A STOCHASTIC APPROACH TO RISK MANAGEMENT AND DECISION MAKING IN DEFINED BENEFIT PENSION SCHEMES

[Presented to the Institute of Actuaries, 27 January 2003]

Introduction to the Paper

The bulk of the work of the Pensions Board is reactive to changes in legislation and other influences. However, the Board also has a longstanding objective of being proactive in commissioning research into areas which will take forward professional thinking and practice. More recently it has adopted an objective to bring together the assets and liabilities in the actuaries work. This paper is therefore entirely consistent with the Board's objectives.

In meeting the terms of reference set by the Board, the authors have brought together funding and asset allocation decision-making; have switched attention from valuation to projection of cash flows up to key decision points and have introduced a range of risk and performance measures. We believe that many of the ideas and their proposed application will be novel to the majority of practising actuaries.

The paper is timely. The Myners proposal of a scheme specific benchmark has been included in the Green Paper recently published for consultation. In pursuit of its public interest duty, the profession needs to respond. Whilst the authors themselves acknowledge that more work needs to be done, the Pensions Board welcomes the authors' contributions towards meeting the challenge posed by scheme specific funding plans.

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A STOCHASTIC APPROACH TO RISK MANAGEMENT AND DECISION MAKING IN DEFINED BENEFIT PENSION SCHEMES

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ABSTRACT

The trustees and sponsors of defined benefit schemes rely on the advice of the Scheme Actuary to make important decisions concerning the funding of the scheme, the investment of its assets, and the use of surplus assets to improve benefits. These decisions have to be made in the face of considerable uncertainty about financial and demographic factors that will affect the future experience of the scheme and its success in meeting various objectives.

The traditional actuarial valuation combined with actuarial judgement has played an important role in guiding decision making; but we argue that stochastic methods can add value in certain crucial areas, in particular the financial risk management of defined benefit schemes. Rather than dealing with risk by incorporating margins in the valuation basis, a stochastic approach allows the actuary to evaluate specific and quantifiable risk and performance measures for alternative funding and investment strategies.

This paper recommends a framework that, when combined with a suitable stochastic model, measures the risks inherent in contribution rate and asset allocation decisions, allowing better decisions to be made. In doing this, we suggest and apply various risk and performance measures that may be thought appropriate, although our intention is to illustrate their use rather than prescribe them as objective standards. The framework provides the means to explore the trade-offs involved in possible contribution and asset allocation decisions, and points to decision strategies expected to give improved outcomes for the same level of risk. A feature of the approach that marks it out from current asset/liability techniques is that it examines the funding and investment decisions together. It does not derive a contribution rate in the traditional way, but leaves this as free variable, in the same way that the investment decision is taken to be a free variable. Another distinctive feature of our framework is that it is based on projection rather than on valuation, involving stochastic simulation of the experience of the scheme over a time horizon reflecting the concerns of the trustees and the sponsoring employer.

The paper provides a case study (based on a model final salary pension scheme) showing the advantages of the framework, and goes on to explain how the results may practically be communicated to trustees and scheme sponsors.

KEYWORDS

Defined Benefit Pension Schemes; Projection; Stochastic Simulation; Risk and Performance Measures; Risk Management; Decision Making

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1. Objectives

1.1.1 In media coverage and around the board tables of companies and at trustee meetings, the words 'pension' and 'risk' are increasingly familiar bedfellows; but how good are our traditional actuarial techniques at unearthing the risk vs. reward trade-offs in decisions that concern funding, investment and benefit design?

1.1.2 Our view is that, although stochastic techniques appear in certain areas, such as investment strategy decisions, their penetration is far too low. In this paper we make what we believe is a strong case for the widespread use of stochastic techniques in all aspects of pensions actuarial work. Our case rests on the premise that understanding the distribution of possible values will lead to better decision making than would be the case if a conventional, deterministic approach were used. We hope that, after you have read this paper, you will agree!

1.2 The Stochastic Valuation Working Party was set up by the Pensions Board in August 2000, with a remit to investigate stochastic valuation methods for defined benefit pension schemes and the assessment of risk. The main objectives of this paper are in line with the detailed terms of reference given to us by the Pensions Board, and are as follows:

(1) to investigate the application of stochastic methods to the provision of advice on the valuation of pension schemes;
(2) to develop methods of decision making based on stochastic projections for the future experience of the scheme;
(3) to address the ways in which actuaries should communicate the risks and rewards of different financial strategies; and
(4) to recommend what should be included in reports to clients on valuations that have adopted a stochastic approach.

1.3 The plan of this paper is as follows. Section 2 considers the areas of actuarial advice for defined benefit pension schemes where a stochastic approach could be introduced. Section 3 discusses the needs and perspectives of the different stakeholders involved in defined benefit pension schemes. Section 4 presents our key recommendations for a stochastic approach to decision making. Thus, Sections 2, 3 and 4 together address objectives (1) and (2). Section 5 presents an illustrative modelling framework, which we use for implementing the stochastic approach, and this is followed by a detailed case study (in Section 6) to exemplify the use of the risk and performance measures that we are advocating. Section 7 then looks at the important issue of presenting and communicating the results arising from this new methodology. Section 8 provides some conclusions. A series of appendices (some technical) supplement the main argument.

1.4 Thus, the new recommendations in the paper are contained in Section 4, and their implementation is discussed via the case study of Section 6. We are proposing a number of new approaches to risk measurement and decision making for pension schemes. Firstly, we suggest that attention is switched from the traditional concept of a valuation, and the associated issues of calculating expected present values, to stochastic projections up to key time horizons, which may coincide with the times at which important decisions are to be taken; for example, the next date for recommending a contribution rate. Secondly, we propose that funding and investment strategies be considered simultaneously rather than separately. Thirdly, we advocate the use of risk and performance measures that recognise the one-sided nature of the solvency and contribution risks facing the scheme stakeholders: specifically, we introduce the mean shortfall and the excess contribution rate measures of risk and the average contribution rate level to measure performance (see Sections 4 and 6).

1.5 Sections 5 and 6 present a modelling framework and a case study that are intended to be for illustrative purposes only. We stress that the particular details of the model scheme are of lesser importance than the results and methods being demonstrated and discussed. We hope that these cautionary comments will persuade readers to consider the general aspects of our proposals rather than debate the choice of stochastic asset model or particular parameter values used.

2. Actuarial Advice where a Stochastic Approach could be of Benefit

2.1 The main subject of this paper is how stochastic techniques can aid the pensions actuary to help his or her client. However, before we consider the details of a stochastic approach, it is perhaps worth setting out the main areas within pensions practice where actuarial input is needed. For this purpose, we will take 'actuarial input' to mean the quantification of, or the giving of opinions on unknown future events, rather than the more general consultancy advice on legislation, etc. The main areas of actuarial input are funding; transfer values; expensing and regulatory requirements; and *ad hoc* calculations (as discussed by Head *et al.*, 2000). Each of these areas is commented on in the following paragraphs.

2.1.1 *Funding and investment*

Many aspects of actuarial input can be grouped under the broad heading of pension scheme funding. This is a wide-ranging topic, and includes the following:

— recommendation of a future contribution rate;
— communication of the likely stability of the chosen contribution rate;
— assessment of benefit security — the measure of benefit security will depend on the scheme circumstances and the party to which the advice is being given; possibilities include the adequacy of assets to meet:
  (a) the cost of buying out the scheme liabilities in the insurance market on scheme wind-up;
  (b) the cash equivalent transfer values;
  (c) the liabilities assessed by reference to the Minimum Funding Requirement (MFR) or its interim and eventual replacements; and
  (d) the ongoing scheme liabilities (with allowance for future salary increases);

— assessment of an appropriate investment strategy, taking account of the trustees' and/or sponsor's attitude to risk;
— communication of the impact of the interaction between the investment strategy and the contribution rate and benefit security referred to above;
— allocation of surplus between members and the scheme sponsor or between categories of member (e.g. active members and pensioners); and
— assessment of the impact of changing benefits and of the effect on the existing benefit structure of a changing economic or demographic background (lower inflation, for example).

2.1.2 *Transfer values*

Transfer values are of two main types:

— *Individual transfer values* for leavers (and divorcees). For the majority of United Kingdom pension schemes (and despite the best efforts of the Pensions Board!), individual transfer values are based on the method and assumptions underlying the MFR. The Government's decision to abolish the MFR is likely to increase attention on the basis underlying the calculation of cash equivalent transfer values.

— *Bulk transfer values*. The payment of a bulk transfer value can arise as a result of:
  (a) a sale or purchase of part or all of a company; or
  (b) a rationalisation or merging of pension schemes.

  The value of the pension asserts and liabilities can be large in relation to the size of a business sale or purchase. Expensing concerns and contribution rate risk, as well as getting the best possible deal, will be in the minds of scheme sponsors. On the other hand, security of the promise to members will be of concern to trustees. The actuary should be able to offer his or her client sound advice on each of these topics.

2.1.3 *Expensing*

2.1.3.1 Expensing refers to the calculation of pension scheme disclosures for inclusion on the sponsor's accounts. The flexibility inherent within the expensing methodology depends on the Accounting Standard which governs the presentation of accounts of sponsors.

2.1.3.2 Within the U.K., Financial Reporting Standard 17 (FRS 17) has increased the focus on pension scheme expensing. Whilst FRS 17 offers restricted flexibility in the calculation of the pension scheme costs and disclosures, the pension scheme valuation can provide a useful tool that assists in the understanding of the likely development of these measures over time. It is worth noting that expensing is not the same as funding, though the former can act as a constraint on the latter, perhaps to the extent where the expensing needs of the sponsor dictate the funding aspects. Of particular interest, therefore, is how funding based on any particular approach is likely to affect the expensing position.

2.1.4 *Regulatory requirements*

Currently, regulations are such that pension scheme valuations are required for the following purposes:
— to meet disclosure requirements;
— to comply with the MFR;
— to comply with surplus regulations; and
— to enable the pension scheme to contract out of the State Earnings Related Pension Scheme.

2.1.5 Ad hoc *calculations*

These include the setting of actuarial factors for early and late retirement and for cash commutation and the costing of benefit improvements.

2.2 Clearly, the reasons for carrying out a pension scheme valuation can be wide ranging. In addition to this, the party to whom the actuary reports is likely to have different priorities in relation to the pension scheme. These factors mean that the actuary should consider closely the objectives behind carrying out a pension scheme valuation in each specific case prior to determining the valuation method and (if applicable) the choice of assumptions.

2.3 In recent history, there have been a number of examples of actuarial work in both pensions and insurance where deterministic approaches have provided inadequate information regarding risk and the key decisions that need to be taken and where, had a stochastic approach been available and used, the outcome may have been vastly different.

2.4 We strongly believe that, for most of the varied pension actuarial activities outlined above, the use of stochastic techniques would substantially improve the level of understanding of risks and the quality of decision making.

3. Needs and Perspectives of Different Stakeholders

3.1 *Stakeholders in a Pension Scheme*

3.1.1 In this section, we consider the needs and perspectives of the various stakeholders in a defined benefit pensions scheme. There are a number of parties with an interest in the financial progress of a given pension scheme. The principal stakeholders in a typical U.K. pension scheme are:

— the sponsoring employer, and its direct stakeholders; and
— the members (including current pensioners), and their dependants;

where we take 'direct stakeholders' to refer to those whose current financial interests are directly affected by a decision affecting the pensions scheme (for example, current and future shareholders of the sponsoring employer).

3.1.2 There are, however, many others who may have an indirect interest in the scheme, including:

— potential future members/employees;
— creditors and debtholders of the sponsoring employer;
— suppliers to the sponsoring employer;
— government, including tax and social security authorities;
— Opra;
— advisers, including actuaries, lawyers, consultants, investment managers; and
— parties involved with other pension schemes who may be influenced by trends.

3.1.3 Each of these stakeholders has different needs and objectives relating to the pension fund, although only some have direct responsibilities or can directly influence decision making in relation to the financial management of the scheme.

3.2 *Different Responsibilities, Needs and Objectives*

The various stakeholders concerned have different responsibilities, and these lead to different needs and objectives, which may be illustrated as follows.

3.2.1 *Trustees*

Trustees are responsible for delivering the benefit promised. This translates to ensuring that the scheme's assets are invested appropriately and future funding is at an appropriate level to meet accruing benefits. In pursuit of benefit security, they will have to balance a desire for the maximum level of funding for accrued benefits against the ability and willingness of the sponsoring employer to support the scheme. Typical practice in the U.K. and in other countries is to fund pension schemes on an ongoing basis, i.e. assuming the scheme continues and taking account of an equity risk premium in assessing the 'right' level of assets to hold. An adequate level of benefit security in an ongoing scenario does not necessarily translate into an adequate level if the scheme is subsequently closed, and this gap presents a significant challenge for trustees.

3.2.2 *Sponsoring employer*

3.2.2.1 From the sponsoring employer's perspective, the pension plan is there to meet certain objectives in terms of attraction, retention, motivation of staff, etc. Although the promise is given in 'good faith', its delivery is not certain, because of uncertainties in investment markets and demography.

3.2.2.2 Against this uncertainty, the employer will have to make choices around the timing of the funding, of the form of 'less today and more tomorrow', or vice versa. Because the employer will generally not normally wish to tie-up capital in its pension scheme which it believes that it could utilise more profitably in its core business operations, its starting point is likely to be one of trying to minimise its funding commitment to the scheme as far as possible, although regulation again may limit what can be achieved in this respect. The employer will, however, also be concerned with the way in which its liability to contribute to the scheme is reflected in its published accounts. FRS 17 has introduced a significant change in the direction by requiring the sponsor to reflect an up-to-date value of the accrued liabilities directly on its balance sheet. Typically, this value may be well in excess of the actual obligation if the employer were to walk away from the pension scheme.

3.2.2.3 The challenge for the sponsoring employer is the balance between stable and affordable cash and accounting costs, on the assumption that everything remains intact, and having sufficient assets to meet the promise if the plan were to wind up. This is the same conundrum facing the trustees, but from the opposite perspective.

3.2.3 *Members*

Members have differing perspectives, depending on factors such as their age and service and whether they are currently receiving benefit. Older, longer serving members who have not yet retired will be most concerned to ensure that adequate security is provided for their interests in the scheme, as their scheme pension is likely to form a very significant part of their personal financial assets, but will remain a lower priority than pensions in payment in the event that the scheme is discontinued. Younger members have less to lose and more to gain from further accrual in the scheme, and so might be more amenable to a more optimistic funding strategy which may, in turn, enable the employer to maintain the scheme without a reduction in the level of accruing benefits.

3.2.4 The needs and objectives of the other parties are less obvious, but are still worthy of consideration.

3.2.4.1 *Potential future members/employees*

Potential future scheme members may be affected by decisions that have been taken in the past or are being taken now, such as alterations to benefit levels (e.g. worsening of terms, a switch from defined to defined contribution) or changes in the funding approach (e.g. a decision to fund aggressively in the past may have led to a weakened level of security for current promises).

3.2.4.2 *Current and future shareholders of the sponsoring employer*

Current shareholders may not fully understand the dynamics behind the current arrangements. The introduction of FRS 17 will increase the profile of pension liabilities and assist with understanding. This should translate into a higher level of risk management in respect of the build up of pension liabilities. The speed at which this happens will vary on a company-by-company basis. This may also lead to an acceptance from shareholders that funding should be at a higher level that perhaps it was in the past. The other perspective, which will be of interest to the shareholders, is ensuring that the business provides sufficient benefits to attract and retain the right calibre of people such that its ability to develop will not be impeded. Thus, what current and future shareholders arguably require is a balance of risk.

3.2.4.3 *Creditors and debt holders of the sponsoring employer*

The interests of creditors and debt holders of the sponsoring employer will be closely aligned with the shareholders, although, in addition, they will be concerned about the possible impact of a pension scheme deficit, in the event of winding up, on the recovery of the monies owed to them. Apart from the shared concern of creditors, suppliers will also be interested in how the costs of pensions impacts upon both the running costs and the survivability of the sponsor, as these will have knock-on effects on the pricing of goods and services exchanged between the two and the likely tenure of their relationship with the sponsor.

3.2.4.4 *Government, including tax and social security*

Government has a number of distinct interests:

— On a general level, governments in many industrialised countries are seeking to encourage pension provision, and, tempering this, they want to ensure that the tax relief, which is afforded to pension schemes, is appropriately balanced.
— At a more detailed level, governments in many industrialised countries are seeking to transfer much of the pension burden away from the state to private pension arrangements. As above, there is a mediating force, in that they would also wish to ensure that any incentives (e.g. national insurance rebates in the U.K.) are appropriately priced.
— These considerations are complicated by government also wishing to ensure that the reasonable expectations of employees are met. Some legislation has been introduced in this area in the U.K., but the level of protection remains relatively low except for those already retired. With recent high profile cases of frustrated expectations, the direction on regulation may be towards more protection. This would shift the relationship towards a need for a higher level of funding.

3.3 *Potential Conflicts*

3.3.1 The main financial management decisions that are used to implement the above objectives are:

— the funding strategy;
— the investment strategy; and
— the benefit strategy (including the use of surplus to improve benefits).

3.3.2 Clearly, there is considerable scope for conflict between the objectives of the different stakeholders in the way in which these decisions are resolved. For example, if the scheme were currently well funded, a natural conflict would tend to arise between the trustees of the scheme, who would wish to follow a conservative investment strategy to maintain that level of security for members' benefits, and the employer, who would wish to minimise the level of its capital commitment to the scheme (and the long-term cost of benefits). The problem is, however, often more complex, in that employers may differ in their ability to deal with the impact of large increases in future contribution requirements or with the impact on their balance sheet of a significant decline in funding level, so that the likely variation in future outcome becomes a key feature.

3.3.3 The key weakness with the current approach to valuations is that these conflicts tend to be addressed in isolation rather than together. This is an inherent feature of using deterministic approaches. It may be, therefore, that the optimal resolution of the problem is not found or that the dangers of certain apparent solutions are not highlighted. As we will demonstrate, stochastic approaches allow one to look at all aspects of the problem.

4. Proposals: A Stochastic Approach to Decision Making

4.1 *Objective of Stochastic Approach*

4.1.1 In Section 4 we consider, in some detail, our proposals for a stochastic approach to decision making for defined benefit pension schemes. The purpose of applying stochastic methods to a pension fund is to assist with decision making in the presence of uncertainty. In ¶2.1.1 we identified the following main areas of decision making in the financial risk management of a defined benefit scheme:
— the funding of the scheme;
— its investment strategy; and
— the use of surplus to improve benefits.

4.1.2 A wide range of alternative strategies is possible for each of the above, and the aim should be to evaluate the potential consequences of different options. In order to do this, we need to devise 'performance measures' for various aspects of the experience of the scheme, which might relate to:
— the solvency of the scheme;
— the risk of having to increase the employer contribution rate;
— the average employer contribution rate; and/or
— the benefits paid by the scheme.

Note that the above list includes two criteria for the employer contribution rate, because the employer does have two distinct concerns in this area: how high the contribution rate will be (on average); and how often it will have to be increased (and by how much).

4.1.3 The advocacy of a stochastic simulation procedure for pension schemes in not new — for example, Bacinello (1988) provides an introduction to the use of simulation, while Ramsay (1993) proposes a new set of funding methods that recognise longevity risk (only) and uses percentiles to measure the probability of adequately covering scheme members' benefit entitlements.

4.2 *Comparison with Deterministic Valuation*

4.2.1 The traditional actuarial valuation is also a method of decision making. The actuary determines the recommended employer contribution rate under a particular set of assumptions, and the surplus, as revealed by successive valuations, will influence mainly the subsequent contribution rate, but possibly also the extent of future benefit improvements (O'Regan & Weeder, 1988).

4.2.2 The actuarial assumptions are arbitrary, to some degree, as it appears that most actuaries would be comfortable with any basis lying within an acceptable range of uncertainty. In such circumstances, it seems inevitable that the choice of assumptions will be influenced by the wishes of the sponsoring employer, who would expect to be consulted on the objectives of the valuation before it is undertaken (Subject 304 Core Reading, Unit 9).

4.2.3 However, there are limits to the extent that any actuary would be prepared to accommodate the wishes of the sponsoring employer. The Scheme Actuary would reject an excessively weak basis because he or she realises that this would involve unacceptable risks for the solvency of the scheme and the payment of the benefits which it has promised. If the employer were to attempt to force the Scheme Actuary to adopt such a basis, he or she might point out the implications for the future solvency of the scheme and the volatility in outcomes, while explaining that the low contribution rate achieved would most likely be a temporary phenomenon, resulting in a much higher contribution rate later on. If, on the other hand, the employer were to request an excessively strong basis, the Scheme Actuary would surely refer to the risk of excessive surpluses leading to enforced benefit improvements and higher overall costs.

4.2.4 It follows that the traditional actuarial valuation relies on the judgement of the actuary to ensure that certain funding strategies are excluded from consideration on the grounds of risk. A deterministic methodology means that it is not possible, however, to quantify the risks associated with different decisions. The only method of allowing for risk is for the actuary to exclude (or advise against) various options because they are 'too risky'. There is also no method of comparing subtle differences in the trade-off between risks for alternative funding strategies lying within the range of acceptability.

4.2.5 Another limitation of the traditional actuarial valuation is that the investment strategy is not treated as a decision to be considered alongside the funding of the scheme. The actuary should simply comment on whether the current asset allocation of the fund is 'appropriate', given the nature of the liabilities. The choice of investment strategy is usually treated as a separate problem, sometimes involving a stochastic asset/liability exercise to illustrate the range of potential outcomes for any given asset allocation. One of the main conclusions of our work is that the investment and funding decisions cannot be separated in this way. For example, suppose that we find that a certain asset allocation increases the long-term risk of insolvency for a given normal contribution rate. We might also find that the same asset allocation reduces the insolvency risk for a different normal contribution rate.

4.2.6 A stochastic approach to decision making for a defined benefit pension scheme should, therefore, simultaneously evaluate all of the options at the disposal of the trustees and employer for controlling the performance of the scheme. The method of evaluation should involve the calculation of explicit and quantifiable performance measures for all possible combinations of strategies relating to funding, investment and the use of surplus to improve benefits.

4.3 *The Important Decisions*

4.3.1 *Funding strategy*

4.3.1.1 The most obvious responsibility of the Scheme Actuary is to calculate the contribution rate required to meet the cost of the promised benefits. A deterministic valuation produces a single answer, but the actuary knows that there is actually a range of reasonable contribution rates because of the uncertainty in the actuarial basis (Subject 304 Core Reading, Unit 7).

4.3.1.2 It sometimes argued, e.g. by Exley *et al.* (1997, ¶3.7.2), that the contribution rate selected at any one valuation does not matter a great deal because it makes no difference to the overall cost of benefit provision; a lower contribution rate today simply means a higher contribution rate tomorrow and vice versa. Paying a higher contribution rate today, however, will certainly have an impact on all of our suggested performance measures: the scheme is less likely to become insolvent; the average contribution rate over any finite time horizon will be higher; the risk of having to increase the current contribution rate will be lower; the likelihood of future benefit improvements may be enhanced.

4.3.1.3 The recommended contribution rate is, therefore, an important decision, but it does not entirely define the chosen funding strategy, because there is no realistic expectation that the current contribution rate can be maintained indefinitely. The funding strategy should encompass both the current contribution rate and the method of changing the future contribution rate if experience deviates from the assumptions, as it surely will.

4.3.1.4 The usual way of dealing with this point is to define the recommended contribution rate as a normal contribution rate plus an adjustment. The normal contribution rate is that which would be recommended if the scheme had no surplus or deficit; the adjustment is the addition (or subtraction) required to amortise any deficit (or surplus) over some fixed time horizon. The choice of this time horizon is an important parameter for the funding strategy, as it has a significant effect on the performance criteria mentioned above (see, for example, the analyses of Dufresne, 1988; and Owadally & Haberman, 1999).

4.3.1.5 In a deterministic valuation, we have the added complication that the actuarial basis might change in future valuations, causing the normal contribution rate to change, so that, in theory, the criteria for changing the assumptions would also be a part of the funding strategy. However, this problem disappears under a stochastic approach, because there is no need for deterministic assumptions. The normal contribution rate and adjustment are simply a method for dividing the total contribution rate into a fixed part and a variable part. It is, therefore, appropriate to assume that the normal contribution rate remains fixed over time by definition.

4.3.1.6 As the surplus or deficit is also a function of the actuarial basis in a deterministic valuation, the adjustment would also be affected by changes in the basis. We can circumvent this problem under a stochastic approach by using an objective measure for the liabilities that does not require arbitrary financial assumptions, i.e. the value of wind-up liabilities using market interest rates. As argued below, this has the added benefit of corresponding to the most obvious funding target for the purpose of measuring solvency risk.

4.3.1.7 Thus, we are now in a position to define the elements of the funding strategy. They are:

— the normal contribution rate; and
— the period over which surpluses and deficits will be amortised.

4.3.2 *Investment strategy*

4.3.2.1 At any point in time, the investment strategy is defined by the allocations given to each of the asset types in the pension fund, such as equities and bonds. As for the contribution rate, however, we also need to understand how these allocations might alter over time: indeed, one of the objectives of a stochastic approach is to determine how these allocations *ought* to change over time. Two important factors that might affect future asset allocations are the maturity of the scheme and the size of the surplus or deficit in the fund — these factors are discussed in the following paragraphs.

4.3.2.2 The maturity of a scheme is reflected in the value of its accrued liabilities. This will be zero for a new scheme in which employees receive no credit for prior service. For a new scheme with a stationary population of active members, we would expect the value of the accrued liabilities to grow until it reaches an upper limit as a multiple of the pensionable payroll, at which point we might say that the scheme is mature. Most actuaries would advise a mature scheme to invest a smaller proportion of its assets in volatile asset classes, possibly matching liabilities for deferred pensions and pensions-in-payment with bonds (Subject 304 Core Reading, Unit 12.) A stochastic approach would allow us to tackle this question directly, because some of the performance measures will depend on the maturity of the scheme. In particular, for any given asset allocation, the expected size of future solvency deficits would increase with the maturity of the scheme.

4.3.2.3 How the current level of surplus (or deficit) should affect asset allocations is less obvious. One point of view is that surplus assets constitute a 'mismatching reserve', which should give the scheme more freedom to invest in riskier asset types, in the hope of maximising returns. Another point of view is that a scheme with surplus assets should preserve its strong financial position by taking fewer investment risks. Perhaps surprisingly, analytical modelling work by Owadally & Haberman (2000) and Cairns (2000) suggests that the second approach is preferable, and the results presented in Section 6 will support this conclusion.

4.3.3 *Benefit improvements*

4.3.3.1 The use of surplus assets to improve benefits is a contentious issue in the management of defined benefit pension schemes, and practice varies significantly from one scheme to another. Legislation also plays an important part. If a scheme has an 'excessive' surplus on the statutory basis prescribed under the Finance Act 1986, and this cannot entirely be removed within the prescribed period through the suspension of employer contributions, it seems likely that the scheme would also have to improve benefits.

4.3.3.2 As far as stochastic modelling is concerned, we need a rule for predicting when benefit improvements are going to be triggered and how the proportion of surplus distributed to the members varies with the total amount of surplus. It may be very difficult to make accurate predictions about what will happen in this regard, but some crude approximation seems to be necessary unless one is certain that the impact of future benefit improvements will be negligible.

4.3.3.3 All the performance criteria mentioned in Section 4.1 are affected by the extent to which surplus assets are used to improve benefits. If a scheme were more inclined in this direction, the benefits paid by the scheme would clearly improve at the expense of the other performance measures. In such circumstances, those representing the members' interests would probably favour riskier investment strategies because they are more likely to create surpluses.

4.4 *Risk and Performance Measures*

4.4.1 *Appropriate definition of risk measures*

4.4.1.1 We require risk measures for the solvency of the scheme and the employer contribution rate. The most straightforward approach would be to estimate the variance of the solvency level and contribution rate at different future time horizons. As a risk measure, the variance has the advantages of being both widely understood and amenable to analytical modelling work of the kind pioneered by Wise (1984) and Dufresne (1988). It also allows optimisation problems to be framed in terms of a mean-variance efficient frontier, a paradigm familiar to those who have studied modern portfolio theory.

4.4.1.2 Actuaries such as Clarkson & Plymen (1988), however, have pointed out the drawbacks of using the variance as a risk measure. Doing so implies that risk is entailed in all deviations from the mean, whatever their sign, which is plainly untrue for solvency and contribution rate risk. We could circumvent this problem by using the semi-variance, where deviations of the 'favourable' sign are excluded. However, risk would still be measured relative to the mean outcome rather than to a benchmark reflecting the genuine concerns of the trustees and sponsoring employer.

4.4.1.3 Another approach to risk evaluation is to estimate the probability of an unwelcome event, such as insolvency, occurring. The value-at-risk (VAR) method, developed in risk management, is based on requiring firms to hold sufficient capital to keep the probability of insolvency at an acceptably low level. There is some evidence of the use of VAR techniques by pension schemes in the United States of America — for example, Dowd (1998) mentions a survey conducted by the Stern Business School at New York University, where 60% of the responding pension funds reported using VAR.

4.4.1.4 Recently, Artzner *et al.* (1997, 1999) have pointed out the drawbacks of the VAR approach. As no account is taken of the size of the financial loss should the unfavourable event occur, a strategy combining a high probability of small gains with a low probability of extreme losses would not be treated with sufficient caution. Other problems are that the choice of the 'acceptable' low probability is arbitrary, and the method cannot be used additively for different risks.

4.4.1.5 A more useful risk measure should allow for both the probability of the event occurring and the magnitude of the resulting loss. Artzner *et al.* (1997, 1999) advocate calculating the conditional tail expectation, that is the expected value of this loss given that the event in question occurs, which suggests a risk measure of the form:

$$\textit{Probability of loss} \times \textit{expected value of loss should event occur.} \qquad (4.1)$$

Appendix A provides a brief review of these risk measures. Equation (4.1) corresponds to the shortfall expectation, defined by equation (A6).

4.4.1.6 On examining the risks relevant to a defined benefit pension scheme, we believe that the above formulation is preferable to a risk measure based purely on probability. Looking at solvency risk, for example, the size of a potential solvency deficit should be of as much concern as the probability of a deficit occurring. A mature scheme, with large accrued liabilities, is clearly more vulnerable to adverse experience than a new scheme, because of the size of potential deficits in relation to its contribution income. This will be taken into account by a risk measure of the form suggested above.

4.4.1.7 Risk measures of the form (4.1) have been widely used in the actuarial literature; see Albrecht *et al.* (2001) for a discussion of equity risk, Haberman *et al.* (2000) for a discussion of process risk; Haberman & Vigna (2002) for a discussion of risk in defined contribution schemes; Wirch & Hardy (1999) for an application to segregated funds; and Wason (2001) for proposals for measuring solvency risk for insurance policies.

4.4.2 *Solvency risk*

4.4.2.1 The most important responsibility of the trustees is ensuring that the benefits promised by the scheme are paid. This seems to be assured as long as both the scheme and the sponsoring employer continue to operate. The circumstance in which some members might not receive their promised benefits is on the wind-up of a scheme with insufficient assets to cover the accrued liabilities. If this occurs as a result of the failure of the employer's business, it is quite likely that the active members and deferred pensioners (at least) will not receive their full entitlement. It follows that the natural measure of solvency concerns the extent to which the market value of the assets covers the value of the benefits payable on a wind-up of the scheme.

4.4.2.2 In devising a suitable measure of solvency risk, we should recognise that the probability of a wind-up occurring at any future duration is difficult to estimate and will vary for different employers. This places a limit on the sophistication of any proposed solvency risk measure. The most practical approach would be to allow the trustees to select some future point in time at which solvency risk would be evaluated. For an employer in financial difficulties, this might be the date of the next actuarial valuation, but for a strong employer we might project forward much further to get a 'long-term' risk measure. The precise duration might not matter too much if the risk measure converges fairly quickly as the time horizon tends to infinity. Thus, in accordance with the generalised risk measure proposed above, our measure of solvency risk becomes:

— the expected value of the solvency deficit, given that a deficit occurs, multiplied by the probability of a deficit occurring.

An equivalent way of representing the above risk measure is:

— the mean solvency deficit, treating surpluses as zero deficits.

4.4.2.3 Hence, what we are proposing is a 'one-sided' risk measure like the semi-variance, in which deficits are included, but surpluses are ignored.

4.4.3 *Contribution rate risk*

4.4.3.1 The possibility of an increase in the required employer contribution rate is an unavoidable aspect of funding a defined benefit pension scheme. In a deterministic valuation, this risk can be controlled by the use of prudent (i.e. somewhat pessimistic) actuarial assumptions. The sponsoring employer is persuaded to pay a contribution rate somewhat higher than the actuary's 'best estimate', on the grounds that this will make a future increase in the contribution rate less likely.

4.4.3.2 As with the solvency risk, both the probability and the expected size of any increase in the contribution rate are important. Unlike the solvency risk, it is clearly incorrect to focus on the increase at a particular point in time; the risk measure should allow for the expected size and frequency of contribution rate increases over some future period. Our proposed risk measure is:

— the expected present value of the excess of future employer contributions over the normal contribution rate.

4.4.3.3 Like the proposed measure for solvency risk, this is a one-sided measure that ignores the 'risk' of a reduction in the employer contribution rate. The formula for this risk measure is:

$$\sum_{t=0}^{T} v^t \mathrm{E}_0[\max(C_t - NC\,,\ 0)] \tag{4.3}$$

where E_0 is the expectation at time zero, C_t is the contribution rate at time t, NC is the normal contribution rate, T is the number of time intervals in the projection period and v is the discount factor for interest net of general salary growth.

4.4.4 *Average employer contribution rate*

4.4.4.1 The expected average employer contribution rate would seem to be a fairly obvious performance measure, as it gives an indication of the expected cost of the scheme to its sponsor. The effect of the current employer contribution rate on this risk measure will reduce as the projection period lengthens. As the projection period tends to infinity, the important factors influencing the expected average contribution rate will be the use of surplus to improve benefits and the investment strategy. A more generous policy on benefit improvements will clearly increase the expected average contribution rate, while investing in assets offering higher expected returns will reduce it.

4.4.4.2 At this point, it is worth mentioning that some actuaries have interpreted the famous 'irrelevancy proposition' put forward by Modigliani & Miller (1958), a central idea in corporate finance, as implying that the investment strategy of a defined benefit scheme has no effect on the cost of benefit provision. Although this is a misleading statement in relation to pension costs as reported in company accounts, there is an important kernel of truth in this argument, which is discussed in Appendix B.

4.4.5 *Benefit value*

4.4.5.1 The purpose of a pension scheme is to pay benefits and, other things being equal, one could argue that the more benefits that a scheme has paid over its lifetime, the better it has performed. The benefit performance of a defined benefit scheme is not entirely predetermined, because of the possibility of benefit improvements on the one hand and the possibility of insolvency on the other. The latter is already taken into account in our insolvency risk measure, so we shall ignore it here.

4.4.5.2 Over the whole lifespan of the scheme, the required performance measure could be defined as the expected present value of the total benefit outgo. As we do not know what the future lifespan of the scheme will be, we, once more, have to select some arbitrary future time period over which to project the benefit outgo and calculate its expected present value. However, we should also allow for benefit improvements that have been promised, but not yet paid, at the end of the projection period, and are therefore reflected in the value of the accrued liabilities. The formula for this performance measure is thus:

$$\sum_{t=0}^{T} v^t \mathrm{E}_0[B_t] + v^T \mathrm{E}_0[AL_T] \tag{4.4}$$

where B_t is the benefit outgo at time t and AL_T is the value of the accrued liabilities at the end of the projection period.

4.5 *Indifference Curves*

4.5.1 Presenting the results of a stochastic modelling exercise in a manner that will help people to make decisions is a considerable challenge. As explained above, the performance of the scheme may be measured according to a number of different criteria, some of which may be in conflict with each other. Furthermore, each of the performance measures is influenced by several different factors under the control of the trustees and sponsoring employer.

4.5.2 The use of indifference curves to model consumer decision making is a standard tool of microeconomics (e.g. Begg *et al.*, 2000). If a consumer can make distinct choices that will affect his or her satisfaction, these choices are represented on the perpendicular axes of a graph. An indifference curve plots those combinations of choices that give the consumer the same level of satisfaction. There is usually an infinite number of non-intersecting indifference curves, each one corresponding to a different level of satisfaction, but it is only necessary to plot a few of these curves to understand the nature of the decision making process. By imposing practical restrictions on the consumer's options, typically in the form of a budget constraint, optimal choices can be determined from the graph.

4.5.3 Indifference curves can be usefully applied to pension scheme decision making if we can define each option by a single variable that can be represented on an axis of a graph. The 'level of satisfaction' from different

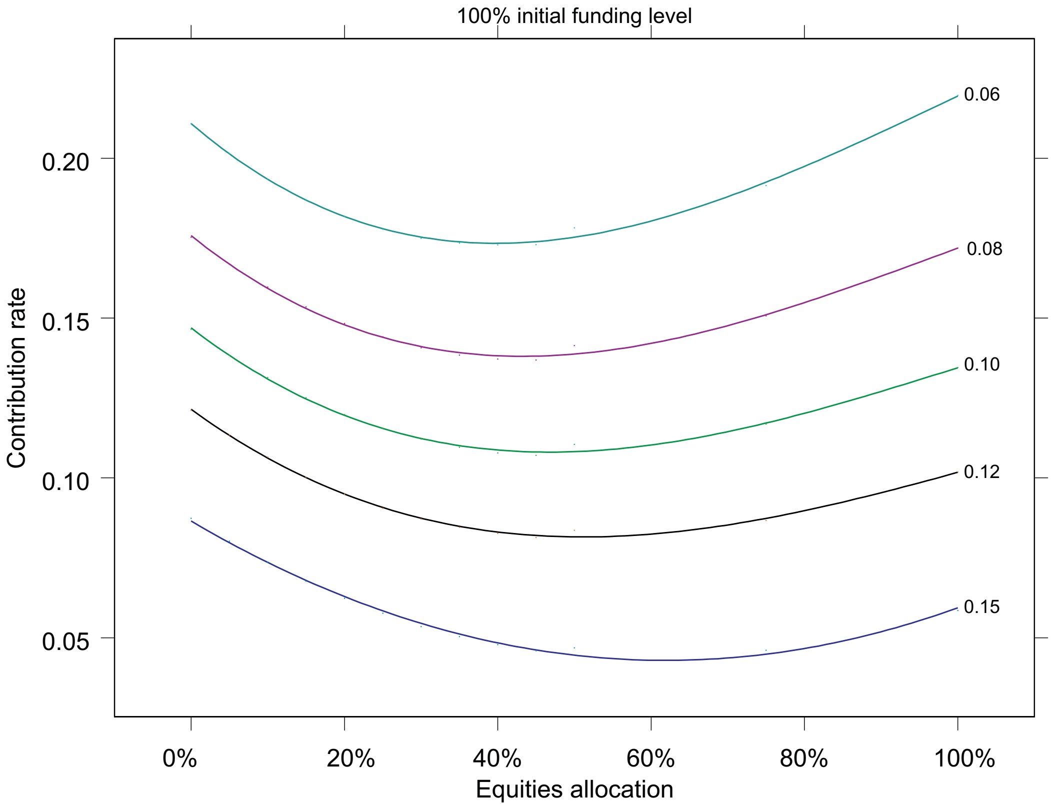

Figure 1. Mean shortfall at the end of three years

combinations of choices would then be one of our performance measures; it follows that a set of indifference curves would be required for each performance measure. A hypothetical set of indifference curves is presented in Figure 1 for a scheme that can freely choose both of the following variables: the proportion of the fund invested in equities (assumed to be held constant over time); and the normal contribution rate (i.e. the recommended contribution rate when the scheme has no surplus or deficit. This is also assumed to be held constant over time.). The performance measure used in this graph is the average solvency deficit at the next actuarial valuation.

4.5.4 Each indifference curve gives the combination of investment and funding choices that produces a fixed level of solvency risk. As one would expect, the solvency risk associated with each curve reduces as you move upwards, because paying a higher normal contribution rate reduces the risk of insolvency, other things being equal. The relationship between solvency and investment, however, is more subtle. Each indifference curve has a minimum, which gives the asset allocation requiring the lowest normal contribution rate to maintain a given insolvency risk. The optimal asset allocation strategy for any fixed normal contribution rate is derived as follows: first draw a horizontal line equal to the contribution rate; and then find the indifference curve for which this horizontal line is a tangent. The optimal asset allocation is found at the minimum of this indifference curve.

4.6 *The Myners Review*

4.6.1 We believe that the recommendations of the Myners Review, published on 6 March 2001, are consistent with the approach to decision making put forward in this paper.

4.6.2 An important conclusion of the review was that trustees of defined benefit schemes should set clear investment objectives, based on the liabilities of the scheme in question, rather than performance targets linked to market or industry benchmarks. The explicit performance measures defined in this section fulfil this requirement. Two of them, dealing with solvency risk and benefit value, are directly concerned with the liabilities of the scheme and the duties of the trustees to protect members' interests. The other two performance measures, dealing with the employer contribution rate, address the concerns of the sponsor.

4.6.3 The proposed replacement of the MFR with a long-term, scheme-specific funding standard, combined with stricter conditions on a voluntary wind-up of the scheme, is very much in line with our proposed measure of solvency risk. The risk measure which we advocate is scheme specific, in that it allows for the future investment and funding strategy of the scheme; it is long term, because the trustees may ask for the risk measure to be evaluated at any future duration which they consider appropriate (although for certain schemes a short duration may be advisable); and it is based on the wind-up liabilities, because it is recognised that the non-payment of promised benefits is most likely to occur on a wind-up of the scheme.

4.6.4 Lastly, the Myners recommendation that more resources be devoted to strategic asset allocation recognises the importance that these decisions have on the performance of the scheme and the need for qualified professional advice which would help the trustees with making them. It is a service, we believe, that actuaries should be well placed to provide.

5. Model Framework

5.1 *The Model Pension Scheme*

5.1.1 *Introduction*

5.1.1.1 In this section we present the model pension scheme which will form the basis of the case study in Section 6, and which will illustrate the ideas put forward in Section 4. Appendix C gives full details of the benefit structure and the initial membership profile for the model pension scheme used in Section 6. In addition, the methodology used to project the future asset and liability cash flows on an annual basis (and to capitalise these future cash flows for valuation purposes) is also outlined.

5.1.1.2 The capital value of the scheme assets is projected forward using a stochastic asset model, allowing for the investment return achieved (divided into income and capital growth), the contribution income received (determined initially by the contribution rate chosen by the sponsoring employer) and the benefit payments made to current pensioners and withdrawals (who are assumed to receive a deferred pension).

5.1.1.3 In addition, the scheme liabilities are projected forward using the same stochastic asset model, and valued on a discontinuance basis at triennial intervals, as discussed in Section 4.4.2. The financial position of the scheme can thus be analysed at regular future intervals. Although we advocate a discontinuance basis for determining the liabilities, other approaches are possible, providing that there is a transparent and explicit method for estimating the valuation parameters.

5.1.1.4 Surpluses and deficits at future valuation dates are amortised (initially over a three year amortisation period), and the initial contribution rate chosen is adjusted accordingly, in order to determine the rate payable until the next valuation date.

5.1.2 *Static and dynamic investment strategies*

5.1.2.1 The projected capital values of the scheme assets will depend on the asset allocation strategy adopted. For simplicity, we assume that there are only two asset classes available to the fund manager, namely long-dated fixed-interest gilts and equities. Further, in the initial analysis a static investment strategy is assumed, whereby the proportion of the fund invested in each asset class is specified at the start of the projection period (and re-balanced annually). This approach may appear to be rather simplistic. However, most U.K. pension funds do invest predominantly in equities and long-dated fixed-interest gilts, and it is likely, in the long term, that other real investments used in practice (for example, overseas equities and property) will behave similarly to equities and that long-dated fixed-interest gilts will behave similarly to undated fixed-interest gilts. Also, in normal circumstances, most large, on-going pension funds in the U.K. are passive investors (i.e. the asset allocation is likely to remain reasonably stable over time). Thus, for the preliminary investigations, it is felt that the use of this simplified approach is appropriate.

5.1.2.2 However, in addition to a static investment strategy, the effect of a simple dynamic investment strategy is also considered. Two variants are explored, namely:

— at each valuation date, the proportion of the fund invested in equities is increased (decreased) by 5% (subject to a maximum allocation of 100%) for each 10% increase (decrease) in the discontinuance funding level (relative to the level at the previous valuation); and
— at each valuation date, the proportion of the fund invested in equities is increased (decreased) by 5% (subject to a maximum allocation of 100%) for each 10% decrease (increase) in the discontinuance funding level (relative to the level at the previous valuation).

5.1.2.3 The former strategy might be thought of as intuitive, i.e. if the financial position of the scheme improves, then a more 'risky' investment strategy is adopted (with the aim of increasing the investment return achieved). Conversely, the latter strategy might be thought of as counter-intuitive (in that a more 'risky' investment strategy is adopted if the financial position of the scheme worsens). In both cases, at each triennial decision point, a decision is made regarding both the required contribution rate and the asset allocation strategy over the period until the next decision point.

5.2 *Comments on and possible Extensions to the Model*

5.2.1 *Projection interval and decision horizon*

5.2.1.1 The model uses annual projections of the assets, the liabilities and the associated cash flows. This is in line with current industry practice, where actuarial valuations and reviews of strategies are carried out every three years. The choice of annual periods is also consistent with the decision horizon that applies to pension schemes.

5.2.1.2 As pension schemes are long-term investment vehicles, projections of 40 or more years would be needed to assess the costs associated with the provision of the promised benefits, and this is very often the horizon used in the traditional, deterministic, actuarial valuations in the U.K. In practice, a shorter period of ten to 15 years is normally used to assess the appropriateness of the funding and investment policies or other strategic decisions, such as changes to benefits or membership profile, usually within the framework of an asset/liability model (ALM) study.

5.2.1.3 In the current environment in the U.K., however, short-term considerations have become much more important as a result of the introduction of the MFR and, more recently, the changes in the way that pension benefits are shown in a company's accounts, as specified in FRS 17. Such a change of focus would have an impact, both on the projection interval and on the decision horizon. In the future, valuations may be performed on a quarterly or even monthly basis, with a much shorter horizon of three to five years, especially as this may be more in line with other management information used for running the sponsor's core business.

5.2.2 *Decision points*

5.2.2.1 In the proposed model, the valuation of the model pension scheme takes place every three years, and the results from the valuation are the current discontinuance funding level and the required contribution rate for the forthcoming inter-valuation period.

5.2.2.2 An alternative approach used by some practitioners in an ALM framework is to shorten the valuation period to one year, and therefore make the decision points more frequent, and to update the required contribution rate every year to reflect the most recent financial results of the scheme. If even shorter projection intervals and decision points, such as quarterly or monthly, were used, then this could lead to undesirable volatility of the contribution rate (e.g. the Russian crisis of August 1998 or the aftermath of the terrorist attacks of 11 September 2001). Another drawback of shortening the interval between decision periods is the need for up-to-date membership and, to a lesser extent, asset data.

5.2.7.3 The valuation basis used is partly fixed (i.e. the parameterisation of the stochastic model), but is partly dynamic, as the yields and returns used are the prevailing yields and returns at the time of the valuation, and will change over time. A good overview of the merits and drawbacks of different valuation methods is given in Head *et al.* (2000).

5.2.3 *Dynamic or static asset allocation*

5.2.3.1 Static asset allocation is when the allocation to the asset classes does not change over the projection period. A static asset allocation may be regarded as unrealistic, as pension schemes can change their asset allocation according to their financial circumstances and the current market and regulatory environment. However, it is very difficult to model and incorporate all the factors that affect the decision making process in practice. On the other hand, dynamic allocations allow us to consider the impact of different ways of controlling asset allocation based on certain rules.

5.2.3.2 Examples of dynamic asset allocation strategies are:

(a) *Threshold strategy*: specify upper and lower thresholds T_U and T_L:
- if funding level is below T_L, then asset allocation follows a strategy with a high equity content;
- if funding level is above T_U, then asset allocation follows a low risk strategy; and
- as funding level increases from T_U to T_L, we change the asset allocation along a smooth trajectory (e.g. linear), in order to blend in with the two endpoints.

The threshold strategy incorporates a simple form of feedback: if the funding level were below a particular target, then the assets would be invested in a high risk strategy in an attempt to benefit from the higher expected returns; if the funding level were higher than a second target, then the assets would be invested in a low risk strategy in order to protect their value.

(b) *Portfolio insurance strategy*, as proposed by Black & Jones (1987):
- proportion of assets allocated to high risk strategy is $\alpha \times \left(1 - \dfrac{\beta}{\text{funding level}}\right)$, where the funding level is the ratio of value of assets to value of liabilities;
- the parameters $\alpha > 1$ and $\beta > 0$ measure the weight to be attached to the funding level being greater than 1;

— when the funding level equals β, the proportion of assets allocated to the high strategy is 0; and
— a simpler version would be to use the 'mirror image' of the threshold strategy in (a).

Strategy (b) moves in the opposite direction to (a), and the weight in the high risk portfolio increases as a hyperbolic function of the funding level. The portfolio insurance strategy tends to penalise asset under-performance, while the threshold strategy gives little credit for asset over-performance, relative to targets.

5.2.3.3 As described in Section 5.1.2, we have effectively used a simpler version of (b), based on a linear relationship, as in (a), but with a change in the sign of the slope.

5.2.4 *Contribution Adjustment Models*

5.2.4.1 *Introduction of a funding corridor*

Let $f(t)$ be the market value of the assets at time t. Then, if we introduce a funding corridor for the size of the assets (or the funding level) at each valuation date $f_{\min}(t)$ and $f_{\max}(t)$, we will need to introduce an adjustment to the contribution rates if $f(t) < f_{\min}(t)$ or $f(t) > f_{\max}(t)$.

5.2.4.2 *Dealing with surpluses or deficiencies*

5.2.4.2.1 There are a number of approaches used for adjusting contribution rates to deal with surpluses or deficiencies. The two most common are:

(a) *spread method or proportional control method* (U.K.):
contribution rate is increased by $k \times$ (unfunded liability at time t), where $0 < k < 1$, and it is customary (but not essential) to use $k = (\ddot{a}_{\overline{m}|})^{-1}$ calculated at the valuation rate of interest; and

(b) *amortisation method* (U.S.A.):
contribution rate is increased by $k \times$(sum of the last m years' intervaluation losses), where $k = (\ddot{a}_{\overline{m}|})^{-1}$ calculated at the valuation rate of interest.

Further adjustments would need to be made to deal with any initial unfunded liability.

5.2.4.2.2 We may regard the choice of k (or m) as being at the disposal of the actuary, so that the selection of the optimal value (or range of values) becomes relevant. This has been discussed by Dufresne (1988) and Owadally & Haberman (1999, 2000), *inter alia*.

5.2.4.2.3 Contribution adjustment means that different simulations could lead to different amounts of contribution being paid during an inter-valuation period. Hence, as discussed in Section 4.4, we need to use performance measures that recognise this feature.

5.3 *Stochastic Asset/Liability Modelling*

5.3.1 *Introduction*

5.3.1.1 The cash flow model described above can be used to project the future cash flows to and from the pension fund on either a deterministic or a stochastic basis. Indeed, within the cash flow model, each of the variables can be considered to be either stochastic or deterministic. As noted earlier, one of the main advantages of a stochastic projection approach is that it allows the level of the funding 'risk' to be quantified, communicated to the scheme sponsors and used in the decision making process.

5.3.1.2 The pooling of demographic risks through the law of large numbers means that, in most cases, the effect of variation in the economic variables (e.g. investment returns, salary growth and price inflation) on the financial position of the fund is likely to be much more significant than the effect of variation in the demographic variables (e.g. pre-retirement and post-retirement mortality rates, withdrawal rates and retirement rates). For large schemes, the pooling of demographic risks through the law of large numbers implies that liability outgo could be dealt with fairly realistically by a deterministic approach using a traditional multiple decrement model (i.e. a service table).

5.3.1.3 For small schemes, demographic variability is an important consideration and very often makes a large contribution to the overall risk of pension scheme; but cost and complexity of the methodology are also important aspects which may argue for approximate methods in the case of small schemes. As a result, in a stochastic asset/liability modelling exercise, it is common to assume that the demographic experience of the fund is deterministic. Elements of stochasticity in demographics are demonstrated in Lee & Wilkie (2000), and Haberman *et al.* (2000) provides a discussion of pooling and process risk, albeit in a non-pensions context.

5.3.1.4 The variability in the economic experience can be modelled using a stochastic asset model. Elements of the demographic experience may well be linked to the economic experience (e.g. withdrawal rates may be expected to rise during periods of recession). However, in the absence of a credible stochastic model, scenario testing could be used in such cases.

5.3.2 *The stochastic asset model used*

5.3.2.1 For the case study discussed in Section 6, the stochastic asset model used is the model proposed by Wilkie (1995). Whilst this model has some acknowledged weaknesses, as discussed by Geoghegan *et al.* (1992), Smith (1996) and Huber (1998), *inter alia*, it remains the most widely-known and used stochastic asset model amongst actuaries in the U.K.

5.3.2.2 It should be noted that the results presented in this paper are intended to be purely illustrative. The authors do not feel that this paper is an appropriate place for a discussion of the merits (or otherwise) of the range of different stochastic asset models now available for actuarial use. Indeed, it was felt that there was a strong argument for using the simplest possible such model when presenting the initial results. The multivariate log-Normal model, proposed by Kemp (1996), was considered. However, this model considers only the total returns achieved on the various asset classes, rather than separating the return into income and capital gains, and so it is not ideal for use in cash flow projections.

5.3.2.3 However, it is acknowledged that the effect on the results of changing the stochastic asset model should be considered to help quantify model sensitivity. This is discussed further in the case study in Section 6.

5.3.3 *Risks involved*

5.3.3.1 The choice of the model and assumptions for the economic and demographic experience of the fund introduces three different types of risk, discussed in the following paragraphs.

5.3.3.2 *Model risk*

- By definition, a 'model' is a simplified version of reality.
- Thus, *model risk* is the risk that the model chosen (e.g. for investment experience, mortality experience, withdrawal experience etc.) does not adequately represent the complexities of the real world.
- As discussed above, the effect of model sensitivity can be explored by considering a number of different models:
  - e.g. with regard to the stochastic asset model used, we may also consider the results obtained using the models proposed by Smith (1996), Yakoubov *et al.* (1999) and Whitten & Thomas (2000).

5.3.3.3 *Parameter risk*

A study by Chopra & Ziemba (1993) examined the impact of parameter risk on optimal asset allocation decisions. More specifically, it considered the importance of the estimates of the mean return, volatility and correlation when deciding on optimal asset allocation. The paper states that, for the average risk averse institutional investor, the mean return parameter is ten times more important than the volatility parameters and 20 times more important than the correlation parameters.

- The parameters used in the chosen model are estimated from a finite sample of real-life observed data.
- Thus, the *parameter risk* is the risk that the parameters used in the model are inappropriate for representing the future experience (even if the underlying model is appropriate):
  - e.g. in the stochastic asset model proposed by Wilkie (1995), the parameters defining the probability distributions and correlations of the future experience of the economic variables are based on historic data.

— An important issue is whether the historic economic data adequately represent the future economic experience.
— In practice, the distinction between model risk and parameter risk is often unclear:
  — e.g. the deterministic model used for the demographic experience could be considered as introducing a model error (as the deterministic model may not adequately represent the future experience) or a parameter error (if a deterministic model is considered to be a stochastic model with the parameter determining the volatility of the model output set equal to zero).

5.3.3.4 *Process risk*
— The outputs from any stochastic model are subject to random fluctuations, even when the model and the parameters are appropriate to describe the future experience.
— Thus, the *process risk* is the risk that the outputs from the model will not adequately represent the range of possible outcomes for the actual future experience.
— Using Monte-Carlo simulation, this risk can be reduced by increasing the number of simulations used.
— The appropriate number of simulations to be used will depend, crucially, on the objectives of the modelling exercise:
  — e.g. adequate estimates of the mean and variance of the future financial position of the fund can usually be obtained from a reasonably small number of simulations (e.g. 100);
  — however, most risk measures will be concerned with the downside tail of the distribution of the future financial position of the fund, requiring a much larger number of simulations (e.g. 1,000, 5,000 or even 10,000); and
  — clearly, the more simulations that are required, the slower will the model be to run (and the results may be more difficult to analyse, interpret and communicate).

5.3.3.5 For further discussion of these types of risk, readers might wish to consult Daykin *et al.* (1994); Cairns (2000a); and Haberman *et al.* (2000).

6. Case Study

6.1 *Introduction*

6.1.1 In this section we present a case study in order to illustrate our proposals. Section 5 has provided details of the model used. In this case study, we carry out projections of a defined benefit scheme over five three-year periods. At the end of every period, we measure, for each initial asset allocation (in equities and bonds) and normal contribution rate, the mean shortfall risk (or average solvency deficit), the excess contribution rate risk and the average contribution rate. These performance measures have already been defined in Section 4.4. However, for the particular calculations we introduce some scaling factors, and these necessitate some changes to the definitions:

— mean shortfall risk $= \dfrac{1}{A(0)} \times \Pr(L - A > 0) \times \mathrm{E}(L - A | L - A > 0)$ (6.1)

where L and A are the discontinuance liability and the market value of assets, respectively, at the end of the period, and $A(0)$ is the market value of assets at the start of the projections, and is used as a scaling factor:

— excess contribution rate risk $= \dfrac{1}{a(T)} \times \sum_{t=0}^{T} v^t \mathrm{E}_0[\max(C_t - NC, 0)]$ (6.2)

where T is the time at the end of the period, and $a(T)$ represents the present value of a T year annuity-due calculated at a real rate of interest. (We take this real rate of interest to be the difference between the long-term mean for the yield on undated fixed-interest gilts and the long-term mean for the force of earnings inflation.)

6.1.2 We amortise all surpluses and deficits over three years at the end of each three-year period. However, for an initially fully-funded scheme the recommended contribution rate is just the normal contribution rate over the first three-year period. Subsequently, the recommended contribution rate is the normal contribution rate plus an adjustment for the surplus or deficit (at the previous valuation). Hence, even though the mean shortfall risks can be calculated at the end of every period, we can only calculate the excess contribution rate risks at the end of the second, third, fourth and fifth (three-year) periods.

6.1.3 We remind readers at this point that the numerical results presented in this section are intended to be illustrative. The precise details will clearly reflect the assumptions made and parameter values chosen and are of lesser importance than the principles and methodology proposed.

6.2 *The Concept of Indifference Curves*

6.2.1 *Mean shortfall risk*

6.2.1.1 Figure 2 shows the mean shortfall risk levels at the end of the second period (i.e. at the end of six years). (We have shown the mean shortfall risk levels at the end of six years instead of at the end of three years in order to match the excess contribution rate risk levels, which can only be calculated from the end of six years.) In this case the scheme is initially fully-funded, and we employ a static asset allocation strategy (with annual

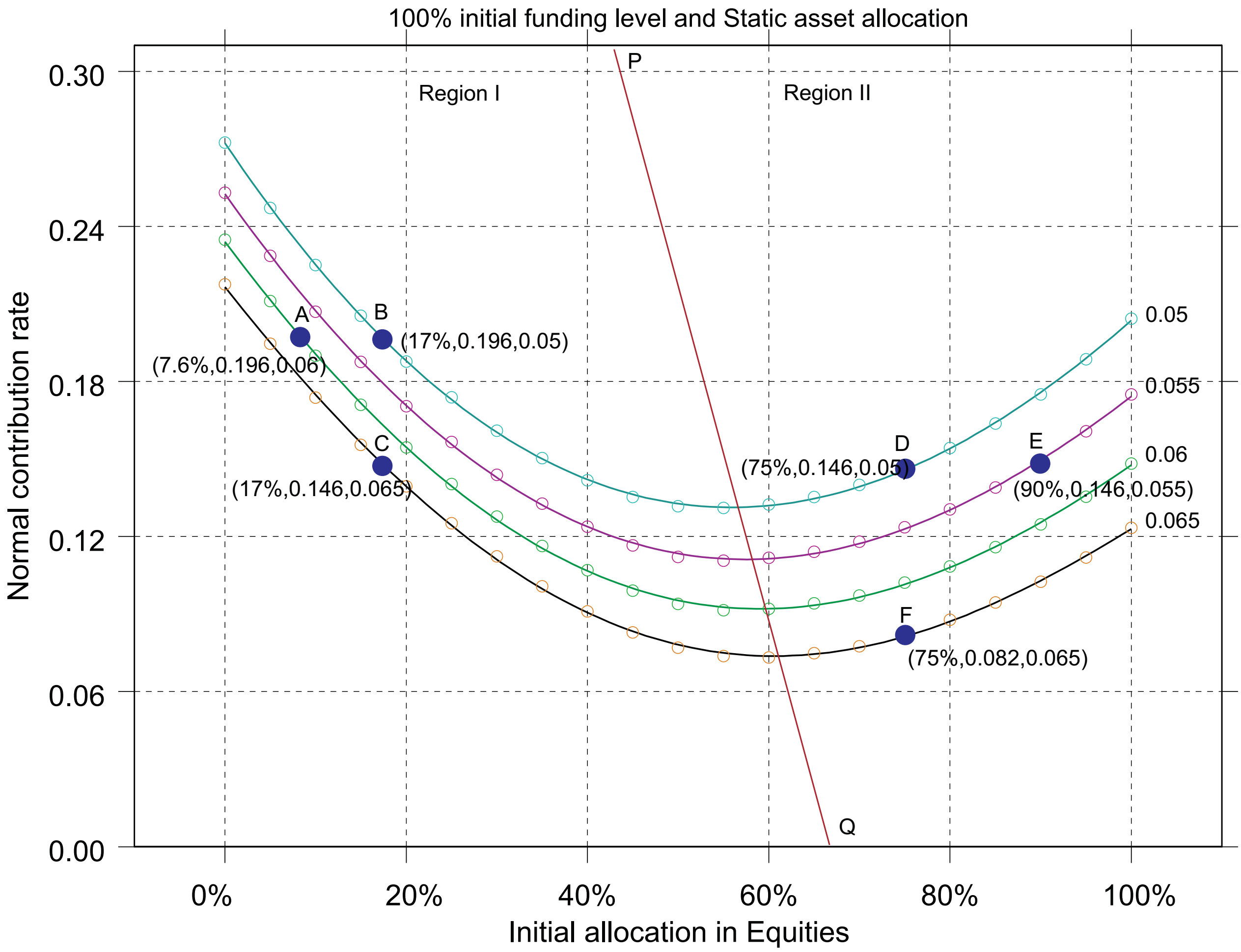

Figure 2. Mean shortfall risk levels at the end of six years

rebalancing). Each curve shows all the combinations of initial allocation in equities and normal contribution rate that lead to a given mean shortfall risk. (We have drawn only four curves for illustration purposes. It is possible to draw a curve through any given point on the graph.) For instance, all combinations along the top curve lead to a mean shortfall risk of 0.05. (It is not possible to consider all combinations of initial allocation in equities and normal contribution rate in the computer simulations. Hence, for a given mean shortfall risk interpolations are carried out, where necessary, to calculate the combinations along the curves. Simulations are carried out for 21 initial allocations in equities, i.e. 0%, 5%, 10% . . . 100%; and 17 normal contribution rates, i.e. 0, 0.02, 0.04 . . . 0.32 in order to reduce the interpolation error. So, for a given normal contribution rate, 21 initial allocations in equities are considered. We fit the curves using third- (or, in some cases, fourth-) order polynomial approximations.) We stress that the normal contribution rate is a free variable, in the same way that the initial asset allocation is a free variable. Thus, the normal contribution rate is not calculated in the traditional way, using an actuarial cost method, for example. Instead, we measure the risks associated with a particular combination of normal contribution rate and initial asset allocation

6.2.1.2 Each point on the graph can be identified by three coordinates (x, y, z), where x is the initial allocation in equities, y is the normal contribution rate, and z is the mean shortfall risk. For example, point A has coordinates (7.6%, 0.196, 0.06) and point D has coordinates (81%, 0.146, 0.05). This means that, if our normal contribution rate is 0.196 and we initially invest 7.6% in equities, then we would expect a mean shortfall risk of 0.06 at the end of the second period. On the other hand, if our normal contribution rate is 0.146 and we initially invest 75% in equities, then we would expect a mean shortfall risk of 0.05 at the end of the second period.

6.2.1.3 This analysis leads to the concept of indifference curves, as explained in Section 4.5. The decision maker would be expected to be indifferent between a combination of 17% initial allocation in equities and normal contribution rate of 0.196 (i.e. point B) and a combination of 75% initial allocation in equities and normal contribution rate of 0.146 (i.e. point D). This is due to the fact that both combinations lead to the same level of mean shortfall risk.

6.2.1.4 We observe from Figure 2 that the two most important concepts of indifference curves hold in our case:

— Firstly, combinations along the lower curves lead to higher mean shortfall risks, and vice versa. For example, point B and point D along the top curve lead to a lower mean shortfall risk of 0.05, while point C and point F along the lowest curve lead to a higher mean shortfall risk of 0.065. Thus, a rational decision maker will, all things being equal, try to reach the highest possible indifference curve (Sloman, 1999, p117).
— Secondly, it is impossible for two indifference curves to intersect, all things being equal (Sloman, 1999, p117). In our case it is impossible for a given combination of initial allocation in equities and normal contribution rate to lead to two different mean shortfall risks.

6.2.1.5 As shown in ¶4.5.4, in order to find the optimal asset allocations we need the minimum points of the indifference curves. Hence, in Figure 2 we are interested in the line PQ which passes through all the minimum points of the indifference curves. In order to draw a line which passes 'exactly' through the minimum points, we would need to consider every possible combination of normal contribution rate and asset allocation. Hence, we construct PQ as a best-fit line which passes either through, or very close to, the minimum points of the indifference curves. Thus, PQ is then a good approximation to the 'curve' which passes through all the minimum points. Hence, we obtain two separate regions, region I and region II. Such a division is very important for decision making, as we will show below.

6.2.1.6 *Decision making*

6.2.1.6.1 Points A and B in Figure 2 have the same normal contribution rate of 0.196, but different initial allocation in equities of 7.6% and 17%, respectively. Furthermore, point A leads to a higher mean shortfall risk than point B, i.e. 0.06 and 0.05, respectively.

6.2.1.6.2 This implies that a decision maker can improve their position by choosing point B (i.e. increasing the initial allocation in equities) rather than point A. This also holds in general; that is, for positions in region I, for a given normal contribution rate a decision maker can improve their position by increasing their normal allocation in equities (i.e. by shifting horizontally from left to right). This means that, for a given normal contribution rate, the decision maker should choose points that are nearer to (or on) the line PQ than, for example, point A.

6.2.1.6.3 We also note that points D and E have the same normal contribution rate of 0.146, but different initial allocations in equities of 75% and 90%, respectively. A decision maker can improve their position by choosing point D (decrease the initial allocation in equities) rather than point E. This is due to the fact that D leads to a lower mean shortfall risk than E.

6.2.1.6.4 Hence, for positions in region II, for a given normal contribution rate, a decision maker can improve their position by decreasing their initial allocation in equities (i.e. by shifting horizontally from right to left). This means that, for a given normal contribution rate, the decision maker should choose points that are nearer to (or on) the line PQ than, for example, point E.

6.2.1.6.5 Other possible movements, like C to B and F to D, lead to lower mean shortfall risk positions (i.e. mean shortfall risk decreases from 0.065 to 0.05), but the normal contribution rate increases from 0.146 to 0.196 and 0.082 to 0.146, respectively. Whilst movements like A to C and E to F lead to higher mean shortfall risk positions (i.e. 0.06 to 0.065 and 0.055 to 0.065, respectively), but the normal contribution rate decreases from 0.196 to 0.146 and 0.146 to 0.082, respectively. Such movements can only be analysed by considering their effect on the excess contribution rate risk. This will be covered in the next section.

6.2.2 *Excess contribution rate risk*

6.2.2.1 Figure 3 shows the excess contribution rate risk levels at the end of six years. Each curve shows all the *combinations* of initial allocation in equities and normal contribution rate that lead to a given excess contribution rate risk. For instance, all combinations along the top curve lead to an excess contribution rate risk of 0.035.

6.2.2.2 Each point on the graph can be identified by three coordinates (x, y, z), where x is the initial allocation in equities, y is the normal contribution rate, and z is the excess contribution rate risk. For example, point A has coordinates (8%, 0.157, 0.04) and point F has coordinates (80%, 0.15, 0.045). This means that, if our normal contribution rate is 0.157 and

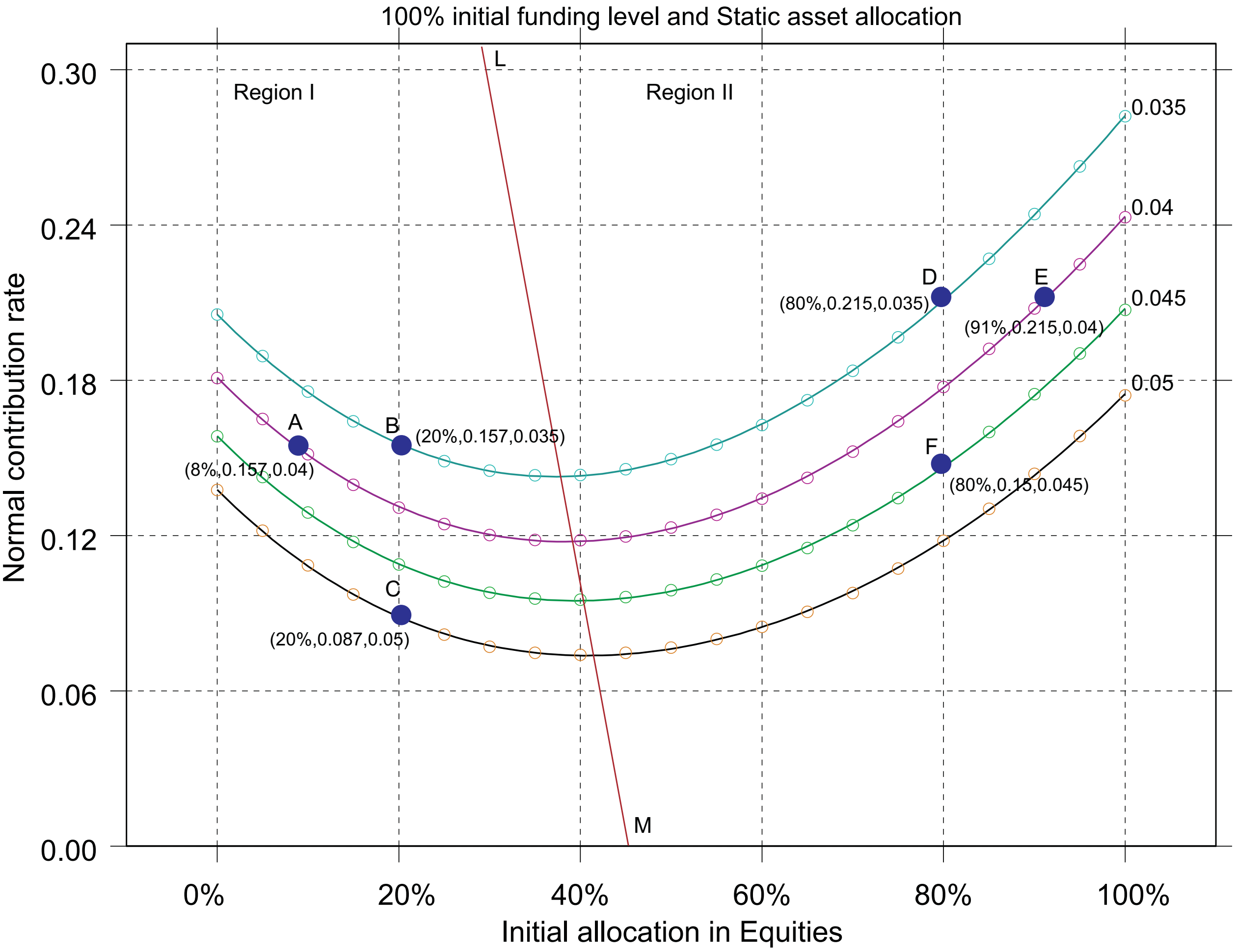

Figure 3. Excess contribution rate risk levels at the end of six years

we initially invest 8% in equities, then we would expect an excess contribution rate risk of 0.04 at the end of the second period. On the other hand, if our normal contribution rate is 0.15 and we initially invest 80% in equities, then we would expect an excess contribution rate risk of 0.045 at the end of the second period.

6.2.2.3 The indifference curve concepts, discussed in Section 6.2.1, also hold in this case. For example, point B and point D along the top curve lead to a lower excess contribution rate risk of 0.035, whilst point C along the lowest curve leads to a higher excess contribution rate risk of 0.05. Furthermore, it is impossible for two indifference curves to intersect, since a given combination of initial allocation in equities and normal contribution rate can only lead to one value of excess contribution rate risk.

6.2.2.4 *Decision making*

6.2.2.4.1 As in the mean shortfall case, let LM be the best-fit line through all the minimum points of the indifference curves in Figure 3. Then, as we observed in the mean shortfall case, for positions in region I, for a given normal contribution rate, the decision maker can improve their position by increasing the allocation in equities, i.e. by moving from left to right towards (or onto) the line LM. While for positions in region II, for a given normal contribution rate, the decision maker can improve their position by decreasing their allocation in equities, i.e. by moving from right to left towards (or onto) the line LM.

6.2.2.4.2 Other possible movements, like C to B and F to D, lead to lower excess contribution rate risk positions (i.e. 0.05 to 0.035 and 0.045 to 0.035, respectively), but the normal contribution rate increases from 0.087 to 0.157 and 0.15 to 0.215, respectively. Whilst movements like A to C and E to F lead to higher excess contribution rate risk positions (i.e. 0.04 to 0.05 and 0.04 to 0.045, respectively), the normal contribution rate decreases from 0.157 to 0.087 and 0.215 to 0.15, respectively. Such movements can only be analysed by considering their effect on the mean shortfall risk. This will be covered in a later section.

6.2.3 *Comparisons at the end of the second period*

6.2.3.1 Figure 4 shows a comparison of mean shortfall and excess contribution rate risk levels at the end of six years. The minimum points for the curves for mean shortfall risk are in the region of 60% initial allocation in equities. On the other hand, the minimum points for the curves for excess contribution rate risk are centred around 40% initial allocation in equities.

6.2.3.2 This means that, for a given normal contribution rate, we would initially invest more in equities if our decision were based only on the mean shortfall risk; and we would initially invest less in equities if our decision were based only on the excess contribution rate risk. This potential conflict needs to be reconciled.

6.2.4 *Comparisons at the end of the third period*

6.2.4.1 Figure 4 also shows a comparison of mean shortfall and excess contribution rate risk levels at the end of 15 years. The minimum points for the curves for mean shortfall risk are in the region of 80% initial allocation in equities. On the other hand, the minimum points for the curves for excess contribution rate risk are centred around 55% initial allocation in equities.

6.2.4.2 This implies that, as we observed in Section 6.2.3, there is a conflict between decisions based only on mean shortfall risk and decisions based only on excess contribution rate risk. Additionally, we observe that the minimum points are at a higher initial allocation in equities for the 15-year projections than for the six-year projections (i.e. 80% versus 60% and 55% versus 40%). This means that, if we were seeking to minimise risk, we would *initially* invest more in equities if our projection period were 15 years than if our projection period were six years. (We stress that these results depend on the assumptions of a modelling framework, described in Section 5. However,

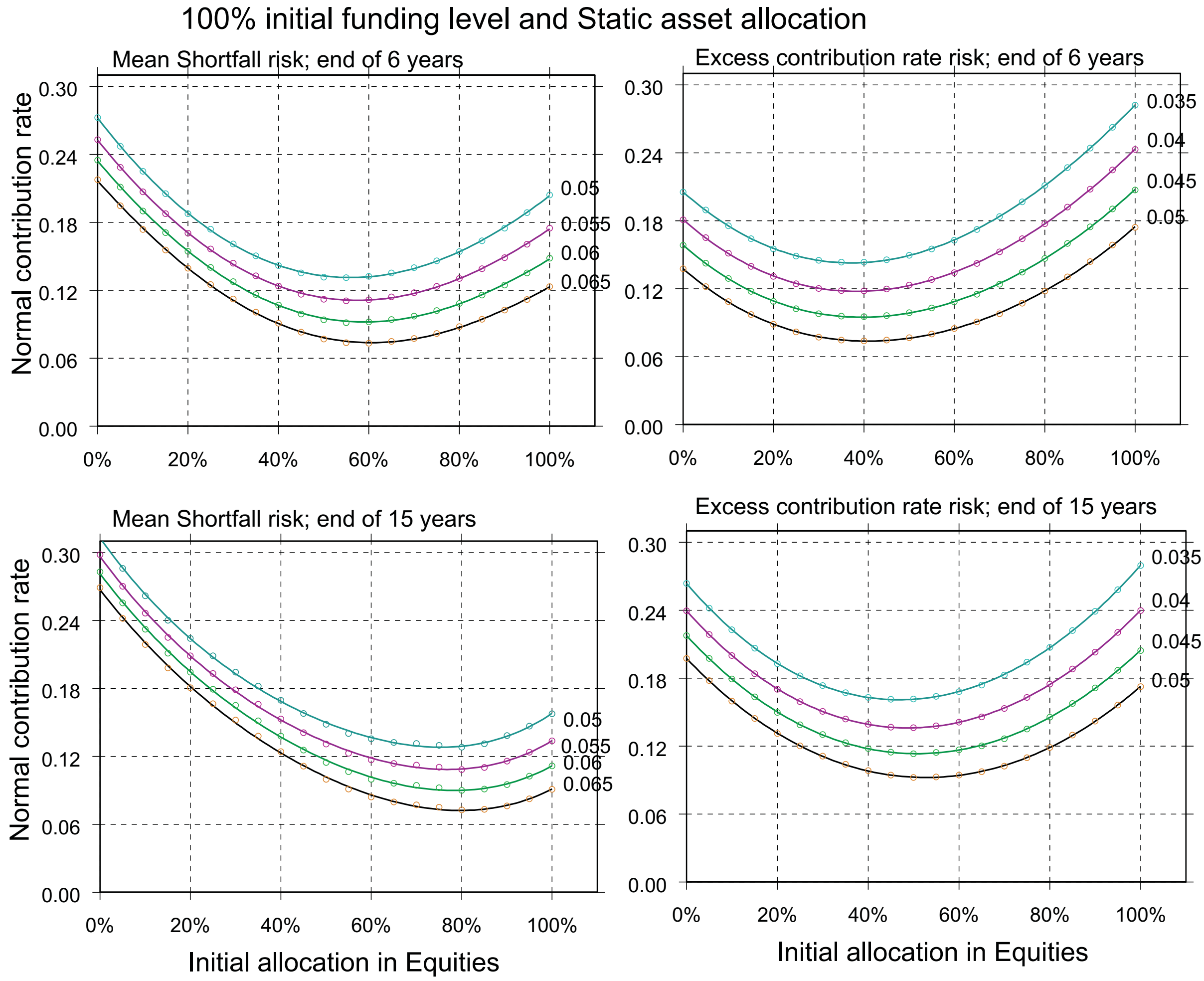

Figure 4. Comparison of risk levels at the end of six and 15 years

the results are similar to those reported elsewhere for optimal investment strategies for defined contribution schemes, defined in terms of risk minimisation (see Haberman & Vigna, 2002).)

6.3 *Decision Making — Reconciling Mean Shortfall Risk and Excess Contribution Rate Risk*

6.3.1 In Section 6.2.1 we showed how the decision maker could choose a combination of initial asset allocation and normal contribution rate by considering only the mean shortfall risk, and in Section 6.2.2 we showed how the decision maker could choose a combination of initial asset allocation and normal contribution rate by considering only the excess contribution rate risk. However, in Sections 6.2.3 and 6.2.4 we observed that there is a conflict between decisions based only on mean shortfall risk and decisions based only on excess contribution rate risk. We will now show how the two risks can be reconciled in the decision making process, so that the decision maker can achieve an efficient combination of initial asset allocation and normal contribution rate.

6.3.2 In Figures 5 and 6 we plot the mean shortfall and excess contribution rate risk levels at the end of 15 years on the same graph. Corresponding figures would emerge for comparisons at other time horizons — these are not shown, but are available from the authors.

6.3.3 The lines LM and PQ are the best-fit lines through the minimum points for the excess contribution rate and the mean shortfall risk levels, respectively. (LM and PQ are not necessarily parallel.) These lines split each of the graphs into three regions. Region I contains all points which are to the left of the minimum points of both the mean shortfall and excess contribution rate risk levels. Points in Region II are to the right of excess contribution rate risk minimum points and to the left of mean shortfall risk minimum points. Meanwhile, points in Region III are to the right of both sets of minimum points.

6.3.4 In Figure 5, we identify points as (x, y) where x is the initial allocation in equities and y is the normal contribution rate. Thus for a given point, the values x and y can be calculated from the x and y axes. In

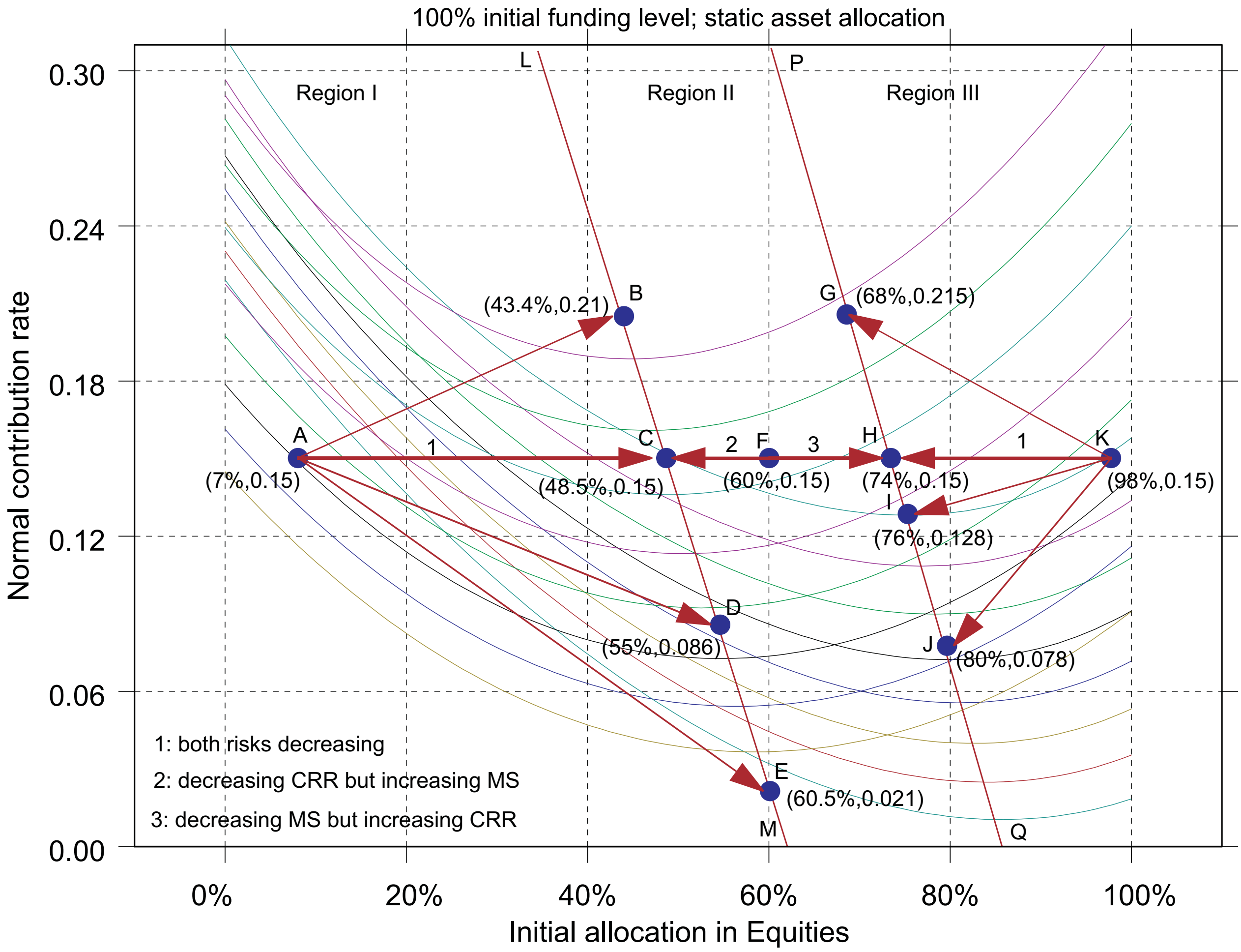

Figure 5. Mean shortfall (MS) and excess contribution rate (CRR) risk levels at the end of 15 years

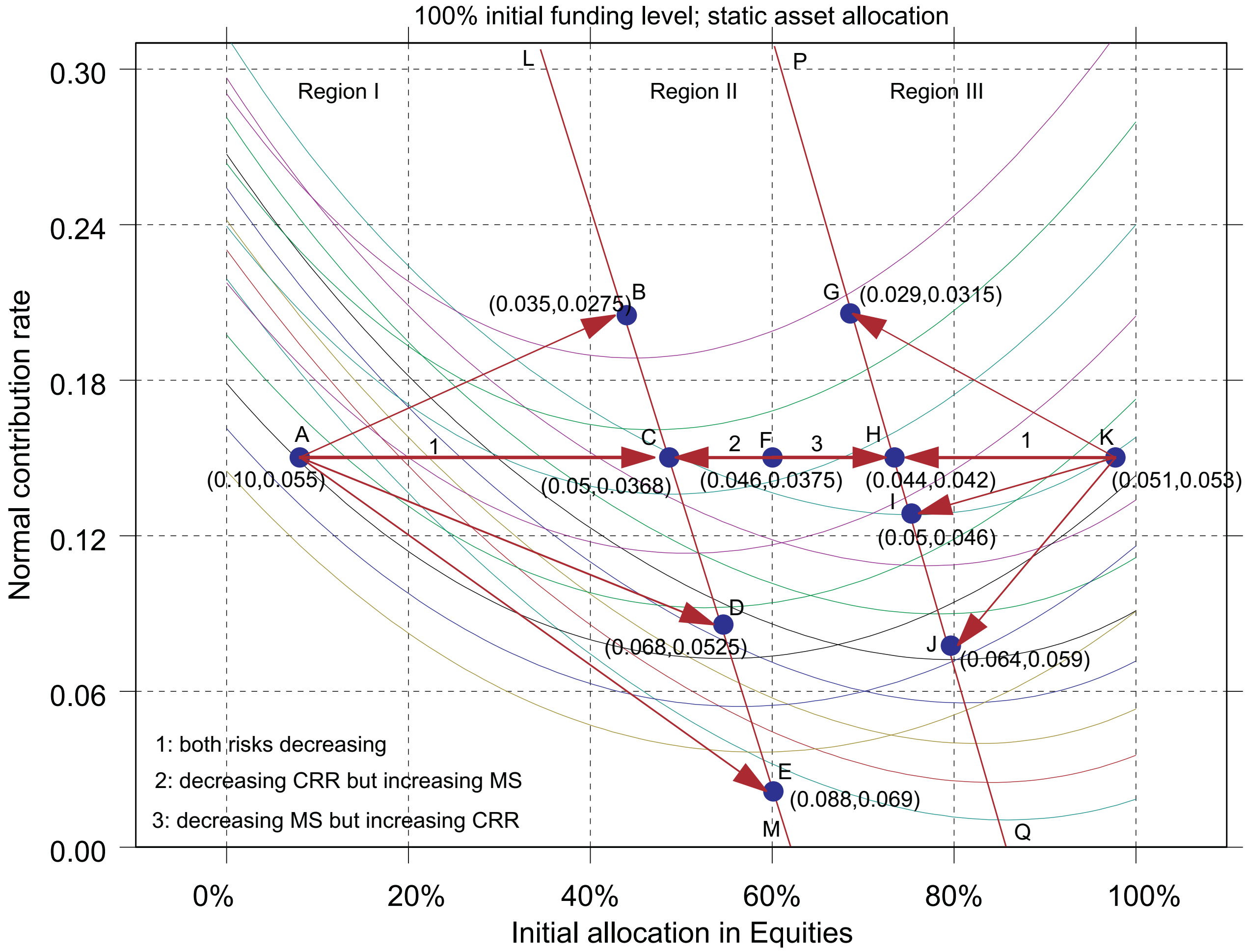

Figure 6. Mean shortfall (MS) and excess contribution rate (CRR) risk levels at the end of 15 years

Figure 6, we identify points as (z, w), where z is the mean shortfall risk and w is the excess contribution rate risk. Figures 5 and 6 must be used concurrently. Obviously, we could have shown all the information on one graph instead of on two. However, we use two graphs in order to show, separately, the choices that are available and the consequences of such choices. Figure 6 shows the consequences of the choices made in Figure 5.

6.3.5 We consider some examples:

— for a normal contribution rate of 0.15 and initial allocation in equities of 7%, we would expect a mean shortfall risk of 0.10 and excess contribution risk of 0.055 after 15 years (point A in Figures 5 and 6);

— for a normal contribution rate of 0.215 and initial allocation in equities of 68%, we would expect a mean shortfall risk of 0.029 and excess contribution risk of 0.0315 after 15 years (G in Figures 5 and 6);

— for a normal contribution rate of 0.021 and initial allocation in equities of 60.5%, we would expect a mean shortfall risk of 0.088 and excess contribution risk of 0.069 after 15 years (E in Figures 5 and 6); and

— for a normal contribution rate of 0.15 and initial allocation in equities of 98%, we would expect a mean shortfall risk of 0.051 and excess contribution risk of 0.053 after 15 years (K in Figures 5 and 6).

6.3.6 *Choosing an efficient combination*

6.3.6.1 We now demonstrate how an efficient combination of initial asset allocation and normal contribution rate can be chosen. We do this by considering the three regions presented in Figure 6.

6.3.6.2 *Region I*

Assume, without loss of generality, that a decision maker chooses point A. Such a decision maker can change their position in several ways:

(1) By moving to higher points, e.g. point B; since we are dealing with indifference curves, this move would lead to a point which has a lower mean shortfall risk and a lower excess contribution rate risk. Our example corresponds to arrow AB in Figure 6. In this case, the mean shortfall risk decreases from 0.10 to 0.035 and the excess contribution rate risk decreases from 0.055 to 0.0275. Moving from A to B can be effected by increasing the normal contribution rate from 0.15 to 0.21 and increasing the initial allocation in equities from 7% to 43.4%.

(2) By moving to points which lie horizontally to the right, e.g. point C; since we are moving horizontally towards the minimum points of both sets of indifference curves, both risks, mean shortfall risk and excess contribution rate risk, will decrease. Our example corresponds to arrow AC in Figure 6. Mean shortfall risk decreases from 0.10 to 0.05 and excess contribution rate risk decreases from 0.055 to 0.0368. We can accomplish this by leaving the normal contribution rate unchanged and increasing the initial allocation in equities from 7% to 48.5%.

(3) By moving to lower points; for example, moving from A to D or moving from A to E; we analyse these cases separately:
   - Moving from A to D, the mean shortfall risk decreases from 0.10 to 0.068, and the excess contribution rate risk decreases from 0.055 to 0.0525. We can accomplish this by reducing the normal contribution rate from 0.15 to 0.086 and increasing the initial allocation in equities from 7% to 55%.
   - Moving from A to E, the mean shortfall risk decreases from 0.10 to 0.088, *but* the excess contribution rate risk increases from 0.055 to 0.069. We can accomplish this by reducing the normal contribution rate from 0.15 to 0.021 and increasing the initial allocation in equities from 7% to 60.5%.

This example shows that we cannot just move arbitrarily to any point lower than A without increasing at least one of the risks. Examples (1), (2), and (3) above show that positions in region I can be improved by moving *towards* region II.

6.3.6.3 *Region III*

The scenario in region III is similar to that in region I. The main difference is that in region III all the movements involve reducing the initial allocation in equities:

(1) moving from K to G is similar to moving from A to B;
(2) moving from K to H is similar to moving from A to C; and
(3) moving from K to I is similar to moving from A to D.

Moving from K to J is also similar to moving from A to E, but, in this case, *both* risks increase. The mean shortfall risk increases from 0.051 to 0.064 and the excess contribution rate risk increases from 0.053 to 0.059.

6.3.6.4 *Region II*

We have observed that positions in regions I and III can be improved by moving *towards* region II. Now we will answer the crucial question: "What happens if a decision maker chooses a position in region II?" Assume, without loss of generality, that the decision maker chooses point F, i.e. a normal contribution rate of 0.15 and 60% initial allocation in equities. Then we would expect, after 15 years, a mean shortfall risk of 0.046 and excess contribution rate risk of 0.0375. Can this position be improved?

(1) We cannot move to lower points (e.g. D, E, I and J) without increasing *both* risks.
(2) Moving to higher points (e.g. B and G) reduces both risks, but involves increasing the normal contribution rate.
(3) Moving horizontally, for example to C or H, maintains the normal contribution rate at 0.15. However, one of the risks decreases whilst the other risk increases:
    — by moving from F to C the mean shortfall risk increases from 0.046 to 0.05 and the excess contribution rate risk decreases from 0.0375 to 0.0368; and
    — by moving from F to H the mean shortfall risk decreases from 0.046 to 0.044 and the excess contribution rate risk increases from 0.0375 to 0.042.

It is important to notice that we cannot only move from regions I and III *towards* region II, but we can actually move *into* region II. For example:

(1) moving from A to F; in this case *both* risks decrease: the mean shortfall risk decreases from 0.10 to 0.046 and the excess contribution rate risk decreases from 0.055 to 0.0375 (this move corresponds to increasing initial allocation in equities from 7% to 60%); and
(2) moving from K to F; again *both* risks decrease: the mean shortfall risk decreases from 0.051 to 0.046 and the excess contribution rate risk decreases from 0.053 to 0.0375 (this move corresponds to decreasing initial allocation in equities from 98% to 60%).

6.3.7 *Conclusions*

This analysis leads us to the following conclusions:

(1) Positions in region I and region III are inefficient. This is because, for a given normal contribution rate, *both* the mean shortfall risk and the excess contribution rate risk can be *reduced* by moving to a position in region II.

(2) Positions in region II cannot be improved without either increasing one of the risks or increasing the normal contribution rate.

6.4 *Cost Measure: Average Contribution Rate*

6.4.1 The sponsor will be interested in the average cost over the funding period. In this section we will show how the average contribution rate can be used as a cost measure in our indifference curves setting that we have constructed.

6.4.2 Figure 7 shows the average contribution rate levels at the end of six years and 15 years. Each curve in Figure 7 shows all the combinations of initial allocation in equities and normal contribution rate that lead to a given average contribution rate.

6.4.3 We observe that combinations along higher curves lead to higher average contribution rates. For example, at the end of six years, combinations along the top curve lead to an average contribution rate of 0.18, whilst combinations along the lowest curve lead to a lower average

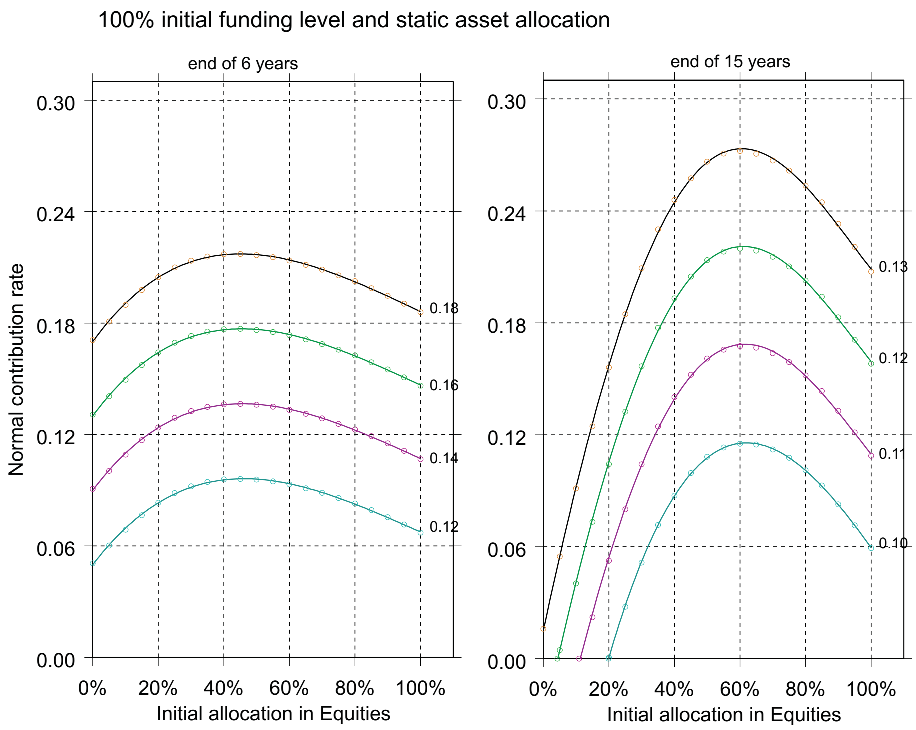

Figure 7. Average contribution rate levels at the end of six and 15 years

contribution rate of 0.12. And at the end of 15 years, combinations along the top curve lead to an average contribution rate of 0.13, whilst combinations along the lowest curve lead to a lower average contribution rate of 0.10.

6.4.4 The curves in Figure 7 are indifference curves, since the decision maker would be indifferent between all combinations along a given curve. These curves can be used as a measure of the cost. In our analysis of mean shortfall and excess contribution rate risks, we have concluded that a decision maker could improve their position by choosing combinations in the efficient region. For every choice of normal contribution rate and initial asset allocation, the decision maker could use Figure 7 to determine the average contribution rate over the projection period.

6.4.5 It is interesting to observe in Figure 7 that, for a given normal contribution rate, if we move horizontally towards the *maximum* point we *reduce* the average contribution rate. The particular shape of the curves in Figure 7 arises from the inclusion of a zero lower bound on the recommended contribution rate. If we remove the zero lower bound, we would obtain the plots of average contribution rate levels which were straight lines with maximum points at 100% equities. (The details are not presented here, but are available from the authors.)

6.5 *Average Contribution Rate Curves and the Efficient Region*

6.5.1 Figure 8 shows the average contribution rate levels and the efficient region at the end of 15 years.

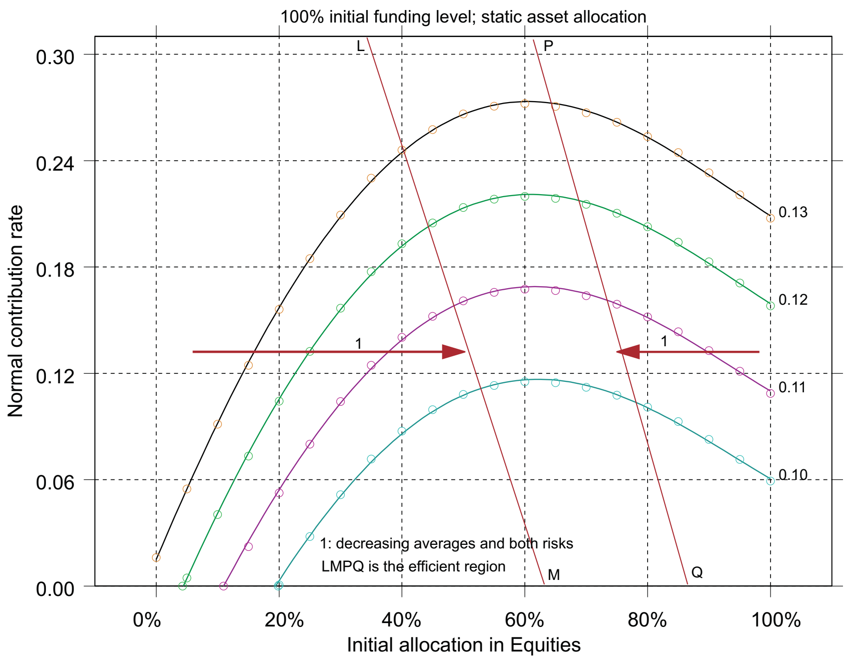

Figure 8. Average contribution rate levels and the efficient region at the end of 15 years

6.5.2 We observe that the maximum points lie in the efficient region LMPQ. In Section 6.3, we have shown that a decision maker could improve their position by moving *towards* or *into* the efficient region. Furthermore, in Section 6.4, we have observed that a decision maker could reduce the average contribution rate by moving horizontally towards the maximum point. Therefore, Figure 8 implies that, by moving towards the efficient region, the decision maker will be *reducing* the mean shortfall risk, the excess contribution rate risk and the average contribution rate.

6.6 *The Effect of Amortisation Periods on the Indifference Curves*

6.6.1 We now investigate the effect of amortisation periods on the mean shortfall and excess contribution rate risk levels. We consider the spread method (or proportional control method — as described in Section 5.2.4.2) and investigate two examples: spreading of surpluses and deficits over three years and over 12 years. In the previous sections we have assumed that surpluses and deficits are spread over three years.

6.6.2 We only examine the indifference curves at the end of the second and fifth projection periods. The curves at the end of the first projection period are equivalent for the two cases of a three-year and 12-year spread period, since the pension scheme is initially fully funded.

6.6.3 *Mean shortfall risk*

6.6.3.1 Figure 9 shows the mean shortfall risk indifference curves at the end of six and 15 years.

(1) *End of six years*

We observe that:

— for a given combination of normal contribution rate and initial allocation in equities, the mean shortfall risk is lower for amortisation over three years than for amortisation over 12 years (except, perhaps, for very low values of initial allocation in equities); and

— the minimum points of the indifference curves are in the region of 60% initial allocation in equities in both cases.

(2) *End of 15 years*

We observe that:

— for a given combination of normal contribution rate and initial allocation in equities, the mean shortfall risk is lower for amortisation over three years than for amortisation over 12 years, especially for initial allocation in equities higher than 20%; and

— the minimum points are in the region of 80% initial allocation in equities for amortisation over three years and 60% for amortisation 12 years.

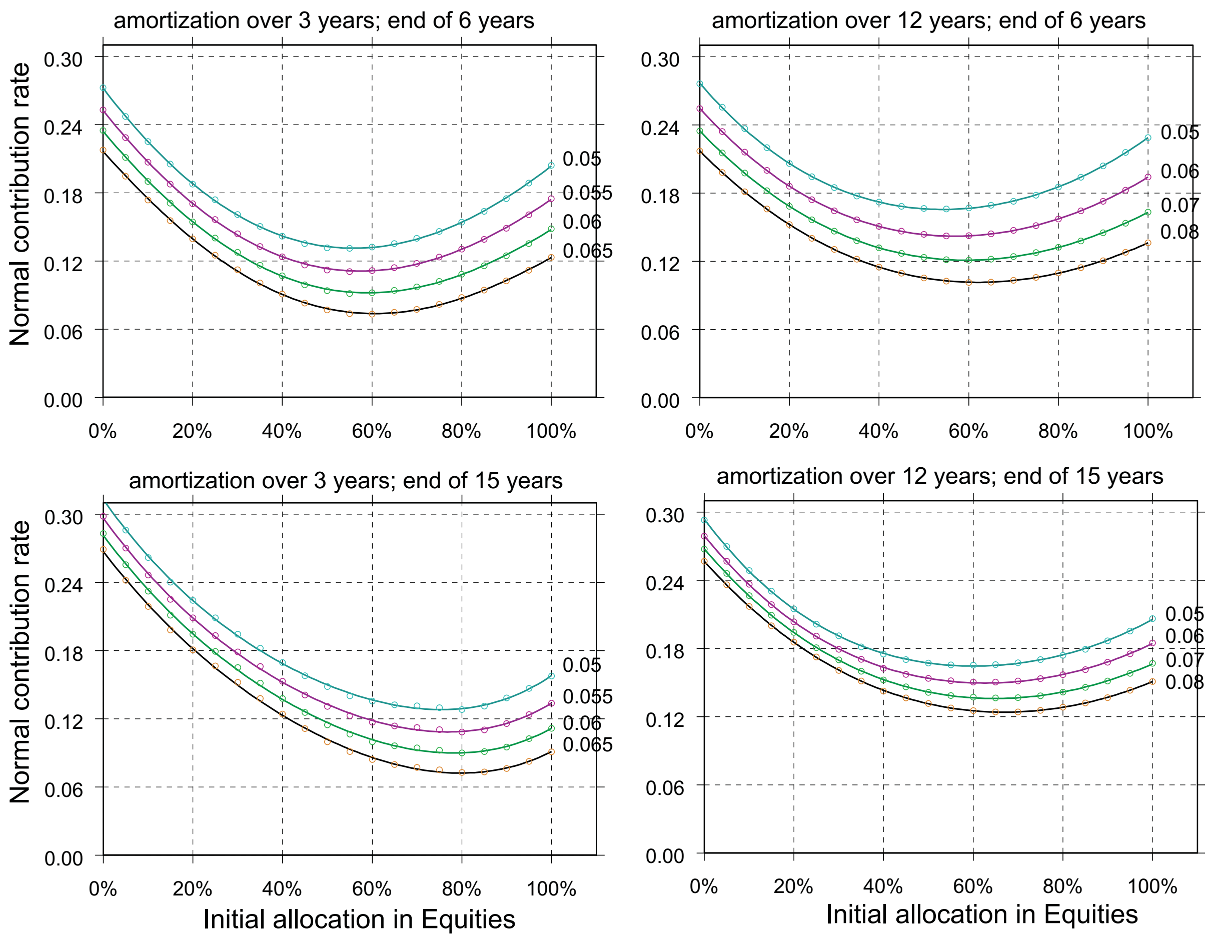

Figure 9. Comparison of mean shortfall risk levels at the end of six and 15 years (100% initial funding level; static asset allocation)

6.6.4 *Excess contribution rate risk*

6.6.4.1 Figure 10 shows the excess contribution rate risk indifference curves at the end of six and 15 years.

(1) *End of six years*

We observe that:

— for a given combination of normal contribution rate and initial allocation in equities, amortisation over three years leads to higher excess contribution rate risk than over 12 years; and

— the minimum points are in the region of 40% initial allocation in equities in both cases.

(2) *End of 15 years*

We observe that:

— for a given combination of normal contribution rate and initial allocation in equities, amortisation over three years leads to higher excess contribution rate risk than over 12 years; and

— the minimum points are in the region of 50% to 60% initial allocation in equities in both cases.

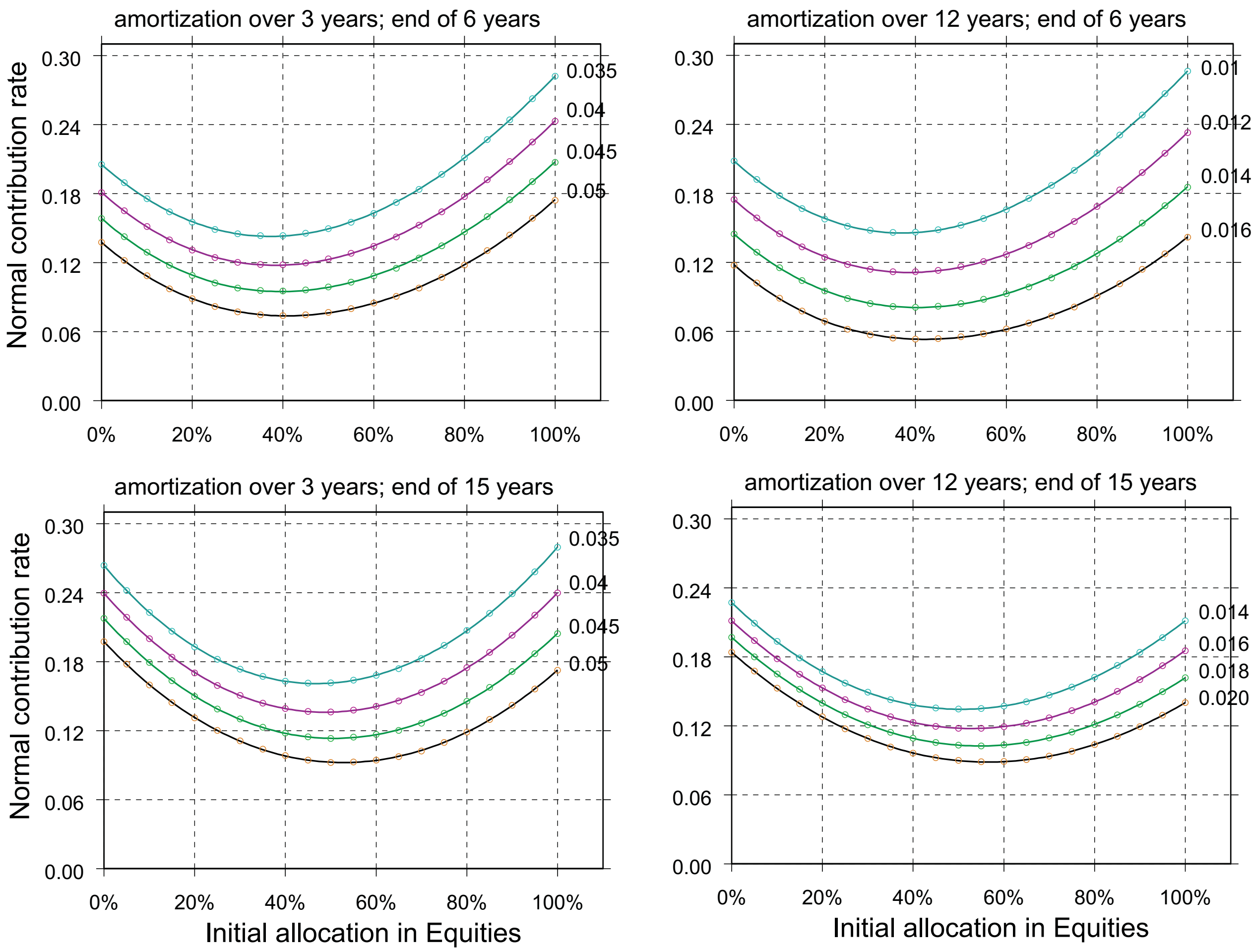

Figure 10. Comparison of excess contribution rate risk levels at the end of six and 15 years (100% initial funding level; static asset allocation)

6.6.5 *How do the amortisation periods compare?*

6.6.5.1 These results are intuitive; if deficits and surpluses are amortised over three years, the scheme would be expected to attain full-funding more quickly than if the amortisation were over 12 years. This explains the better mean shortfall risk position for amortisation over three years than for the 12 years' case.

6.6.5.2 However, the 'penalty' for this is the poor excess contribution rate risk positions for amortisation over three years: we would expect higher recommended contribution rates for amortisation over three years than over 12 years, all things being equal.

6.6.6.1 Figure 11 shows the efficient regions for projections over 15 years for amortisation over three and 12 years. ABCD is the efficient region for amortisation over 12 years, whilst LMPQ is the efficient region for amortisation over three years. The efficient region is much smaller when amortisation takes place over 12 years rather than over three years. Further calculations indicate that this is a general result — the region becomes more tightly defined as we increase the spread period.

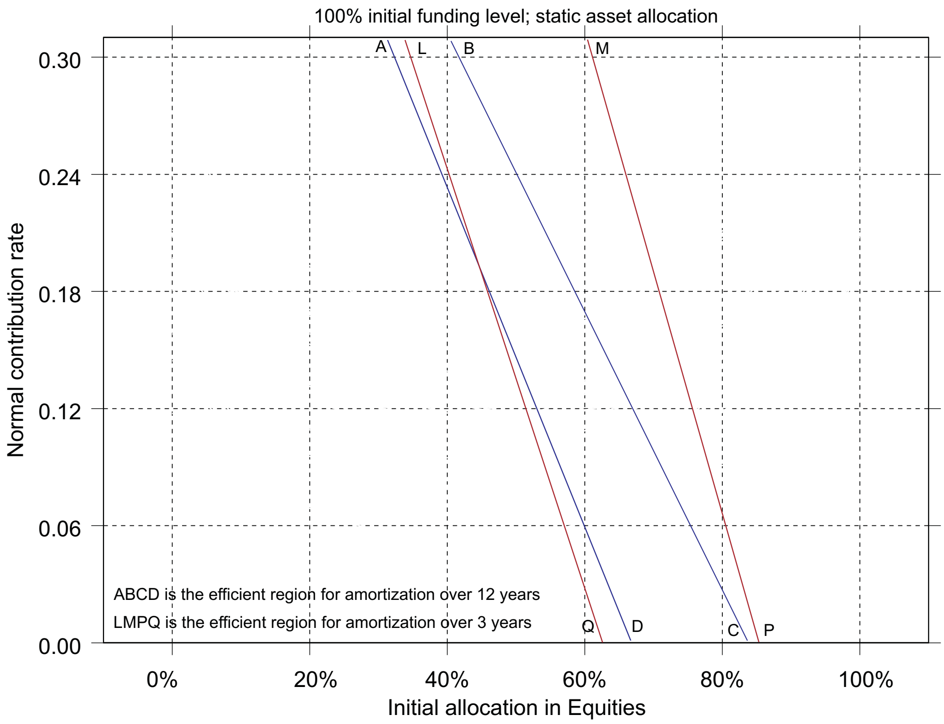

Figure 11. The efficient regions for 15-year projections for amortisation over three and 12 years

6.7 *Effect of Investment Strategies on the Indifference Curves*

6.7.1 We now investigate the effect of investment strategies on the indifference curves. We consider a static strategy and two dynamic strategies (as introduced in ¶5.1.2.2):

— *Static asset allocation.* This is the case we have been considering in the previous sections. The asset allocations are fixed and annually rebalanced.
— *Dynamic asset allocation (DS1).* In this case the asset allocations are changed at the end of every three years, as follows: for each 10% increase in the funding level (relative to the initial value), the allocation in equities is decreased by 5%, and vice versa.
— *Dynamic asset allocation (DS2).* In this case, for each 10% increase in the funding level (relative to the initial value), the allocation in equities is increased by 5%, and vice versa.

6.7.2 We note that DS1 is a counter-intuitive strategy, where funds are switched into equities as the funding level becomes more adverse and into bonds as the funding level becomes more favourable.

6.7.3 *Mean shortfall risk*

Figure 12 shows the mean shortfall risk indifference curves at the end of six and 15 years. (At the end of three years the indifference curves are

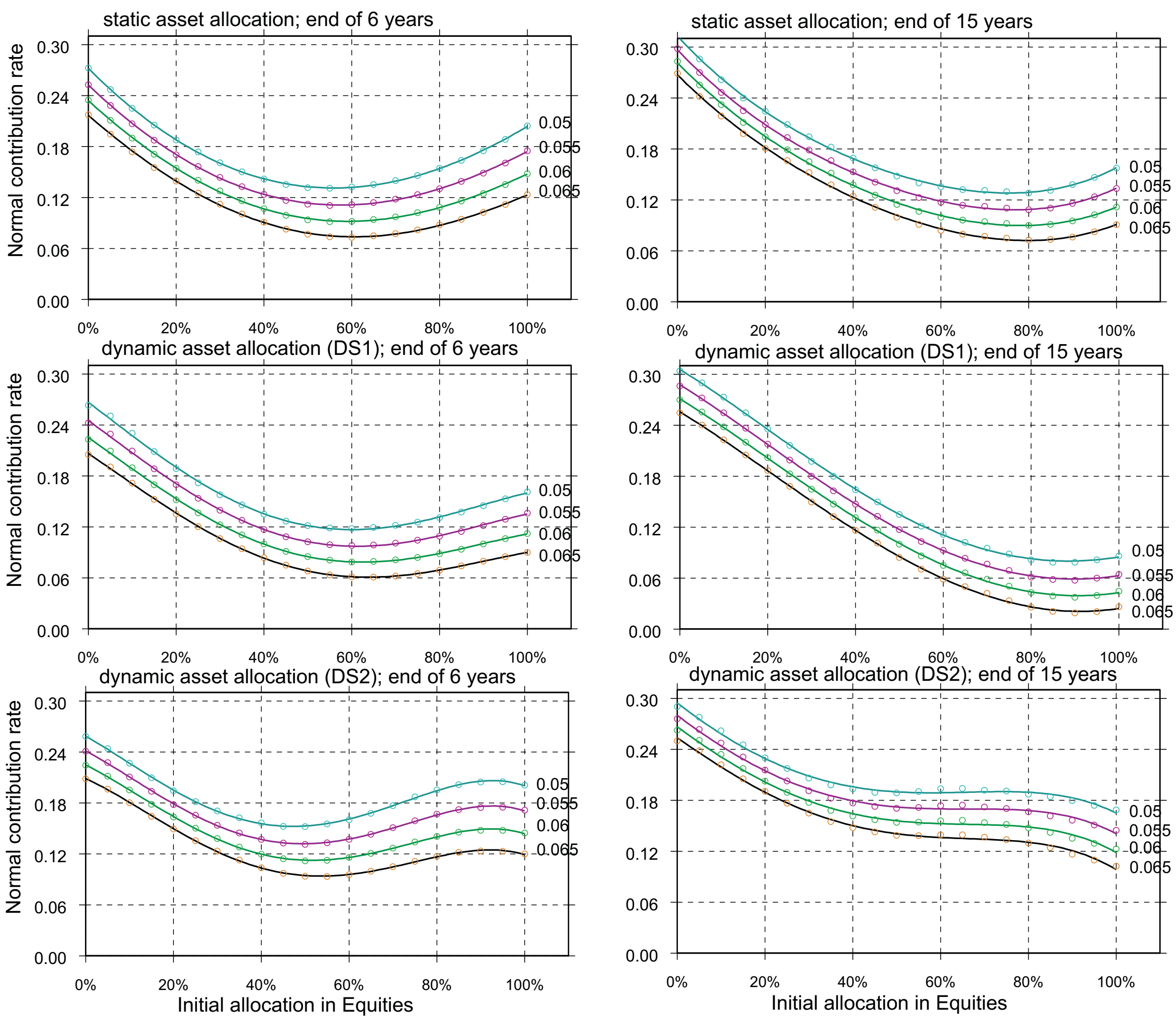

Figure 12. Comparison of mean shortfall risk levels at the end of six and 15 years (100% initial funding level)

similar, since all asset allocations in the first period are as in the static case.) We observe that, for a given combination of normal contribution rate and initial allocation in equities, the mean shortfall risk is lowest under DS1 and highest under DS2 (except for very low values of initial allocation in equities).

6.7.4 *Excess contribution rate risk*

Figure 13 shows the excess contribution rate risk indifference curves at the end of nine and 15 years. (At the end of six years the indifference curves are similar for all three asset allocations.) We observe that, for a given combination of normal contribution rate and initial allocation in equities, the excess contribution rate risk is lowest under DS1 and highest under DS2.

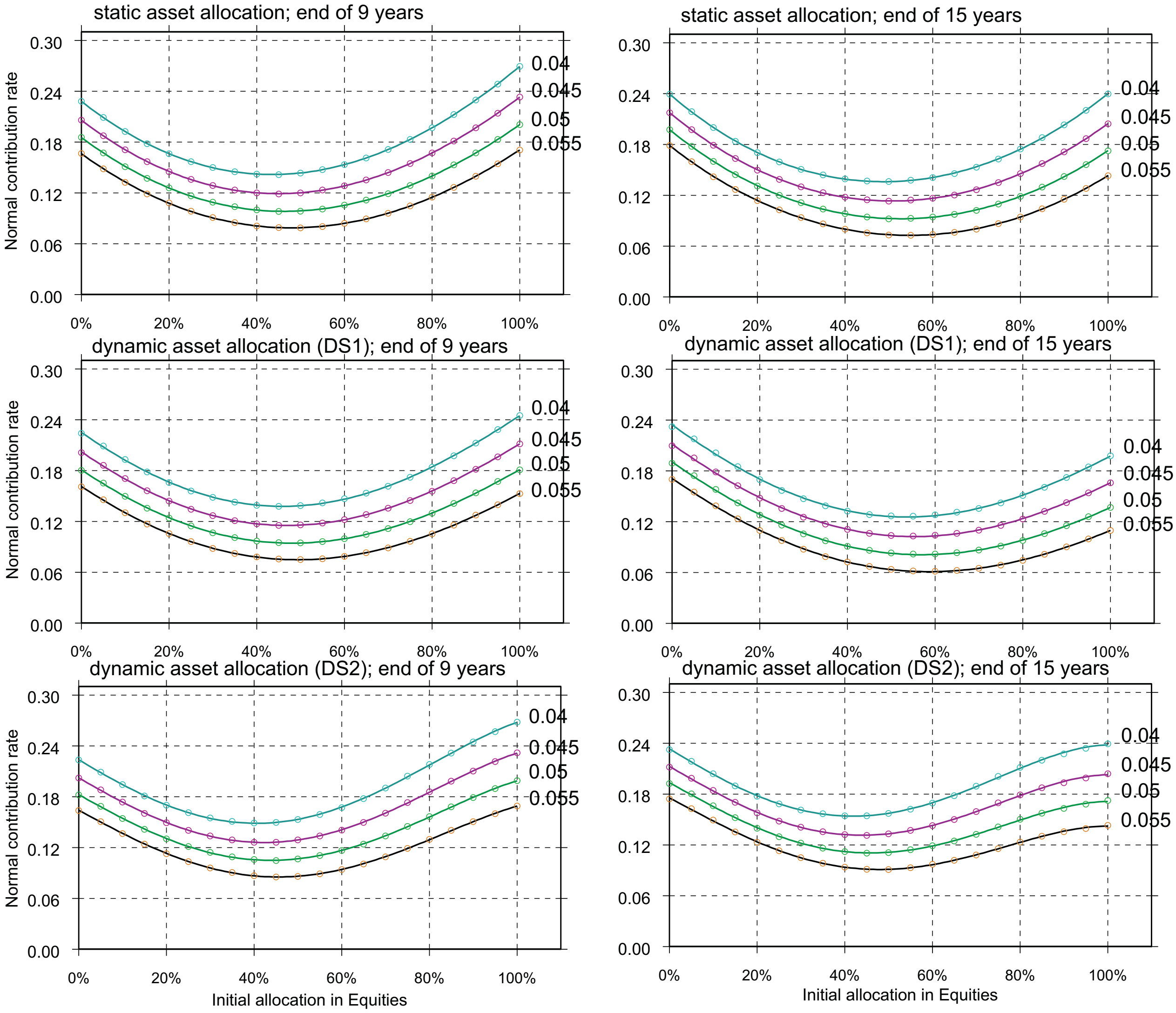

Figure 13. Comparison of excess contribution rate risk levels at the end of nine and 15 years (100% initial funding level)

6.7.5 *How do the investment strategies compare?*

6.7.5.1 We have observed that the mean shortfall and excess contribution rate risks are lowest under DS1 and highest under DS2, for a given combination of normal contribution rate and initial asset allocation. This holds for all projection periods.

6.7.5.2 Hence, we conclude that dynamic strategy DS1 performs consistently better than the other two strategies; whilst the static strategy performs better than dynamic strategy DS2. These results are consistent with the results emerging from theoretical models, as discussed in ¶4.3.2.3 and presented in more detail in Appendix D.

6.8 *Effect of the Initial Funding Level on the Indifference Curves*

6.8.1 We have so far analysed the results for an initially fully-funded scheme. We now consider the effect of other initial funding levels: a scheme initially in deficit and a scheme initially in surplus. We compare the results for the 100% initial funding level with the results for 80% and 120% initial funding levels.

6.8.2 We spread any surpluses and deficits, including the initial surplus or deficit, over three years. In our stochastic pension fund simulations, we use a recommended contribution rate, which is just the normal contribution rate plus an adjustment generated by the spread formula (as discussed in Section 4.3.1). A problem arises, however, if the adjustment is negative and larger, in absolute value, than the normal contribution rate. In practice, this would mean a refund of contribution to the sponsor. In stochastic simulations, however, a refund of contributions would introduce unnecessary complications, and, hence, we impose a lower bound of zero on the recommended contribution rate.

6.8.3 This feature has a significant impact on the results of a scheme which is initially in surplus. When the initial surplus, 20% in our case, is amortised over three years, the consequential adjustment is so large and negative that, in the first three-year period, we recommend a contribution rate of zero for most of the choices of the normal contribution rate. This has three significant implications.

6.8.3.1 Mean shortfall risk indifference curves would be impossible to calculate at the end of three years. The mean shortfall risk would be the same, for a given asset allocation, for all normal contribution rates where the recommended contribution rate is zero. This complication only arises in the first three-year period, due to the initial surplus. However, this does affect the excess contribution rate risk calculations at the end of the second three-year period, as we shall show below.

6.8.3.2 Excess contribution rate risk indifferences curves would be impossible to calculate at the end of three years. (For a scheme which is initially in deficit, we can calculate the excess contribution rate risk at the end of three years; but we cannot derive the indifference curves, because this risk level is the same for all choices of the normal contribution rate, because there is only one value for the adjustment factor.) This is due to the fact that there is no excess contribution in the first three-year period, since the scheme has an initial surplus. Thus, whatever the initial funding position of the scheme, the excess contribution rate risk indifference curves cannot be calculated at the end of three years.

6.8.3.3 Excess contribution rate risk indifference curves would be impossible to calculate at the end of six years for a scheme which has a (significant) initial surplus. This demands careful explanation. We have noted above that, at the end of three years, the mean shortfall risk is the same, for a given asset allocation, for all normal contribution rates where the recommended contribution rate is zero in the first three years. This implies that, when we spread surpluses and deficits at the end of three years, the adjustments will be the same wherever we had a zero recommended

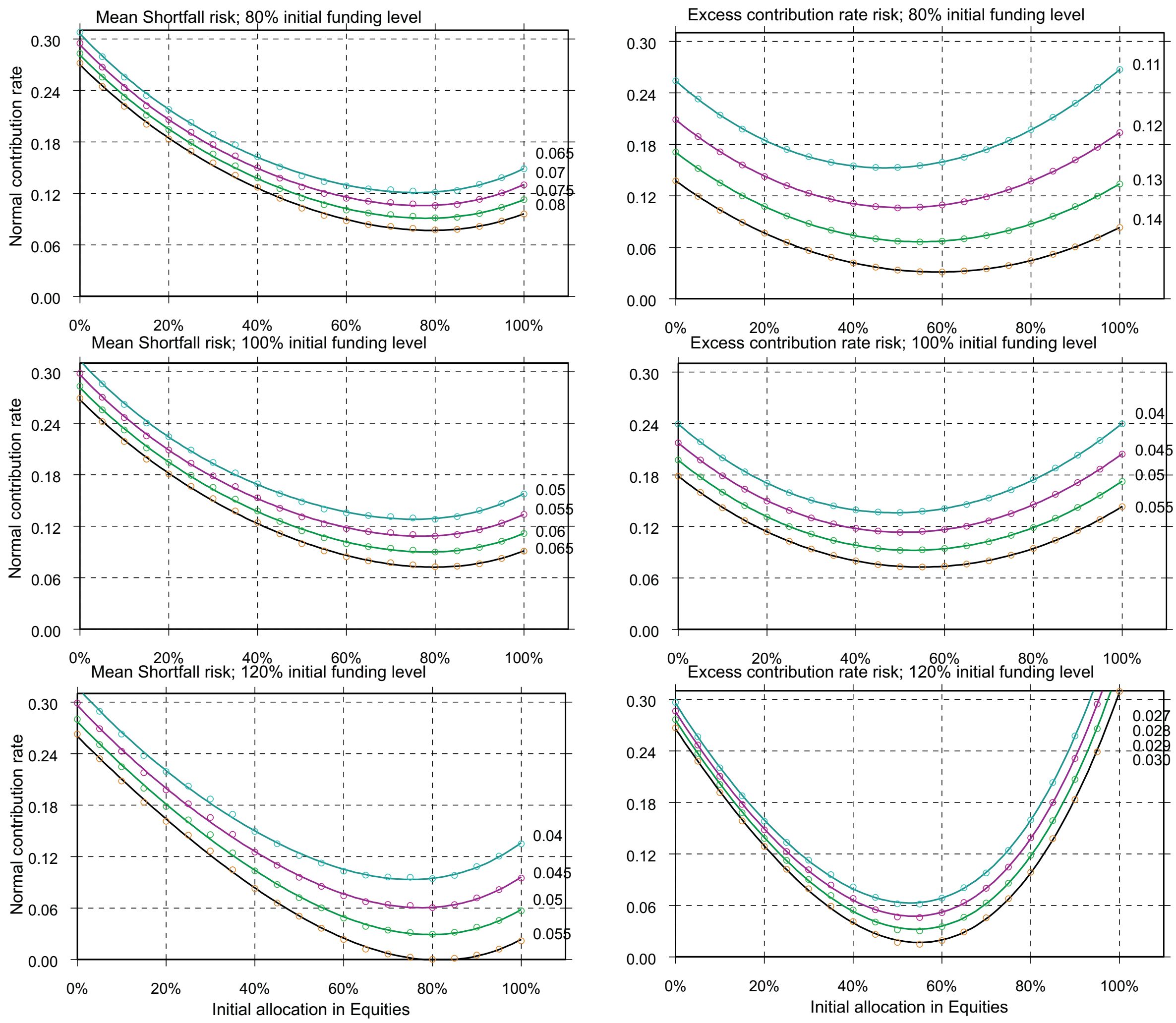

Figure 14. Mean shortfall and excess contribution rate risk levels at the end of 15 years (static asset allocation)

contribution rate. In such cases, differences will only arise due to the initial asset allocation. Thus, at the end of six years, we would calculate the excess contribution rate risks, but we would not be able to derive the indifference curves. (However, it is important to note that some of these problems arise due to the amortisation period for the initial surplus or deficit; and/or the amount of the initial surplus or deficit.)

6.8.4 Figure 14 shows the mean shortfall risk and the excess contribution rate risk indifference curves at the end of the 15 years for the three initial funding levels. From Figure 14, we observe the following features.

6.8.5 *Mean shortfall risk*

— For a given combination of normal contribution rate and initial allocation in equities, the mean shortfall risk is highest for the 80% case and lowest for the 120% case.

— The minimum points are in the region of 70% to 85% initial allocation in equities.

6.8.6 *Excess rate risk*

— For a given combination of normal contribution rate and initial allocation in equities, the excess contribution rate risk is highest for the 80% initial funding case and lowest for the 120% initial funding case.
— The minimum points are in the region of 50% to 60% initial allocation in equities.

6.8.7 *The efficient regions*

As observed in ¶¶6.8.5 and 6.8.6, the minimum points in Figure 14 lie in similar regions for the three initial funding cases. Thus, the efficient regions for the three cases are similar. (The details are not shown here, but are available from the authors.)

7. Presentation of Results

7.1 In this section we consider the important issue of presenting and communicating the results to the client. We believe that a stochastic approach, based around our new performance measures, will help trustees and scheme sponsors (and Scheme Actuaries) to understand the risk trade-offs that can be gained when decisions are taken in the management of defined benefit schemes. Equally, we are aware that some of the concepts underpinning our work look complex, and concede that the range of outputs can be bewildering at first sight. How can we break all this down to the points that the client really needs to know?

7.2 We suggest that a suitable way of approaching client communication would be as set out in the following paragraphs.

7.2.1 Firstly — meet the clients.

7.2.2 As with asset/liability modelling, it is vital to collect the clients' opinions on what issues matter most to them. We suggest a pre-meeting that aims to do two things:

(1) Educate the clients as to what they can expect from the work, using some high-level generic examples.
(2) Collect views on what concerns the clients. These would include:
    — the solvency level they might be able to tolerate (Although our case study is based on a pension scheme that is fully funded relative to bond based wind-up liabilities, many life schemes are (in reality) currently very far away from this position. How comfortable are trustees with this? Would they want to target something less than 100% solvency on this basis in their solvency performance measure?);
    — the average contribution rate that is acceptable (There is likely to be a range of rates of input that will not be affordable.);

— the size of an increase in the contribution rate that would cause them difficulties; and
— the time horizon over which the results should have most weight (This might be relatively short (three years, say) if, for example, the trustees consider that the employer may not be able to support the scheme in the medium or long term. Longer periods, say 12 or 15 years, might be more appropriate for very stable employers.)

7.2.3 Secondly — prepare an initial analysis and a report for the clients.

7.2.4 Our firm view is that the written report needs to be kept short, concentrating on the results that matter to the clients and, where possible, limiting the amount of detail provided on the stochastic methodology underlying the results.

7.2.5 We believe that the report should contain the following information:

— a note of the objectives of the stochastic approach;
— an explanation of the risk and performance measures used;
— a discussion of the input gained from the client at the pre-meeting and how this fits in with the risk and performance measures used;
— a description of how to interpret the indifference curves (perhaps the profession should come up with a better name?); and
— the results presented as indifference curves. The following graphs are the ones to focus on (using the examples from our case study in Section 6):
  (1) *Solvency risk* — based on the mean shortfall risk at the end of the period. Figure 15 shows the mean shortfall risk levels for projections

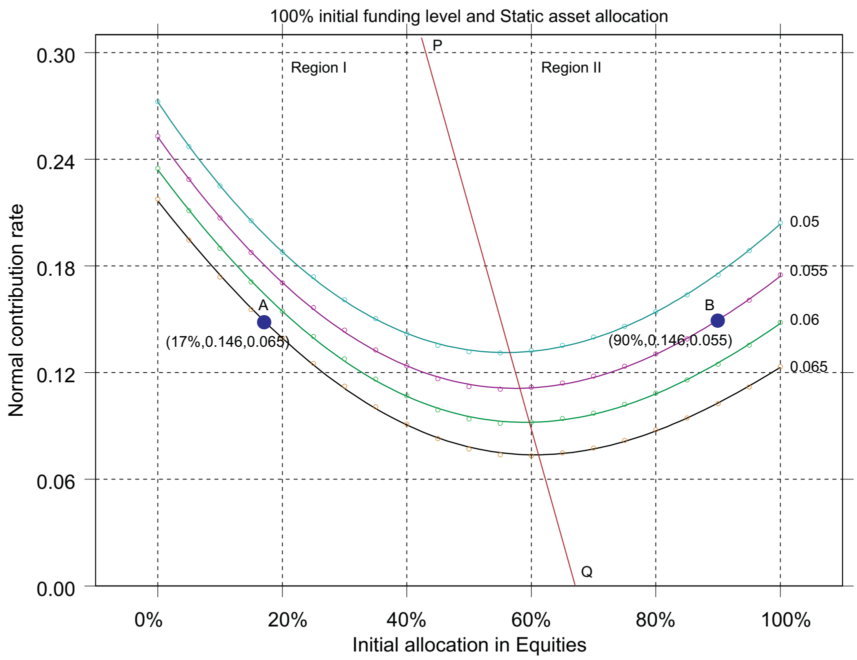

Figure 15. Mean shortfall risk levels at the end of six years

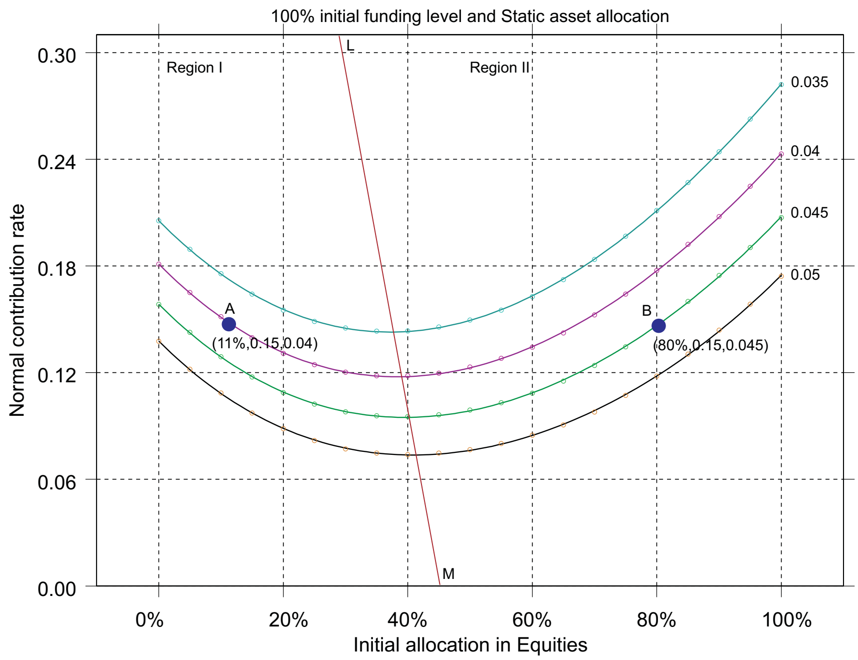

Figure 16. Excess contribution rate risk levels at the end of six years

over six years, as an example. This would be accompanied by a narrative explaining briefly the derivation of the PQ line and what it means to clients.

(2) *Contribution rate risk* — based on the excess contribution rate risk at the end of the period. Figure 16 shows the excess contribution rate risk levels for projections over six years. Again, this would be accompanied by narrative explaining the significance of the LM line.

(3) *The combined solvency and contribution rate risk* overlayed with any information the client has given on acceptable bounds. The area of interest is within the region lines and between the maximum normal contribution rate bound gleaned at the pre-meeting and the clients' solvency bound. (See Figure 17, which shows the efficient zone XYWZ for projections over 15 years. In this case we have assumed that the client suggests a maximum normal contribution rate of 0.18 and a maximum mean shortfall risk of 0.085.) This should be accompanied by a narrative that explains the risk trade-offs as one moves around the efficient zone.

(4) *The average contribution level* over the period. Figure 18 shows the average contribution rate levels and the efficient zone for projections over 15 years.

It should be explained that the clients then have an introduction through (1) and (2) to the individual risks involved, with the information combined into (3) to show how changing the asset mix and contribution rate affects the risks. Figure 18 gives a check that the average contribution rate for a given

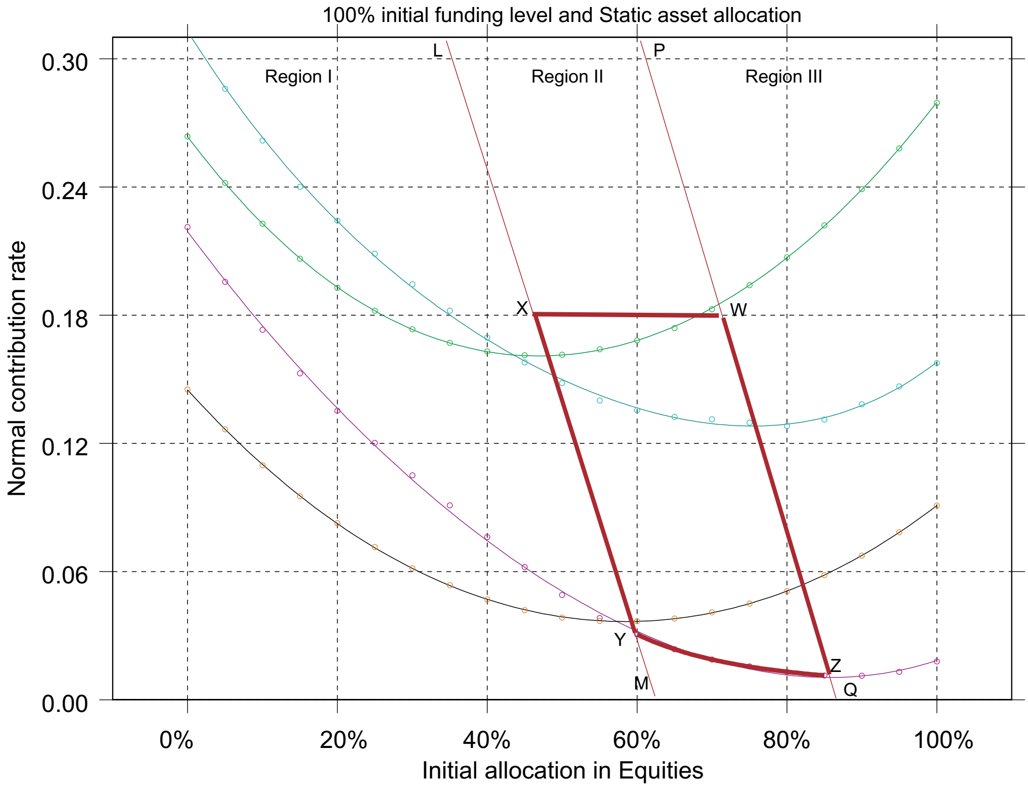

Figure 17. The efficient zone for projections over 15 years

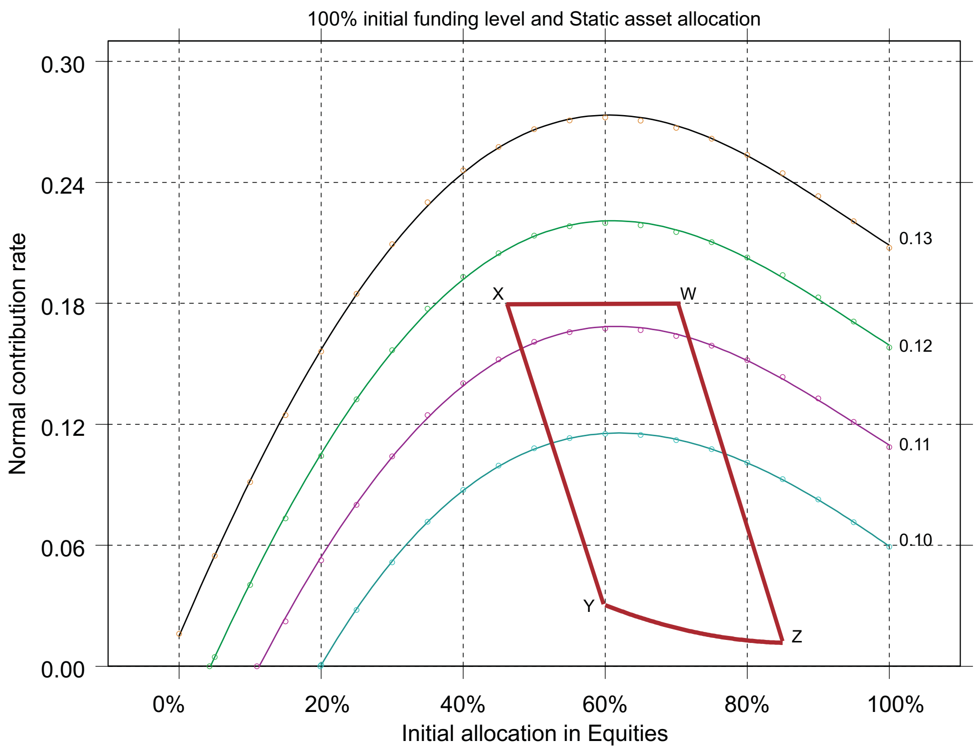

Figure 18. Average contribution rate levels and the efficient zone for projections over 15 years

risk profile derived from (3) is acceptable. Alternatively, Figure 18 can be used to select a contribution rate and the results read back into Figure 17 as a check on the magnitude of the risks.

- The report should contain appendices explaining:
  - the model used;
  - the parameters within the model;
  - key assumptions, including those about any flow of new entrants assumed; and
  - an explanation of the dynamic decision rules used in the model. This is to allow the clients the opportunity to check that the model provides a reasonable representation of the way that the clients would react to the events being modelled.

- The report body should finish with an invitation to meet and discuss the report.

7.2.6 Thirdly — meet the clients again.

7.2.7 This is where the practising actuary can add considerable value. We can see many ways where a laptop based presentation could be used at a meeting to guide the clients through the analysis. For example, it would be possible to overlay graphs, fading them in and out to demonstrate the effect of different choices. The results at different time points within the main period could also be explored.

7.2.8 In the background, the actuary may well have carried out sensitivity analysis (similar to that shown in Appendix E). We would not normally recommend that this be shown to the client unless the results throw up something very significant. Rather, we feel that the extra understanding the actuary will thereby develop of the model will help in the explanation of the main results.

8. Conclusions and Recommendations

8.1 The comfortable surpluses in defined benefit pension schemes are running out, and life is becoming much tougher for scheme sponsors and trustees. Risk management is now very much to the fore. Risk management needs risk measurement, which, in turn, needs a proper recognition of the stochastic nature of variability. In some cases, the goal of managing risk has led to perfectly good final salary scheme being closed or wound up. We wonder how much of this is an over-reaction caused by insufficient information — in the absence of guidance on the effects of other possible decisions, it may be easier just to play safe and cut the exposure. How can actuaries give that help?

8.2 We see two main areas where traditional funding approaches could be improved:

— The traditional deterministic valuation approach, even when supplemented by sensitivity testing, cannot quantify the risk inherent in the chosen funding strategy. We appreciate that risk *is* limited to an (unknown) extent, because Scheme Actuaries use their judgement or instinct to steer clients away from the riskier funding strategies. However, actuaries cannot use the traditional techniques to help sponsors or trustees test and understand how different decisions can reduce the risks which they run.
— Investment allocation is sometimes analysed through stochastic asset/liability studies, but it is often treated as a separate problem. We believe that an integrated view, taking investment and other decision making together, offers a much more informative way of helping clients manage their schemes.

8.3 From the work which we have done, we are convinced that better decisions would be made if traditional valuation approaches were supplemented by stochastic analyses. We believe that any stochastic analysis must look in an integrated way at the interaction of the main points of concern to trustees and scheme sponsors. We have developed four risk and performance measures that achieve this objective, and we would recommend that stochastic modelling of defined benefit pension schemes be based around these measures and others like them. We have also outlined methods of presentation that would reduce the results of the analysis to the main points of concern to the client.

8.4 Our case study, while general in nature, demonstrates that our proposal of stochastic analysis centred on multiple performance measures enables more effective decisions on funding and investment strategy to be made. This flows from the advantages of the method:
— It measures risk and looks at trade-offs.
— It looks at the interaction of more than one decision.
— It reveals variability, including the risk of underfunding or overfunding.

8.5 The key to securing client acceptance from this new work lies in the communication of the results. It is crucial that the presentation is clear and that the focus is on the principal issues. We recommend the use of a limited set of indifference curves as part of this presentation.

8.6 At the same time, the abundance of information that can be obtained from the stochastic methodology proposed, which is far more than would ever be presented to a client, allows the actuary a much better understanding of the dynamics of the particular scheme, or pension schemes generally. As can be seen in our case study, the results are not always as expected. It is intriguing to wonder how different funding levels, investment mix and benefit design would now be had our proposals been adopted ten years ago.

8.7 Of course, further work will need to be done before the actuary can start to help clients. Preparations for implementation would include building a cash flow model, estimating parameters, selecting the stochastic inputs and deciding on the treatment of other model features, including demographic variability.

8.8 Someone will have to pay for all of this new development! It would be a disappointment to us if only the larger clients with the bigger budgets were able to benefit from our proposals — in many ways it is the small schemes that are more in need of advice on risk. To help with this, we suggest that the profession commissions and makes available the results of some generic modelling of, say, closed or maturing small schemes.

8.9 We opened our paper by holding out the hope of better decision-making. We hope you have decided that we are right.

Acknowledgements

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APPENDIX A

RISK MEASURES

Consider a random variable X, which could represent loss in a particular time period in an insurance context, or size of fund at a particular time or level of contribution for a particular time period for a defined benefit pension scheme. Then, as in Albrecht *et al.* (2001), we can define a number of possible risk measures.

Firstly, we define the shortfall probability with respect to a deterministic benchmark x as:

$$\mathrm{SP}(x) = \Pr(X \leq x). \tag{A1}$$

Then the quantile reserve or VAR for X at the $100q$th percentile is defined as:

$$\begin{aligned} \mathrm{VAR}_q &= \inf\{x : \Pr(X \leq x) \geq q\} \\ &= \inf\{x : \mathrm{SP}(x) \geq q\}. \end{aligned} \tag{A2}$$

A risk measure which allows for both the probability of a shortfall and the extent of a shortfall is the unconditional shortfall expectation, defined as follows:

$$\mathrm{SE}(x) = \mathrm{E}[\max(x - X, 0]. \tag{A3}$$

We can also define the conditional mean excess shortfall:

$$\mathrm{MES}(x) = \mathrm{E}[x - X \mid X < x]. \tag{A4}$$

Artzner *et al.* (1997, 1999) advocate the conditional tail expectation as a risk measure, defined as:

$$\mathrm{CTE}(x) = \mathrm{E}[X \mid X < x]. \tag{A5}$$

It is common to choose $x = \mathrm{VAR}_q$ in the definitions of $\mathrm{MES}(x)$ and $\mathrm{CTE}(x)$: Artzner *et al.* (1997), Wirch & Hardy (1999). It is clear that the following deterministic relationship holds:

$$\mathrm{MES}(x) = x - \mathrm{CTE}(X)$$

and it is also clear that:

$$\mathrm{SE}(x) = \mathrm{SP}(x).\mathrm{MES}(x). \tag{A6}$$

The shortfall expectation is a measure of the unconditional average shortfall. It is the sum of shortfalls weighted by their probabilities and it can be regarded as "the price of an insurance contract which would cover the shortfall". (Maurer & Schlag, 2002).

APPENDIX B

MODERN FINANCIAL THEORY AND DEFINED BENEFIT SCHEMES

Introduction

Some members of the profession, impressed with the elegance and logical consistency of pricing methods developed by financial economists, have recently challenged the traditional approach taken by actuaries to both the valuation of pension liabilities and the formulation of investment strategies. Exley *et al.* (1997) and Chapman *et al.* (2001) have provoked much debate within the profession, and have encouraged many actuaries to learn more about modern financial theory.

A concise summary of the main arguments put forward by these papers is listed below:

(1) The value of an uncertain future payment is the price at which this payment would be traded in a competitive market.
(2) This price can be evaluated in the absence of a market for the liability by using mathematical models developed by financial economists.
(3) These models are built upon on the principle of no-arbitrage, or more generally on the concept of equilibrium.
(4) The value of defined benefit pension liabilities, and hence the cost of meeting such liabilities, is independent of the assets used to fund them.
(5) The effect of any decision taken in a defined benefit pension scheme is to change the distribution of wealth between the interested parties, who are the members of the scheme, the shareholders of the company, the Government, and various 'agents' acting on behalf of these parties.
(6) A 'scheme-centred' decision making process is therefore flawed.
(7) A company does not create value for its shareholders by switching its pension fund into assets expected to yield a higher return — the consequences for the shareholders are determined by second-order factors such as taxation, rather than by the expected return on the assets.

Can Pension Liabilities be Priced?

Is it really possible to price liabilities for which there is no market? The examples used to convince people that this is possible are usually very simple ones. For example, suppose we have to pay a lump-sum benefit of £1 due in one year. Can the price of this liability differ from the market price of a zero-coupon bond maturing in one year? The answer is 'no', because of the principle of no-arbitrage: if it were possible to sell the liability for a different price, someone would be able to make a risk-free profit by taking an opposite position in the zero-coupon bond.

Pension liabilities, however, cannot always be replicated in this way by marketable securities. The liabilities are often too long in duration; they are usually indexed to price or wage inflation; they depend on contingencies linked to human life. In such circumstances, applying the principle of no-arbitrage can at best produce a range of prices. The concept of equilibrium is now required to complete the journey to a unique 'market-consistent' price, and 'state price deflators' have been invoked as the mathematical tool underpinning equilibrium pricing (Chapman *et al.*, 2001).

But what does equilibrium mean in the context of financial markets? The concept of equilibrium used in the pricing of financial assets is inseparable from mathematical utility theory, as proposed by von Neumann & Morgenstern (1944). Under this theory, the behaviour of every investor is guided by a subjective utility function for wealth or consumption. Equilibrium describes a state of affairs in which all investors have optimised their portfolios, where 'optimisation' means the maximisation of expected utility.

We should note that the assumptions required to make this theory work are far more problematic than the 'no free lunch' requirement embodied in the principle of no-arbitrage. Economists such as Allais (1953), Tversky (1969) and Shoemaker (1980) have questioned whether investors do, in fact, maximise their expected utility in the manner assumed by equilibrium pricing theory. In spite of these reservations, we do not dispute that utility-based models have been of value in understanding both prices observed in the market and some of the factors influencing the behaviour of individuals. One of the triumphs of utility theory, for example, is its explanation of the existence of insurance contracts (Bowers *et al.*, 1997).

Using utility-based models to price liabilities for which no market exists, however, may be stretching equilibrium pricing theory beyond the limits of its reliability. To be confident of the results obtained, we would have to believe, not only in the existence of market equilibrium and the validity of mathematical utility theory, but also in what the financial literature refers to as *homogeneous expectations*. This requires that the statistical parameters used in our pricing model are agreed upon by all the investors in the market. Put simply, everyone must agree about the means, variances and covariances of the returns on risky assets. Under this assumption, the millions of stocks that change hands every day are traded by investors who are in complete agreement on their merits. The 'bulls' and 'bears' of stockmarket folklore simply do not exist.

As one would expect, financial economists have attempted to devise equilibrium pricing models relaxing the unrealistic assumption of homogeneous expectations. Lintner (1971), for example, develops a model in which investors use subjective statistical parameters for asset returns. In this model, equilibrium asset prices depend on complex weighted averages of the investors' subjective statistical parameters. Whatever insights can be gained from such models, it should be clear that there is no question of using them as a practical pricing tool, because we could never hope to acquire the countless parameters required. The aim is to further our fundamental understanding rather than to price real assets.

It follows that equilibrium pricing models have been used mainly as a tool for understanding and explaining the factors that influence prices actually observed in the market. Using such models to price pension liabilities for which no market exists involves making many assumptions that are open to question. We also have no way of testing the accuracy of the 'prices' derived, unless the liabilities are actually sold.

Can a Scheme Centred Approach be Justified?

Despite the above-mentioned reservations, we assume that the pricing problem referred to above can be solved, and we have confidence in our model for valuing the pension liabilities under different strategies for managing the scheme. Can these 'prices' help the trustees and sponsoring employer manage the scheme?

Chapman *et al.* (2001) contend that pricing pension liabilities in this way allows us to quantify the transfers of wealth that occur as a result of different decisions. For example, if a scheme were to follow a stronger funding policy, more surplus would be distributed to members and more tax relief would be earned on contributions. A suitable stochastic model can be used to project how the random cash flow streams paid to each party will change, and these random cash flows can then be priced using the state price deflators. The implied transfers of wealth from shareholders to members and from government to shareholders can thus be quantified.

This approach treats the decision-making process as a zero-sum game. There is a fixed amount of wealth available for distribution, equal to the assets of the business plus the assets of the pension fund, which is allocated in a complex way between shareholders, pension scheme members and other interested parties. Any decision that changes the nature of the pension liabilities, or the risks associated with them, will change the share of wealth held by each party. The implication is that the pension scheme should be run in such a way as to minimise the wealth allocated to the Government (and presumably also to 'agents' such as company directors, fund managers and actuaries), leaving the shareholders and members to haggle over the spoils.

However there are two problems with this approach.

Firstly, the transfers of wealth that occur are not really transfers of wealth. They are the transfers of wealth that *would* occur if it were possible to sell the liabilities at the prices produced by the model. Even if these prices were correct, the option of selling off the liabilities does not usually exist. So, the actual wealth that *will* be transferred as result of any decision is unknown — it is a random variable that depends on the future experience of the scheme. Nor is the total amount of wealth that will be available for allocation a constant — it is also a random variable that will depend quite significantly on decisions made today. Hence, the case for basing decision making on these hypothetical market prices is rather weak, for what use is a market price when there is no market?

A simple example might be instructive. Suppose that we are about to buy £1,000 worth of equities when the Bank of England offers us £1,000 worth of gilts at a 10% discount to market value. Obviously, we buy the gilts for £900, sell them in the market for £1,000 and buy the equities, which we originally wanted, leaving us with a profit of £100. But, suppose the Bank of England decrees that, after our transaction, the gilt-edged market will be closed for ten years. Should we still buy the gilts for £900, knowing that we will not be able to sell them in the market for ten years? The answer, we would suggest, is 'not necessarily'.

The second problem is the legal position of the trustees. Under trust law, they are required to protect the interests of the members (Lee, 1986); the interests of the shareholders and 'agents' should be an irrelevance to them. It follows that a scheme centred approach, based on analysing different outcomes rather than attempting to price them, may be a more practical method of decision making. For the trustees, indeed, it appears to be the only approach consistent with their legal responsibilities. The interests of the shareholders should be left to the sponsoring employer, and this issue is discussed below.

It is noteworthy that Hairs *et al.* (2001), in reviewing and commenting on the proposals for the valuation of insurance liabilities put forward by the International Accounting Standards Board, advocate an 'entity-specific' approach (rather then a market-based approach) to valuation. Thus, the valuation should take account of 'entity-specific circumstances while making assumptions about the future which are market based'.

Maximising Shareholder Value

A simplistic argument could be made that the interests of members and shareholders are necessarily in conflict, as benefit improvements inevitably result in smaller profits. However, this ignores the fact that companies run pension schemes for the same reason that they pay their employees: namely, to recruit and retain suitably qualified people. The effect of benefit improvements on shareholder value is therefore complex and judgemental. Sponsoring employers should assess them in the same way that they would assess pay increases. (In the discussion following Exley *et al.* (1997), Mr A. Judes made some pertinent observations on this matter).

The issue that seems to have caused most controversy within the actuarial profession, however, is the impact of pension fund investment on shareholder value. The principle that the value of a pension liability should not depend on how it happens to be funded is an important idea from financial economics (Treynor *et al.*, 1978), and is now enshrined in pension accounting standards such as SFAS 87 and FRS 17. However, Exley *et al.* (1997) also argue that the cost of the scheme is largely independent of its investment strategy, and, in particular, that investing in assets expected to produce higher returns does not reduce pension costs. Actuaries who have determined pension costs in accordance with these standards, however, are confronted with the fact that higher investment returns create surplus, and surplus inevitably reduces reported pension costs (unless benefits are improved). In FRS 17, moreover, expected investment returns in excess of the discount rate must be anticipated.

The point, however, is rather more subtle. Any reduction in pension costs created by investing in higher-returning assets is an illusion as far as the shareholders are concerned. The extra profits created are fictitious profits, because the shareholders could have invested in these assets within their own portfolios, and secured the benefits of these higher returns without having to share them with the members of the scheme. It has also been pointed out that, in a tax-exempt pension fund, bonds are more tax-efficient than equities. These considerations would lead to the conclusion that pension funds should invest in bonds to maximise shareholder value and eschew equities altogether.

There are two problems with the arguments outlined in the above paragraph. The first is again the legal position — the trustees are supposed to be in charge of investment policy and their duty is to protect the interests of members, not shareholders. Does this mean that whatever is bad for the shareholders is good for the members? Or is the whole approach of looking at the problem as a competition between members and shareholders an unhelpful one for the purpose of decision-making? The second problem is that the shareholders are not clients of the actuary — any advice concerning their interests must be mediated through the sponsoring employer. So, even if the concept of artificial profits created by equity investment were theoretically sound, it would be quite irrelevant if the employer were to insist that the size of reported pension costs should be a major factor in decision-making.

We would agree, however, that the effect of pension fund investment returns on reported pension costs creates problems of interpretation for the shareholders. In particular, any anticipation of the equity risk premium in valuation interest rates appears to be inconsistent with fundamental concepts of prudent accounting (although it is not clear that the actuarial valuation used for funding purposes should be similarly constrained). However, no rational system of accounting can avoid recognising *realised* equity gains, either through a reduction in pension costs or some other adjustment. *Ex post*, a pound of profit remains a pound of profit, whatever its source.

APPENDIX C

THE PENSION SCHEME MODEL

C.1 *Benefits Provided*

The model pension scheme used in this paper has the following simplified benefit structure:

- a pension on retirement at the normal retirement age (NRA) of 65 of one-sixtieth of final pensionable salary at the date of retirement for each year of pensionable service:
  - pensions in payment are assumed to be increased in line with Limited Price Indexation (LPI) (i.e. in line with increases in the Retail Prices Index, subject to a maximum increase in any year of 5%);

- a deferred pension on withdrawal prior to NRA of one-sixtieth of final pensionable salary at the date of withdrawal for each year of service:
  - this deferred pension is revalued in the period up to retirement in line with statutory requirements; and

- no benefit is provided on death in service.

C.2 *Initial Membership Profile*

The initial membership profile is chosen so as to be representative of the working population as a whole. For the analysis presented here, the scheme membership is assumed to remain stable over time (with respect to age, pensionable salary in real terms and past pensionable service). Thus, the number of active members at each age prior to NRA is based on the service table shown at the end of this appendix.

However, to prevent the scheme from being very bottom heavy (i.e. with a large number of members at the younger ages), we assume that the normal age retirements and deaths amongst active members (prior to NRA) are replaced by new entrants at the youngest age and withdrawals prior to NRA are replaced by new entrants at the same age.

On grounds of simplicity, ill-health and early retirements are ignored in the projections of the model scheme. Whilst the incidence of such retirements will impact on the timing of benefit outgo, it is unlikely to affect significantly the cost of the benefit provision (as the benefits provided in this case will often be determined such that the effect on the financial position of the scheme is broadly neutral).

As a result, the average past pensionable service at each age will be calculated recursively as a weighted average of the past pensionable service for those active members remaining in the scheme from the previous age and the zero past pensionable service assumed for the new entrants at the current age.

It would be straightforward to adjust the new entrant assumption to replicate schemes in different stages of maturity (e.g. for a closed scheme, future new entrants would simply be ignored).

The pensionable salary at each age is assumed to follow the salary scale shown in the service table.

We have assumed that the number of current pensioners at each age after NRA is in accordance with the profile of the PA(90) standard mortality table. Pensions in payment to current pensioners are calculated in accordance with the rules outlined above.

For simplicity, we also assume that the scheme has no deferred pensioners. Instead, we assume that, on withdrawal prior to NRA, members are assumed to receive a transfer value, and the scheme is left with no further liability. This will lead to a slight increase in the volatility of the scheme outgo in respect of withdrawals (as the benefit payments are made in a single lump sum, rather than in the form of a deferred pension payable from retirement).

C.3 *Initial Scheme Valuation*

The nominal liabilities in respect of active members and current pensioners are valued on a discontinuance basis.

For simplicity, we assume that this requires:

- — an immediate annuity to be purchased in respect of each current pensioner:
  - — as pensions in payment are granted LPI increases, we use an interest rate in possession equal to the current real yield available on long-dated index-linked bonds (in practice, it may be appropriate to adjust this real yield to allow for the fact that pension increases are capped at 5% p.a., however, on grounds of prudence, this adjustment has been ignored); and

- — a deferred annuity to be purchased in respect of each current active member:
  - — as the accrued pension at the date of leaving is to be increased in line with price inflation prior to retirement, we use an interest rate in deferment equal to the current real yield available on long-dated index-linked bonds (similar comments apply to those above, as pension increases in deferment are also capped); and
  - — as for current pensioners, we also use an interest rate in possession equal to the current real yield available on long-dated index-linked bonds.

In practice, GN19 requires that the benefits provided in the event of a wind-up are subject to a minimum of the MFR liabilities which have been calculated in accordance with the guidelines laid down in GN27.

Current MFR legislation allows for returns on equity holdings to be used to value benefits in respect of current active members. Thus, the value placed on the discontinuance liabilities in this case can be considered prudent.

For the initial analysis, the initial market value of the assets is chosen such that the discontinuance funding level is 100%. However, the effect of starting with both a well-funded position (with an initial discontinuance funding level of 120%) and a poorly-funded position (with an initial discontinuance funding level of 80%) are also considered.

The normal contribution rate to be paid is a free variable which must also be chosen. A range of possible contribution rates is considered from 0% to 32%.

Where the initial discontinuance funding level differs from 100%, this normal contribution rate is adjusted by spreading the surplus (or deficit) over a three-year period. A three-year spread period is used, as scheme valuations are assumed to be triennial. However, many actuaries advocate the use of a longer spread period, with the aim of reducing the volatility of the sponsor's contribution rate. Thus, the effect of using a 12-year spread period is also considered.

Clearly, choosing to pay a lower contribution rate initially will, all other things being equal, require the payment of a higher contribution rate in future. We note that this flexibility in the pace of funding is one of the key advantages of a defined benefit pension scheme from the point of view of the sponsoring employer.

C.4 *Modelling the Future Cash Flows*

For the analysis presented here, the annual future cash inflows and outflows are projected forward for a period of 15 years, and the resulting capitalised values of the assets and liabilities used to assess the financial position of the model scheme at the triennial valuation dates.

C.4.1 *Valuation of liabilities in respect of active members*

Let $N(x, t)$ denote the number of active members at age x at time t, for all $x < NRA$.

Then, as the scheme is assumed to be stationary, we have $N(x, t) = N(x, t+1)$ for all $t \geq 0$.

Let $S(x, t)$ denote the salary for an active member of age x at time t.

Then, as salaries are assumed to grow in line with the salary inflation process generated by the Wilkie stochastic asset model $W(t)$, we have:

$$S(x, t) = S(x-1, t-1) \times \frac{s_x}{s_{x-1}} \times \frac{W(t)}{W(t-1)} = S(x, t-1) \times \frac{W(t)}{W(t-1)} \qquad \text{(C1)}$$

Then, the total salary roll at time t is given by:

$$TS(t) = \sum_{\text{all ages } x<NRA} N(x, t) \times S(x, t) = TS(0) \times \frac{W(t)}{W(0)}. \tag{C2}$$

Let $n(x, t)$ denote the average number of years of past pensionable service for an active member of age x at time t.

Then, as the scheme is assumed to be stationary, we have $n(x, t) = n(x, t+1)$ for all $t \geq 0$.

Now, using the method outlined in Section 5.1.1, the value of the discontinuance liability at time t is respect of an active member at age x is given by:

$$V_L(x, t) = \left(\frac{n(x, t)}{60} \times S(x, t)\right) \times {}_{NRA-x|}\ddot{a}_x(t) \quad \text{for } x < NRA \tag{C3}$$

where the deferred annuity is evaluated using the current real yield index-linked bonds (as generated by the Wilkie stochastic asset model), given by $R(t)$.

Mortality pre-retirement is assumed to follow that used in the construction of the service table and mortality post-retirement is assumed to follow the PA(90) standard mortality table.

Then, the total discontinuance liability at time t (in respect of active members only) is given by:

$$\sum_{\text{all ages } x<NRA} N(x, t) \times V_L(x, t).$$

C.4.2 *Valuation of liabilities in respect of current pensioners*

Similarly, let $N(x, t)$ denote the number of current pensioners at age x at time t, for $x \geq NRA$.

Then, as the scheme is assumed to be stationary, we have $N(x, t) = N(x, t+1)$ for all $t \geq 0$.

Let $B(x, t)$ denote the current amount of the pension benefit for a pensioner of age x at time t.

Then, as pensions are assumed to grow in line with $Q(t)$, the price inflation process generated by the Wilkie stochastic asset model (subject to a maximum of 5% p.a.), we have:

$$B(x, t) = B(x-1, t-1) \times \min\left(\frac{Q(t)}{Q(t-1)}, 1.05\right). \tag{C4}$$

In addition, it is highly unlikely that, in event of negative inflation, pensions in payment would be reduced. Thus, we ensure that $B(x, t) \geq B(x-1, t-1)$.

Note that, using the notation defined above, on retirement at time t, the initial amount of pension benefit received is given by $B(65, t) = \frac{n(65, t)}{60} \times S(65, t)$. This assumes that, for simplicity, none of the pension benefit is commuted for a cash lump sum on retirement.

Now, using the method outlined in Section 5.1.1, the value of the discontinuance liability at time t is respect of a current pensioner at age x is given by:

$$V_L(x, t) = B(x, t) \times \ddot{a}_x(t) \quad \text{for } x \geq NRA$$

where the immediate annuity is evaluated using the current real yield index-linked bonds (as generated by the Wilkie stochastic asset model), given by $R(t)$.

As for active members, mortality post-retirement is assumed to follow the PA(90) standard mortality table.

Then, the total discontinuance liability at time t (in respect of current pensioners only) is given by:

$$\sum_{\text{all ages } x \geq NRA} N(x, t) \times V_L(x, t).$$

Then, the total value of the discontinuance liabilities at time t is given by:

$$V_L(t) = \sum_{\text{all ages } x} N(x, t) \times V_L(x, t). \tag{C5}$$

C.4.3 *Valuation of the scheme assets*

In addition, the nominal market value of the fund at the end of each projection year will depend on the actual performance of the underlying assets over the year.

We assume that there are two asset classes available, long-dated fixed-interest gilts and equities. In the initial analysis discussed in the case study, a static investment strategy is assumed, whereby the proportion of the fund invested in each asset class is specified at the start of the projection period (and re-balanced annually).

Let $f(t)$ be the market value of the assets held at time t.

Thus, the funding level at time t is given by $FL(t) = \frac{f(t)}{V_L(t)}$ and the surplus is given by $S(t) = f(t) - V_L(t)$.

We assume that the fund held at time t, $f(t)$, is invested at z% in U.K. equities and $(100 - z)$% in U.K. long-dated fixed-interest bonds, and that the fund is re-balanced in the specified proportions annually.

For simplicity, we assume that these are the only asset classes available. However, the approach used can be readily extended to include other asset classes.

Using the notation of Wilkie (1995), we have:

— $PR(t)$ is the accumulated amount at time t for an initial investment of 1.0 made at time 0 in U.K. equities, assuming that the investment income received is reinvested in the fund and tax is ignored.

— $CR(t)$ is the accumulated amount at time t for an initial investment of 1.0 made at time 0 in U.K. long-dated fixed-interest bonds, assuming that, again, the investment income received is reinvested in the fund and tax is ignored.

Then, for a mixed fund with z% of the assets in U.K. equities and $(100 - z)$% in U.K. long-dated fixed-interest bonds, we define the accumulation of the fund using the following recursive relationship:

$$MR(t) = MR(t-1) \times \left[z \times \frac{PR(t)}{PR(t-1)} + (1-z) \times \frac{CR(t)}{CR(t-1)} \right] \tag{C6}$$

where $MR(0) = 1.0$.

Then, at time $(t+1)$, the accumulation of the fund held at time t, $f(t)$, is given by:

$$f(t) \times \frac{MR(t+1)}{MR(t)}.$$

However, in order to find the market value of the assets held at time $(t+1)$ we must also allow for the cash flows into and out of the fund during the year.

For the purposes of projecting the future cash flows, we use time units of one year. Thus, at the start of each projection year there is a cash inflow in respect of the contributions received (based on the contribution rate calculated at the previous valuation and the total pensionable salary roll over the year). We assume, for simplicity, that the contributions are paid at the start of the scheme year.

As the scheme is assumed to remain stationary with respect to pensionable salary, the total pensionable salary roll in nominal terms will increase each year in line with the actual salary growth experienced.

If the contribution rate payable at time t is c% of total salary roll, then, at time $(t+1)$, the accumulation of the total contributions received at time t is given by:

$$\frac{c}{100} \times TS(t) \times \frac{MR(t+1)}{MR(t)}.$$

At the initial valuation at time 0, the contribution rate is specified. However, at future triennial valuations, the required contribution rate is calculated by adjusting the chosen normal contribution rate in order to allow for the spreading of any valuation surplus (or deficit), given by $S(t) = f(t) - V_L(t)$, over the chosen period. We consider only the spread method ((a) in ¶5.2.4.2.1) and, as discussed above, amortisation periods of three years and 12 years are considered.

Also, at the end of each projection year, there is a cash outflow in respect of the benefit payments (both to current pensioners, and to those active members withdrawing during year and receiving a transfer value). In addition, we must allow for cash outflows in respect of transfer values paid to withdrawing members during the year and pensions paid to current pensioners. We assume, for simplicity, that membership movements occur at the end of each year.

Considering the group of $N(x, t)$ active members of age x at time t (for $x < NRA$), the number of withdrawals at the end of the year $(t, t+1)$ is $N(x, t) \times \frac{w_x}{l_x}$.

Also, we note from earlier that the transfer value paid on withdrawal is calculated in the same way as the current discontinuance liabilities. However, we must allow for the fact that the cohort of active lives is now at age $(x+1)$ and the appropriate current yield on long-dated index-linked bonds is $R(t+1)$.

Thus, at time $(t+1)$, the total amount of the benefits paid (in respect of withdrawals among current active members) is:

$$\sum_{\text{all ages } x<NRA} N(x, t) \times \frac{w_x}{l_x} \times V_L(x+1, t+1). \tag{C7}$$

On withdrawal, the transfer value paid is calculated in accordance with the discontinuance liabilities outlined above (i.e. using a rate of interest, both in deferment and in possession, equal to the current real yield available on long-dated index-linked bonds at the date of exit). In practice, many schemes will pay a lower transfer value on withdrawal (equal to the accrued benefits valued in accordance with the MFR basis laid down in GN27, which allows a higher equity-based discount rate to be used). Thus, as for the discontinuance liabilities, the method used to value the benefit payments in respect of withdrawals can be considered to be prudent.

From above, we have $N(x, t)$ current pensioners of age x at time t (for $x \geq NRA$), each receiving a benefit payment at time t of $B(x, t)$. Assuming that membership movements occur at the end of the year, the cash outflow at time $(t+1)$ in respect of the pension benefits payable to this group of lives is given by:

$$N(x,t) \times \left[B(x,t) \times \min\left(\frac{Q(t+1)}{Q(t)}, 1.05\right)\right] = N(x,t) \times B(x+1,t+1) \quad \text{(C8)}$$

Thus, at time $(t+1)$, the total amount of the benefits paid (in respect of current pensioners) is:

$$\sum_{\text{all ages } x \geq NRA} N(x,t) \times B(x+1,t+1).$$

Then, the total market value of the fund at time $(t+1)$, denoted by $V_A(t+1)$, is calculated recursively using:

$$\begin{aligned} f(t+1) = f(t) \times \frac{MR(t+1)}{MR(t)} + \frac{c}{100} \times TS(t) \times \frac{MR(t+1)}{MR(t)} \\ - \left(\sum_{\text{all ages } x < NRA} N(x,t) \times \frac{w_x}{l_x} \times V_L(x+1,t+1) \right. \\ \left. + \sum_{\text{all ages } x \geq NRA} N(x,t) \times B(x+1,t+1) \right). \end{aligned} \quad \text{(C9)}$$

Every three years, a valuation of the scheme takes place to determine both the current discontinuance funding level and the required contribution rate over the forthcoming inter-valuation period.

Whilst the scheme membership profile is assumed to remain stable, the nominal 'market' value of the liabilities at each triennial valuation date will depend on:

— the actual salary growth experienced since the previous valuation date; and
— the current market conditions prevailing at the valuation date.

The required contribution rate at the valuation date is then calculated by adjusting the initial contribution rate to allow for the spreading, over a three-year (or 12-year) period, of the current surplus (or deficit) on a discontinuance basis.

Thus, the annual cash flows of the scheme can be projected forward recursively, with a decision point occurring at each triennial valuation basis, when the contribution rate payable over the three-year period until the next decision point is determined.

Table C. The service table and salary scale used to construct the membership profile for the model pension scheme

Age, x	l_x	w_x	d_x	i_x	r_x	s_x
16	1,000,000	100,000	500	0	0	1.000
17	899,500	89,950	450	0	0	1.200
18	809,100	80,910	405	0	0	1.404
19	727,786	72,779	364	0	0	1.606
20	654,643	65,464	327	65	0	1.810
21	588,786	58,879	236	118	0	2.013
22	529,554	52,955	212	106	0	2.218
23	476,281	47,628	191	95	0	2.421
24	428,367	40,695	171	86	0	2.626
25	387,415	34,867	155	116	0	2.828
26	352,277	29,944	106	106	0	3.026
27	322,122	25,770	97	97	0	3.226
28	296,159	22,212	89	118	0	3.433
29	273,739	19,162	82	109	0	3.635
30	254,386	16,789	76	102	0	3.845
31	237,418	14,720	95	119	0	4.043
32	222,485	12,904	89	133	0	4.249
33	209,358	11,305	84	147	0	4.454
34	197,823	9,891	79	178	0	4.654
35	187,674	8,633	94	206	0	4.848
36	178,741	7,507	107	214	0	5.041
37	170,912	6,495	120	222	0	5.226
38	164,076	5,579	115	246	0	5.398
39	158,136	4,744	127	269	0	5.553
40	152,997	4,131	138	306	0	5.692
41	148,422	3,562	134	341	0	5.813
42	144,385	3,032	144	375	0	5.922
43	140,833	2,535	155	408	0	6.011
44	137,735	2,066	165	441	0	6.101
45	135,062	1,621	176	473	0	6.192
46	132,793	1,195	186	505	0	6.285
47	130,908	785	223	550	0	6.387
48	129,350	517	259	621	0	6.482
49	127,953	256	281	717	0	6.579
50	126,699	0	317	824	0	6.676
51	125,559	0	352	942	0	6.784
52	124,265	0	385	1,081	0	6.885
53	122,799	0	430	1,253	0	6.986
54	121,117	0	472	1,453	0	7.099
55	119,191	0	513	1,681	0	7.203
56	116,998	0	550	1,930	0	7.309
57	114,517	0	584	2,199	0	7.426
58	111,734	0	626	2,481	0	7.535
59	108,628	0	673	2,781	0	7.656
60	105,174	0	726	3,103	36,811	7.768
61	64,535	0	497	2,194	5,163	7.881
62	56,681	0	493	2,228	4,534	8.006
63	49,426	0	484	2,244	3,954	8.123
64	42,743	0	474	2,248	3,419	8.252
65	36,601	0	0	0	36,601	8.371

APPENDIX D

DYNAMIC APPROACHES TO ASSET ALLOCATION AND FUNDING

As an illustration of possible dynamic approaches to the problem of choosing optimal asset allocation and funding strategies, we present two examples based on a simplified model of a defined benefit pension scheme.

D.1 *A Simplified Model of a Defined Benefit Pension Scheme*

In constructing the model, we follow Owadally & Haberman (2000) closely.

Consider a strictly defined benefit pension plan in which no discretionary or *ad hoc* benefit improvement is allowed except for benefit indexation.

Assume that provision is made only for a retirement benefit at normal retirement age based on final salary, and that actuarial valuations are carried out with the following features:

(1) Actuarial valuations take place at regular intervals of one year.
(2) The actuarial valuation basis is invariant in time.
(3) Pension fund assets are valued at market value without smoothing (market value $f(t)$).
(4) An 'individual' pension funding method is used, generating an actuarial liability (or standard fund) $AL(t)$ and a normal cost $NC(t)$.

A simple model for such a pension plan may be projected forward based on the following:

(1) The pension plan population is assumed to be stationary from the start.
(2) Mortality and other decrements are assumed to follow a given survival model represented by the life table l'_x.
(3) A salary scale is assumed to reflect exactly promotional, merit-based or longevity-based increases in salaries (and may be incorporated in the life table $l_x = s_x l'_x$).

Economic wage inflation (the general increase in wages is measured by a national wages index) may be distinguished from the salary scale. The actuarial valuation basis includes l_x as well as a valuation discount rate and an assumption as to wage inflation (in order to value final salary benefits). Actual experience is assumed to be in accordance with the actuarial valuation assumptions except for inflation and the returns on plan assets.

Suppose that cash flows occur at the start of each scheme year. The unfunded liability at the start of year $(t, t+1)$ is the excess of the actuarial liability over the value of plan assets: $ul(t) = AL(t) - f(t)$. The outcome of an actuarial valuation at the start of year $(t, t+1)$ is to recommend a contribution:

$$c(t) = NC(t) + adj(t) \tag{D1}$$

where $adj(t)$ is a supplementary contribution (or contribution adjustment) paid to amortise past and present experience deviations from actuarial assumptions. These deviations result in actuarial gains or losses, and it is assumed that these are spread by paying a proportion k of the unfunded liability in each year, so that:

$$\mathrm{adj}(t) = k(AL(t) - f(t)). \tag{D2}$$

k is often written as $1/\ddot{a}_{\overline{m}|}$ so that (D2): "may be interpreted as spreading the unfunded liability over a period of m years" (Dufresne, 1988). This approach is commonly used in the U.K., and has been discussed widely in the literature: see Trowbridge & Farr (1976), Bowers *et al.* (1979), McGill *et al.* (1996), Owadally & Haberman (1999).

On the grounds of mathematical tractability, we assume that pensions in payment are indexed with *wage* inflation. All monetary quantities may then be considered net of wage inflation. Under the assumptions made earlier, the payroll, actuarial liability, normal cost and benefit outgo each year are constant, after deflating by wage inflation. Alternatively, we could disregard inflation on salaries and just use nominal quantities throughout.

Assume that the real rate of investment return on pension scheme assets, net of wage inflation, may be represented by a random variable which is independent and identically distributed from year to year. Then if the real rate of return for the year $(t, t+1)$ is $i(t+1)$:

$$f(t+1) = (1 + i(t+1))(f(t) + c(t) - B) \tag{D3}$$

where B is the annual benefit outgo, which is assumed to be constant, as noted above.

For simplicity and convenience, we consider two possible investments: a risk free asset offering a deterministic real rate of return r, and a risky asset offering a random real rate of return $r + \alpha(t+1)$ for year $(t+1)$. Hence, $\alpha(t+1)$ may be interpreted as a random risk premium. Let $y(t)$ be the proportion of the fund invested in the risky asset at time t, so that:

$$\begin{aligned} i(t+1) &= y(t)(r + \alpha(t+1)) + (1 - y(t))r \\ &= r + y(t)\alpha(t+1). \end{aligned} \tag{D4}$$

D.2 *Optimal Asset Allocation for Minimising the Expected Valuation Deficit* (due to R. Gerrard, S. Haberman and M. Z. Khorasanee)

Consider a time horizon of T years.

Our aim is to choose the set of $y(t)$ values in order to minimise the following expression:

$$\mathrm{E}\left[\max(AL - f(T),\ 0 \middle| \Im_t\right] \tag{D5}$$

where the expectation is taken conditional on the information available at time t, $\Im_t$.

The solution uses the stepwise approach of dynamic programming. To illustrate this, we consider $t = T - 1$ and the optimal choice of $y(T-1)$.

From (D1) – (D4), we note that:

$$\begin{aligned} AL - f(T) &= AL - (1 + r + y(T-1)\alpha(T))(f(T-1) + c(T-1) - B) \\ &= AL - X(T-1)(1 + r + y(T-1)\alpha(T)) \end{aligned} \tag{D6}$$

where $X(T-1) = f(T-1)(1-k) + k.AL + NC - B$ is the level of the fund immediately after the cash flows at time $T-1$. Note that $f(T-1)$ is the fund immediately before the cash flows at time $T-1$.

We define the intermediate parameter w_T such that:

$$AL - f(T) = 0 \text{ when } \alpha(T) = w_T.$$

Hence:

$$w_T = \left(\frac{AL}{X(T-1)} - (1+r)\right)\frac{1}{y(T-1)} = \frac{D(T-1)}{y(T-1)}, \text{ say.} \tag{D7}$$

We can interpret $D(T-1)$ as the projected deficit at time T as a function of the fund at time $T-1$, after adjusting for cash flows (i.e. $X(T-1)$), assuming a risk-free return.

Then $\mathrm{E}\left[\max\,(AL - f(T), 0) \middle| \Im_{T-1}\right] = E_{T-1}$ say $= \int_{-\infty}^{w_T} [AL - X(T-1)\ (1 + r + y(T-1)z)]\, g(z)\, dz$, where $g()$ is the pdf of the risk premium $\alpha()$, which we assume to be independent of time.

If we define $I_n(x) = \int_{-\infty}^{x} z^n g(z) dz$, then it is clear that:

$$E_{T-1} = (AL - X(T-1)(1+r))\, I_0\,(w_T) - y(T-1)\, X(T-1)\, I_1\,(w_T).$$

If we wish to choose $y(T-1)$ to minimise E_{T-1}, we need to consider $\dfrac{\partial E_{T-1}}{\partial y(T-1)}$, and it is straightforward (but laborious) to show that:

$$\frac{\partial E_{T-1}}{\partial y(T-1)} = -X(T-1)\, I_1\left(\frac{D(T-1)}{y(T-1)}\right). \tag{D8}$$

The properties of $I_1(x)$ are such that $I_1(x)$:
- is negative and decreasing for $x < 0$;
- has a minimum at $x = 0$;
- is increasing for $x > 0$; and
- is positive for $x >$ some x^* providing that $\int_{-\infty}^{\infty} zg(z) = M > 0$. We expect that $M > 0$ because there should be a positive risk premium on the risky asset.

We can now comment on the sign of $\frac{\partial E_{T-1}}{\partial y(T-1)}$ in (B8).

Assume that $X(T-1) > 0$, $M > 0$ and define x^* such that $I_1(x^*) = 0$. We expect $X(T-1) > 0$ because pension funds cannot have negative market values:

(1) If $D(T-1) > 0$, then:

$$\frac{\partial E_{T-1}}{\partial y(T-1)} = -X(T-1).M < 0 \text{ when } y = 0$$

$\frac{\partial E_{T-1}}{\partial y(T-1)} \to -X(T-1).I_1(D(T-1))$ as $y \to 1$, which is positive if $D(T-1) < x^*$ and negative otherwise.

So, $D(T-1) < x^*$ implies that E_{T-1} is a decreasing function of y, which is hence minimised at $y = 1$; and $x^* > D(T-1) > 0$ implies that there is some $y \in (0, 1)$ at which E_{T-1} has a minimum.

(2) Similarly, if $D(T-1) < 0$, then:

$$\frac{\partial E_{T-1}}{\partial y(T-1)} = 0 \text{ when } y = 0$$

$\frac{\partial E_{T-1}}{\partial y(T-1)}$ is a positive increasing function for $y > 0$, so the minimum of E_{T-1} is achieved at $y = 0$.

Thus, the optimal investment strategy is degenerate for most values of $f(T-1)$ in that $y(T-1)$ takes the values 0 or 1. There is a potentially small range of possible values of $f(T-1)$ for which a non-degenerate strategy is optimal, viz.:

$$\frac{AL}{1+r+x^*} - k\,AL - NC + B < (1-k)f(T-1) < \frac{AL}{1+r} - kAL - NC + B. \tag{D9}$$

Hence:

$$y^*(T-1) = 0 \text{ if } f(T-1) > F_1 = \left(\frac{AL}{1+r} - kAL - NC + B\right)(1-k)^{-1}$$

$$y^*(T-1) = 1 \text{ if } f(T-1) < F_2 = \left(\frac{AL}{1+r+x^*} - kAL - NC + B\right)(1-k)^{-1}$$

and $y^*(T-1)$ takes a non degenerate value if, as in (D9),

$$F_2 < f(T-1) < F_1. \tag{D10}$$

So, a 'high' level of the market value of the assets at time $T-1$ leads to a 100% investment in the riskless asset for the year $(T-1, T)$, and a 'low' level of the market value leads to a 100% investment in the risky asset for the year $(T-1, T)$.

As a numerical illustration, assuming that the return on the risky asset is log normal with parameters μ and σ, and taking reasonable parameter values of $\mu = 0.10$, $\sigma = 0.20$ and $r = 0.05$ we find that $x^* \cong 0.3$. Then, using a valuation rate of interest corresponding to the expected return on a balanced portfolio, we can calculate the range of values in (D10). For $m = 5$, $k = 0.24$ and for $m = 10$, $k = 0.14$ (see (D2)). Then:

$$\begin{aligned} F_1 - F_2 &= \left(\frac{AL}{1+r} - \frac{AL}{1+r+x^x}\right)(1-k)^{-1} \\ &= 0.28\,AL \text{ for } m = 5 \\ &= 0.25\,AL \text{ for } m = 10. \end{aligned}$$

This represents the range of values for $f(T-1)$ over which a non-degenerate investment strategy is optimal for $(T-1, T)$. The actual values of F_1 and F_2 depend on the characteristics of the pension fund, as represented by the numerical values of AL, NC and B.

D.3 *Optimal Funding and Asset Allocation for Minimising Contribution and Solvency Risk*

This section of Appendix D is based closely on Owadally & Haberman (2000).

Unlike Section D2, we demonstrate how optimisation of the contribution strategy and asset allocation can be implemented simultaneously and we use a different optimisation criterion, which is described below.

In addition to the earlier assumptions, we specify the mean $\alpha > 0$ and variance σ^2 of the IID sequence $\{\alpha(t)\}$.

The independence assumption for $\{\alpha(t)\}$ means that $f(t)$ is Markovian, i.e.

$$\mathrm{E}\big[f(t+1)\big|\Im_t\big] = \mathrm{E}\big[f(t+1)\big|f(t),\ y(t),\ c(t)\big]$$

where $\Im_t$ represents the information available up to time t.

We assume that the objectives of the funding and asset allocation processes are to stabilise contributions, defray any unfunded liabilities and pay off actuarial losses and gains as they emerge. The performance of the pension fund may be judged in terms of the deviations in the values of plan assets and contributions from their desired levels (say FT_t and CT_t respectively) relating to the actuarial liability and normal cost. The 'cost' incurred for any such deviation at time $0 \le t \le T-1$ may be defined as:

$$C(f(t),\ c(t),\ t) = \theta_1(f(t) - FT_t)^2 + \theta_2(c(t) - CT_t)^2. \tag{D11}$$

Different weights ($\theta_1 > 0$ and $\theta_2 > 0$) are placed on these twin long-term objectives of fund security and contribution stability. The cost in equation (D11) reflects a quadratic utility function. Minimising the cost also minimises the risks of contribution instability and of fund inadequacy. It is reasonable to assume that $FT_t > 0$ and $CT_t < B$, as otherwise there is no reason to fund the retirement benefits in advance.

The performance of the fund may be given different importance over time by introducing a discount factor β. At the end of the given control period N, a closing cost is incurred if an unfunded liability still exists: $\Omega_T = \theta_0(f(T) - FT_T)^2$. For $0 \le t \le T-1$, the discounted cost incurred from time t to T is:

$$\Omega_t = \sum_{s=t}^{T=1} \beta^{s-t} C(f(s),\ c(s),\ s) + \beta^{T-t}\Omega_T. \tag{D12}$$

An objective criterion for the performance of the pension funding system over period T may therefore be defined to be:

$$\mathrm{E}[\Omega_0|\Im_0] = \mathrm{E}\left[\beta^T\Omega_T + \sum_{s=0}^{T-1}\beta^s C(f(s),\ c(s),\ s)\big|\Im_0\right]. \tag{D13}$$

The value function $J(f(t), t)$ is defined as the minimum, over the remaining asset allocation and distribution decisions, of the expected discounted cost incurred from time t given information at time t: $J(f(t), t) = \min_\pi \mathrm{E}[\Omega_t | \Im_t]$, where:

$$\pi = \{c(t),\ y(t),\ c(t+1),\ y(t+1), \ldots,\ c(T-1),\ y(T-1)\}.$$

Objective criterion (D12) may be minimised using the Bellman optimality principle (see e.g. Bertsekas, 1976): the minimising values of $c(t)$ and $y(t)$ (say, $c^*(t)$ and $y^*(t)$ respectively) in the optimality equation:

$$J(f(t), t) = \min_{c(t), y(t)} \{C(f(t), c(t), t) + \beta \mathrm{E}[J(f(t+1), t+1) | f(t), c(t), y(t)]\} \quad \text{(D14)}$$

with boundary condition $J(f(T), T) = \Omega_T = \theta_0 (f(T) - FT_T)^2$, are then the optimal contribution and asset allocation controls.

The pension planning objectives above have been set over a finite period T. The plan is assumed to remain solvent and not discontinued during these T years, so that the funding process does not terminate unexpectedly. When the pension plan is regarded as a going concern, an infinite planning horizon may be usefully envisaged as a reasonable approximation to long-term funding: Owadally & Haberman (2000) discuss how this problem may be formulated and solved, under certain conditions, for the case $T \to \infty$.

Owadally & Haberman (2000) show that the solution to the Bellman equation (D14) in the finite-horizon case is:

$$J(f(t), t) = P_t f(t)^2 - 2Q_t f(t) + R_t \quad \text{(D15)}$$

where:

$$P_t = \theta_1 + \theta_2 \beta \sigma^2 (1+r)^2 \tilde{P}_{t+1} P_{t+1} \quad \text{(D16)}$$

$$\tilde{P}_{t+1} = [\theta_2(\alpha^2 + \sigma^2) + \beta\sigma^2(1+r)^2 P_{t+1}]^{-1} \quad \text{(D17)}$$

$$Q_t = \theta_1 FT_t + \theta_2 \beta \sigma^2 (1+r) \tilde{P}_{t+1} [Q_{t+1} - P_{t+1}(1+r)(CT_t - B)] \quad \text{(D18)}$$

with boundary conditions $P_T = \theta_0$ and $Q_T = \theta_0 FT_T$ (R_t represents some additional terms independent of $f(t)$ and so we omit the details of the recursion corresponding to (D16) and (D17)).

Writing $H_t = \theta_2(\alpha^2 + \sigma^2)\tilde{P}_t$, Owadally & Haberman (2000) also prove that, for $0 \le t \le T-1$, the optimal contribution is:

$$c^*(t) = H_{t+1}\, CT_t + (1 - H_{t+1})\left[B - f(t) + Q_{t+1} P_{t+1}^{-1} (1+r)^{-1}\right] \quad \text{(D19)}$$

and the optimal amount invested in the risky asset is given by:

$$y^*(t)(f(t)+c^*(t)-B) = \left[Q_{t+1}P_{t+1}^{-1}(1+r)^{-1} - (f(t)+c^*(t)-B)\right]\alpha(1+r)(\alpha^2+\sigma^2)^{-1}. \tag{D20}$$

It is straightforward to show that $P_t > 0$, $\tilde{P}_t > 0$, $0 < H_t < 1$ and $Q_t > 0$ for $t \in [1, T]$.
Then we can show that:

$$\frac{\partial c^*(t)}{\partial f(t)} < 0 \text{ and } \frac{\partial y^*(t)}{\partial f(t)} < 0 \tag{D21}$$

so that the proportion invested in the risky asset decreases as $f(t)$ increases. (This holds regardless of the planning horizon, i.e. as $T \to \infty$). A similar result is obtained by Boulier *et al.* (1995), Cairns (1997) and Siegmann & Lucas (1999).

The optimal asset allocation strategy is counterintuitive, in that it involves a higher proportion of the riskless asset as the funding level improves. However, it can be justified because, firstly, liabilities need to be hedged so as to minimise the volatility of both surpluses and contributions, and secondly, any available surpluses should be 'locked in' by being invested in less risky assets (Exley *et al.*, 1997). Conversely, the optimal strategy requires that an under-funded plan take a riskier investment position than an over-funded plan, all other things being equal. This contrarian strategy is a consequence of the quadratic utility function implied by criterion (D10), which is simplistic, in that it is symmetric (treating upside and downside risks in a similar manner) and continuous and does not admit solvency and full funding constraints. We note, however, the similarity of these conclusions to those of Section D.2: see (D9).

We note also that the optimal contribution is linear in $f(t)$, whatever the planning horizon: from equation (D18), $c^*(t)$ may be written as $c_0(t) - (1 - H_{t+1})f(t)$, where $1 - H_{t+1} > 0$. The optimal contribution at the start of year $(t, t+1)$ is therefore similar to the contribution calculated when gains and losses are spread as in equation (D2), in that they both depend in a decreasing linear way on the current market value of assets. This result is based on the assumption of serially independent rates of return and a Markovian funding process. The market value of plan assets represents the state of the funding process, and, conditional on knowing the current state of funding and current funding decisions, the future evolution of the fund is statistically independent of its past history so that $c^*(t)$ is a function of the current state only.

APPENDIX E

SENSITIVITY ANALYSIS

In Section 6 we have demonstrated how crucial decisions in a defined benefit scheme could be made using an indifference curve analysis. In this appendix, we carry out sensitivity analyses of some of the results in Section 6.

In a sensitivity analysis, one checks how much the results vary when a single parameter or an assumption is changed. The sensitivity analysis should enable the actuary to gain further insight into the problem and its solution. Thus, the actuary would be better able to advise the scheme trustees and/or the scheme sponsors.

In our stochastic projections, various sensitivity analyses can be carried out on the asset model and/or on the liability model. In this appendix we consider only two sensitivity analyses for reasons of space. Firstly we analyse the sensitivity of our results to the risk premium on equities. Secondly, we analyse the sensitivity to the inflation scenario.

E.1 *The Effect of the Equity Risk Premium on the Indifference Curves*

In this section, we investigate the sensitivity of our results in the equity risk premium. This investigation is carried out by considering three different values of the long-term mean rate of real dividend growth (We adopt Wilkie's notation and signify the long-term mean rate of real dividend growth as DMU in Figures 19, 20 and 21.) in the Wilkie Model. Wilkie (1995, p845) comments that, using his original model: "a 95% confidence interval for [the mean rate of real dividend growth] could be from about −0.9% to about +4.0%". We thus consider the following dividend growth rates: 0, 0.02, and 0.04.

We would expect that the asset value would be higher if the mean real dividend growth rate were high than if the mean real dividend growth rate were low (all other things constant). Hence, there are three expected results:

— Firstly, for a given normal contribution rate and initial equities allocation we would expect the mean shortfall risk, the excess contribution rate risk and the average contribution rate to be lower in the high mean real dividend growth rate case than in the low (or zero) dividend growth case.
— Secondly, if we initially allocate 0% to equities (i.e. 100% to gilts), then reducing (or increasing) the mean real dividend growth rate has no effect on both the risk levels and the average contribution rate.
— Thirdly, if the mean real dividend growth rate were high, then we would expect the decision maker to opt for a higher initial equities allocation. This is due to the fact that we would expect the efficient region to shift towards a higher initial equities allocation in the high dividend growth case.

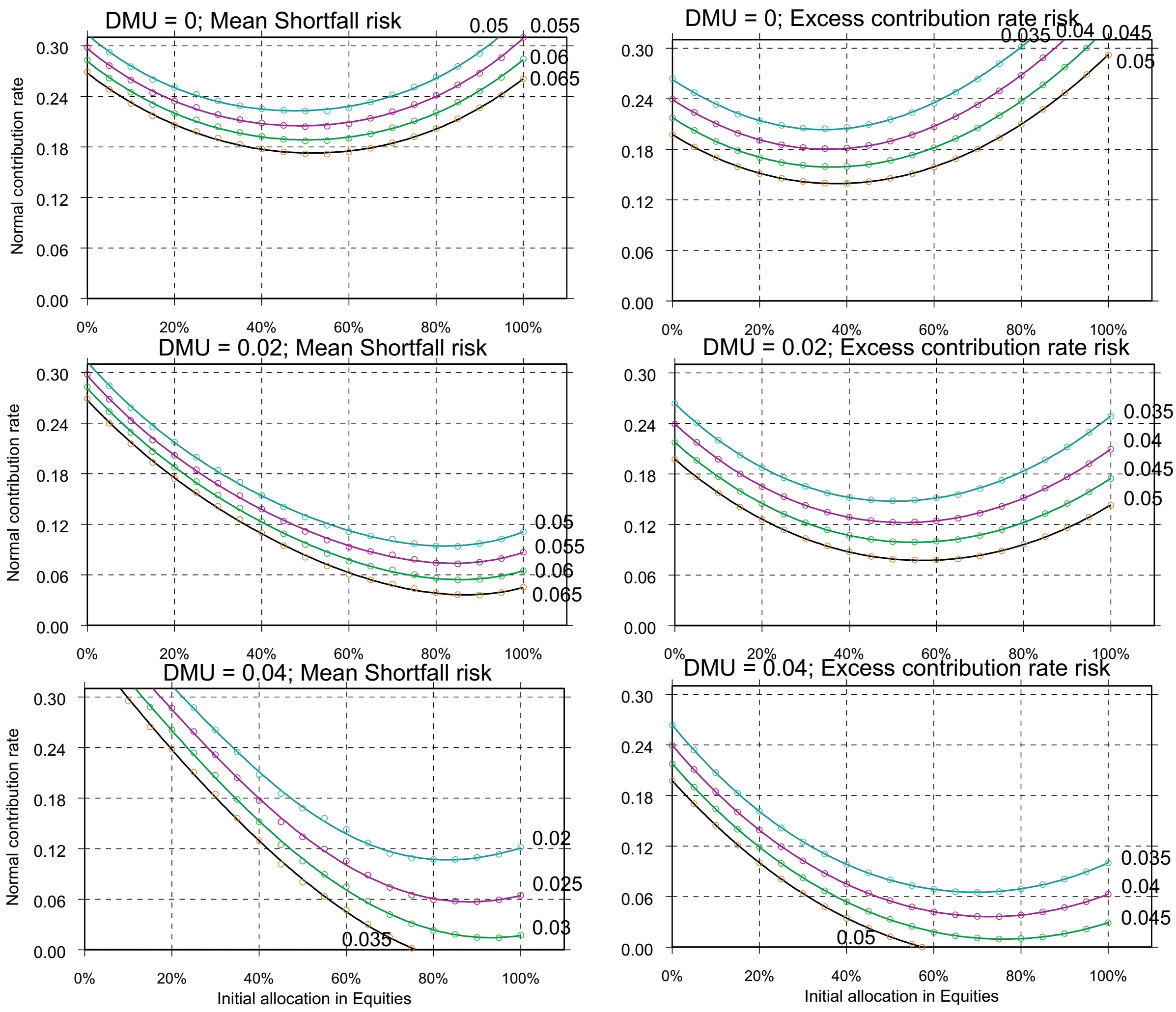

Figure 19. Mean shortfall and excess contribution rate risk levels at the end of 15 years (static asset allocation)

E.1.1 *Mean shortfall and excess contribution rate risk*

Figure 19 shows the mean shortfall and the excess contribution rate risk levels at the end of 15 years.

As expected, for a given normal contribution rate and initial allocation in equities, a higher mean real dividend growth rate leads to lower risk (except at 0% equities). Furthermore, the reduction in the risk level is very considerable, since, for instance, by doubling the dividend growth rate the mean shortfall risk is almost halved.

For example:

— for a normal contribution rate of 0.18 and 50% initial allocation in equities:
  (1) the mean shortfall risk is:
    (a) 0.062 for dividend growth rate of 0;
    (b) 0.037 for dividend growth rate of 0.02; and
    (c) 0.018 for dividend growth rate of 0.04; meanwhile,

(2) the excess contribution rate risk is:
(a) 0.042 for dividend growth rate of 0;
(b) 0.029 for dividend growth rate of 0.02; and
(c) 0.019 for dividend growth rate of 0.04.

— for a normal contribution rate of 0.12 and 100% initial allocation in equities:
(1) the mean shortfall risk is:
(a) 0.104 for dividend growth rate of 0;
(b) 0.048 for dividend growth rate of 0.02; and
(c) 0.020 for dividend growth rate of 0.04; meanwhile,

(2) the excess contribution rate risk is:
(a) 0.084 for dividend growth rate of 0;
(b) 0.054 for dividend growth rate of 0.02; and
(c) 0.032 for dividend growth rate of 0.04.

Thus, both the mean shortfall risk and the excess contribution rate risk levels are very sensitive to the changes in the dividend growth rate.

Minimum points of the risk levels

As expected, for a given normal contribution rate the location of the minimum point shifts towards higher initial equities allocation as the dividend growth rate is increased. However, for the mean shortfall risk, the change in the position of the minimum points is much greater when the dividend growth rate is increased from 0 to 0.02 than when the growth rate is increased from 0.02 to 0.04. That is, the position of the minimum points is more sensitive as we increase the dividend growth rate from 0 to 0.02 than if we increase the dividend growth rate from 0.02 to 0.04.

For the excess contribution rate risk, the position of the minimum points does not change very much as the dividend growth rate is increased. For instance, the minimum points are in the region of 40% for dividend growth rate of 0 and in the region of 70% for dividend growth rate of 0.04.

Thus, though the position of minimum points is fairly sensitive in the mean shortfall risk case, there is little sensitivity in the excess contribution rate risk case.

E.1.2 *The efficient regions*

Figure 20 shows the efficient regions at the end of 15 years for different estimates of the mean real dividend growth rate. ABCD is the efficient region if the mean real dividend growth rate is 0; EFGH is the efficient region if the mean real dividend growth rate is 0.02; and JKLM is the efficient region if the mean real dividend growth rate is 0.04.

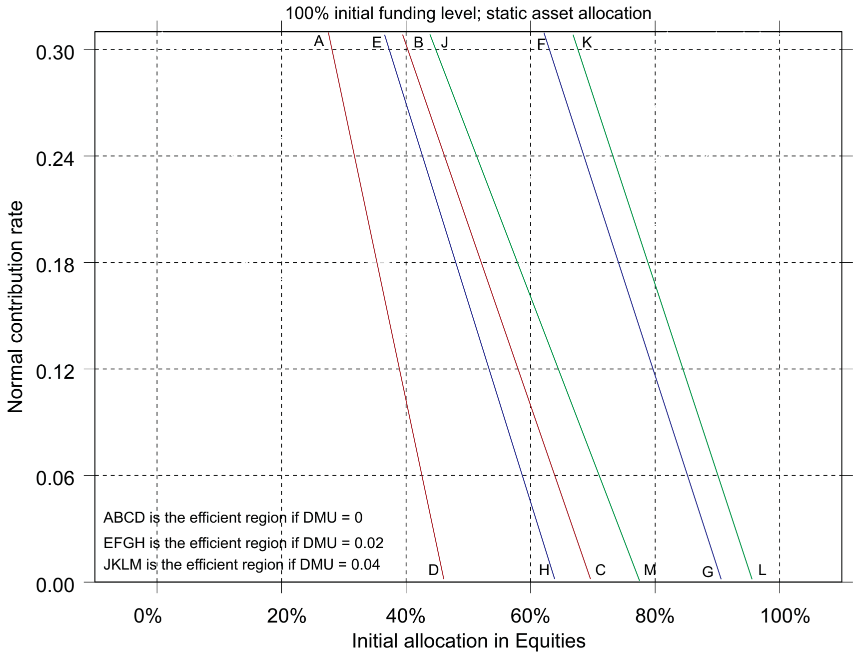

Figure 20. The efficient regions for a 15-year projection period for different estimates of the dividend growth rate (DMU)

As expected, the efficient region shifts towards a higher initial equities allocation as the mean real dividend growth rate increases. However, despite the differences in the dividend growth rates, the three efficient regions overlap to a significant extent. As such, there is not much difference in the positions of, for instance, regions ABCD and JKLM. This means that the position of the efficient region is not very sensitive to the changes in the dividend growth parameter.

E.1.3 *Average contribution rate levels*

Figure 21 shows the average contribution rate levels at the end of six and 15 years for the different estimates of the mean real dividend growth rate.

As expected, for both projection periods, for a given normal contribution rate and initial allocation in equities, the average contribution rate decreases as the mean real dividend growth rate increases. However, the average contribution rate levels for projections over six years are less sensitive than for projections over 15 years.

For example:

— for a normal contribution rate of 0.18 and 50% initial allocation in equities:
  (1) the average contribution rate at the end of 6 years is:
    (a) 0.174 for dividend growth rate of 0;
    (b) 0.159 for dividend growth rate of 0.02; and
    (c) 0.146 for dividend growth rate of 0.04; meanwhile,

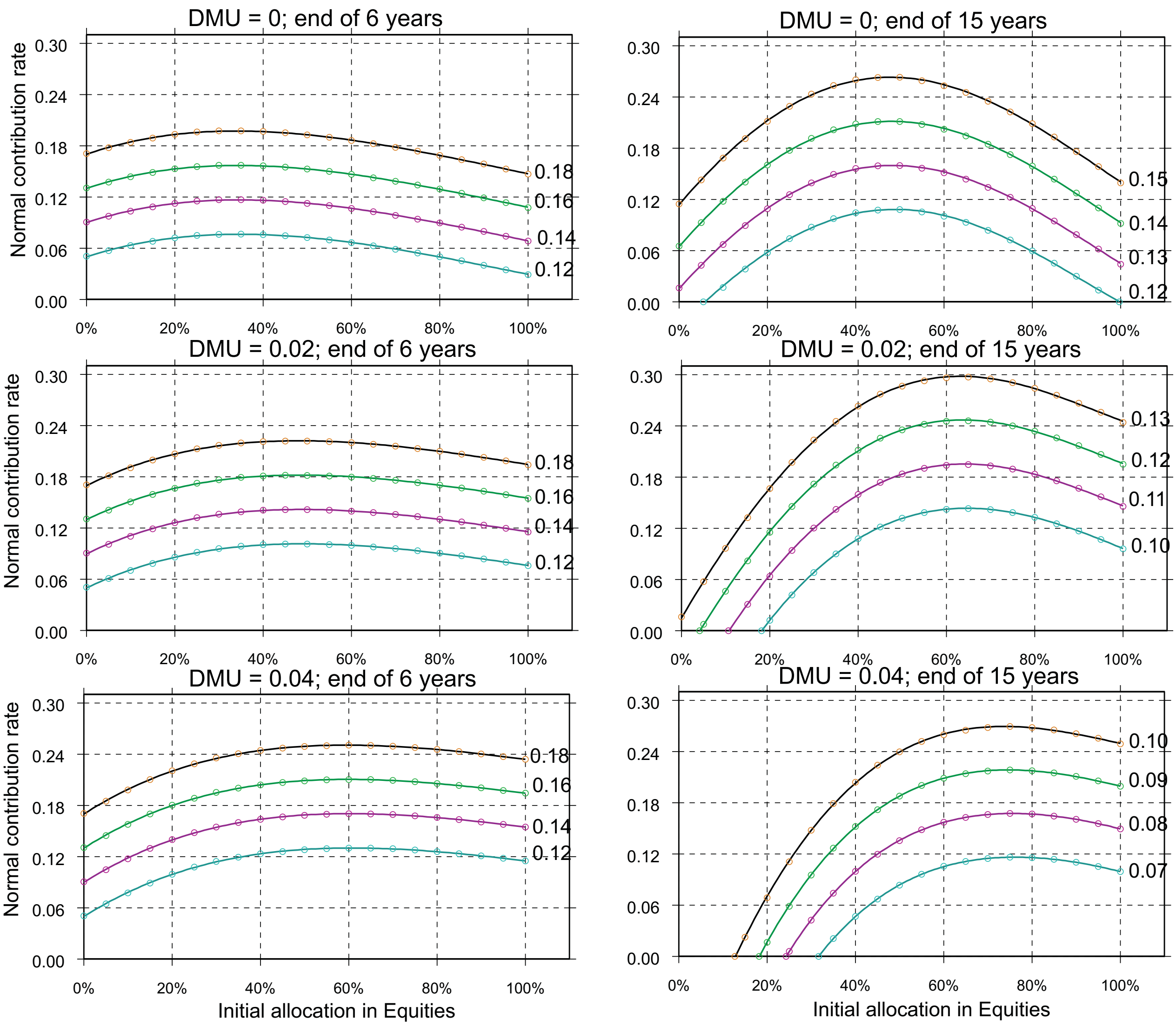

Figure 21. Average contribution rate levels at the end of six and 15 years

(2) the average contribution rate at the end of 15 years is:
   (a) 0.134 for dividend growth rate of 0;
   (b) 0.109 for dividend growth rate of 0.02; and
   (c) 0.088 for dividend growth rate of 0.04.

— for a normal contribution rate of 0.12 and 100% initial allocation in equities:
(1) the average contribution rate at the end of 6 years is:
   (a) 0.166 for dividend growth rate of 0;
   (b) 0.142 for dividend growth rate of 0.02; and
   (c) 0.122 for dividend growth rate of 0.04; meanwhile,

(2) the average contribution rate at the end of 15 years is:
   (a) 0.146 for dividend growth rate of 0;
   (b) 0.105 for dividend growth rate of 0.02; and
   (c) 0.074 for dividend growth rate of 0.04.

This implies that the average contribution rate levels are reasonably sensitive to changes in the dividend growth rate.

E.1.4 *Conclusion*

We have shown that the mean shortfall risk, the excess contribution rate risk and the average contribution rate are sensitive to the risk premium on equities. However, the efficient region is not very sensitive to the risk premium.

Therefore, although the risk levels are sensitive to the risk premium, our asset allocation decisions in this case study would not be substantially sensitive to the risk premium.

E.2 *The Effect of Low and Stable Inflation on the Indifference Curves*

In this section we consider the effect of low and stable inflation on the indifference curves. Instead of considering the effect of an entirely different investment model, we have decided to consider only different estimates for the parameters in Wilkie's autoregressive inflation model.

Wilkie (1995) estimated the parameters by considering the economic data for the period 1923-1994. The parameter estimates for the long-term inflation mean and the inflation volatility are 0.047 and 0.0425, respectively. (We adopt Wilkie's notation and refer to the long-term inflation mean as QMU and the long-term inflation volatility as QSD.)

In this sensitivity analysis we consider a lower long-term inflation mean of 0.025 and a lower inflation volatility of 0.01548. The 0.01548 inflation volatility estimate is based on economic data for the period 1982-1994 (see Khorasanee, 1999).

For comparison purposes we will also consider two intermediate cases:

Case 1: $QMU = 0.047$ and $QSD = 0.0425$ (standard case)
Case 2: $QMU = 0.047$ and $QSD = 0.01548$ (intermediate case)
Case 3: $QMU = 0.025$ and QSD = 0.0425 (intermediate case)
Case 4: $QMU = 0.025$ and QSD = 0.01548 (low and stable inflation case).

In Cases 1 to 4, all other factors are kept the same.

In Case 1 versus Case 2 and Case 3 versus Case 4, we leave the long-term inflation mean unchanged, whilst the long-term inflation volatility is reduced from 0.0425 to 0.01548. Lower inflation volatility implies lower uncertainty in the inflation. Thus, we would expect lower mean shortfall risk and lower excess contribution rate risk in Cases 2 and 4 than in Cases 1 and 3, respectively. Furthermore, in the cases where inflation volatility is low, we would expect fixed-interest bonds to provide a better match for the liabilities than equities. Hence, we would expect lower initial allocation in equities in Cases 2 and 4 than in Cases 1 and 3, respectively. (In other words, the efficient region would shift towards a lower equities allocation in Cases 2 and 4).

In Case 1 versus Case 3 and Case 2 versus Case 4, we leave the long-term inflation volatility unchanged, whilst the long-term inflation mean is reduced from 0.047 to 0.025. In this situation, we would expect our results to be complicated because of the definition of Limited Price Indexation (LPI). In the model, we have assumed that pensions-in-payment are increased at the lower of 5% and the Retail Price Index (RPI) with a lower bound of 0. In the cases where the inflation mean is low and the inflation volatility is high, for instance Case 3, we would expect to obtain more scenarios of negative inflation. With the current definition of the LPI, the scheme would not 'benefit' from negative inflation, since the pensions-in-payment would be kept constant whilst the inflation is negative. However, this could be a short-term problem only, since over the long-term we expect the inflation to revert towards the mean (given the properties of the Wilkie model).

E.2.1 *Mean shortfall risk*

Figure 22 shows the mean shortfall risk indifference curves at the end of three years.

For the intermediate cases, we observe that:

— as expected, leaving the long-term inflation mean unchanged whilst decreasing the volatility (i.e. Case 1 versus Case 2 and Case 3 versus Case 4) implies that the low inflation volatility leads to a lower mean shortfall risk (except for higher equities allocation); and

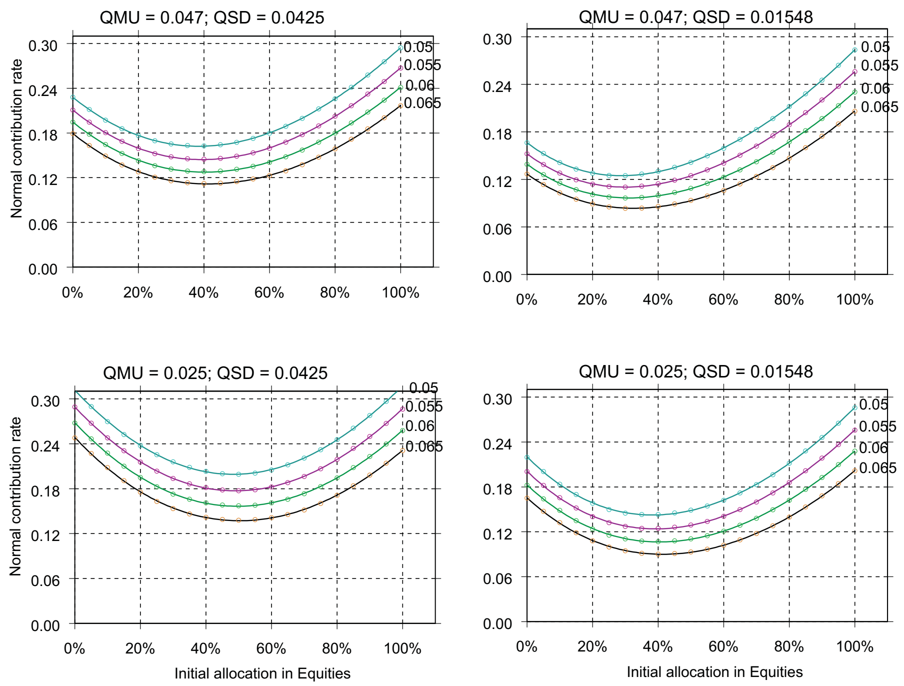

Figure 22. Mean shortfall risk levels at the end of three years

— leaving the inflation volatility unchanged whilst decreasing the long-term inflation mean (i.e. Case 1 versus Case 3 and Case 2 versus Case 4) implies, surprisingly, that the low long-term inflation mean leads to higher mean shortfall risk.

For Cases 1 and 4, we observe that, for a given normal contribution rate and asset allocation, the low and stable inflation case (Case 4) leads to lower mean shortfall risk than in the standard inflation case (Case 1). However, the differences in the risk levels are not very considerable. This implies that, for three-year projections, the risk levels are not very sensitive to the change from the standard inflation case to the low and stable inflation scenario.

Figure 23 shows the mean shortfall risk indifference curves at the end of 15 years.

For the intermediate cases we observe that:

— as expected, leaving the long-term inflation mean unchanged whilst decreasing the volatility (i.e. Case 1 versus Case 2 and Case 3 versus Case 4) implies that low inflation volatility leads to a lower mean shortfall risk (this is more obvious for lower initial allocations in equities); whilst

— leaving the inflation volatility unchanged whilst decreasing the long-term inflation mean (i.e. Case 1 versus Case 3 and Case 2 versus Case 4) implies that low long-term inflation mean leads to a lower mean shortfall risk (this is more evident for higher initial allocations in equities).

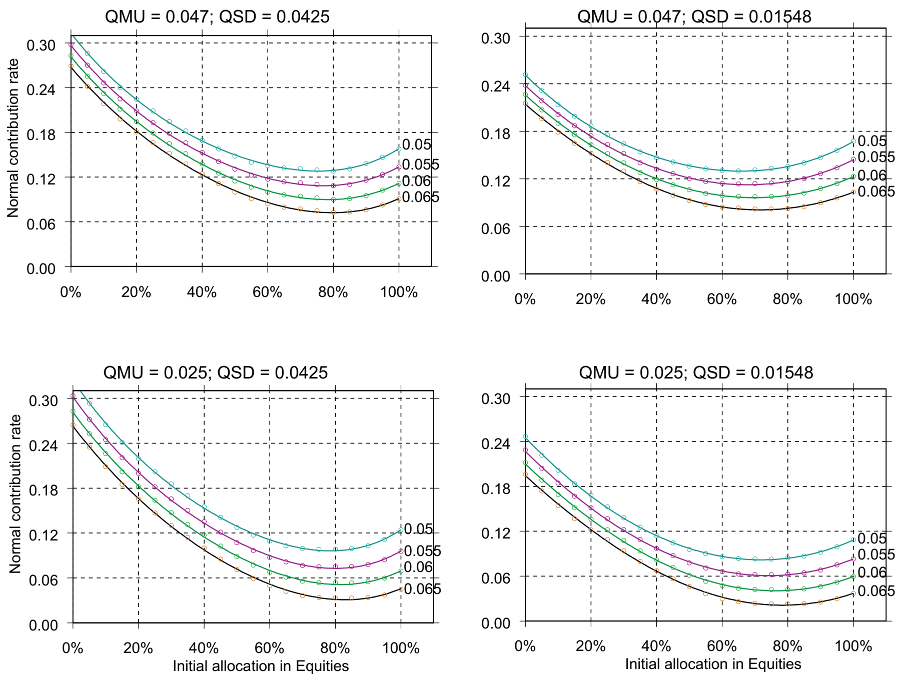

Figure 23. Mean shortfall risk levels at the end of 15 years

For Cases 1 and 4, we observe that, as for the three-year projections, the low and stable inflation case leads to lower mean shortfall risk than in the standard inflation case. Furthermore, there are substantial differences in the risk levels. This means that, for projections over 15 years, the risk levels are fairly sensitive to the change from the standard inflation case to the low and stable inflation scenario.

E.2.2 *Excess contribution rate risk*

Figure 24 shows the excess contribution rate risk indifference curves at the end of 15 years. For the intermediate cases, we observe that:

— as expected, leaving the long-term inflation mean unchanged whilst decreasing the volatility (i.e. Case 1 versus Case 2 and Case 3 versus Case 4) implies that low inflation volatility leads to a lower excess contribution rate risk; and

— leaving the inflation volatility unchanged whilst decreasing the long-term inflation mean (i.e. Case 1 versus Case 3 and Case 2 versus Case 4) implies that low long-term inflation mean leads to a higher excess contribution rate risk.

For Cases 1 and 4, we observe that the low and stable inflation case leads to higher excess contribution rate risk than in the standard inflation case. Nevertheless, the differences in the risk levels are not very considerable. This implies that, compared to the mean shortfall risk, the excess contribution rate risk levels are not very sensitive to the change from the standard inflation case to the low and stable inflation scenario.

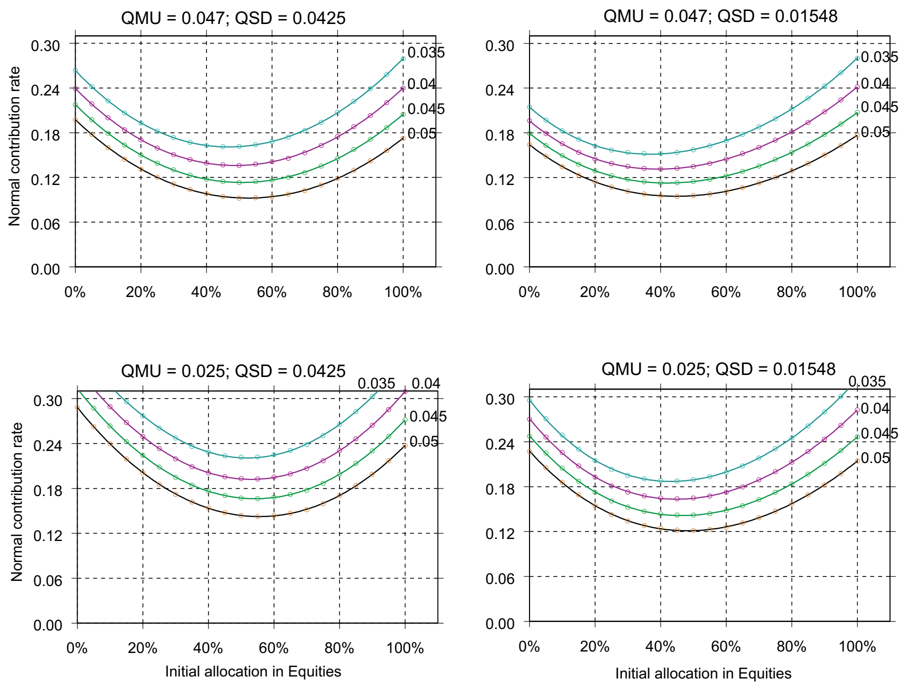

Figure 24. Excess contribution rate risk levels at the end of 15 years

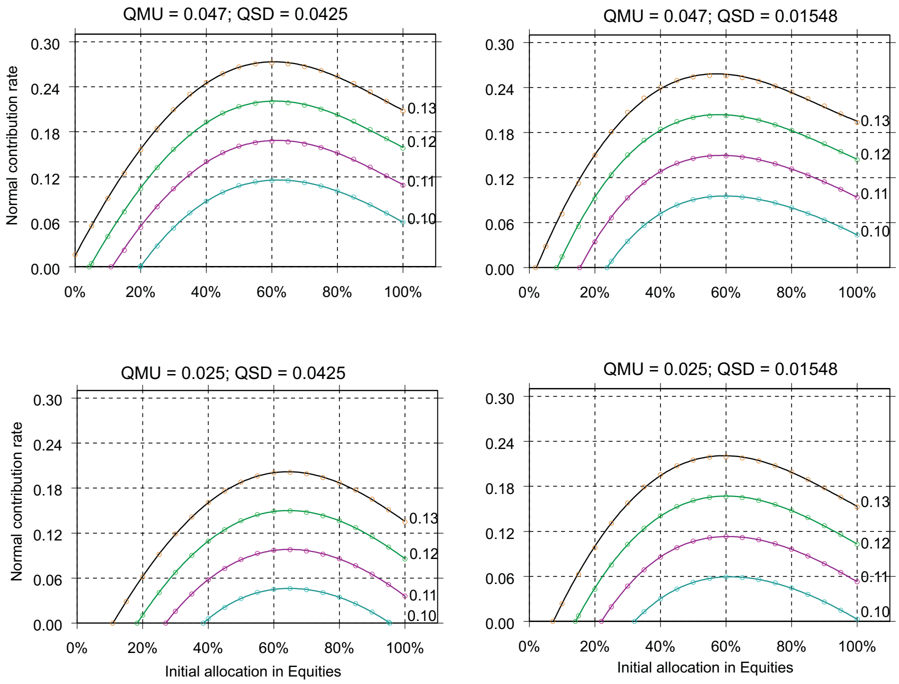

Figure 25. Average contribution rate levels at the end of 15 years

E.2.3 *Average contribution rate levels*

Figure 25 shows the average contribution rate levels at the end of 15 years.

For the intermediate cases, we observe that:

— leaving the long-term inflation mean unchanged at 0.047 whilst decreasing the volatility (i.e. Case 1 versus Case 2) implies that low volatility leads to slightly higher average contribution rate; however

— leaving the long-term inflation mean unchanged at 0.025 whilst decreasing the volatility (i.e. Case 3 versus Case 4) implies that low volatility leads to a lower average contribution rate; and

— leaving the inflation volatility unchanged whilst decreasing the long-term inflation mean (i.e. Case 1 versus Case 3 and Case 2 versus Case 4) implies that low long-term inflation mean leads to a higher average contribution rate.

For Cases 1 and 4, we observe that the low and stable inflation case leads to a higher average contribution rate than in the standard inflation case. Nevertheless, the differences in the average contribution rate levels in the two cases are not significant. This implies that the average contribution rate levels are not very sensitive to changes in the inflation scenario.

E.2.4 *The efficient regions*

Figure 26 shows the efficient regions for projections over 15 years for the standard case (Case1) and the low and stable inflation case (Case 4). LMPQ

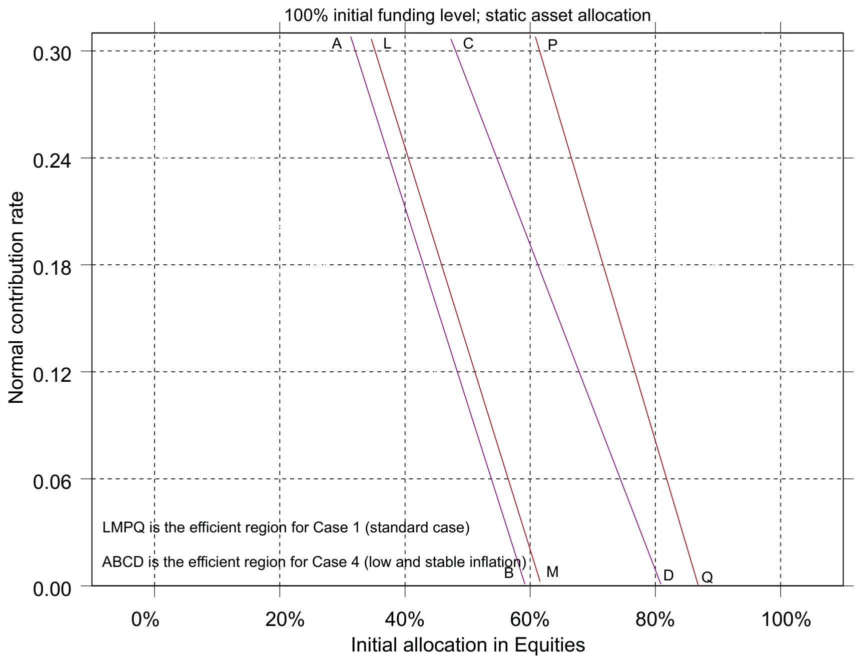

Figure 26. The efficient regions for projections over 15 years for Case 1 (standard case) and Case 4 (low and stable inflation)

is the efficient region for the standard case whilst ABCD is the efficient region for the low and stable inflation case.

We observe that in a low and stable inflation scenario the efficient region shifts towards a lower initial allocation in equities as compared to the standard inflation scenario. Thus, as expected, in the low and stable inflation case we obtain a shift in asset allocation towards fixed-interest bonds. Furthermore, in the low and stable inflation case, we obtain a 'smaller' efficient region (i.e. the efficient region 'shrinks' slightly). This result follows from the fact that there is reduced uncertainty when the inflation is low and stable.

However, the difference in the positions of the efficient regions is not substantial. This implies that the asset allocation decisions will be similar in the two inflation scenarios. Thus, the efficient region is not very sensitive to the change from the standard inflation case to the low and stable inflation scenario.

In the intermediate cases (Cases 2 and 3), we also conclude that the efficient regions are not very sensitive. This is because Figure 23 shows that the mean shortfall risk minimum points are located in a similar region. Furthermore, Figure 24 shows that the minimum points for the excess contribution rate risk curves are also in a similar region.

E.2.5 *Conclusions*

For projections over three years, we get the surprising result that reducing only the long-term inflation mean leads to a higher mean shortfall risk. This result is due to the LPI. We have carried out a further sensitivity analysis of the effect of the LPI by increasing pensions in payment at only the RPI. [A summary of this sensitivity analysis is shown below.] This analysis shows that keeping the inflation volatility unchanged at 0.0425 whilst decreasing the long-term inflation mean from 0.047 to 0.025 leads to a lower mean shortfall risk (except for a very low initial allocation in equities).

For projections over 15 years, the low and stable inflation scenario leads to a lower mean shortfall risk than in the standard inflation scenario; the excess contribution rate risk is higher in the low and stable inflation case than in the standard inflation case. Also, the average contribution rate is higher in the low and stable inflation scenario. However, although the mean shortfall risk is fairly sensitive to the inflation scenarios, the excess contribution rate risk and the average contribution rate are not very sensitive to these scenarios. Furthermore, the asset allocation decisions are not very sensitive to the inflation scenarios because of the lack of sensitivity of the efficient regions.

The effect of pension increases at retail price index (RPI)

Because of space constraints, we have not endeavoured to include a detailed section on the sensitivity analysis of the results to changes in the definition of the LPI. Thus, we will only summarise the results which we have obtained when pensions-in-payment are increased at the RPI.

We compare results from two new cases: in the first case we set the inflation mean and volatility at 0.047 and 0.0425, respectively; whilst in the second case we set the inflation mean and volatility at 0.025 and 0.0425, respectively. In both cases, pensions-in-payment are increased at the RPI only. The following results were obtained:

— For projections over three years: the mean shortfall risk is higher in the second case (the low inflation mean case) than in the first case only for low initial allocation in equities (i.e. for equities allocations less than approximately 40%); whilst the mean shortfall risk is lower in the low inflation mean case (second case) than in the first case for initial allocation in equities greater than approximately 40%.

— For projections over 15 years: the mean shortfall risk is lower in the low inflation mean case than in the high mean case; whilst both the excess contribution rate risk and the average contribution rate are higher in the low inflation mean case than in the high mean case.

These results shows that our conclusions are similar in the RPI case and LPI case except for projections over three years. Thus, in the three-year projections the LPI complicates the results. However, these results also show that, as observed above, this problem occurs in the short term only.

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