Faculty of Actuaries **Institute of Actuaries**

CERTIFICATE IN DERIVATIVES: MATHEMATICS AND BASIC PRINCIPLES

EXAMINERS' REPORT

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1 (i) A spread trading strategy involves taking a position in two or more options of the same type (e.g., two or more calls or two or more puts).

A combination is an option trading strategy that involves taking a position in both calls and puts on the same stock.

Strips, straps, strangles and straddles are examples of combinations.

(ii) This involves buying a call and a put with the same strike price and expiration date.

Let the total amount paid for the call (c) and the put (p) be $c + p$.

If the options expire at-the-money the investor loses $c + p$ on the strategy.

If the underlying price at expiration (S_T) is above the strike price (X) the call option is worth $S_T - X$ but the put option is worthless

If the underlying price at expiration (S_T) is below the strike price (X) the put option is worth $X - S_T$ but the call option expires worthless

The upside break-even point is therefore where:

$$S_T - X - c - p = 0$$

or

$$S_T = X + c + p$$

By symmetry the downside break-even point is:

$$S_T = X - c - p$$

It was perfectly acceptable for candidates to draw a diagram and argue the case geometrically.

2 (i) Absolute price risk or delta: the change in the value of the portfolio due to changes in the prices of the underlying instruments.

Convexity risk or gamma: the change in the delta arising from changes in the price of the underlying instruments.

Volatility risk or vega: the change in the value of the portfolio arising from changes in implied volatility of the underlying instrument.

Time-decay risk or theta: the change in value of the portfolio arising from the passage of time.

Interest rate risk or rho: the change in the value of the portfolio arising from changes in interest rates used to discount future cash flows

Basis risk: the change in the value of the portfolio arising from changes in correlated variables.

(ii) G30 recommended that market risk was best measured as "value-at-risk" or VaR for short.

VaR is the maximum expected loss from an adverse market movement with a specified probability over a specified period of time.

For example, derivatives participants can determine with, say, 97.5% probability that an adverse change in the value of the portfolio over one day will not exceed a calculated amount. Put another way, there is only a 2.5% chance of an adverse change in the value of the portfolio in excess of the calculated amount.

VaR should take into account changes in all the major market risk components and be calculated to a common time horizon and confidence interval.

3 (i) To suffer a credit loss on a derivatives transaction the counterparty must default and the derivatives contract must have a positive mark-to-market value to the non-defaulting party.

Both conditions must be satisfied simultaneously for a credit loss to occur.

A forward contract may be either an asset or a liability at any point in time.

Suppose you had entered into, say, a long forward contract, if the counterparty were to default and the contract had a negative mark-to-market value to the non-defaulting counterparty at the time of default both conditions would not be satisfied simultaneously. So the non-defaulting party would not suffer a credit loss.

On the other hand, if the contract had a positive mark-to-market value to the non-defaulting counterparty at the time of default both conditions would be satisfied simultaneously. So a credit loss would occur.

(ii) In an exchange traded derivative contract, the performance of the parties is guaranteed by a central counterparty[1] whereas in an OTC contract each party is dependent on the other for performance.

As a general rule, exchange traded contracts are marked-to-market each day with the counterparty suffering a loss making payment equal to the loss to the exchange and the exchange transferring the amount to a corresponding counterparty making a profit.

OTC contracts don't usually involve daily close out and settlement.

However, OTC contracts may also be marked-to-market by the posting of collateral though special arrangements and additional legal documentation work must be undertaken for such transfers of collateral.

Exchange traded contracts require initial margin to be put up by both counterparties in order to effect a contract. This is not generally the case with OTC contracts.

Closing out positions is much easier in the case of exchange-traded derivatives compared with OTC derivatives. OTC contracts can usually only be closed out by mutual agreement between the parties — sometimes on penal terms to the party seeking to close out — or by assigning the contract to a third party — but this tends to be very rare. This relative difficulty in closing out OTC contracts compared to exchange traded contracts adds to the credit risk of OTC derivatives.

[1] In reality only general clearing members of the exchange get the guarantee of the clearing house. Most participants on the exchange including many institutional investors deal through a futures broker. Their credit exposure is to the broker and not the clearing house. In the UK and US the regulatory environment facilitates segregation of accounts which reduces the risk somewhat.

4 (i) Estimating volatility from historical data

To estimate the volatility of a stock price empirically, the stock price is usually observed at fixed intervals of time like every day, every week, or every month.

If we are estimating volatility for use in a model which assumes that stock prices follow a lognormal distribution we might want to calculate the standard deviation of the natural logs of the returns i.e.

$$\ln \frac{S_i}{S_{i-1}}$$

The standard deviation calculated in this way is an estimate of $\sigma\sqrt{\tau}$

Where τ is the length of the time interval in years between the time at which S_i and S_{i-1} are observed. Equally spaced intervals assumed.

Therefore σ can be estimated as $\hat{e}$ where

$$\hat{e} = s/\sqrt{\tau}$$

Choosing an appropriate number of observations is not an easy task:

- more data generally lead to more accuracy but
- σ does change over time and data that are too old may not be relevant for predicting the future

The compromise usually chosen is to use closing prices from daily data over the most recent 90 to 180 days

A rule of thumb is to set the time period over which volatility is to be measured equal to the time period over which it is to be applied

Empirical research suggests that time for the purpose of measuring volatility should be measured over trading rather than calendar days

<u>Implied volatility</u>

Implied volatility is the volatility implied by an option price observed in the market.

Implied volatility is the market's opinion about the volatility of a particular stock — which changes over time

Several implied volatilities could be obtained simultaneously from different options on the same stock and a composite implied volatility could be calculated

Empirical research suggests that one should use the implied volatility from the option whose price is most sensitive to σ

(ii) *A large number of different points could be made in answer to this question. Many points are given but any four of the points would have scored full marks.*

The length of the life of the option compared with the historical period of time over which the historical volatility was calculated or the features of the option used to estimate implied volatility. Let's say we are pricing a 50-day option. Then it would be important to examine the historical pattern of 50-day volatility (both historical and implied) so as to get some idea of the range within which 50-day volatilities moves.

If the current level of volatility is significantly different from its historical average then we need to decide how quickly it will return to its historical average or what else it might do

Look at historical volatility over a short period of time, a somewhat longer period of time and over a very long period of time. For example, look at 10 day, 50 day and 250 day volatility. This will give an idea how volatility is trending and will help in deciding whether volatility is moving away from or back to its historical average for the period of the option.

Some weight must be given to implied volatility in arriving at an estimate; opinion as to the weight varies among traders

The volatility of similar commodities/instruments may also be considered in arriving at an estimate. For example, if treasury bonds were to become more volatile, in all likelihood so would treasury notes and Eurodollar bonds. Thus some weight might be given to the volatility of similar commodities

Graphs of historical volatility over time can be useful for making inferences about the characteristics of volatility

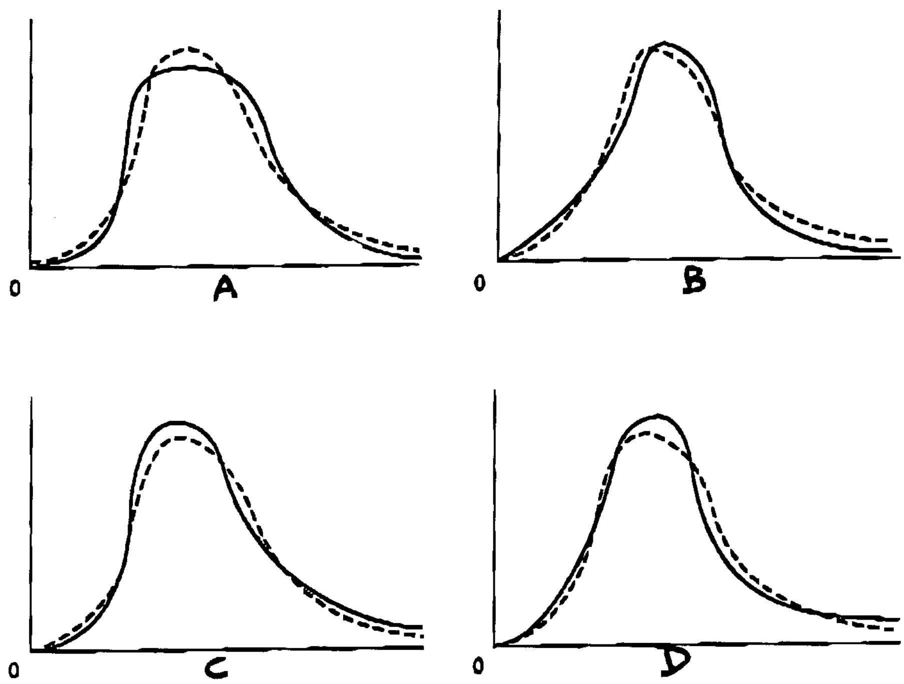
Graphs of implied volatility are also useful as, in the pricing of some options, there is a need to make a judgement about how implied volatility is likely to fluctuate

Implied volatilities depend on strike price (the volatility smile effect) and term to maturity (the term structure of volatility)

The smile effect and the term structure of volatility lead us to the construction of a two dimensional volatility matrix — strike price and term to maturity — when a new option is to be valued one looks up the relevant entry in the table or interpolates between entries

5 Figure 1 shows four ways in which the terminal stock price can differ from a lognormal distribution and yet have a stock price return with the same mean and standard deviation.

Figure 1
Various Terminal Stock Price Distributions



______________________ True Distribution
---------------------------- Lognormal Distribution

True Distribution has same Mean & Standard Deviation as that of the Lognormal Distribution

In figure 1 (A), both tails are thinner than those of the lognormal distribution; in figure 1 (D), both tails are fatter than those of the lognormal distribution; in figures 1 (B) and 1 (C), one tail is thinner and the other fatter.

[Candidates were required to draw the diagrams and clearly label them]

The pricing bias can be seen if one tries to price an out-of-the-money call option (*any other clearly presented demonstration of pricing bias was also acceptable as a solution)*

Such an option will show a positive payoff only if there is a large increase in stock price.

Its value is determined by the thickness of the right tail of the terminal stock price distribution.

The thicker the tail the more valuable such an option will be.

The Black-Scholes model will tend to underprice out-of-the-money call options in figure 1 (C) and (D) and overprice out-of-the-money call options in figure 1 (A) and (B).

In the case of out-of-the-money put options they will acquire value only if there is a large decrease in the price of the underlying stock.

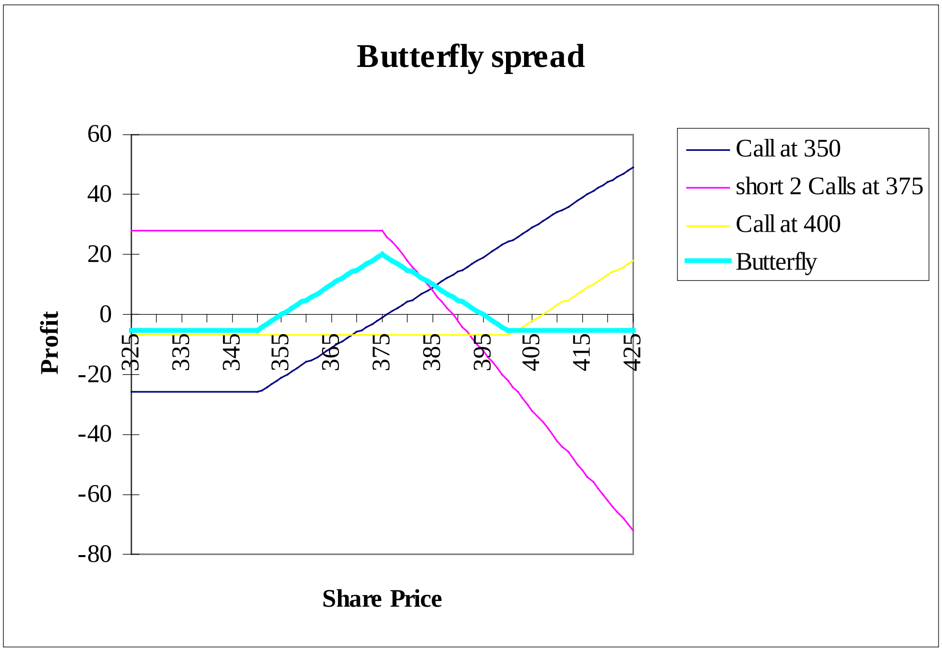
The left tail of the terminal stock price distribution drives the value of such options.

The Black-Scholes model will tend to underprice out-of-the-money put options in figure 1 (B) and (D) ...

... and overprice out-of-the-money put options in figure 1 (A) and (C).

6 (i) A long butterfly spread is constructed by buying one call with a strike at 350 and one call with a strike at 400 and selling 2 calls with a strike at 375.

(ii) The payoff profile and the contributions from the call positions are shown in the diagram below:



The maximum profit is when the share price is 375p at the expiry date of the options.

In this case the contributions to the profit are:

Position	*payoff*	*cost*	*Profit*
long 350 call	25	26	–1
2 Short 375 calls	0	–28	28
long 400 call	0	7	–7
		Total profit	20

The maximum loss occurs when the share price at the expiry date is below 350p or above 400p.

When the share price is below 350p

Position	*payoff*	*cost*	*Profit*
long 350 call	0	26	–26
2 Short 375 calls	0	–28	28
long 400 call	0	7	<u>–7</u>
		Total profit	–5

Similarly if the share price is $T > 400$

Position	*payoff*	*cost*	*Profit*
long 350 call	T–350	26	T–376
2 Short 375 calls	–2(T–375)	–28	–2T+778
long 400 call	T–400	7	<u>T–407</u>
		Total profit	–5

(iii) In practice the trader would consider a long butterfly strategy if he/she expected the volatility to be low so that the share price would end in the range 355–395 where the contract would deliver a profit.

A trader might choose a long butterfly spread over, say, a short straddle, because of its limited risk.

7 (i) An arbitrage is where a trader can lock in a riskless profit by simultaneously entering into transactions in two or more markets or two or more assets.

More formally an arbitrage is such that at time $t = 0$ a portfolio can be created with net value zero. But at some future time $t = T$, the portfolio has value > = 0 with probability 1 and has value > 0 with a non-zero probability.

The no-arbitrage principle says that prices in the market are set such that no arbitrage opportunities occur.

(ii) There exists an arbitrage opportunity in this economy.

This can be exploited by:

- Selling (shorting) 2 units of asset 1, to generate income of 2 and buying 1 unit of each of assets 2 and 3 at a cost of 2

- Net cost of initial position is zero.

- At time $t = 1$, must repurchase 2 units of asset 1, this will cost 2.1 whatever state the economy is in.
  - If the economy is in state 1 then the income from selling 1 unit of each of assets 2 and 3 is 2.12 (= 1.03 + 1.09)
  - If the economy is in state 2 then the income from selling 1 unit of each of assets 2 and 3 is 2.11 (= 1.07 + 1.04)
- Hence whichever state the economy is in at time 1 the portfolio has positive value. Hence an arbitrage opportunity exists

Marks were awarded for writing down equations even if they were not solved.

At time 0 investor buys x_i units of asset i such that $\sum_{i=1}^{3} x_i = 0$.

Let $P_i^{[j]}$ be the price of asset i in state j at time 1. Then an arbitrage opportunity exists if:

$$1.05x_1 + P_2^{[j]}x_2 + P_3^{[j]}x_3 > 0$$

The strict mathematical definition requires that $1.05x_1 + P_2^{[j]}x_2 + P_3^{[j]}x_3$ *is not negative and is positive for at least one j, Full marks were awarded for the above somewhat looser expression.*

Note that asset 1 is the risk-free asset as there is no uncertainty about its value at time 1.

8 (i) The forward price is F, which is given by:

$$F = Se^{(r-q)(T-t)}$$

where $r = 7\%$ is the risk-free rate

$q = 3.5\%$ is the dividend yield

$(T - t) = 1$ is the time to maturity of the forward contract in years

$S = 3250$ the index level

$$F = 3250e^{(.07-.035)}$$
$$F = 3365.76$$

(ii) Value of the forward price with 9 months to maturity

$$F = 3900e^{(.065-.031)\times 0.75}$$
$$F = 4000.73$$

Value of the forward position, *f*, is given by

$$f = Se^{-q\times(T-t)} - Ke^{-r\times(T-t)}$$

where $K = 3365.76$ the delivery price

$$f = 3900e^{-0.031\times 0.75} - 3365.76e^{-0.065\times 0.75}$$
$$f = 3810.37 - 3205.61$$
$$f = 604.76$$

(iv) The published yield is an historic yield and so may not be applicable to the period for which the forward is held.

Also dividends are not paid continuously as assumed. Actual dividend payments are spread unevenly throughout the year.

An estimate of the dividends to be paid in the period of the forward contract would be required.

9 (i) A probability measure is a collection of probabilities on the set of all possible outcomes describing the likelihood of each outcome. There may be multiple measures for the same experiment, for example a coin when tossed once has the set of outcomes {Heads, Tails}. One measure is given by P = {½, ½} indicating that the coin is unbiased. A second measure Q could be {1/3,2/3} indicating that Tails are twice as likely as heads.

(ii) Two probability measures P and Q are equivalent if they operate on the same sample space and agree on what is possible. For P and Q to be equivalent all events that have a positive probability under P must have a positive probability under Q. Similarly if an outcome is impossible under P then it must be impossible under Q. If A is an event from the sample space $P(A) > 0 \Leftrightarrow Q(A) > 0$.

(iii) C-M-G Theorem as per Baxter and Rennie section 3.4 page 74.

If we have a Brownian Motion under one measure P and the drift term (which can be a function of time) satisfies boundedness conditions we can create zero drift Brownian Motion under a change of measure. The new measure Q is equivalent to P, and the Radon-Nikodym of Q with respect to P is known.

(iv) $dX_t = X_t\,(\sigma dW_t + \mu dt)$

add and subtract $r_t dt$ within the brackets r_t is a time dependent parameter.

$dX_t = ((\mu_t - r_t)\; dt + r_t dt + \sigma d\; W_t)$

$$= X_t\left(\sigma d\tilde{W}_t + r_t\; dt\right) \text{ where } d\tilde{W}_t = \left(\frac{\mu_t - r_t}{\sigma}\right) dt + dW_t$$

$\dfrac{\mu_t - r_t}{\sigma}$ must be a pre-visible process and satisfy the C–M–G boundedness conditions.

Hence there is a Q measure such that $\tilde{W}_t$ is a Q Brownian Motion.
The solution to $dX_t = X_t\left(\sigma d\tilde{W}_t + r_t dt\right)$ is:

$$\frac{dX_t}{X_t} = \sigma d\tilde{W}_t + r_t\; dt$$

Using Ito's lemma we have:

$$d(\ln X_t) = \frac{dX_t}{X_t} + \left[\tfrac{1}{2} * \left(\frac{-1}{X_t^2}\right) * X_t^2\; \sigma^2\; dt\right]$$

$$= \sigma d\tilde{W}_t + r_t\; dt - \tfrac{1}{2}\sigma^2\; dt$$

$$\ln X_t = 1nX_0 + \left\{\sigma\tilde{W}_t + \int_0^t r_s ds - \tfrac{1}{2}\sigma^2 t\right\}$$

$$X_t = X_0 \exp\left\{\sigma\tilde{W}_t + \int_0^t r_s ds - \tfrac{1}{2}\sigma^2 t\right\}$$

In marking the papers, two other perfectly acceptable solutions emerged. Full marks were given for these solutions.