EXAMINATIONS

19 April 2004 (am)

Certificate in Derivatives: Mathematics and Basic Principles

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You have 15 minutes at the start of the examination in which to read the questions. You are strongly encouraged to use this time for reading only, but notes may be made. You then have three hours to complete the paper.*

3. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

4. *Mark allocations are shown in brackets.*

5. *Attempt all 8 questions, beginning your answer to each question on a separate sheet.*

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available Actuarial Tables, Derivatives Formula Sheet and your own electronic calculator.

NOTE: In this examination, you are never required to prove the use of an arbitrage-free methodology unless clearly stated in the question.

1 Consider a stock S whose current value at time t is $S_t = s$. Over the next possible time step, the stock can change to two possible values, viz: $S_{t+\Delta t} = s_1$ or s_2, where Δt is the width of the time step. There is also a cash bond B that can accumulate risk free over the time step from its current value $B_t = b$ to a future value $B_{t+\Delta t} = b e^{r\Delta t}$.

At the end of the time step, a derivative on S pays off an amount:

$$f(S) = \begin{cases} f(s_1) \text{ if } S_{t+\Delta t} = s_1 \\ f(s_2) \text{ if } S_{t+\Delta t} = s_2 \end{cases}$$

(i) Derive the quantities ϕ of stock S and ψ of bond B which ensure that the portfolio (ϕ, ψ) of stock and bond accumulates to exactly the same amount as the derivative payoff, no matter which value of $S_{t+\Delta t}$ the stock takes on. [4]

(ii) Hence write down an expression for the value of the derivative at the current time t, and explain why this value is the unique arbitrage-free value. [4]

[Total 8]

2 You are asked to value options on a non-dividend paying stock using the Black-Scholes model.

(i) Write down the equation for the fair price of a European put option, explaining all the terms you use. [2]

(ii) By considering the value of a European put option against its intrinsic value, prove that an American put option is worth more than a European put option with the same strike. [3]

(iii) Draw an approximate diagram of American and European put option values against varying stock prices, illustrating what you have proved in (ii). [3]

[Total 8]

3 (i) A price x of a non-dividend paying asset follows this geometric Brownian motion process:

$$dx = \mu x \, dt + \sigma x \, dW_t$$

where μ, σ are positive constants and W_t is a standard Brownian motion.

Using Ito's Lemma, find the processes followed by:

(a) $\ln x$ (= $\log_e x$)

(b) x^2 [5]

(ii) Consider two currencies, A and B. The price of currency A expressed in terms of currency B follows the same geometric Brownian motion process as in (i) above, with a growth rate $\mu = r_B - r_A$, the difference between the interest rates in currencies B and A respectively.

Show that the process followed by the price of currency B expressed in terms of currency A is also a geometric Brownian motion, but that the growth rate is not $r_A - r_B$. [5]

[Total 10]

4 (i) Define the terms "gamma" and "vega" as they would apply to a portfolio of options on a commodity, and explain what is meant by a "gamma hedge". [2]

(ii) Describe, using diagrams, the way gamma and vega change with the price of the commodity and time to maturity. [6]

(iii) Explain the problems which a trader might encounter in respect of gamma and vega when hedging a short-dated option using a longer-dated option. [3]

[Total 11]

5 (i) Explain the differences between a forward contract and a future in terms of their characteristics and settlement bases. [4]

(ii) A UK company transacts a large proportion of its business in US dollars. It wishes to enter into a forward exchange rate contract to reduce the uncertainty relating to the pound sterling value of certain assets nine months hence.

Set up two portfolios, one involving a forward exchange rate contract, that replicate the worth of a dollar in sterling in nine months time, and then derive a formula for the arbitrage-free forward exchange rate. Define any symbols you use. [4]

(iii) An alternative strategy to the one adopted in (ii) above would be to hedge using currency futures. Describe which contracts to use, and give three reasons why the futures hedge might be more problematic for the company. [4]

[Total 12]

6 (i) Define the following terms in relation to bonds and options on bonds:

(a) duration (D)
(b) forward price volatility (σ_P)
(c) forward yield volatility (σ_Y)

[2]

(ii) Prove that the following relationship is approximately true:

$$\sigma_P = Dy\,\sigma_Y$$

where y is the forward yield on the bond. [3]

(iii) Consider a bond with five years to redemption. It has an annual coupon of 4% and is callable at par (100%) in one year's time.

A non-callable bond from the same issuer, with identical coupon and life, would currently trade at a continuously compounded yield of 4.70% per annum. It would have a one-year-forward yield volatility of 20% and a duration of 4.41. The one-year risk free rate is 3% per annum, continuously compounded.

Calculate the price of the embedded option, stating clearly any assumptions you make, and thereby derive the price of the callable bond.

[10]

(iv) Describe, with reasons, what happens to the price of the callable bond as yields fall from their current levels, and explain how this behaviour might differ from that of a non-callable bond. [2]

[Total 17]

7 (i) Define what is meant by:

(a) a filtration $\mathbf{F}_t$

(b) two probability measures **P** and **Q** being equivalent to each other [3]

(ii) Let W_t be a Brownian motion process and $f_p(x)$ be the Normal density function

$$f_P(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

both under a probability measure **P**.

Develop an expression for the joint likelihood function $f_P^n(x_1, x_2 . \cdots x_n)$ for the **P**-Brownian motion process W_t. [4]

(iii) Suppose **Q** is a probability measure equivalent to **P**. Explain what is meant by the Radon-Nikodym derivative of **Q** with respect to **P** over the time interval (0, T]. [5]

(iv) Let X be a stochastic process with increment:

$$dX_t = \sigma_t dW_t + \mu_t dt$$

where W_t is a Brownian motion process under a probability measure **P**, and the suffix $_t$ denotes a time-dependent variable.

Explain how to find if there is an alternative measure **Q**, such that the drift of the process X under **Q** is $\nu_t dt$ instead of $\mu_t dt$ for some ν_t. [5]

[Total 17]

8 A stock of price s is assumed to follow the process:

$$ds = \mu s \, dt + \sigma s \, dB_t$$

where μ, σ are positive constants and B_t is a standard Brownian motion.

The continuously compounded risk free interest rate r is constant for all maturities.

You are asked to construct a recombining Binomial tree algorithm to value a path dependent option. Using an "up step" u, the top nodes of the tree will be placed at $S_0 u^k$ for $k = 0, 1, 2 \ldots$ for each small time interval t, where S_0 is the price at time 0.

(i) Specify fully the first step of the binomial process, giving formulae for the up and down probabilities and step size u. [6]

(ii) If t is small enough, u can be approximated by $e^{\sigma\sqrt{t}}$.
If the initial stock price is 100, σ is 10%, and the continuously compounded risk free interest rate is zero, draw three steps of the tree for quarter-year time steps and calculate the stock prices at each node. [3]

(iii) From the tree in (ii), evaluate a nine-month average rate call option with a strike price of 100, where the average is computed on the basis of the price every three months, including the start and end dates. [8]

[*An average rate option on a stock is an option whose payoff is the average of the stock price at certain points over the option's life less the exercise price.*]

[Total 17]

END OF PAPER