text{-}\gamma(\mathrm{W_T} - W_t)\right) \\
&= \exp\left(-\gamma W_t - \tfrac{1}{2}\gamma^2 t\right)
\end{aligned}$$

So the left hand side of (*1) equals

$$\exp\left(\theta\gamma t - \tfrac{1}{2}\gamma^2 t\right) E_P\left[\exp\left((\theta - \gamma)(W_{t+s} - W_s)\right) | F_s\right] \text{ ……… ………} \quad (*2)$$

Again, since $W_{t+s} - W_s$ is $N$(0, t), and independent of $F_s$ …
… the expectation factor in (*2) is equal to $\exp\left(\tfrac{1}{2}(\theta - \gamma)^2 t\right)$.

Thus the left hand side of (*1) becomes:

$$\exp\left(\theta\gamma t - \tfrac{1}{2}\gamma^2 t\right)\exp\left(\tfrac{1}{2}(\theta - \gamma)^2 t\right) = \exp\left(\tfrac{1}{2}\theta^2 t\right) = RHS \text{ of (*1). as required.}$$

(iii) <u>Significance</u>

The effect of changing the probability measure from **P** to **Q** is to introduce a constant drift into the process, i.e. $W_t$ has no drift under **P** but a constant drift of -γ under **Q**.

*[A more general result than this, for an* **F***-previsible process* $\gamma_t$*, is in fact the basis of the Cameron-Martin-Girsanov theorem.]*

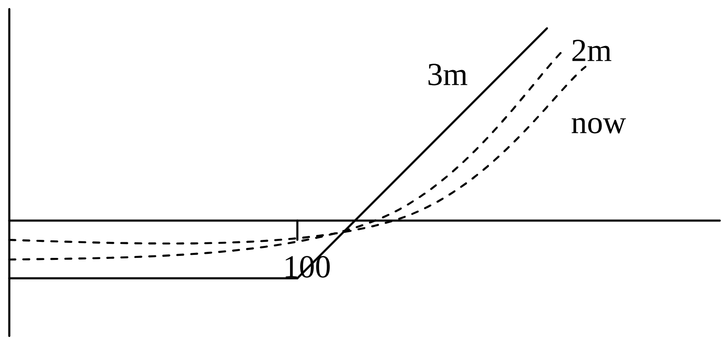
**QUESTION 4** ***N.B. Requires Page View for this question***

[*These diagrams are for illustration only. They will be replaced with accurate ones in due course. At present, they would not in themselves attract full marks!*]

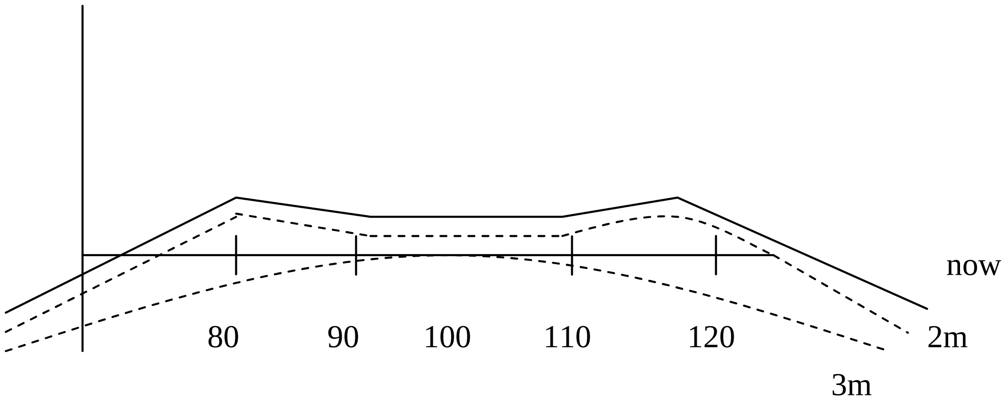
***Syllabus: 1.1, 1.4, 7.4***
***Reading: Hull Ch 8, 13.6***

(i)

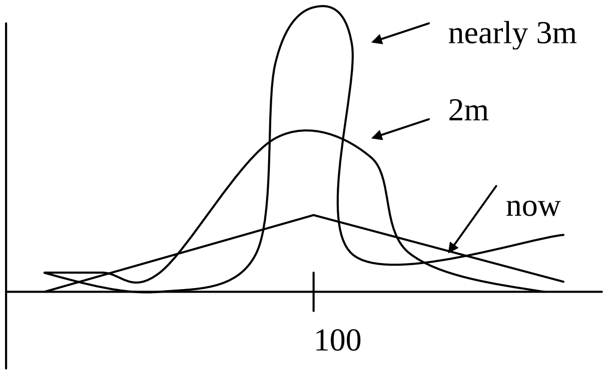
(a) Buy 1 European at-the-money call option




(b) Sell 2 European put options strike 80
Sell 2 European call options strike 120
Buy 1 European call option strike 110
Buy 1 European put option strike 90



(ii) Gamma for option (i) (a)



**QUESTION 5**
***Syllabus: 7.4***
***Reading: Hull Ch 13.4***

(i)

(a) [*One of these three definitions will suffice*]

The delta $\Delta$ of a derivative is defined as the rate of change of the value of the derivative with respect to the price of the underlying security.

***OR***

If we consider the value of the option as a function of the underlying security $V_A(S)$, then $\Delta$ is the slope of the tangent to this function evaluated at the point S.

***OR***

In algebraic terms:

$$\Delta = \frac{\partial V_A}{\partial S}(S_0)$$

(b) A delta hedge of an option position is an equal and opposite position in the underlying security such that the net instantaneous sensitivity to movements in the price of the underlying security is nil.

(ii)

(a) <u>Option value</u>

The value of Put option is given by Black-Scholes:

$$V_P = Xe^{-rT} N(-d_2) - SN(-d_1)$$

where

$$d_1 = \frac{\ln(S/X) + (r + \tfrac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

When $S$ = 100, $X$ = 93, $r$ = 0.05, $\sigma$ = 0.20, $T$ = 3

$$d_1 = 0.81571$$
$$d_2 = 0.46930$$
$$N(-d_1) = 0.20733$$
$$N(-d_2) = 0.31943$$

and hence $V_P$ = 4.83556 per option, per share.

On 2,000 shares, the option is worth:

$$2{,}000 \times 4.83556 = 9{,}671.1$$

Delta

Since $\Delta$ is the coefficient of the $S$ term in Black-Scholes, hence

$$\Delta = -N(-d_1) = -0.20733$$

(b) Delta Hedge

As $\Delta$ is negative, a short position in the option means the overall effect of the position is long, so it should be hedged with a short position in the underlying stock.

This involves selling 0.20733 shares for each option on one share, so on 2,000 shares it involves selling 414.665 shares (which can be rounded to 415 shares).

These currently are worth –100*414.665 = -41,466.5 (i.e. a short position).

The short position in Put options and the short position in shares is matched on the balance sheet by a long position in risk free 3-year zero coupon bonds. This requires an investment in:

$Xe^{-rT}N(-d_2) = 25.56882$ of zero coupon bonds per option per share.

With 2,000 shares, this means taking a long position in:

100 x 20 x 25.56882 = 51,137.6 of 3-year zero coupon bonds.

*Summary (not required in the solution):*

The <u>delta neutral balance sheet</u> would therefore look like:

| **Assets** | **Liabilities** |
|---|---|
| Risk Free 3 Year Zero Coupon Bonds<br>51,137.6 | Short Put Options<br>9,671.1 |
| Short Position in Shares<br>-41,466.5 | |
| Sub-Total Assets<br>9,671.1 | Sub-Total Liabilities<br>9,671.1 |

[*Note: when* $\Delta$ *is negative, a long position in the option would have been hedged with a long position in the shares; but this is not relevant to the investor in question.*]

(c) <u>Change in delta</u>

<u>Underlying price increases</u>: The delta of a Put option increases (from a negative value). When the price of the underlying is zero, the delta is –1. As the price of the underlying increases indefinitely, the delta asymptotically approaches zero.

<u>Term decreases</u>: The magnitude of the Put option delta varies as time progresses and the time to expiration of the option reduces, whilst always remaining negative. When the option is at or out of the money, the magnitude of the delta will tend to increase as the option expires. If the option is deeply in the money, then the magnitude of the delta can decrease as time progresses, although close to maturity it may still increase. (A plot of magnitude of delta against time may have a turning point.)

(iii)

(a) The futures price is given by

$$F = Se^{rT^*}$$

When

$S = 100$, $r = 5\%$, $T^* = 3/12 = 0.25$

then $F = 101.2578$

(b) When the stock price changes by $\Delta S$ , the futures price increases by $\Delta S e^{rT^*}$ .

The delta of the futures contract is therefore $\dfrac{dV}{dS} = \lim_{\Delta S \to 0} \dfrac{\Delta S e^{rT^*}}{\Delta S} = e^{rT^*}$.

Thus $e^{rT^*}$ futures contracts (on 1 share) have the same sensitivity to small stock price movements as one share of the stock. So if $H_A$ is the volume of shares needed to delta hedge the option position, then $H_F = H_A e^{rT^*}$ is the volume of futures contracts needed to do the same job.

(c) As $e^{rT^*}$ = 1.012578 and as hedging the option position involves shorting 414.665 shares worth 41,466.5, this can also be achieved by shorting:

$$1.012578 \times 41{,}466.5 = 41{,}988.1$$

worth of futures contracts (assuming the contracts are divisible).

As each contract is on 100 shares (worth £100 per share, or £10,000 per contract), the delta hedge could be achieved by shorting 4.2 futures contracts.

This hedge will not be very precise, as you cannot short less than 1 contract.

[Equivalently $e^{rT^*}$ = 1.012578 and as hedging the option position involves shorting 414.665 shares, we can also short 1.012578 × 414.665 = 419.88 shares (3 months future) = 4.2 futures contracts.

**QUESTION 6**
***Syllabus: 8.3***
***Reading: Hull Ch 4.3 to 4.5, 5.3***

(i)

Discount factors – use $d_n = (1 + z_n)^{-n}$ to 5dp:

| | |
|---|---|
| $d_1$ | 0.96852 |
| $d_2$ | 0.92991 |
| $d_3$ | 0.88772 |
| $d_4$ | 0.84663 |
| $d_5$ | 0.80824 |

(ii)

Useful to calculate the sum of the discount factors i.e. $s_n = d_1 + d_2 + ... + d_n$:

| | |
|---|---|
| $s_1$ | 0.96852 |
| $s_2$ | 1.89844 |
| $s_3$ | 2.78615 |
| $s_4$ | 3.63279 |
| $s_5$ | 4.44102 |

Then bond prices $b_n = 6 \,.\, s_n + 100\, d_n$ to 2dp:

| | |
|---|---|
| $b_1$ | 102.66 |
| $b_2$ | 104.38 |
| $b_3$ | 105.49 |
| $b_4$ | 106.46 |
| $b_5$ | 107.47 |

The yields from these bonds will be slightly below the zero coupon (spot) yields, since the curve is upward sloping, except in year 1 when they will be the same. The reason is that the income from the coupon bond will be reinvested roughly at the average yield up to that maturity, which is lower than the spot yield.

(iii)

Let $f_{\text{n}}$ be the floating coupon rates, and $X$ be the fixed coupon.

Then the floating side of a par swap is valued at $F = f_1 + f_2 + f_3 + f_4 + f_5$. It would be tedious to calculate the floating coupons. However, if we add 100 $d_5$ to $F$, the resulting cash flows are a (zero margin) floating rate note, i.e. present value 100.

Hence $F = 100 \,.\, (1 - d_5)$.

The fixed side is valued at $X \,.\, s_5$, and since the fixed and floating sides have the same value, $X = 100 \,.\, (1 - d_5) \,/\, s_5 = 4.3180$, i.e. the 5-year swap rate fixed side is 4.318%.

The formula demonstrated above for $X$ is the same as that for a 5-year par bond coupon (i.e. $100 = X \,.\, s_5 + 100\, d_5$).

**QUESTION 7**
***Syllabus: 5.3***
***Reading: Baxter and Rennie Ch 2, Hull Ch 9 (Equation 9.2)***

(i)

Let $x$ = quantity of stock and $y$ = quantity of par bond.

At time 0, the portfolio is worth $xs_0 + y$.

At time 1, the portfolio is worth $\begin{cases} xs_1 + ye^{r\delta t} & \text{if the stock jumped up} \\ xs_2 + ye^{r\delta t} & \text{if the stock jumped down} \end{cases}$

If this portfolio of stock and bonds replicates the derivative, then we must have

$$\pi_1 = xs_1 + ye^{r\delta t} \qquad (1)$$
$$\pi_2 = xs_2 + ye^{r\delta t} \qquad (2)$$

These two equations can be solved to give

$$x = \frac{\pi_1 - \pi_2}{s_1 - s_2}$$
$$y = e^{-r\delta t}\left(\pi_1 - \frac{(\pi_1 - \pi_2)s_1}{s_1 - s_2}\right)$$

(ii)

Firstly, $MV_\Pi$ is arbitrage free

$$\Leftrightarrow MV_\Pi \text{ equals the value of the replicating assets}$$
$$\Leftrightarrow MV_\Pi = \left(\frac{\pi_1 - \pi_2}{s_1 - s_2}\right)s_0 + e^{-r\delta t}\left(\pi_1 - \frac{(\pi_1 - \pi_2)s_1}{s_1 - s_2}\right)$$
$$\Leftrightarrow MV_\Pi = e^{-r\delta t}\left(\frac{\pi_1 - \pi_2}{s_1 - s_2}\right)s_0 e^{r\delta t} + e^{-r\delta t}\left(\pi_1 - \frac{(\pi_1 - \pi_2)s_1}{s_1 - s_2}\right)$$
$$\Leftrightarrow MV_\Pi = e^{-r\delta t}\left[\left(\frac{\pi_1 - \pi_2}{s_1 - s_2}\right)s_0 e^{r\delta t} + \left(\pi_1 - \frac{(\pi_1 - \pi_2)s_1}{s_1 - s_2}\right)\right]$$
$$\Leftrightarrow MV_\Pi = e^{-r\delta t}\left[\left(\frac{\pi_1 - \pi_2}{s_1 - s_2}\right)s_0 e^{r\delta t} + \left(\frac{\pi_1(s_1 - s_2)}{s_1 - s_2} - \frac{(\pi_1 - \pi_2)s_1}{s_1 - s_2}\right)\right]$$
$$\Leftrightarrow MV_\Pi = e^{-r\delta t}\left[\left(\frac{\pi_1 s_0 e^{r\delta t} - \pi_2 s_0 e^{r\delta t} + \pi_1 s_1 - \pi_1 s_2 - \pi_1 s_1 + \pi_2 s_1)}{s_1 - s_2}\right)\right]$$

$$\Leftrightarrow MV_{\Pi} = e^{-r\delta t}\left[\left(\frac{\pi_1 s_0 e^{r\delta t} - \pi_2 s_0 e^{r\delta t} - \pi_1 s_2 + \pi_2 s_1)}{s_1 - s_2}\right)\right]$$

$$\Leftrightarrow MV_{\Pi} = e^{-r\delta t}\left[\left(\frac{\pi_1 s_0 e^{r\delta t} - \pi_2 s_0 e^{r\delta t} + \pi_2 s_2 - \pi_2 s_2 - \pi_1 s_2 + \pi_2 s_1)}{s_1 - s_2}\right)\right]$$

$$\Leftrightarrow MV_{\Pi} = e^{-r\delta t}\left[\pi_1\left(\frac{s_0 e^{r\delta t} - s_2}{s_1 - s_2}\right) + \pi_2\left(\frac{(s_1 - s_2) - (s_0 e^{r\delta t} - s_2)}{s_1 - s_2}\right)\right]$$

$$\Leftrightarrow MV_{\Pi} = e^{-r\delta t}[\pi_1 q_1 + \pi_2(1 - q_1)]$$

where $q_1 = \dfrac{s_0 e^{r\delta t} - s_2}{s_1 - s_2}$ is thus a probability measure.

Thus far we have shown that the market value is arbitrage free <u>if and only if</u> the market value equals $e^{-r\delta t}\mathrm{E}_{\mathbf{Q}}[\Pi(S, \delta t)]$.

Secondly,

$$q_1 = \frac{s_0 e^{r\delta t} - s_2}{s_1 - s_2}$$
$$\Leftrightarrow q_1(s_1 - s_2) = s_0 e^{r\delta t} - s_2$$
$$\Leftrightarrow q_1 s_1 - q_1 s_2 = s_0 e^{r\delta t} - s_2$$
$$\Leftrightarrow q_1 s_1 + (1 - q_1)s_2 = s_0 e^{r\delta t}$$
$$\Leftrightarrow s_0 = e^{-r\delta t}[q_1 s_1 + (1 - q_1)s_2]$$
$$\Leftrightarrow s_0 = e^{-r\delta t}\mathrm{E}_{\mathbf{Q}}[S(\delta t)]$$

Thus $MV_{\Pi}$ is arbitrage free

<u>if and only if</u> $MV_{\Pi} = \mathrm{E}_{\mathbf{Q}}[\Pi(S, \delta t)]e^{-r\delta t}$

<u>if and only if</u> $s_0 = e^{-r\delta t}\mathrm{E}_{\mathbf{Q}}[S(\delta t)]$. QED.

(iii)

<u>Comment on significance</u>

The probability measure **Q** is said to be a risk neutral probability measure since the arbitrage free market value equals the expected value of the discounted derivative cashflows when expected values are calculated using the measure **Q** and when discounting is at the risk free rate.

The process S is said to be a martingale with respect to **Q** since its current market value equals its expected value (under measure **Q**) discounted at the risk free rate.

The above results show that, under the binomial model, the probability measure **Q** is risk neutral if and only if the stock price process is a martingale under measure **Q**.

(iv)

<u>Trading strategy</u>

The derivative is above the replicating portfolio price.

The trader should thus sell the derivative and invest in the replicating portfolio ...
… and hence take the residual value as arbitrage profit.

At time $\delta t$ use the proceeds from the replicating portfolio to meet the obligations under the derivative.