EXAMINATIONS

April 2000

Certificate in Derivatives: Mathematics and Basic Principles

EXAMINERS' REPORT

1 (i) A futures contract is an exchange traded contract that involves two parties having agreed to buy or sell an asset at a certain future time for a certain price. As such, a futures contract is itself an asset, and if someone has the right (but not the obligation) to buy or sell this futures contract for a specified price at or before a specified time, then the option on this futures contract is referred to as a futures option.

The futures contract normally expires shortly after the expiration of the option.

Futures options are now available for most of the assets on which futures contracts are traded.

(ii) Call Futures Option

When the holder of a call futures option exercises it the writer of the option delivers:

1. A long position in the underlying futures contract and
2. An amount of cash equal to the excess of the of the futures price over the strike price of the option.

The payoff from a call futures option is the same as the payoff from a call option on a stock with the stock price replaced by the futures price.

(iii) (a) The company would receive $\frac{\$1 \text{ million}}{1.6020} = £624{,}220$.

(b) If the company didn't have a future's contract, it would have been able to receive $\frac{\$1 \text{ million}}{1.65} = £606{,}061$. Since the company does have a future's contract, the payoff is therefore £624,220 – £606,061 = £18,159.

(c) It may be convenient to first express the spot rates as pounds per dollar.

Let:

S_{90} = Spot exchange rate in 90 days time expressed as \$ per £

S_{90}^{*} = Spot exchange rate in 90 days time expressed as £ per \$

= S_{90}^{-1}

K = Delivery exchange rate under the futures contract (expressed as \$ per £) = 1.6020

K^{*} = Delivery exchange rate under the futures contract (expressed as £ per \$)

$= K^{-1} = 1.6020^{-1} = 0.62422$

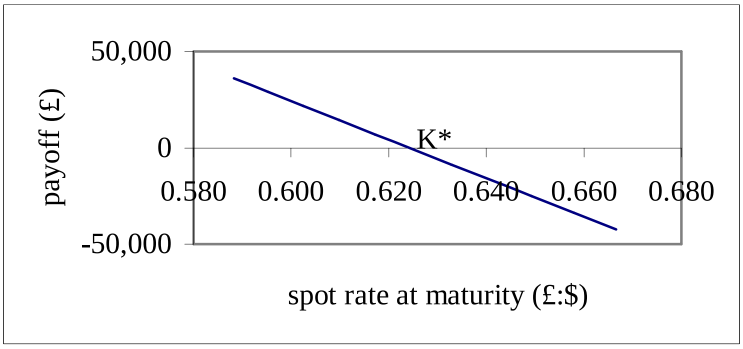
$P(S_{90})$ = payoff of the short futures position

Then

$P(S_{90}) = (-S_{90}^{*} + K^{*}) * 1 \text{ million}$

$= (-S_{90}^{-1} + K^{-1}) * 1 \text{ million}$

The sketch of the payoff, as a function of S_{90}^{*} and K^{*} is given below.



Note that marks are also available for plotting the payoff as a function of S_{90} and K (notwithstanding that the question asked for the plot as a function of spot and delivery prices expressed as pounds per dollar). If a plot of the payoff as a function of the dollar per pound spot and delivery prices are given, then the sketch would have to show a horizontal intercept of $K = 1.6020$, a positive gradient and some non-linearity.

It is important to show the units on the horizontal axis (i.e. either pounds per dollar or dollars per pound).

(iv) Range-Forward contract

Let: $B = X_1^{*} = X_1^{-1} = 1/1.55 = 0.64516$

$A = X_2^{*} = X_2^{-1} = 1/1.65 = 0.60606$

The payoff function for the range forward contract is

$$P(S_{90}^*) = 1 \text{ million} \times \begin{cases} -S_{90}^* + X_2^* & \text{if } S_{90}^* < X_2^* \\ 0 & \text{if } X_2^* \le S_{90}^* \le X_1^* \\ -S_{90}^* + X_1^* & \text{if } S_{90}^* > X_1^* \end{cases}$$

2 (i) Entering into an OTC derivatives transaction almost always involves having a credit exposure to one's counterparty.

Credit exposure on a derivatives transaction varies over time as the variables that drive the value of the underlying contract change.

In assessing credit risk, one must look at two issues:

1. The replacement cost of the derivative if the counterparty were to default immediately and

2. The potential size of the replacement cost if the counterparty were to default at some future time during the life of the derivative contract.

The first of these issues is simply the current mark-to-market value of the derivative contract which can be either positive or negative.

In relation to the second issue, one needs to know what the replacement cost of the derivative transaction might be in the future if the variables that determine the value of the underlying contract were to move adversely.

(ii) Simulation techniques are used by derivatives dealers to assess the potential exposure on a derivatives transaction.

Two measures of potential exposure are usually examined:

1. Expected potential exposure and
2. Maximum or 'worst case' potential exposure.

The first measure at any point in time during the life of the derivatives transaction is the average of all the possible probability-weighted replacement costs.

The first measure of potential exposure is useful for pricing and capital-allocation purposes.

The second method of measuring potential exposure needs to be defined in relation to a specific probability.

For example:

What is the level of exposure such that there is only a 2.5% chance that actual exposure to the counterparty will ever exceed this level?

Maximum or worst-case exposure so defined is used in making credit allocation decisions.

3 (i) $N(x)$ is the cumulative probability that a variable with a standardised normal distribution will be less than x.

$N'(x)$ is the probability density function for a standardised normal variable distribution so:

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

(ii)
$$d_1 = \frac{\ln S - \ln X + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

$$d_1 - d_2 = \sigma\sqrt{T-t}$$

$$\frac{\partial d_1}{\partial S} - \frac{\partial d_2}{\partial S} = 0 \qquad \therefore \frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$$

$$c = SN(d_1) - Xe^{-r(T-t)}N(d_2)$$

$$\frac{\partial c}{\partial S} = N(d_1) - Xe^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial S} + SN'(d_1)\frac{\partial d_1}{\partial S}$$

as $SN'(d_1) - Xe^{-r(T-t)} N'(d_2) = 0 \quad \frac{\partial c}{\partial S} = N(d_1)$

(iii) $c = SN(d_1) - Xe^{-r(T-t)} N(d_2)$

$$\frac{\partial c}{\partial t} = SN'(d_1) \frac{\partial d_1}{\partial t} - rXe^{-r(T-t)}N(d_2) - Xe^{-r(T-t)} N(d_2) \frac{\partial d_2}{\partial t}$$

Assuming $SN'(d_1) = Xe^{-r(T-t)} N'(d_2)$ we have

$$\frac{\partial c}{\partial t} = -rXe^{-r(T-t)} N(d_2) + SN'(d_1) \left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right)$$

$$d_1 - d_2 = \sigma\sqrt{T-t} \quad \text{so } \frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} = -\frac{\sigma}{2\sqrt{T-t}}$$

$$\frac{\partial c}{\partial t} = -rXe^{-r(T-t)} N(d_2) - \frac{SN'(d_1)\sigma}{2\sqrt{T-t}}$$

4 The table gives the differences between an exchange traded option on a stock and a warrant on the same stock:

Exchange Traded Option versus Warrant

Characteristic	Exchage Traded Option	Warrant
Writer or Issuer	Side-bets on the price of the stock by people outside the company	Issued by the company
No. of Contracts Outstanding	Fluctuates daily	Determined by the size of the original issue and changes only when an option exercises or expires
Trading	Traded on futures and option exchanges	Tend to be traded on stock exchanges if traded at all
Upon Exercise	No change in the number of shares outstanding No new money raised by the company	Number of shares outstanding increases Warrant exercise price received by company
Term to Expiry	Terms beyond 18 months are unusual	Tend to be very long term at issue - like 5 - 7 years
Strike Price	Tight range around current price	Usually substantially above the current price

5 First suppose $F_t > S_t e^{(r+c)(T-t)}$.

The company could then:

- borrow an amount $S_t e^{c(T-t)}$ at time t
- purchase the gold at time t for S_t
- take a short position on a gold forward / futures contract at time t with a delivery price of F_t
- pay for all storage costs as they fall due

At time T the company could then:

- sell the gold under the terms of the forward/ futures contract for F_t
- repay the loan with interest; the repayment being $S_t e^{c(T-t)} e^{r(T-t)}$
- realise a profit of $F_t - S_t e^{(r+c)(T-t)}$

The impact of people taking advantage of such arbitrage opportunities would lead to a tendency for either the S_t to increase or F_t to decrease (or both) until $F_t = S_t e^{(r+c)(T-t)}$.

Next suppose $F_t < S_t e^{(r+c)(T-t)}$.

In this case investors with gold in the market could:

- sell their gold for S_t
- save storage costs of $S_t\,(e^{c(T-t)} - 1)$
- accumulate the proceeds of such sales and savings at the risk free rate of r until time T
- enter into a long position on a futures contract with price F_t

and at time T:

- the accumulated proceeds and savings would be $S_t e^{c(T-t)}\, e^{r(T-t)}$,
- the investors could close out the futures contract for the price F_t agreed at time t
- realise a profit of $S_t e^{(r+c)(T-t)} - F_t$

The actions of such investors would lead to tendencies for the price S_t to decrease and the price F_t to increase until $F_t = S_t e^{(r+c)(T-t)}$.

6 (i) A continuous stochastic process X_t is a time varying function which can be written as

$$X_t = X_0 + \int_0^t \sigma_s \, dW_s + \int_0^t \mu_s \, ds$$

or in differential form as

$$dX_t = \sigma_t \, dW_t + \mu_t \, dt$$

where:

- W is a Brownian Motion
- σ and μ are random **F**-previsible processes (i.e. their values can depend on the history $\mathbf{F}_t$ of the Brownian Motion W but not on its future values and they may have some jump discontinuities)
- $\int_0^t \left(\sigma_s^2 + \|\mu_x\|\right) ds$ is finite for all t (with probability 1).

(ii) (a) We are told that X_t and Y_t are two stochastic processes adapted to the same **P**-Brownian motion process W_t, with:

$$dX_t = \sigma_t\, dW_t + \mu_t\, dt$$
$$dY_t = \rho_t\, dW_t + \nu_t\, dt\ . \tag{1}$$

Now if

$$dX_t = \sigma_t\, dW_t + \mu_t\, dt \text{ and}$$

$$dY_t = \rho_t\, dW_t + \nu_t\, dt$$

Then

$$X_t = X_0 + \int_0^t \sigma_s\, dW_s + \int_0^t \mu_s\, ds \quad \text{and}$$

$$Y_t = Y_0 + \int_0^t \rho_s\, dW_s + \int_0^t \nu_s\, ds$$

which implies

$$X_t + Y_t = X_0 + Y_0 + \int_0^t (\sigma_s + \rho_s)\, dW_s + \int_0^t (\mu_s + \nu_s)\, ds \tag{2}$$

or

$$d(X_t + Y_t) = (\sigma_t + \rho_t)\, dW_t + (\mu_t + \nu_t)\, dt$$

$$= dX_t + dY_t \tag{3}$$

(b) We are required to show that

$$d(X_tY_t) = X_t\, dY_t + Y_t\, dX_t + \sigma_t\, \rho_t\, dt\ .$$

Note that $X_t\, Y_t = \frac{1}{2}[(X_t + Y_t)^2 - X_t^2 - Y_t^2]$ (4)

Let $Z_t = X_t + Y_t$ and consider $f(Z_t) = Z_t^2$. From Newtonian calculus we have

$$f'(Z_t) = 2Z_t \text{ and}$$

$$f''(Z_t) = 2$$

Itô's formula tells us that

$$d(Z_t^2) = (\sigma_t + \rho_t) \times 2Z_t\, dW_t + [(\mu_t + \nu_t) \times 2Z_t + \frac{1}{2}(\sigma_t + \rho_t)^2 \times 2]\, dt$$

$$\Rightarrow d((X_t + Y_t)^2) = (\sigma_t + \rho_t) \times 2(X_t + Y_t)\, dW_t + [(\mu_t + \nu_t) \times 2(X_t + Y_t)$$

$$+ \frac{1}{2}(\sigma_t + \rho_t)^2 \times 2]\, dt \qquad (5)$$

Itô's formula also tells us that

$$d(X_t^2) = \sigma_t \times 2X_t\, dW_t + [\mu_t \times 2X_t + \frac{1}{2}\sigma_t^2 \times 2]\, dt \text{ and}$$

$$d(Y_t^2) = \rho_t \times 2Y_t\, dW_t + [\nu_t \times 2Y_t + \frac{1}{2}\rho_t^2 \times 2]\, dt \qquad (6)$$

The result that $(d(X_t + Y_t) = dX_t + dY_t$ tells us that

$$d((X_t + Y_t)^2 - X_t^2 - Y_t^2) = d((X_t + Y_t)^2) - d(X_t^2) - d(Y_t^2) \qquad (7)$$

Substituting equations (5) and (6) into (7) gives

$$d((X_t + Y_t)^2 - X_t^2 - Y_t^2)$$

$$= (\sigma_t + \rho_t) \times 2(X_t + Y_t)\, dW_t + [(\mu_t + \nu_t) \times 2(X_t + Y_t)$$

$$+ \frac{1}{2}(\sigma_t + \rho_t)^2 \times 2]\, dt$$

$$- [\sigma_t \times 2X_t\, dW_t + (\mu_t \times 2X_t + \frac{1}{2}\sigma_t^2 \times 2)]\, dt$$

$$- [\rho_t \times 2Y_t\, dW_t + (\nu_t \times 2Y_t + \frac{1}{2}\rho_t^2 \times 2)]\, dt$$

$$= \sigma_t\, 2Y_t\, dW_t + \rho_t\, 2X_t\, dW_t + \mu_t\, 2Y_t\, dt + \nu_t\, 2X_t\, dt + 2\sigma_t\, \rho_t\, dt$$

$$= 2X_t\, (\rho_t\, dW_t + \nu_t\, dt) + 2Y_t(\sigma_t\, dW_t + \mu_t\, dt) + 2\sigma_t\, \rho_t\, dt$$

$$= 2X_t\, dY_t + 2Y_t\, dX_t + 2\sigma_t\, \rho_t\, dt \qquad (8)$$

Now from equation (4)

$$d(X_tY_t) = \frac{1}{2}\, d[(X_t + Y_t)^2 - X_t^2 - Y_t^2] \qquad (9)$$

Substituting (8) into (9) gives

$$d(X_t Y_t) = X_t\, dY_t + Y_t\, dX_t + \sigma_t\, \rho_t\, dt$$

7 (i) The nodes are numbered for ease of reference (see Figure 3 below).

j i	**Expectation**	**Filtration**	**Node No.**	**Value**	**S_i at Node**
(2 0)	$E_Q(S_2 \mid \mathbf{F}_0)$	{1}	1	$\left(\frac{1}{3}\right)\left(\frac{2}{5}\right)(180)+\left(\frac{1}{3}\right)\times\left(\frac{3}{5}\right)\times(80)$ $+\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)(72)+\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)(36)=80$	80
(2 1)	$E_Q(S_2 \mid \mathbf{F}_1)$	{1, 3}	3	$\left(\frac{2}{5}\right)(180)+\left(\frac{3}{5}\right)(80)=120$	120
		{1, 2}	2	$\left(\frac{2}{3}\right)(72)+\frac{1}{3}(36)=60$	60
(2 2)	$E_Q(S_2 \mid \mathbf{F}_2)$	{1, 3, 7}	7	180	180
		{1, 3, 6}	6	80	80
		{1, 2, 5}	5	72	72
		{1, 2, 4}	4	36	36
(1, 0)	$E_Q(S_1 \mid \mathbf{F}_0)$	{1}	1	$\frac{1}{3}(120)+\frac{2}{3}(60)=80$	80
(1, 1)	$E_Q(S_1 \mid \mathbf{F}_1)$	{1, 3}	3	120	120
		{1, 2}	2	60	60
(0, 0)	$E_Q(S_0 \mid \mathbf{F}_0)$	{1}	1	80	80

(ii) Given the history of the process up until the current time i, the expected value of the process at any future point in time j is simply its current price. The measure Q is used for calculating expected values for this purpose.

Another way of saying this is that the process has no drift and no upward or downward bias in its expected future value relative to its current price (where the measure Q is used for calculating expected values).

This defines the process to be a Martingale with respect to measure Q and filtration $\mathbf{F}$.

8 (i) (a) The portfolio is self-financing if and only if the change in its value depends only on the change of the underlying asset prices.

If $V_t = \phi_t S_t + \psi_t B_t$ is the value of the portfolio at each time t, then, if at the next instant of time the change in the value of the portfolio is exactly matched by the change in the value of the underlying assets (i.e. the stock and the cash), then the portfolio will be self-financing since it will not require an injection or release of capital (or transfer of losses or profits) to continue to match the value of the underlying assets.

This necessary and sufficient condition can be written mathematically as

$$\text{portfolio } (\phi, \psi) \text{ is self-financing} \Leftrightarrow dV_t = \phi_t dS_t + \psi_t dB_t$$

(b) A replication strategy for X is:

- a self-financing portfolio ϕ of S and ψ of B such that
  - $\int_0^T \sigma_t^2 \phi_t^2 dt < \infty$
  - $V_T = \phi_T S_T + \psi_T B_T = X$

(c) If a replication strategy for X can be found, then the value at time t of X will be the value of the portfolio underlying the replication strategy; i.e.

$$V_t = \phi_t S_t + \psi_t B_t$$

If the value of X were less than V_t then a market participant could buy an additional unit of the derivative (X) and sell ϕ units of S and ψ units of B.

Since the price of X at time t is lower than $V_t = \phi_t S_t + \psi_t B_t$, the amount V_t realised from selling the stock and bond is greater than the amount outlayed to buy the derivative.

By the replication property of the portfolio, this short position can be maintained without additional capital support until time T, at which time the value of the portfolio equals the value of the derivative X.

But this means that the excess funds realised at time t crystallise a risk-free profit. Such arbitrageurs could deal in arbitrarily large volumes of the portfolio in order to maximise their profits. Their actions would lead to the price of the derivative at time t increasing (due to the increased demand) and the price of the portfolio assets to decrease (due to the increased supply). This eliminates the possibility of the price X remaining less than the value of the portfolio for any significant period of time.

Conversely, if the derivative price had been higher than the portfolio value at time t, then market participants could have sold the derivative and bought the portfolio, which due to the disparate values, would lead to a positive net cash flow at time t.

By the replication property of the portfolio, this long position can be maintained without additional capital support until time T, at which time the value of the portfolio equals the value of the derivative X. This would turn the positive net cash flow realised at time t into a risk-free profit.

The action of such market participants (who can deal in arbitrarily large volumes) would, by the laws of supply and demand, quickly eliminate such arbitrage opportunities at time t.

Thus the fair (no arbitrage) price for the derivative at any time t is the value of the replication portfolio V_t.

(ii) (a) We have

$$Z_t = B_t^{-1} S_t \qquad (1)$$

$$\Rightarrow \quad dZ_t = S_t d(B_t^{-1}) + B_t^{-1} d(S_t) \qquad (2)$$

Also, S_t is exponential Brownian

$$\Rightarrow \quad dS_t = \sigma S_t dW_t + (\mu + \tfrac{1}{2}\sigma^2)\, S_t dt \qquad (3)$$

While, $B_t^{-1} = e^{-rt}$

$$\Rightarrow \quad d(B_t^{-1}) = -re^{-rt}\, dt = -rB_t^{-1} dt \qquad (4)$$

Substituting (3) and (4) into (2) gives

$$dZ_t = S_t(-rB_t^{-1})\, dt + B_t^{-1}(\sigma S_t dW_t + (\mu + \tfrac{1}{2}\sigma^2)\, S_t dt)$$

$$= dW_t[B_t^{-1}\sigma S_t] + dt[-rS_t B_t^{-1} + (\mu + \tfrac{1}{2}\sigma^2)\, S_t B_t^{-1}]$$

$$= S_t B_t^{-1}(\sigma dW_t + (\mu - r + \tfrac{1}{2}\sigma^2) dt)$$

$$= Z_t(\sigma dW_t + (\mu - r + \tfrac{1}{2}\sigma^2) dt)$$

<u>Alternatively</u>

Using the definitions of Z_t, B_t^{-1} and S_t, it can be seen that

$$Z_t = B_t^{-1} S_t = e^{-rt} e^{\sigma W_t + \mu t} = e^{\sigma W_t + (\mu - r)t}$$

is an exponential Brownian motion process with drift parameter $\nu = \mu - r$ and volatility parameter σ.

Using the result for the stochastic differential equation for exponential Brownian motion processes gives

$$dZ_t = Z_t(\sigma dW_t + (\nu + \tfrac{1}{2}\sigma^2)dt)$$

$$= Z_t(\sigma dW_t + (\mu - r + \tfrac{1}{2}\sigma^2)dt)$$

(b) Let $\gamma = \dfrac{\mu - r + \frac{1}{2}\sigma^2}{\sigma}$.

The Cameron-Martin-Girsanov (CMG) theorem says that there exists a measure **Q** (equivalent to **P**) such that

$$dZ_t = \sigma Z_t d\tilde{W}_t$$

where

$$\tilde{W}_t = W_t + \gamma \text{ is } \mathbf{Q}\text{-Brownian.}$$

In order to apply the CMG theorem, one needs to note that since γ is constant and finite,

$$E_P[e^{\frac{1}{2}\int_0^T r^1 ds}] < \infty.$$

Since σ is constant and finite, we also have $E_P[e^{\frac{1}{2}\int_0^T \sigma^2 ds}] < \infty$.

These facts establish that Z_t is driftless and a martingale under **Q**.

The next step involves forming the process

$$E_t = E_Q\,[B_T^{-1}X\,|\,\mathbf{F}_1] = E_Q[f(Z_t)\,|\,\mathbf{F}_t]$$

for some filtration $\mathbf{F}_t$ which is known at time t.

Since $E[E[f(Z_T)\,|\,\mathbf{F}_t]\,|\,\mathbf{F}_s] = E[f(Z_T)\,|\,\mathbf{F}_s]$, the Tower Law tells us that E_t is a Martingale

Since both E_t and Z_t are Martingales, with Z having positive (non-zero) volatility, the Martingale Representation Theorem tells us that there exists a pre-visible process ϕ which enables us to express E_t in terms of Z_t; i.e.

$$E_t = E_Q(X) + \int_0^t \phi_s dZ_s$$

or

$$dE_t = \phi_t dZ_t \ .$$

The replication strategy then involves holding

ϕ_t units of stock; and
$\psi_t = E_t - \phi_t Z_t$ units of the cash bond