EXAMINATIONS

10 April 2000 (pm)

Subject 103 — Stochastic Modelling

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Write your surname in full, the initials of your other names and your Candidate's Number on the front of the answer booklet.*

2. *Mark allocations are shown in brackets.*

3. *Attempt all 11 questions, beginning your answer to each question on a separate sheet.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet and this question paper.

In addition to this paper you should have available Actuarial Tables and an electronic calculator.

1 Let $X_n = Y_1 + Y_2 + ... + Y_n$ be a simple random walk with step distribution

$$\mathbf{P}[Y_j = 1] = p = 1 - \mathbf{P}[Y_j = -1].$$

Derive an expression, for each $\lambda > 0$, for the two values of γ such that $M_n = e^{-\lambda n + \gamma X_n}$ is a martingale with respect to the natural filtration of X_n. [4]

2 Let X and Y be jointly normal random variables with means μ_X, μ_Y, variances σ_X^2, σ_Y^2 and correlation coefficient ρ. Derive an expression for the values of the coefficients α and β, given that the conditional expectation $\mathbf{E}[X | Y]$ is of the form $\alpha + \beta Y$.

[5]

3 The total number of claims received by an insurance company is described by an inhomogeneous Poisson process with rate $\lambda(t)$.

Write down the Kolmogorov forward equations for this process and show that, as in the homogeneous case, the solution is of the form

$$P_{0j}(s, t) = \frac{(m(s,t))^j e^{-m(s,t)}}{j!}$$

where $m(s, t) = \int_s^t \lambda(x)dx$. [5]

4 According to a model used in econometrics, three economic time series, X, Y and Z, are related to one another by the equations

$$X_n = X_{n-1} + \theta_X Z_{n-1} + e_{1,n}$$

$$Y_n = Y_{n-1} + \theta_Y Z_{n-1} + e_{2,n}$$

$$Z_n = \theta_Z Z_{n-1} + e_{3,n}$$

where each of θ_X, θ_Y and θ_Z lies in the interval $(-1, +1)$ and the random variables $\{e_{i,n} : n \geq 1, i = 1, 2, 3\}$ may be assumed to be uncorrelated and to have mean zero.

State, giving your reasons:

(a) whether X, Y and Z are $I(0)$ or $I(1)$

(b) whether each of X, Y and Z individually satisfies the Markov property

(c) whether the vector-valued process $\{(X_n, Y_n, Z_n) : n \geq 1\}$ satisfies the Markov property

(d) whether X and Y are cointegrated [5]

5 Let B_t be a standard Brownian motion, and let $F_t = \sigma(B_s, 0 \le s \le t)$ be its natural filtration.

(i) Derive the conditional expectations $\mathbf{E}[B_t^2 | F_s]$ and $\mathbf{E}[B_t^4 | F_s]$, where $s \le t$.

You may assume that the fourth moment of a random variable with distribution $N(0, \sigma^2)$ is $3\sigma^4$. [4]

(ii) Hence construct a martingale out of B_t^4. [3]

[Total 7]

6 (i) Apply the inverse transform method to generate an observation from the density

$$f_1(x) = \frac{1}{(1+x)^2} \qquad (x > 0)$$

using a pseudo-random number u in the range $0 < u < 1$. Explain how this can be extended to generate an observation from the symmetrised form of the same density

$$f_2(x) = \frac{1}{2(1+|x|)^2} \qquad (x \in \mathbf{R})$$ [3]

(ii) The Cauchy distribution has density function

$$f(x \mid \theta) = \frac{\theta}{\pi(x^2 + \theta^2)} \qquad (x \in \mathbf{R})$$

where θ is a positive parameter.

Show that

$$f(x \mid \theta) \le C f_2(x) \qquad \text{for all } x \in \mathbf{R}$$

as long as $C \ge \frac{2}{\pi}(\theta + \theta^{-1})$. Hence devise a method based on Acceptance-Rejection sampling for generating observations from the Cauchy distribution. [4]

[Total 7]

7 (i) (a) Calculate the autocovariance function $\{\gamma_k : k \geq 0\}$ and autocorrelation function $\{\rho_k : k \geq 0\}$ of a first-order Moving Average process

$$X_t = \mu + e_t + \beta_1 e_{t-1} ,$$

where $\{e_t : t \geq 0\}$ is a sequence of uncorrelated, zero-mean random variables with common variance σ_e^2.

(b) State the conditions on the values of the parameters such that the process is invertible. [5]

(ii) A sequence of observations $x_1, x_2, ..., x_n$ has sample variance $\hat{\gamma}_0 = 14.5$, sample lag-1 autocovariance $\hat{\gamma}_1 = 5.0$. Show that there is more than one first-order moving average process which can be fitted to these data, but verify that only one of the fitted processes is invertible. [4]

[Total 9]

8 The members of a health insurance scheme are classified as contributors or beneficiaries; a member who is a contributor in one period becomes a beneficiary in the next period if he or she becomes seriously ill, and this happens with probability 0.1. The probability of a serious illness continuing into the next period is 0.2. The rules of the scheme specify that any member who is a beneficiary for three successive periods must become a contributor for the next period; if the illness still persists the member may thereafter revert to being a beneficiary.

(i) (a) Construct a discrete time Markov chain to model this health scheme, introducing if necessary various classes of beneficiaries and contributors (a five state model is suggested).

(b) Draw the transition graph.

(c) Write down the transition matrix of the chain. [6]

(ii) Explain whether the above Markov chain is irreducible, periodic or both. [2]

(iii) (a) Calculate the stationary probability distribution of the chain.

(b) Determine the proportion of beneficiaries among the membership in the stationary régime. [4]

(iv) Let b be the average gross payout per beneficiary and c the average gross payout per contributor per period; this means that the nett payments are $b - f$ and $c - f$ respectively, where f is the membership fee per period (assumed to be uniform over members and over time).

(a) Explain how b, c and f should be related if the scheme is to be viable.

(b) Calculate the average profit per period per member in the stationary régime if $b = 600$, $c = 150$ and $f = 300$. [3]

[Total 15]

9 A stationary second-order autoregressive process X, which may be assumed to be in equilibrium at time 0, is defined by

$$X_t = \mu + \alpha_1(X_{t-1} - \mu) + \alpha_2(X_{t-2} - \mu) + e_t ,$$

where $\{e_t : t \geq 1\}$ is a sequence of independent, zero-mean Normal random variables, each with variance σ_e^2.

(i) (a) Obtain an equation for γ_1 in terms of γ_0 and γ_2 by substituting for X_t in the equation $\gamma_1 = \text{Cov}(X_t, X_{t-1})$.

(b) Derive similar equations for γ_2 and γ_0.

(c) State the autocorrelation function ρ_k of X for $k = 0, 1, 2$. [5]

(ii) Suppose that the equations derived in (i) for ρ_1 and ρ_2 are used as the basis of an estimation procedure: estimates $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are defined to be the solutions of those equations when ρ is replaced by a suitably-defined sample autocorrelation function r.

Solve these equations. [3]
[Total 8]

10 (i) (a) Define standard Brownian motion B_t, $t \geq 0$ and give its transition probability density.

(b) Write down the transition probability density of general Brownian motion $W_t = \sigma B_t + \mu t$. [4]

Let S_t defined represent a share price at time t.

(ii) Solve the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t .$$ [5]

(iii) Calculate, given the parameters $\mu = 25\%$ p.a., $\sigma = 20\%$ on an annual basis, the probability that the share price will exceed 45 in four months' time given that its current price is 38. [4]

(iv) Calculate the probability that the share price will exceed 45 *at any stage* during the next four months given that its current value is 38.

[You may use the formula

$$\mathbf{P}[\max_{0 \leq s \leq t}(B_s + \lambda s) > y] = G\left(\frac{\lambda t - y}{\sqrt{t}}\right) + e^{2\lambda y} G\left(\frac{-y - \lambda t}{\sqrt{t}}\right),$$

where $y \geq 0$ and G denotes the normalised Gaussian probability distribution function.] [4]
[Total 17]

11 Patients arriving at the Accident and Emergency department (state A) wait for an average of one hour before being classified by a junior doctor as requiring in-patient treatment (I), out-patient treatment (O) or further investigation (F). Only one new arrival in ten is classified as an in-patient, five in ten as out-patients.

If needed, further investigation takes an average of 3 hours, after which 50% of cases are discharged (D), 25% are sent to receive out-patient treatment and 25% admitted as in-patients.

Out-patient treatment takes an average of 2 hours to complete, in-patient treatment an average of 60 hours. Both result in discharge.

It is suggested that a time-homogeneous Markov process with states A, F, I, O and D could be used to model the progress of patients through the system, with the ultimate aim of reducing the average time spent in the hospital.

(i) Write down the matrix of transition rates, $\{\sigma_{ij} : i, j = A, F, I, O, D\}$, of such a model. [2]

(ii) Calculate the proportion of patients who eventually receive in-patient treatment. [1]

(iii) Derive expressions for the probability that a patient arriving at time $t = 0$ is:

(a) yet to be classified by the junior doctor at time t, and

(b) undergoing further investigation at time t [4]

(iv) Let m_i denote the expectation of the time until discharge for a patient currently in state i.

(a) Explain in words why m_i satisfies the following equation:

$$m_i = \frac{1}{l_i} + \sum_{j \notin \{i,D\}} \frac{s_{ij}}{l_i} m_j$$

where $\lambda_i = \sum_j \sigma_{ij}$.

(b) Hence calculate the expectation of the total time until discharge for a newly-arrived patient. [4]

(v) State the distribution of the time spent in each state visited according to this model. [1]

The average times listed above may be assumed to be the sample mean waiting times derived from tracking a large sample of patients through the system.

(vi) Describe briefly what additional feature of the data might be used to check that this simple model matches the situation being modelled. [2]

(vii) The hospital management committee believes that replacing the junior doctor with a more senior doctor will save resources by reducing the proportion of cases sent for further investigation. Alternatively, the same resources could go towards reducing out-patient treatment time.

(a) Outline briefly the calculations that would need to be performed to compare the options.

(b) Discuss whether the current model is suitable as a basis for making decisions of this nature. [4]

[Total 18]