EXAMINATIONS

7 April 2003 (am)

Certificate in Derivatives: Mathematics and Basic Principles

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You have 15 minutes at the start of the examination in which to read the questions. You are strongly encouraged to use this time for reading only, but notes may be made. You then have three hours to complete the paper.*
3. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
4. *Mark allocations are shown in brackets.*
5. *Attempt all 7 questions, beginning your answer to each question on a separate sheet.*

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available Actuarial Tables, Derivatives Formula Sheet and your own electronic calculator.

NOTE: In this examination, you are never required to prove the use of an arbitrage-free methodology unless clearly stated in the question.

1 Let z_m, z_n be the annually compounded yields on m and n year zero-coupon bonds respectively, with $n > m$.

(i) Explain what is meant by the annually compounded forward interest rate for the period (m, n), and show algebraically how you would calculate this forward rate. [3]

(ii) Show that the continuously compounded forward interest rate for the time interval (m, n) is given by

$$\frac{n\delta_n - m\delta_m}{n-m}$$

where δ_m, δ_n are the continuously compounded forces of interest for the zero-coupon bond yields of term m and n respectively. [4]

(iii) If a 14-year zero-coupon bond yields 4.9% p.a. and a 15-year zero-coupon bond yields 5.0% p.a., both annually compounded, calculate the one-year forward interest rate for year 15. Comment on the level of this forward rate relative to the zero-coupon rates. [3]
[Total 10]

2 (i) Outline the use and significance of Ito's lemma for the valuation of derivatives in a Black-Scholes-Merton economic world. [2]

(ii) Use Ito's lemma to determine the differential equation for the value of the following derivatives under the stock price process $ds = \mu s dt + \sigma s dz$, where z is a standard Brownian motion, and μ and σ are constants:

(a) European Call option, strike price k, period to maturity T.

(b) American Call option, strike price k, period to maturity T, where the payoff is the excess (if positive) of the square of the stock price above the strike price.

In each case, state precisely the boundary conditions which apply, without attempting to solve the equations. [11]

(iii) Compare in general terms the instantaneous stock hedges for the options in (ii). In particular, compare the way in which they vary with the underlying stock price. [2]

(iv) Describe how your answers to (ii) and (iii) would change if instead the stock price followed the process $ds = \mu s dt + \sigma_0 e^{-\beta t} s dz$ for some constants σ_0 and β.
[3]
[Total 18]

3 (i) (a) State the three necessary and sufficient conditions for the continuous process W_t to be a **P**-Brownian Motion.

(b) State the requirements for a probability measure **Q** to be equivalent to **P**. [3]

(ii) Let W_t be a continuous **P**-Brownian Motion process.

Consider the "change of measure" process $\varsigma_t = \mathrm{E}_{\mathbf{P}}\left[\frac{d\mathbf{Q}}{d\mathbf{P}} \mid \mathbf{F}_t\right]$, where **Q** is a measure equivalent to **P** and $\frac{d\mathbf{Q}}{d\mathbf{P}} = \exp\left(-\gamma W_T - \frac{1}{2}\gamma^2 T\right)$ for some time horizon T, constant γ and given filtration $\mathbf{F}_t$.

[$\mathrm{E}_{\mathbf{P}}$[…] represents expectation with respect to measure **P**.]

Show that $\tilde{W}_t = W_t + \gamma t$ is a continuous **Q**-Brownian Motion process. [11]

(iii) Comment on the significance of the result in (ii). [2]

(Hint: You may use the fact that a random variable X is a normal $N(\mu,\sigma^2)$ under measure **P** if and only if $E_{\mathbf{P}}\left(\exp(\theta X)\right) = \exp\left(\theta\mu + ½\theta^2\sigma^2\right)$ for all real θ.) [Total 16]

4 Rivendale is a non-dividend paying equity, with current price 100. You are considering some strategies based upon buying and selling options on Rivendale. All these options expire three months from now.

(i) Draw approximate aggregate Profit/Loss diagrams for the following two option strategies:

(a) Buy 1 European at-the-money call option.

(b) Sell 2 European put options strike 80.
Buy 1 European put option strike 90.
Buy 1 European call option strike 110.
Sell 2 European call options strike 120.

Diagrams are required for each strategy as at now, two months from now and three months from now. [8]

(ii) Draw an approximate diagram for the Gamma of the option strategy in (i) (a) above, as at now, two months from now and **nearly** three months from now. [4]

(Note: Sketch all the time curves for a single strategy on the same graph, plotted against the Rivendale stock price using a suitable scale. You should allow for the estimated purchase price of the underlying options where appropriate. Any curves may of course be drawn freehand.) [Total 12]

5 (i) Explain what is meant by:

(a) the delta of a derivative
(b) a delta hedge of an option position [2]

(ii) A non-dividend paying stock is currently priced at \$100 per share.

An investor has sold a 3-year European Put option on 2,000 shares, with a strike of \$93 per share. The value of each option contract is given by the Black-Scholes formula with the volatility parameter $\sigma = 20\%$ per annum and risk free rate of $r = 5\%$ per annum.

(a) Calculate the value of the investor's overall option position, together with its delta.

(b) Describe the portfolio of assets that delta hedges the investor's option position.

(c) Describe briefly how the delta of the option would vary as:

- the value of the stock increases
- the term of the option decreases [10]

(iii) The investor in (ii) is considering using futures to delta hedge the option position, rather than investing directly in the underlying asset. Suppose the maturity of the futures contract is $T^* = 3$ months.

(a) Calculate the futures price F.

(b) If H_A is the amount of the underlying asset needed to delta hedge the option position, derive a formula in terms of H_A for the equivalent futures hedge amount H_F.

(c) If each futures contract is for 100 shares, how many futures contracts are required to delta hedge the option position? Comment on the precision of this hedge. [6]

[Total 18]

6 An interest rate market consists of the following zero coupon bonds, with their corresponding annually compounding yields per annum:

Term	*Zero Coupon Yield % p.a.*
1 year	3.25
2 year	3.70
3 year	4.05
4 year	4.25
5 year	4.35

(i) Calculate the annual discount factors applicable to this yield curve out to five years. (A discount factor for year n is the present value of one unit of cash payable at the end of year n.) [2]

(ii) Using the values in (i), calculate the prices of bonds with annual coupons of 6%, for each maturity from 1 to 5 years. Comment on where you would expect the yields of these coupon bonds to lie on the curve compared with the zero coupon yields. [5]

(iii) Find the fixed coupon of a 5-year par value fixed to floating interest rate swap. Demonstrate that this coupon is the same as would apply to a par coupon bond of the same maturity. [4]

[Total 11]

7 A stock price follows a process S, whose current value is $S(0) = s_0$.

Over a small time period δt, the stock can move up to a value $S(\delta t) = s_1$, or down to value $S(\delta t) = s_2$, with ($s_2 < s_0 < s_1$). The probability of an upward movement is p_1 and the probability of a downward movement is $p_2 = 1 - p_1$.

A trader is considering replicating a derivative, whose payoff function is:

$$\Pi(S, \delta t) = \begin{cases} \pi_1 & \text{if stock is up at time } \delta t \\ \pi_2 & \text{if stock is down at time } \delta t \end{cases}.$$

The trader can also invest in risk free cash bonds, which earn continuously compounding interest at rate r per annum.

(i) Derive expressions for the quantity of stocks and bond in which the trader should invest in order to replicate the derivative. [4]

(ii) Let $\mathbf{Q} = \{q_1, q_2\}$ be an alternative probability measure equivalent to $\mathbf{P} = \{p_1, p_2\}$ with $q_1 = \dfrac{s_0 e^{r\delta t} - s_2}{s_1 - s_2}$ and $q_2 = 1 - q_1$.

Show that these three statements are equivalent:

(1) the current market value of the derivative is arbitrage free

(2) the current market value of the derivative equals the expected value of the discounted payoffs of the derivative

(3) the current market value of the stock equals the expected value of the discounted future stock price process

with discounting at the risk free rate and expectations in each case calculated with respect to probability measure **Q**.

(In algebraic terms, you are required to show that:

$$MV_\Pi = \text{arbitrage free}$$
$$\Leftrightarrow MV_\Pi = e^{-r\delta t} \mathrm{E}_{\mathbf{Q}}[\Pi(S, \delta t)]$$
$$\Leftrightarrow s_0 = e^{-r\delta t} \mathrm{E}_{\mathbf{Q}}[S(\delta t)]$$

where MV_Π denotes the arbitrage free market value of the derivative and $\mathrm{E}_{\mathbf{Q}}[\ldots]$ denotes an expectation with respect to measure **Q**.) [6]

(iii) Comment briefly on the theoretical significance of the result in (ii). [3]

(iv) The trader notices that the derivative is being traded at a market price more than the value of the replicating assets. Suggest a trading strategy to take advantage of the arbitrage opportunity. [2]

[Total 15]