The Actuarial Profession
making financial sense of the future

Analysing Dependent Data
A Case Study of Equity Returns

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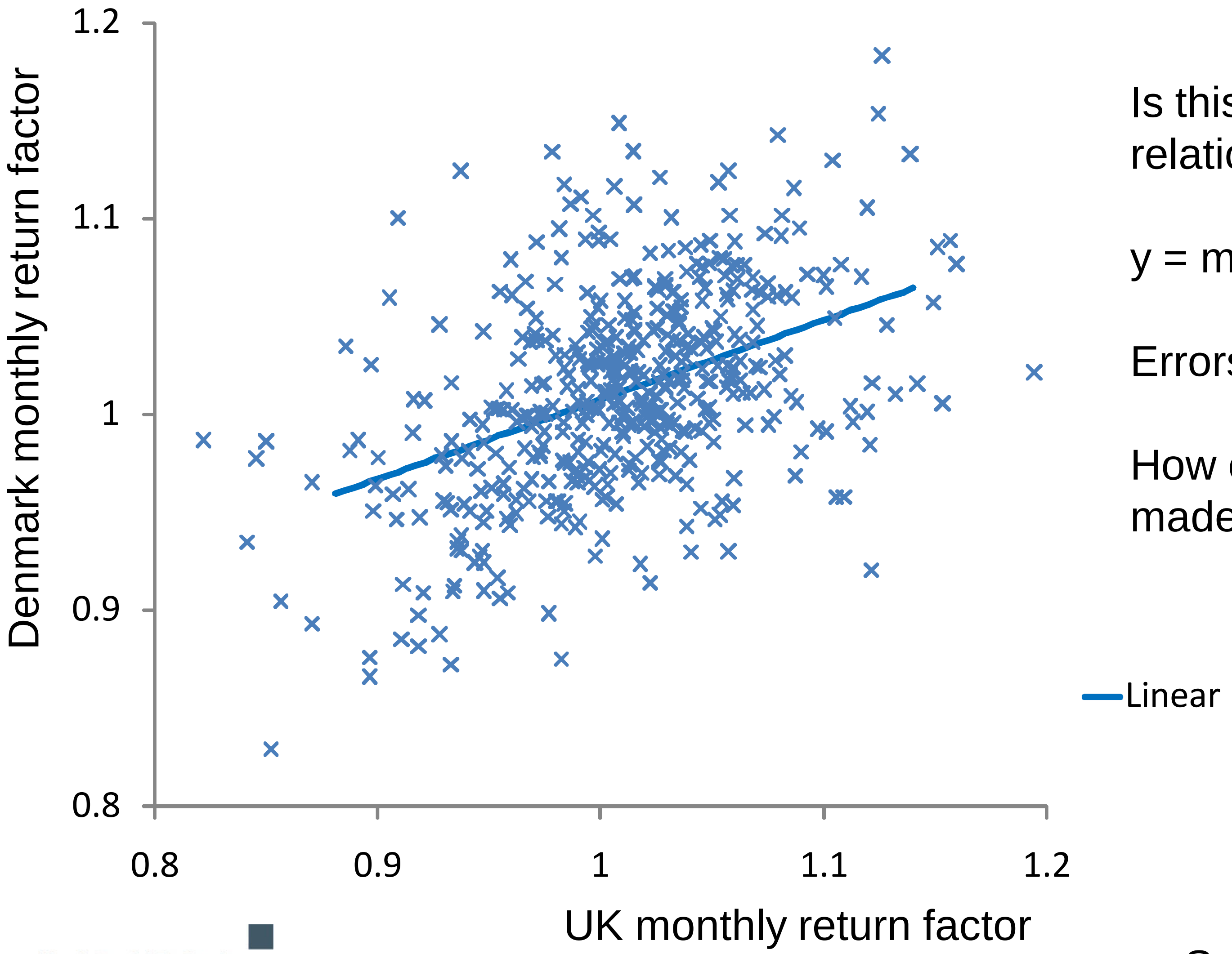
Workshop Overview

- Empirical analysis of non-linear dependence in UK and Danish stock market returns
- Fitting copula functions
- Capital requirements and quadrant correlations
- Analysis of transformed and standardised data
- Classification of dependency models
- Multiple comparisons: stock market stylised facts
- Questions and answers

Empirical Evidence for non-linear dependency in UK and Danish Stock Markets

Linear Regression: Denmark vs UK



Is this a simple linear relationship of the form

$y = mx + c + \text{error}$

Errors independent of x.

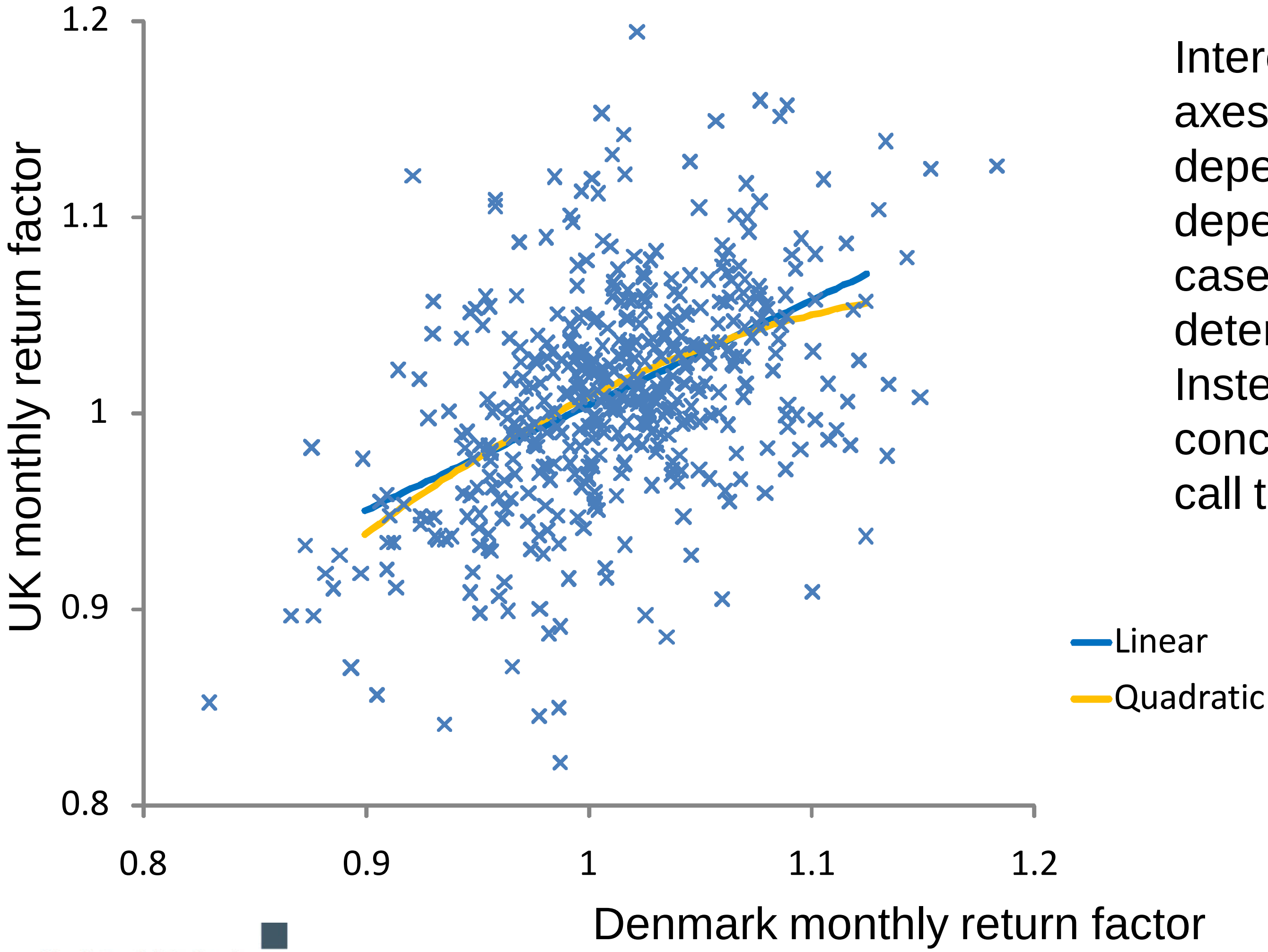
How could this dependency be made non-linear?

Source: Datastream

Quadratic Regression: Denmark vs UK

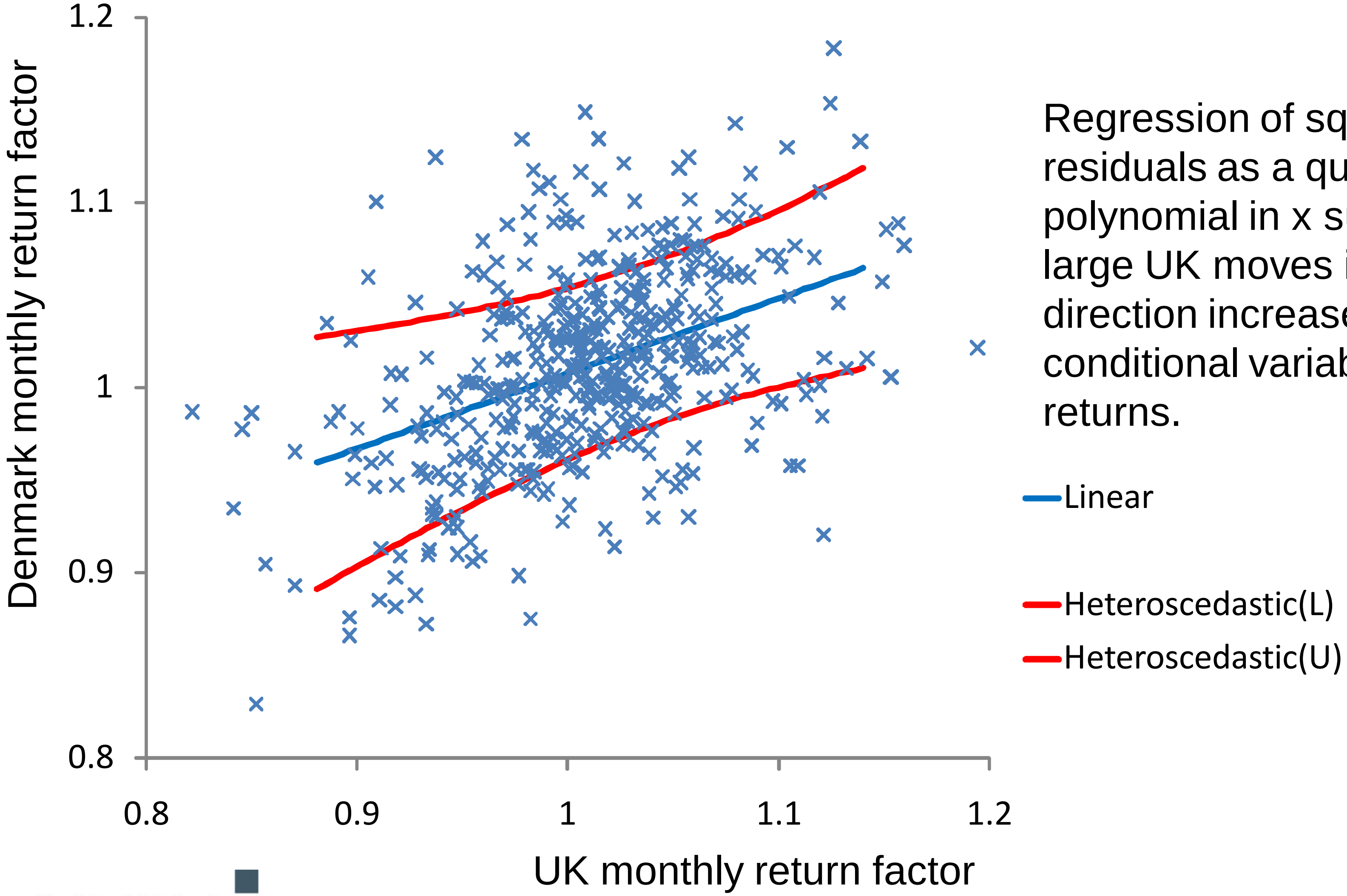
Quadratic regression suggests a convexity effect. For large UK moves in either direction, the corresponding expected Danish return is a little lower than predicted by linear regression.

Quadratic Regression: UK vs Denmark



Interchanging the X and Y axes does not turn concave dependency into convex dependency as would be the case for increasing deterministic functions. Instead, the dependency is concave for both orders. We call this “two-way concave”.

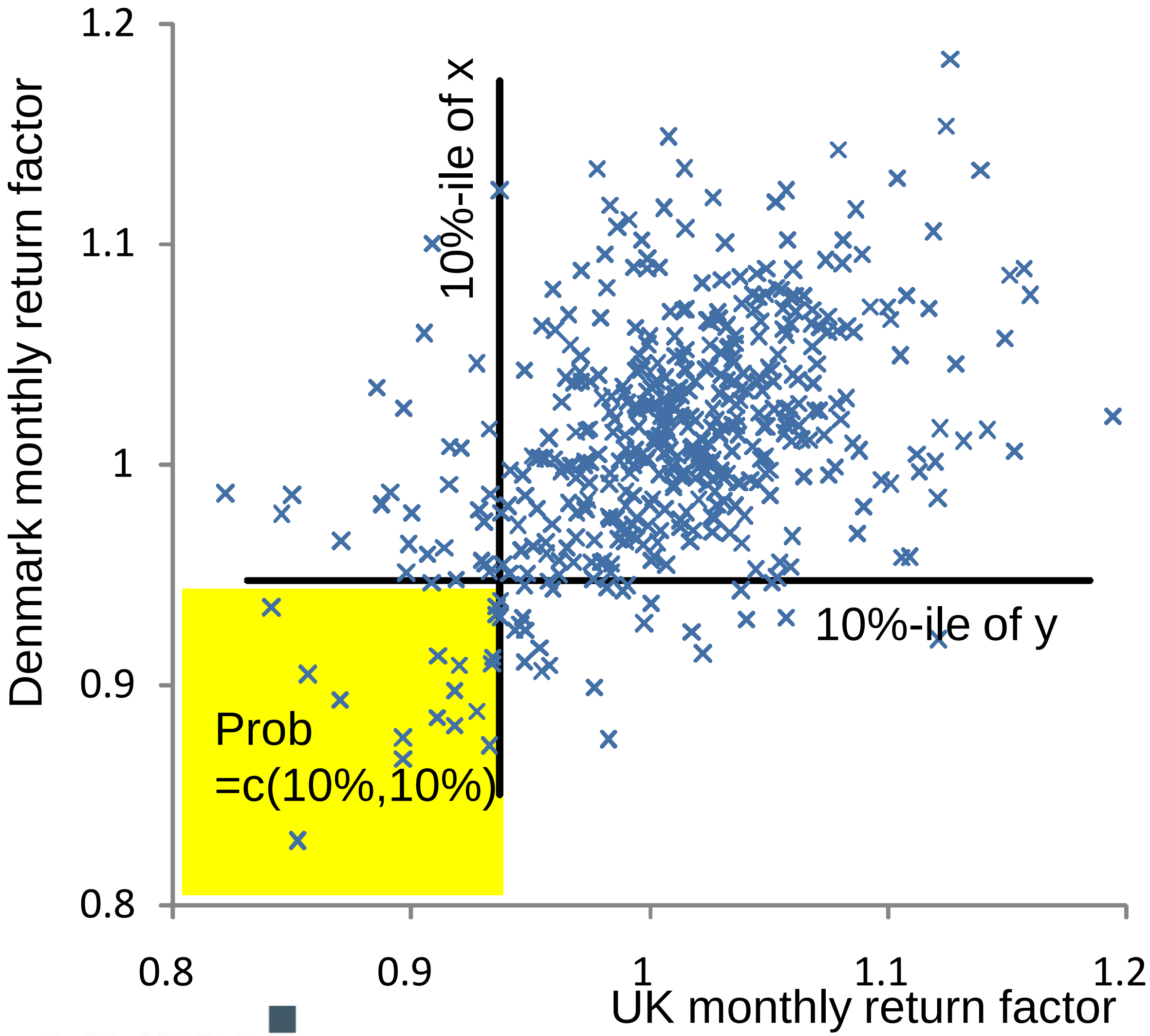
Heteroscedastic Regression: Denmark vs UK



Regression of squared residuals as a quadratic polynomial in x suggests that large UK moves in either direction increase the conditional variability of Danish returns.

Fitting Copula Functions

Copula Function Definition



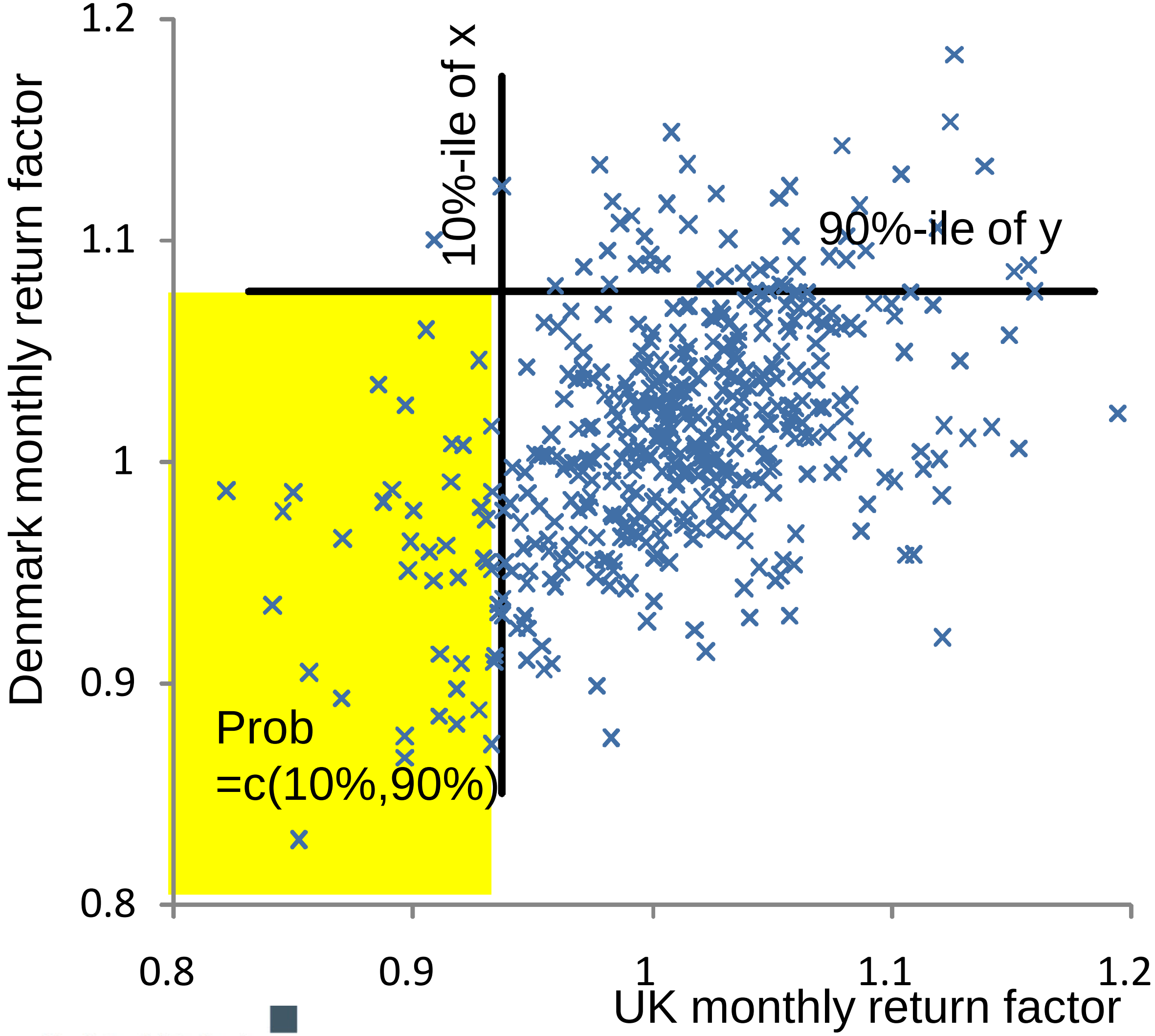
Let 0<a,b<1. The copula function c(a,b) is the proportion of the data points for which

- x lies below its *a*'th quantile
- and y below its b'th quantile.

The chart shows the calculation of c(10%,10%).

If x and y are independent then c(10%,10%)=1%.

Definition of c(10%, 90%)



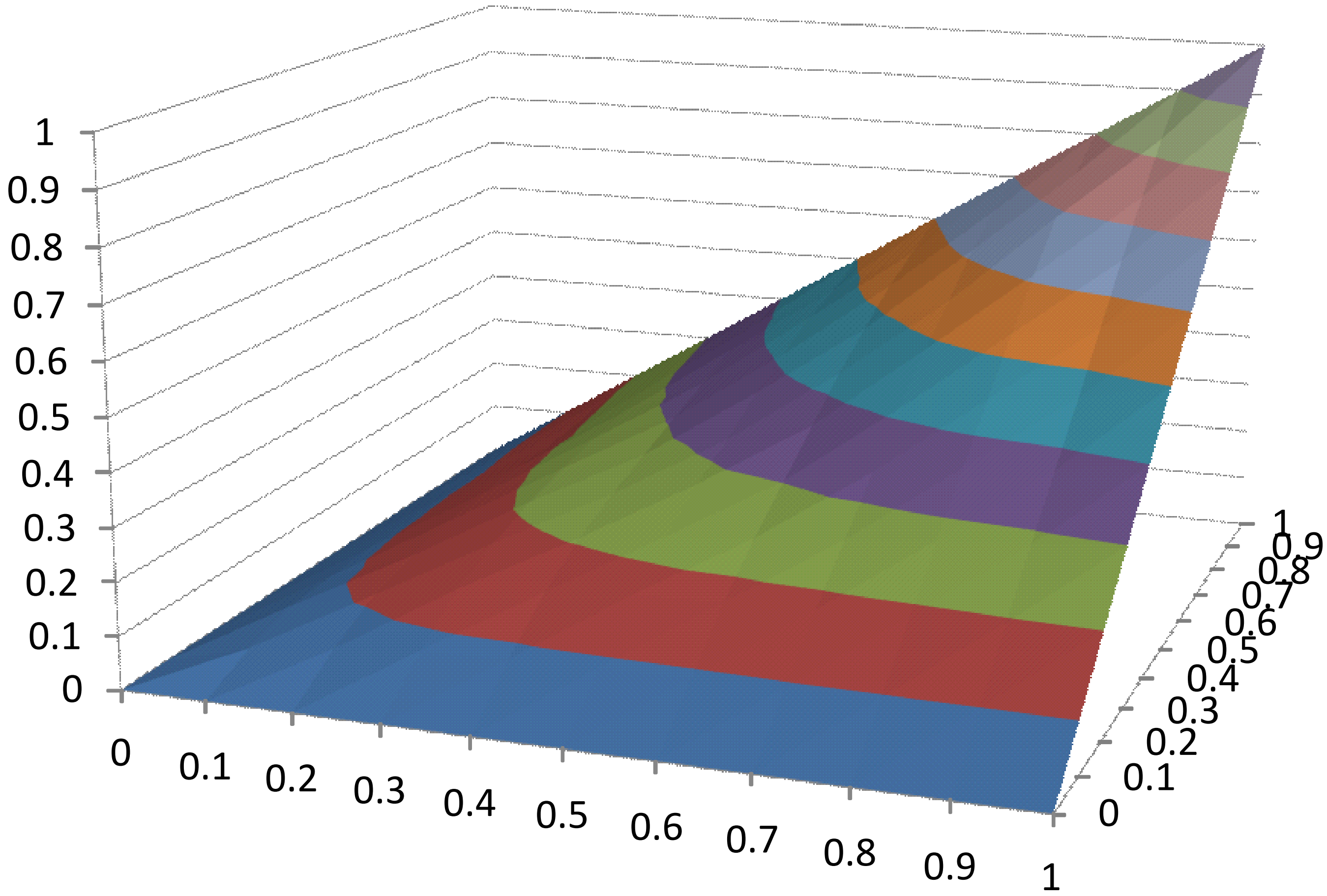
Let 0<a,b<1. The copula function c(a,b) is the proportion of the data points for which

- x lies below its *a*'th quantile
- and y below its b'th quantile.

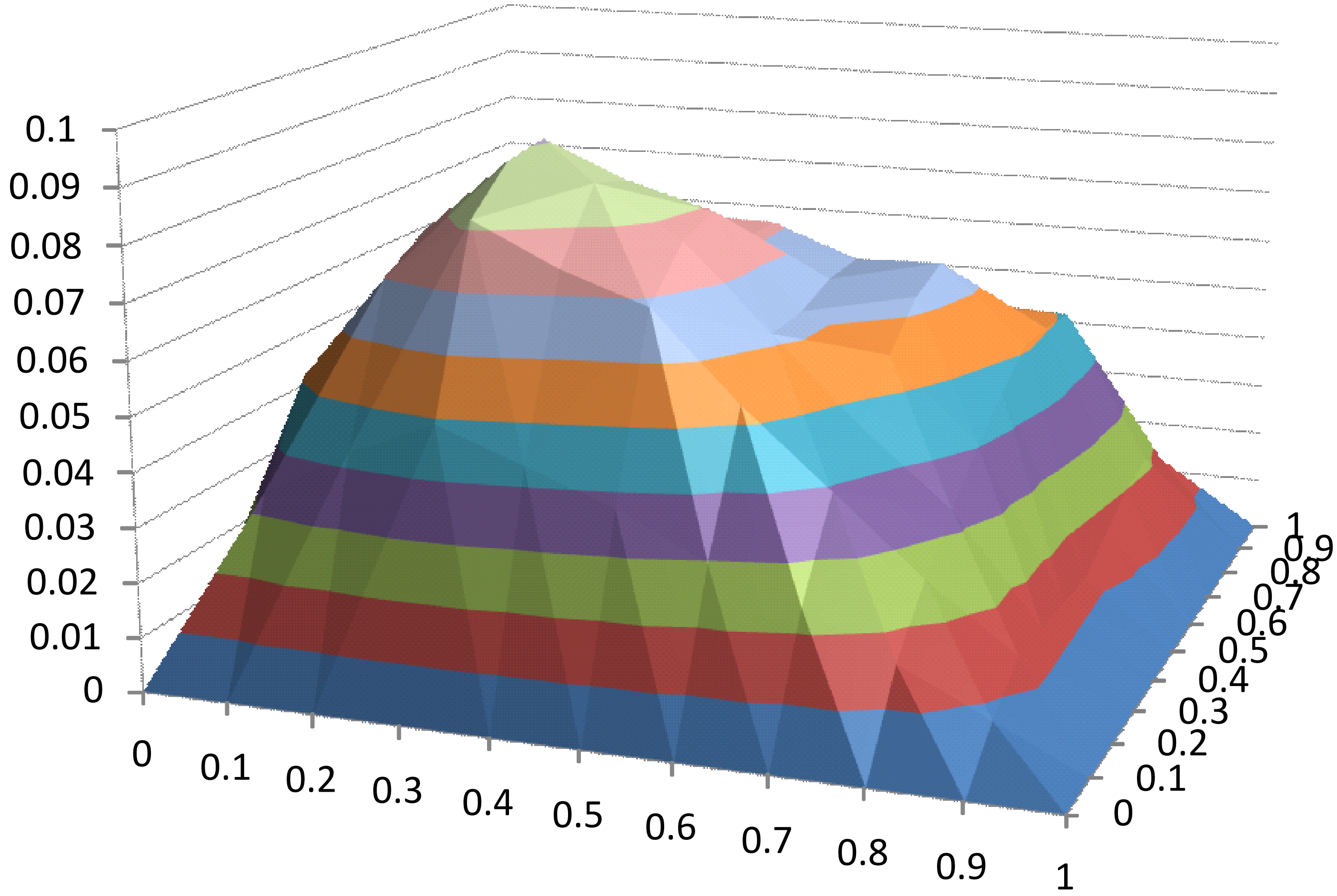
The chart shows the calculation of c(10%,90%).

If x and y are independent then c(10%,90%)=9%.

Empirical Copula Function



Empirical Copula minus Independent Copula



Copula Approaches: Strengths and Weaknesses

Strengths

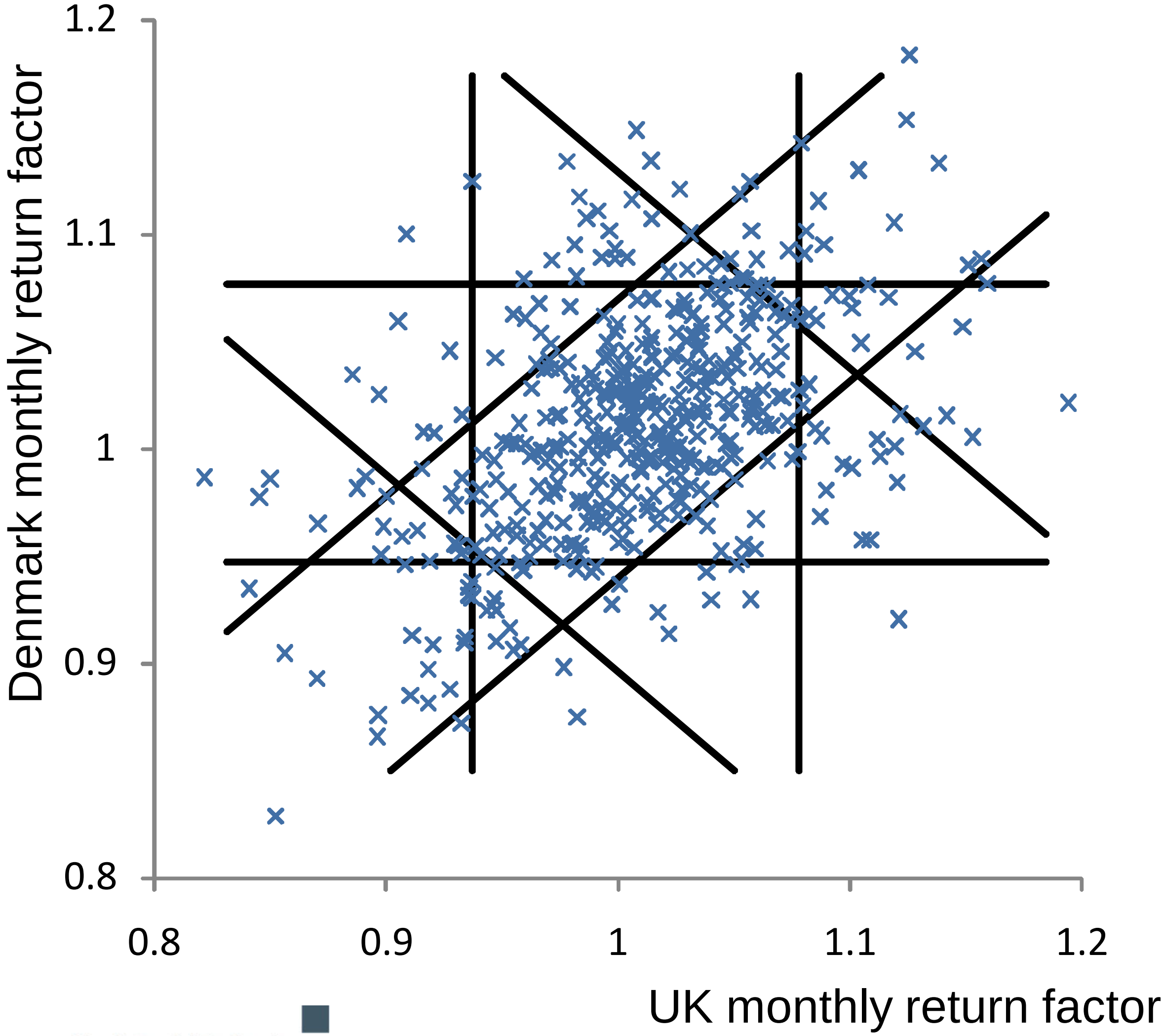
- Invariant under increasing transforms of x and y (for example, taking logs)
- Captures all the information in the dependency structure without reference to marginal distributions
- Allows unconstrained choice of marginal distributions
- Suitable for Monte Carlo

Weaknesses

- May be difficult to find copula functions to capture specific data features
- For example, two-way concave or heteroscedastic
- Seldom amenable to analytical calculations

Capital Requirements and Quadrant Correlations

90%-ile lines

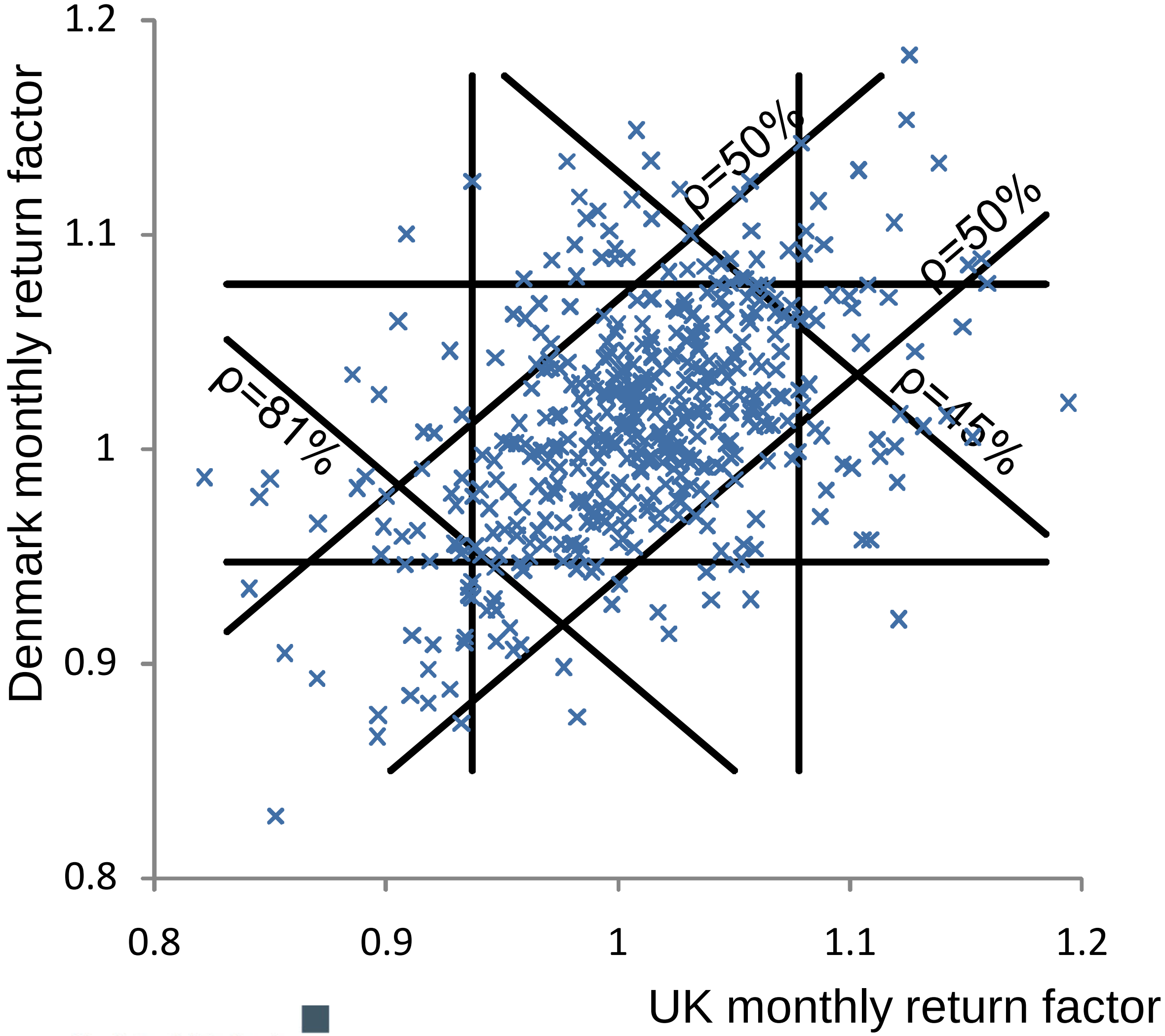


Each line separates 90% of the data from 10% of the data.

Aggregation Formula

- Consider a confidence level α with $\frac{1}{2} < \alpha < 1$
- Denote quantiles by $q_{1-\alpha}$ and q_α
- Let X and Y be risk factors
  - With $q_{1-\alpha}(X) = q_{1-\alpha}(Y) = -1$
  - And $q_\alpha(X) = q_\alpha(Y) = 1$
- Then, for elliptical distributions with correlation ρ:
  - Sums: $q_\alpha(X+Y) = -q_{1-\alpha}(X+Y) = \sqrt{(2+2\rho)}$
  - Differences: $q_\alpha(X-Y) = -q_{1-\alpha}(X-Y) = \sqrt{(2-2\rho)}$
- This gives four ways (for any α) for estimating correlation ρ

Solved Values of Quadrant Correlations



These are the correlations which, when substituted into a "correlation and square root" aggregation formula, gives the correct capital requirement.

Note the higher correlation in the South-West corner. Some would interpret this as correlations increasing in adverse situations, ie equity market falls.

For context, the Pearson correlation between UK and Denmark returns is 46%.

Half Space Depth Approach
Strengths and Weaknesses

Strengths

- Measures regions that likely correspond to insurers' ruin regions (half-spaces rather than quadrants)
- Captures different correlation effects in four quadrants
- Formulas easy to calculate
- Visual representation assists communication

Weaknesses

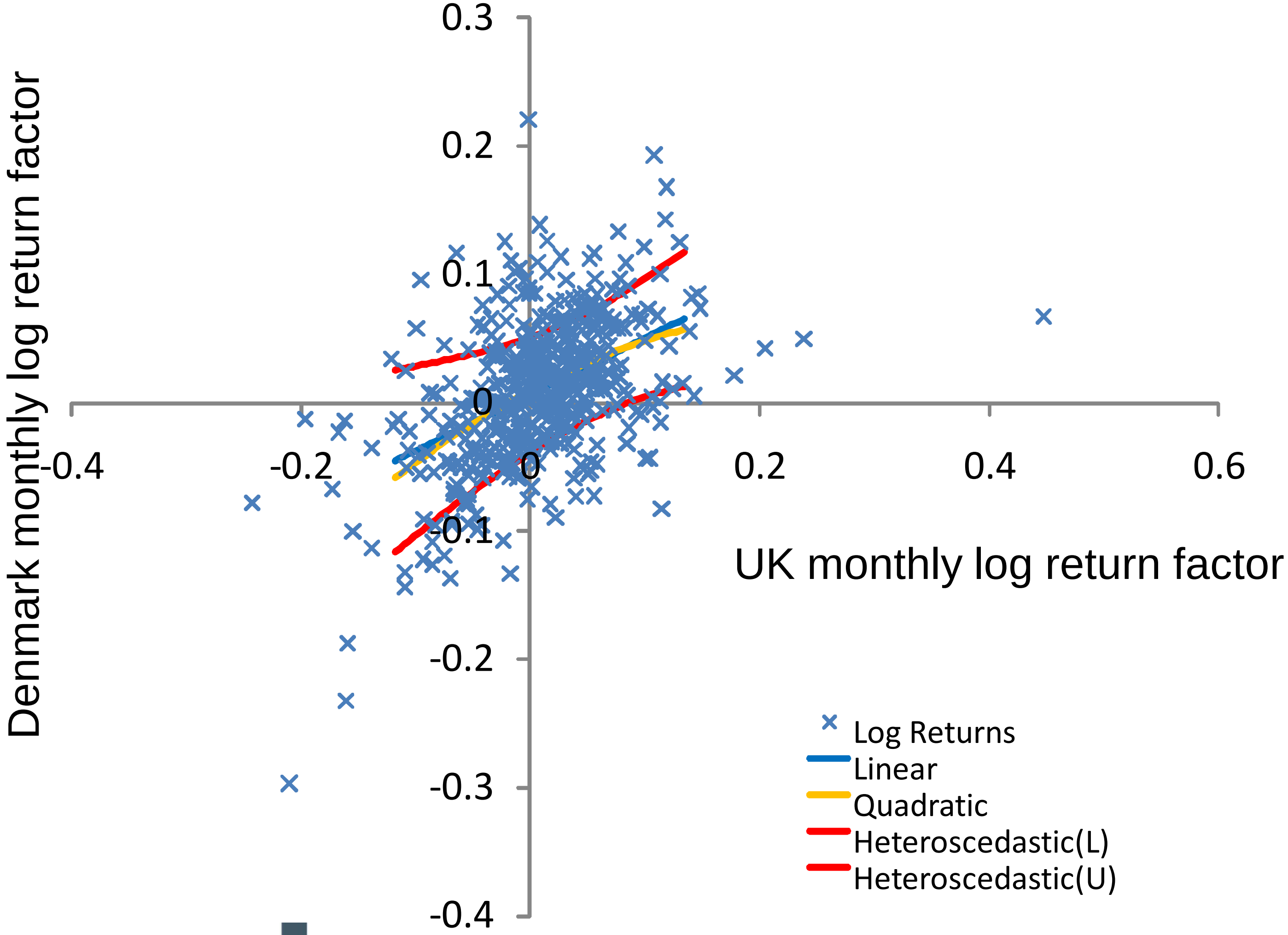
- Relies on firm's risks exposures being linear
  - This is about a firm's asset and liability valuation function, and is nothing to do with linear/nonlinear dependency in risk drivers
- Extension to other linear combinations or multiple risks involves interpolation
- Not easy to implement with Monte Carlo

Transformed and Standardised Data

Thought Experiment

- Suppose that $\ln X$ and $\ln Y$ are bivariate normal, with common standard deviation σ, mean $-\frac{1}{2}\sigma^2$ and correlation ρ.
- Then (properties of lognormal distribution) $\mathbf{E}(X)=\mathbf{E}(Y)=1$
- Also $\mathbf{E}(Y|X) = X^{\rho}\exp\{\frac{1}{2}\rho(1-\rho)\sigma^2\}$
- For $0<\rho<1$, the power X^{ρ} is concave.
  - Same also applies with *X* and *Y* interchanged
- Could the two-way concave effect simply be a consequence of a logarithmic transformation?
- Check this by analysing log transformed data

Analysis of Logarithmic Data

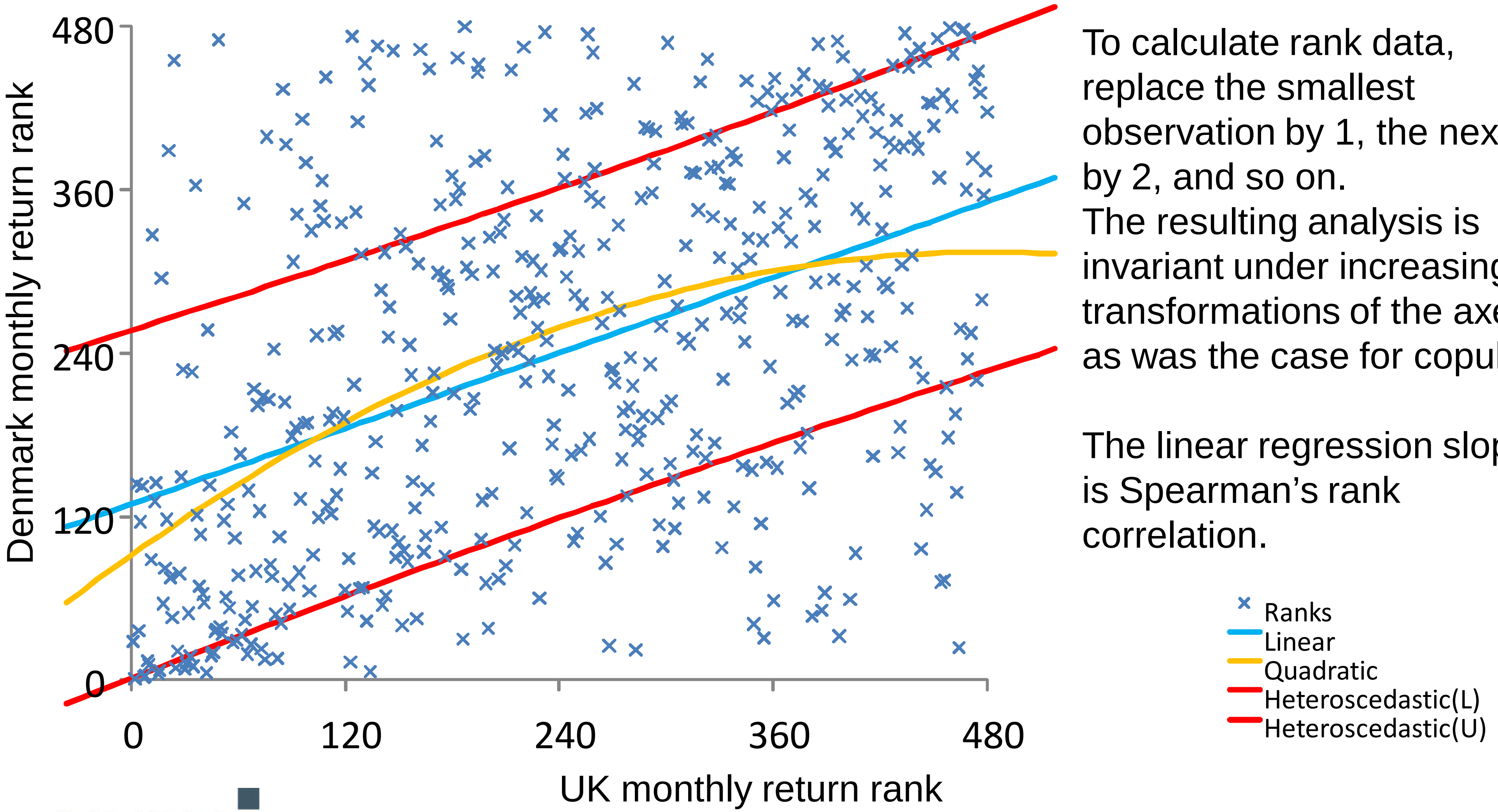


Taking logs does not remove the two way concave effect.

We could try many other monotone functions in an effort to get eliminate the effect.

Better still is to define dependency measures that are invariant under monotone axis transformations.

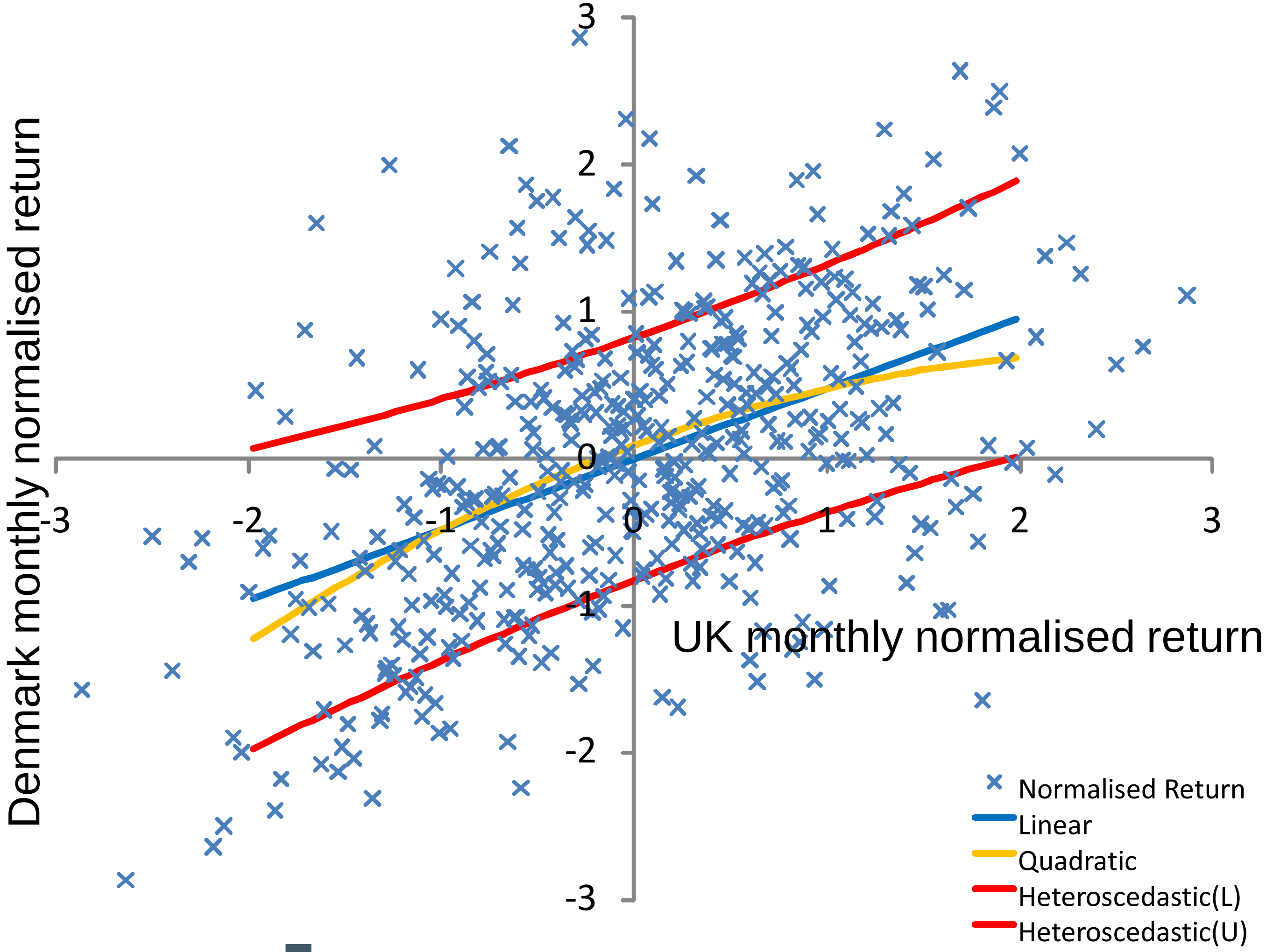
Analysis of Rank Data



To calculate rank data, replace the smallest observation by 1, the next by 2, and so on.
The resulting analysis is invariant under increasing transformations of the axes, as was the case for copulas.

The linear regression slope is Spearman's rank correlation.

Analysis of Normalised Data
Replace rank r by Normsinv(r/481)



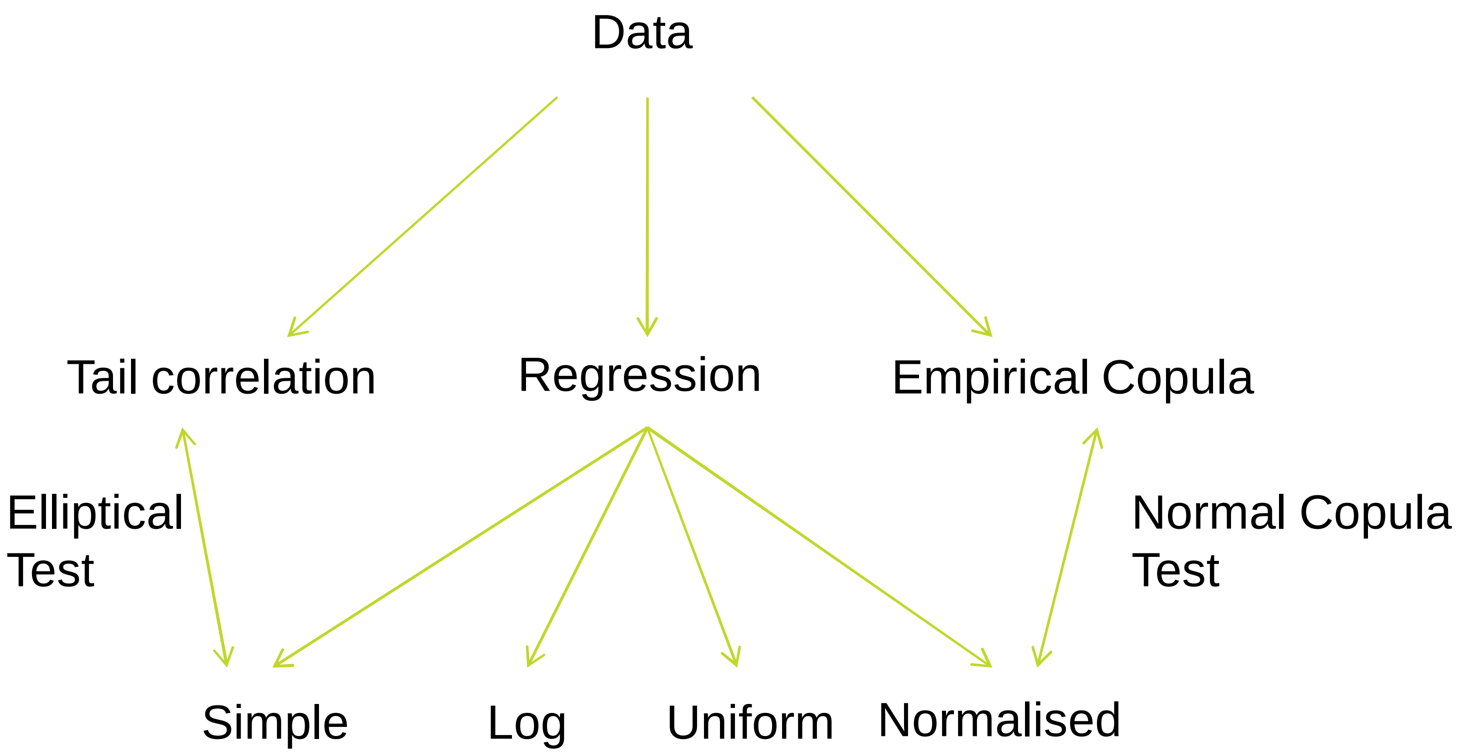
Like the rank analysis (and the copula construction) the normalised analysis is invariant under increasing transformations of the axes.

If returns were related by a Gauss copula then the normalised data should show no signs of non-linear dependence.

Empirical data does not support this, as there is evidence of concave and heteroscedastic effects.

Classification of Dependency Models

Methods of Analysis



Elliptical Distributions vs Gauss Copula

Elliptical Distributions

- Test tail correlations all equal
- Or, test zero convexity in quadratic regression, and symmetry in heteroscedastic regression
- Two-way concave effect not captured
- Models underlying the "correlation and square root method" for risk aggregation.

Gauss Copula

- Test by transforming risks to normal marginals, then verifying that non-linearity (convexity, heteroscedasticity) are absent
- Two way concave-effect and observed heteroscedasticity not captured
- Capital calculations by Monte Carlo

Multiple Comparisons and Stylised Equity Facts

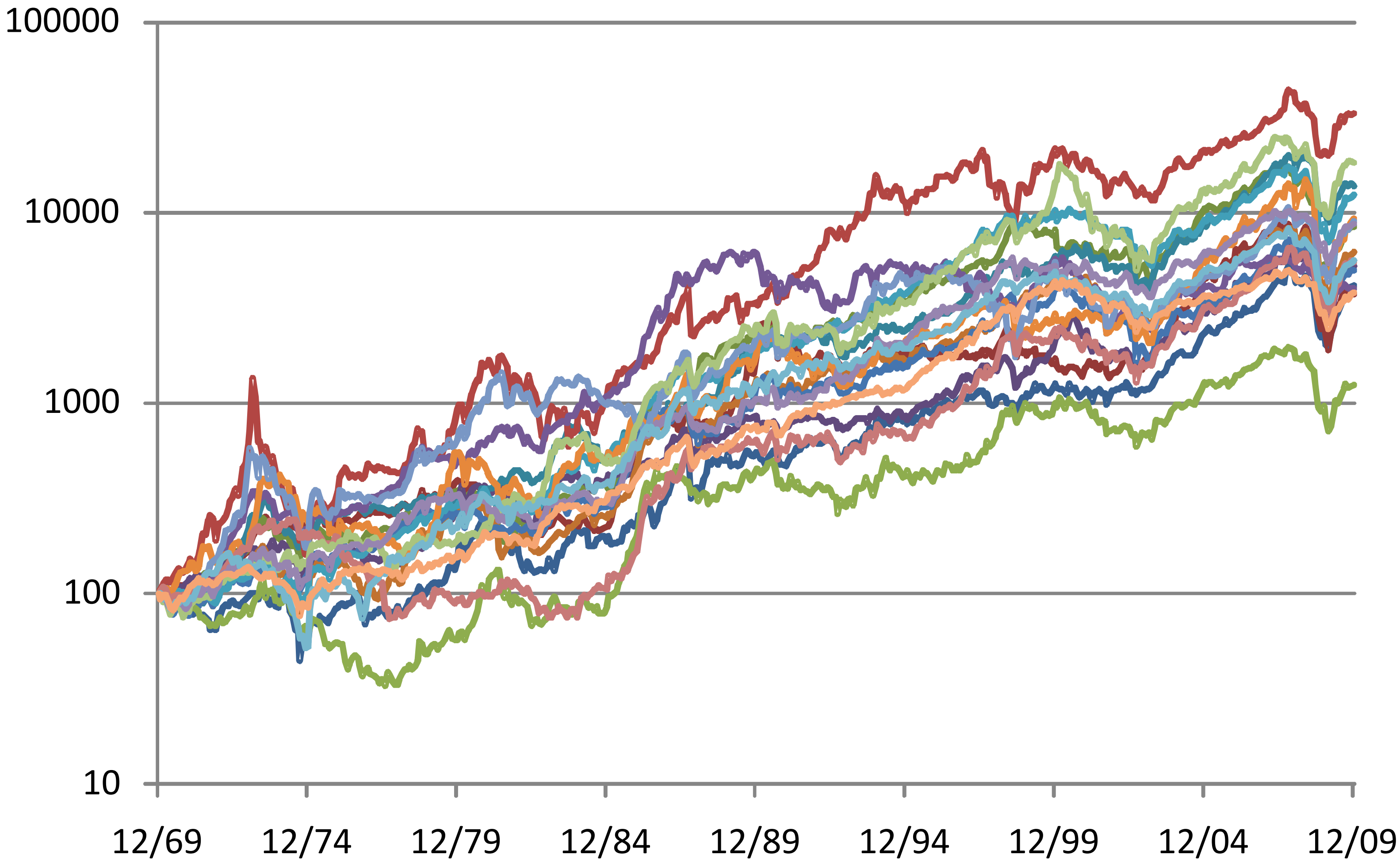
Our Chosen Data Set

- MSCI equity indices
- 31/12/1969 – 31/12/2009
- Monthly total return indices, coverage for 480 months
- In US Dollars
- 18 series representing different countries

  Countries represented: Australia, Austria, Belgium, Canada, Denmark, France, Germany, Hong Kong, Italy, Japan, Netherlands, Norway, Singapore, Spain, Sweden, Switzerland, UK, US

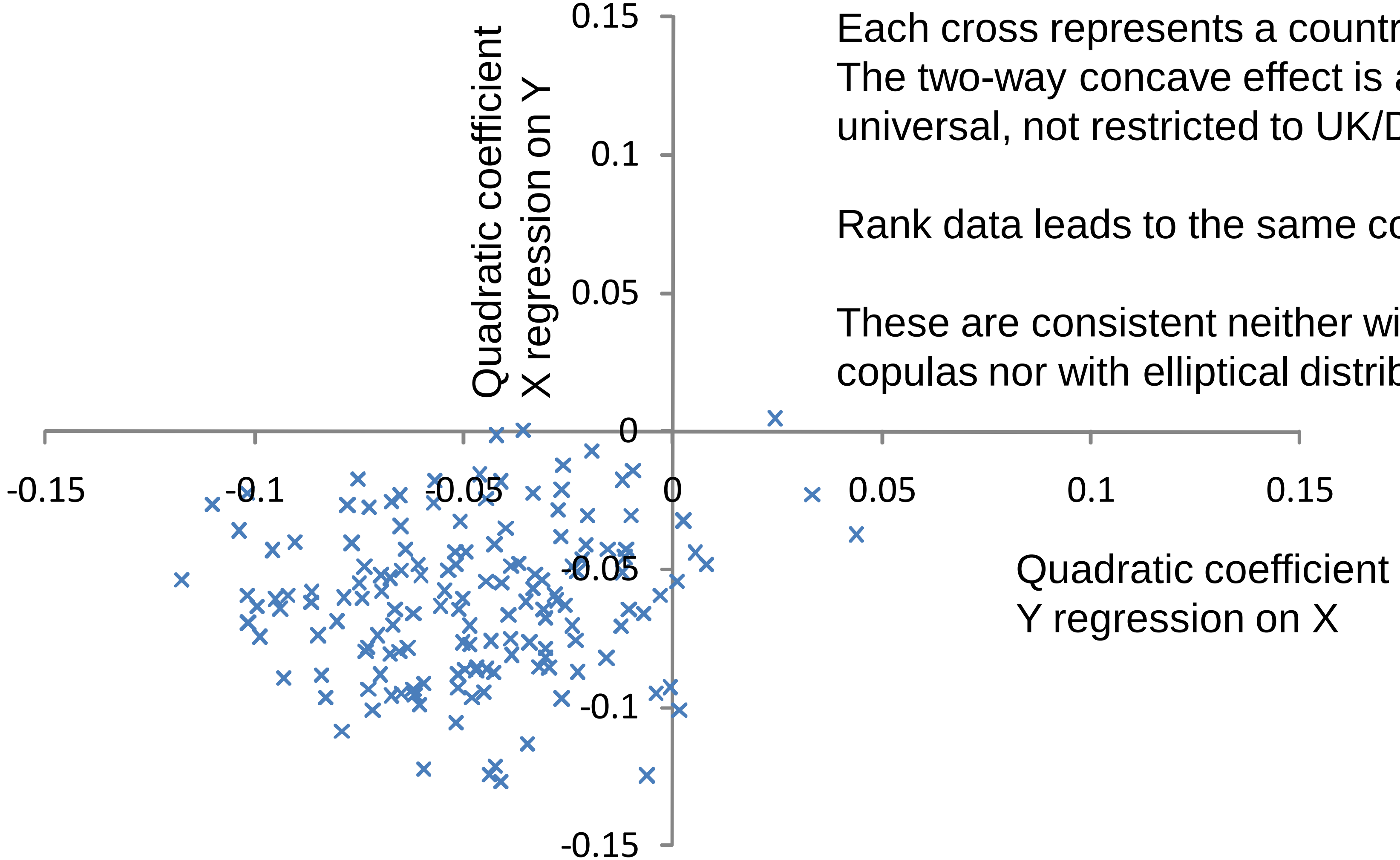
- In this presentation we analyse only two-dimensional dependency. There are 153 pairs of countries for which this can be analysed. In the charts that follow, each country pair is represented by one point.

Equity Total Return Data 1970-2010



Source: Datastream

Two Way Concave Effect

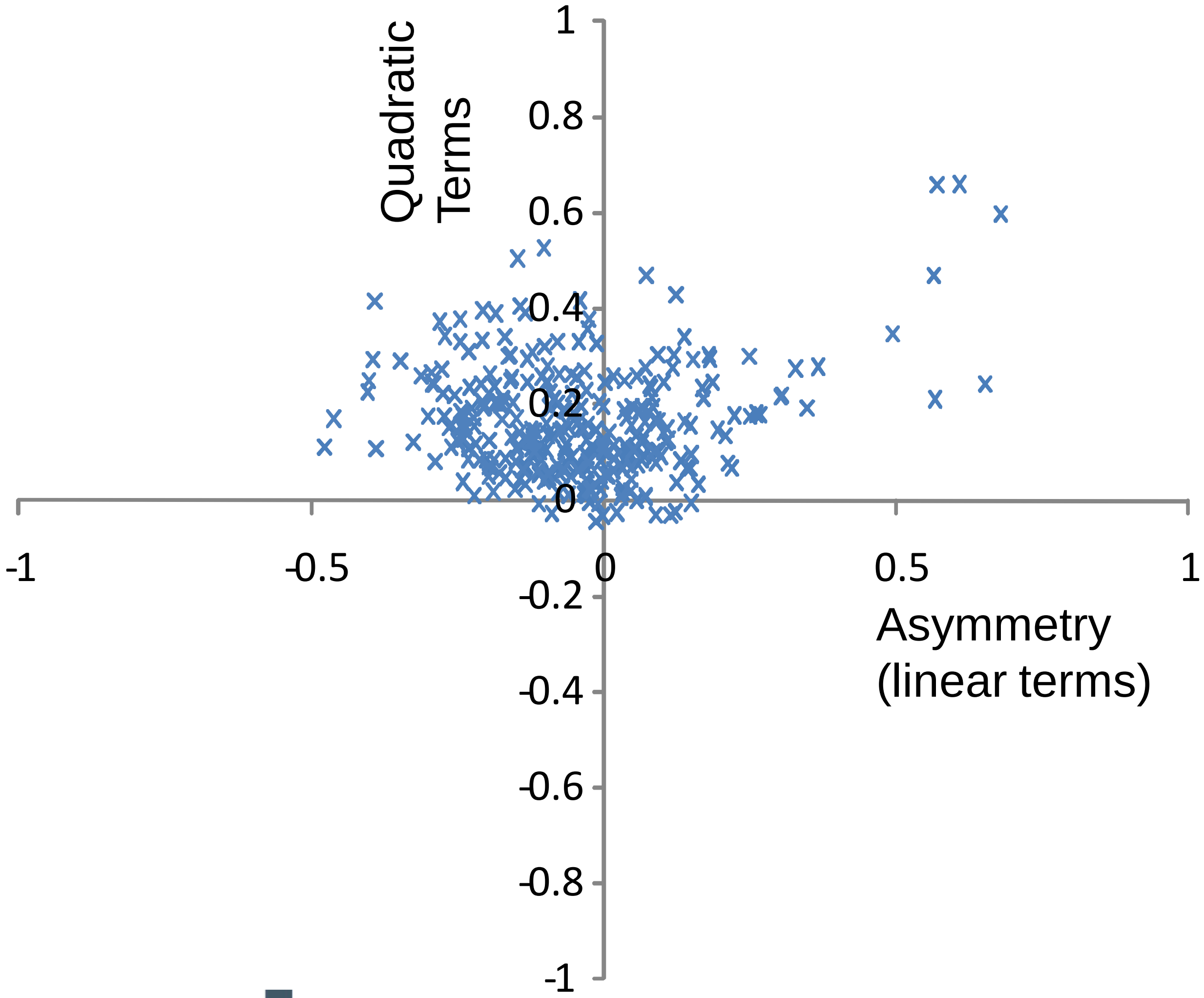


Each cross represents a country pair.
The two-way concave effect is almost universal, not restricted to UK/Denmark.

Rank data leads to the same conclusions.

These are consistent neither with Gauss/T copulas nor with elliptical distributions.

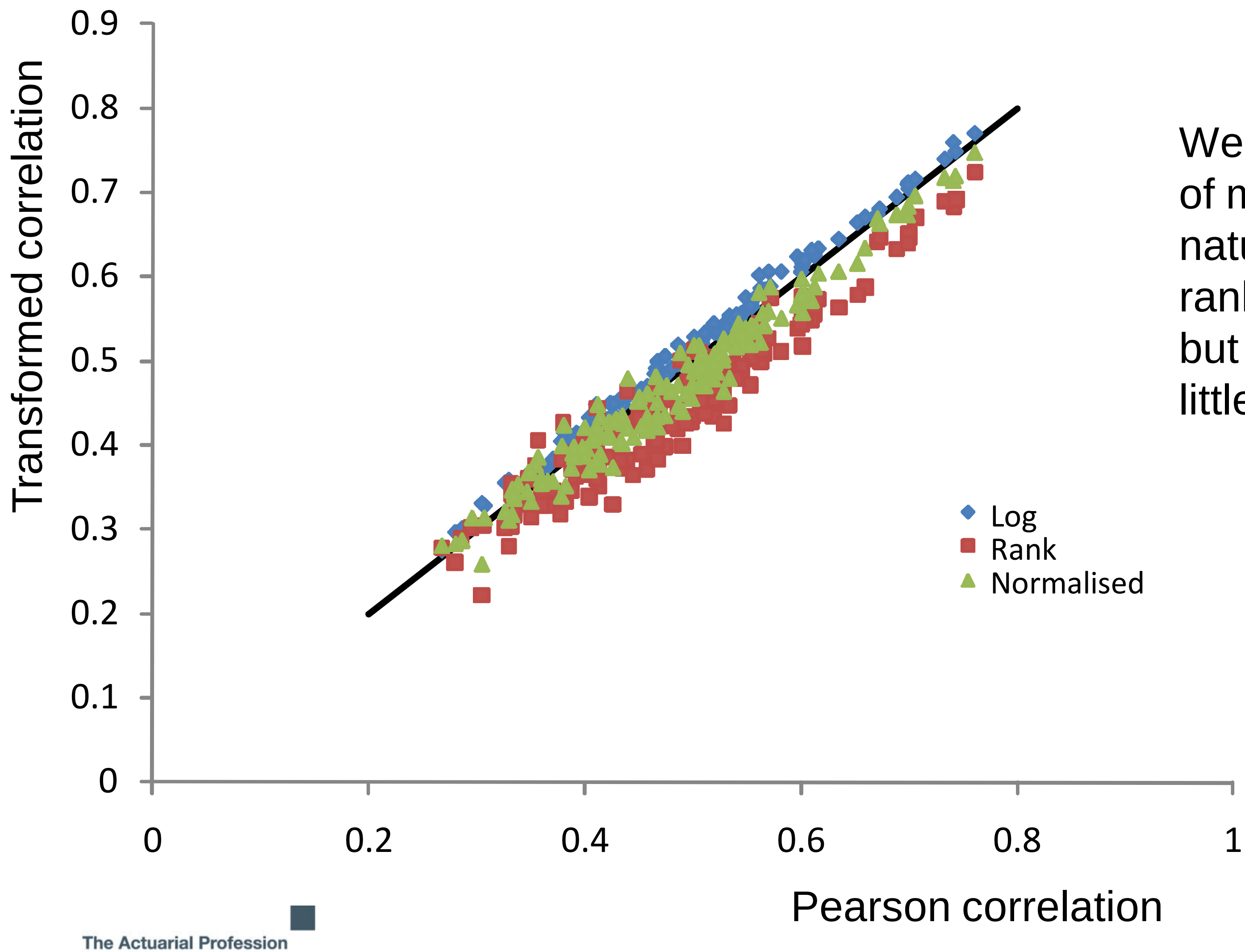
Heteroscedasticity



There is a clear bias to higher conditional variance of Y given extreme value of X. There is little evidence of systematic asymmetry.

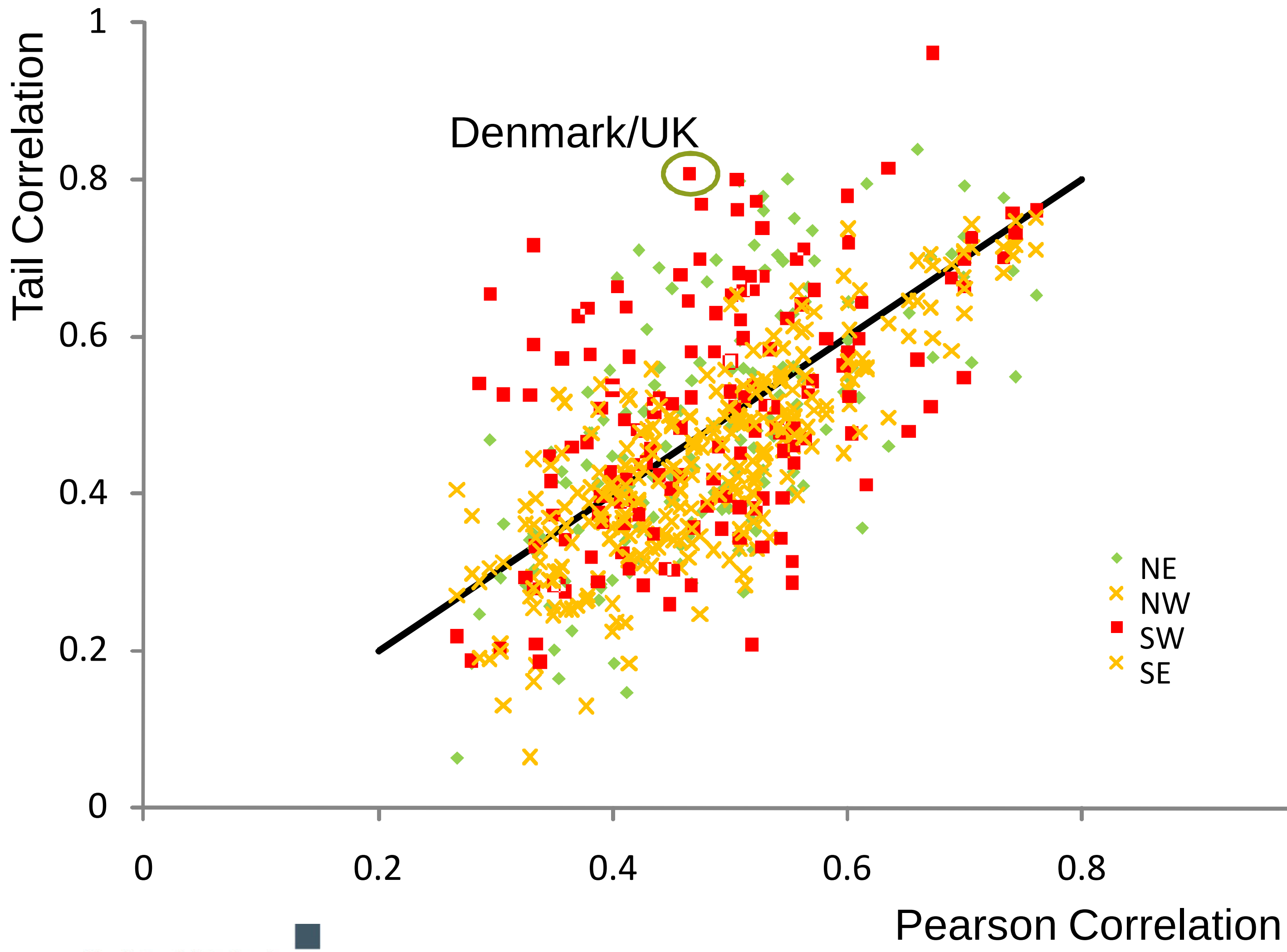
This is consistent with elliptical distributions but not with bivariate normal distributions.

Axis Transformation Effect on Correlations



We can debate relative merits of measuring correlations for natural returns, log returns, ranks or normalised returns, but numerically this makes little difference.

Tail Correlations (at 90% confidence)



Denmark/UK is an outlier. In general this plot offers little evidence that correlations increase specifically in bad outcomes.

Are Pearson correlations a good enough guide to tail correlations for risk aggregation?

Questions or comments?

Expressions of individual views by members of The Actuarial Profession and are encouraged.

The views expressed in this presentation are solely those of the presenters.

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