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THE RATIONALE OF THE USE OF THE GEOMETRIC AVERAGE AS AN INVESTMENT INDEX

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THE Actuaries' Investment Index is computed on the basis of the geometric average of the market prices of the securities constituting the index. Thus, if the index at time 0 is 1, then the index at time t is

$$\left(\frac{P_{1,t}\times P_{2,t}\times P_{3,t}\times \ldots \times P_{n,t}}{P_{1,0}\times P_{2,0}\times P_{3,0}\times \ldots \times P_{n,0}}\right)^{1/n},$$

where $P_{1,t}, P_{2,t}, P_{3,t}, \ldots, P_{n,t}$ are the prices at time t of the n securities comprised in the index.

The choice of the geometric average rather than, say, the arithmetic average as the basis of the index was made on various grounds (see for instance A. C. Murray, *The compilation of price index numbers and yield statistics relative to Stock Exchange securities*, *T.F.A.* Vol. XIII, p. 97), but the choice has generally been regarded as an arbitrary one. The arithmetic average

$$\frac{1}{n}\left(\frac{P_{1,t}}{P_{1,0}}+\frac{P_{2,t}}{P_{2,0}}+\frac{P_{3,t}}{P_{3,0}}+\ldots+\frac{P_{n,t}}{P_{n,0}}\right)$$

can be truly said to depict over the course of time the result of investing at time 0 equal amounts in each of the n securities and of continuing to hold without change the investments so made. But, so far as is known, it has never been suggested that the geometric average has any such practical meaning.

It can, however, be shown that the geometric average has a real interpretation. It represents, in fact, the result of investing at time 0 equal amounts in each of the n securities and of keeping the market values of the n investments so made mutually equal by following the rule that, on a change in prices which makes the values unequal, then purchases and sales are carried out in order to restore the equality. For instance, if one of the securities rises in price while the others remain unchanged, then a part of the holding of the security which has appreciated (to the value of $(n-1)/n$ of the appreciation that has occurred) is sold and the proceeds are invested equally among the remaining $n-1$ securities. It is necessary to assume that all movements in price are continuous and that all purchases and sales are carried out continuously. The proof is outlined below.

2. Consider the position when the price (denoted by P) of one of the securities changes while the prices of the others remain constant. The value of the whole portfolio, V, may be regarded as a function of P.

Suppose that P and V change to $P+\delta P$ and $V+\delta V$ respectively, where δP and δV are small.

The value of each holding is V/n. When the price of one of the securities changes to $P+\delta P$ the value of the holding of this security becomes

$$\frac{1}{n}V\frac{P+\delta P}{P}=\frac{1}{n}V+\frac{1}{n}V\frac{\delta P}{P}.$$

Thus the increase in the value of the portfolio of investments is $\frac{1}{n}V\frac{\delta P}{P}$,

or
$$\delta V=\frac{1}{n}V\frac{\delta P}{P},$$
i.e.
$$\frac{1}{V}\frac{\delta V}{\delta P}=\frac{1}{nP}.$$
Proceeding to the limit as $\delta P\to 0$, we have
$$\frac{1}{V}\frac{dV}{dP}=\frac{1}{nP},$$
therefore
$$\frac{d}{dP}(\log V)=\frac{1}{nP},$$
whence, by integration,
$$\log V=\log P^{1/n}+c,$$
therefore
$$V=kP^{1/n}.$$

If, therefore, the price of one security changes from P to P′ while the prices of the others remain unchanged, then the value of the whole portfolio will change in the ratio $(P'/P)^{1/n}$.

We may now contemplate a series of changes of price—first of one security, then of another, in any order, upwards or downwards, large or small. Reverting to the original notation, the result at time t will be that the value of the portfolio has been altered in the ratio

$$\left(\frac{P_{1,t}}{P_{1,0}}\right)^{1/n}\times\left(\frac{P_{2,t}}{P_{2,0}}\right)^{1/n}\times\left(\frac{P_{3,t}}{P_{3,0}}\right)^{1/n}\times\ldots\times\left(\frac{P_{n,t}}{P_{n,0}}\right)^{1/n},$$

i.e. in the ratio of the index computed on the basis of the geometric average of the prices.

3. It can easily be seen that if the amounts originally invested in the n securities are in the ratios $w_1:w_2:w_3:\ldots:w_n$, and if the values of the holdings are kept in these ratios, then the value of the portfolio at time t will be

$$\left[\left(\frac{P_{1,t}}{P_{1,0}}\right)^{w_1}\times\left(\frac{P_{2,t}}{P_{2,0}}\right)^{w_2}\times\left(\frac{P_{3,t}}{P_{3,0}}\right)^{w_3}\times\ldots\times\left(\frac{P_{n,t}}{P_{n,0}}\right)^{w_n}\right]^{1/(w_1+w_2+w_3+\cdots+w_n)}$$

This justifies the use of a weighted geometric average.

4. It has been held to be one of the superiorities as an index of the geometric average over the arithmetic average that the ratio between the geometric averages at any times t and t' depends only on the prices of the securities at those two times and is independent of the prices at time 0, i.e. at the base-date. This can be seen from the algebraic formula for the index. It can also be seen from the representation of the index as a portfolio of n investments which are kept mutually equal in value, for the prices at which the investments were originally made obviously have no bearing on the ratio in which the value of the portfolio changes between any two subsequent dates. With the arithmetic average, on the other hand, the ratio depends on the relative amounts held of the n securities, and these cannot be divorced from the original purchase prices.

5. Another interesting exercise is to consider the case where an amount $1/n$ is invested in each of n securities, the value of each holding being kept at $1/n$ by selling some or buying more of a security if its price rises or falls. It will be found that at time t the net cash profit or loss that has been realized is the natural logarithm of the geometric index. According as the index is greater or less than unity, then its logarithm is positive or negative, representing either the net profit realized or net additional sum that has been invested.