EXAMINATIONS

11 September 2002 (pm)

Subject 103 — Stochastic Modelling

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available Actuarial Tables and your own electronic calculator.

1 Consider the Poisson process X_t defined as a Markov jump process with state space $\{0, 1, 2, \ldots\}$ and transition probabilities

$$P_{ij}(t) = \frac{(\lambda t)^{j-i}}{(j-i)!} e^{-\lambda t} \qquad (j \geq i).$$

(i) Prove that X_t has independent increments. [3]

(ii) Hence prove that $X_t - \lambda t$ is a martingale with respect to the natural filtration $\mathcal{F}_t$ associated with X_t. [3]

[Total 6]

2 Y_t, $t = 1, 2, \ldots$, is a time series defined by

$$Y_t - \alpha Y_{t-1} = Z_t + (1-\alpha) Z_{t-1}$$

where Z_t, $t = 0, 1, \ldots$, is a sequence of independent zero-mean variables with common variance σ^2 and where $|\alpha| < 1$.

(i) State, giving your reasons, the values of p, d and q for which Y is an ARIMA(p, d, q) process. [2]

(ii) Derive the autocorrelation function ρ_k, $k = 0, 1, 2, \ldots$. [6]

[Total 8]

3 Consider the accident proneness model in which the cumulative number of accidents X_t suffered by a driver is a Markovian birth process with linear transition rates given by

$$\sigma_{i,j} = \begin{cases} (i+1)\beta & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

(i) Denoting by $a_i(t)$ the probability that a driver who has had no accidents at time 0 has had at least i accidents by time t, explain why

$$a_i(t + dt) = a_i(t) + (a_{i-1}(t) - a_i(t))i\beta dt + o(dt)$$

as $dt \to 0$ and hence derive a differential equation satisfied by $a_i(t)$. [3]

(ii) Show that

$$a_i(t) = (1 - e^{-\beta t})^i$$

and deduce the value of $P[X(t) = i | X(0) = 0]$. [4]

(iii) A colleague suggests that a better model would involve transition rates $\sigma_{i,i+1}(t)$ which are dependent on t as well as on i. Comment on this suggestion. [1]

(iv) The same colleague proposes the model $\sigma_{i,i+1}(t) = \frac{(i+1)\beta}{t}$. Comment on the proposed model. [1]
[Total 9]

4 An investor believes that the price of gold increases when the volatility of the equities market is high and decreases when the volatility is low. The investor therefore wishes to model the price of gold X_t, as an Itô process defined by $dX_t = V_t dB_t + V_t^2 dt$, where B_t is standard Brownian motion and V_t is a measure of market volatility calculated from the equity price information available at time t.

(i) Comment briefly on the suitability of this model, mentioning in particular its behaviour when V_t is large and when V_t is small. [2]

(ii) (a) State Itô's Lemma.

(b) Use Itô's Lemma to find an expression for dM_t, where $M_t = e^{-2X_t}$.

(c) Deduce that M_t is a martingale. [5]

(iii) Jensen's Inequality implies that

$$e^{-2E(X_t)} \le E(e^{-2X_t}).$$

Show that $E(X_t) \ge X_0$ whatever the value of X_0 and comment briefly again on the suitability of the model. [2]
[Total 9]

5 (i) The classification of stochastic models according to discrete or continuous time variable, discrete or continuous state space gives rise to a four-way classification. Give four examples, one of each type, of stochastic models which may be used to model observed processes. [2]

(ii) For each of the following observed processes, identify a type of model which could be used to model the process, stating which features of the process lead you to make this choice:

(a) a monthly index of food prices

(b) an index of prices of shares on the London Stock Exchange, constantly updated

(c) the status (active, retired, dead) of a member of a pension scheme [3]

(iii) (a) Discuss briefly the importance of model verification.

(b) A simple random walk model, whose single parameter θ is the probability of an upwards jump, is to be fitted to a set of observations $\{x_1, x_2, \ldots, x_n\}$ which have the property that $|x_{i+1} - x_i| = 1$ for each $1 \leq i < n$. Write down the estimate for the parameter of the model and describe **one** test which you would use in the process of model verification. [5]

[Total 10]

6 The members of a disability insurance scheme are classified as "active" (A), "temporarily disabled" (T), "permanently disabled" (P) or "dead" (D). Members are entitled to benefits when they are in state T or P. For the purpose of analysis the state of each member is recorded on 1 April each year. It is found that the history of a typical member evolves in time as a discrete-time Markov chain with transition matrix

$$\begin{array}{c} \\ A \\ T \\ P \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & T & P & D \end{array} \\ \begin{pmatrix} 0.75 & 0.1 & 0.05 & 0.1 \\ 0.5 & 0.3 & 0.1 & 0.1 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

(i) Draw the transition graph of the chain and find its stationary probability distribution. [3]

(ii) Calculate the mean duration (in years) of a permanent disability benefit. [2]

(iii) Calculate the probability that a member, initially active, is either temporarily or permanently disabled three years after the start of the scheme. [3]

(iv) Calculate the probability that a member, initially active, will **never** draw any benefit from the scheme. [2]

[Total 10]

7 (i) Describe the three elements of a linear congruential generator, and set out the recursive relationship used to generate the pseudo-random number sequence. [3]

(ii) The first three numbers produced by a linear congruential generator are 0.954, 0.462 and 0.628. Use these to generate three pseudo-random numbers from the Pareto distribution with $\alpha = 2$ and $\lambda = 1$.

[The density of the Pareto distribution is $f(x) = \dfrac{\alpha\lambda^{\alpha}}{(\lambda + x)^{\alpha+1}}$ $(x > 0)$.] [4]

(iii) An exponential random variable T with rate parameter λ may be simulated using the formula

$$T = -\frac{1}{\lambda}\log U$$

where U is uniformly distributed on [0, 1]. Explain how this can be used to simulate a path of a Markov jump process $\{X_t: t \geq 0\}$ which has two states, H (healthy) and S (sick), with transition rates σ from H to S and ρ from S to H. Assume $X_0 = H$. [4]

[Total 11]

8 An analyst is investigating the extent to which the price of a stock at the beginning and end of a time interval can be used to provide information about its price during the interval. The model used is

$$S_t = S_0 + \mu t + \sigma B_t,$$

where B_t is a standard Brownian motion. The analyst wishes to investigate the difference between S_t and the price $\hat{S}_t$ which would be predicted given only S_0 and S_T, given by

$$\hat{S}_t = \frac{(T-t)S_0 + tS_T}{T}, 0 \le t \le T.$$

(i) Write down $E(S_t - S_0)$, $\text{Var}(S_t - S_0)$, $E(S_T - S_t)$, $\text{Var}(S_T - S_t)$ and $\text{Cov}(S_t - S_0, S_T - S_t)$, for $0 \le t \le T$. [3]

(ii) Calculate the expectation and variance of $S_t - \hat{S}_t$. [4]

(iii) Find the value of $t \in [0, T]$ where $\text{Var}(S_t - \hat{S}_t)$ is greatest. Comment on your answer. [2]

(iv) Give **two** reasons why the model

$$S_t = S_0\exp(\mu t + \sigma B_t)$$

might be more suitable than the one used by the analyst and discuss whether the results of (iii) might have been substantially different if the analyst had used this model in the first place. [3]

[Total 12]

9 Let $\{x_t\colon 1 \le t \le n\}$ be a time series to which an ARMA(1, 1) model is to be fitted.

(i) Write down the defining equation of an ARMA(1, 1) process, identifying the parameters. [2]

(ii) (a) Outline the Method of Moments parameter estimation technique.

(b) State the assumptions underlying the Maximum Likelihood parameter estimation technique.

(c) Describe briefly the technique known as backforecasting and explain why it, or something similar, is necessary in this case. [5]

(iii) The model fitted to the data is

$$x_t = 5.67 + 0.61x_{t-1} + e_t - 0.23e_{t-1}$$

The most recently observed value in the series is $x_{20} = 8.2$, with estimated residual $\hat{e}_{20} = -1.38$.

(a) Evaluate estimates $\hat{x}_{20}(1)$ and $\hat{x}_{20}(2)$ for x_{21} and x_{22}.

(b) The simplest form of the method of exponential smoothing used at time 19 gave a forecast for x_{20} of 8.37. Assuming the smoothing parameter is equal to 0.2, find the forecast of x_{21}.

(c) Give an example of a circumstance in which a form of exponential smoothing might be expected to outperform Box-Jenkins forecasting in the prediction of future values of the time series. [5]

[Total 12]

10 The ticket office at a train station has a single ticket machine that is used by travellers to purchase tickets.

The machine has a tendency to break down, at which point it must be repaired. The time until breakdown and the time required to effect repairs both follow the exponential distribution.

Let $P_{1i}(t)$, $i = 0, 1$, be the probability that at time t ($t > 0$) there are i ticket machines working at the ticket office, given that the ticket machine is working at time $t = 0$.

(i) Derive the Kolmogorov forward differential equations for $P_{1i}(t)$ $i = 0, 1$ in terms of:

- σ, where $1/\sigma$ is the mean time to breakdown for a machine; and
- ρ, where $1/\rho$ is the mean time to repair a machine [3]

(ii) Show that $P_{10}(t) = \dfrac{\sigma}{\sigma+\rho}\left(1-e^{-(\sigma+\rho)t}\right)$ deduce the value of $P_{11}(t)$. [4]

(iii) The station manager is considering adding a second identical ticket machine, though there is only one repair team to work on the machines in the event that both are out of action simultaneously. Assuming that a second machine is added and operates independently of the first one:

(a) Write down the generator matrix of the Markov jump process X_t which counts the number of working ticket machines at time t.

(b) Derive the Kolmogorov forward differential equations for $p_i(t)$, $i = 0$, 1, 2, the probability that i ticket machines are working.

(c) Given that, for some t, $p_0(t) = \dfrac{2\sigma^2}{2\sigma^2+2\rho\sigma+\rho^2}$,

$p_1(t) = \dfrac{2\rho\sigma}{2\sigma^2+2\rho\sigma+\rho^2}$ and $p_2(t) = \dfrac{\rho^2}{2\sigma^2+2\rho\sigma+\rho^2}$,

show that $\dfrac{d}{dt}p_i(t) = 0$ for $i = 0, 1, 2$.

(d) State what conclusions you draw from part (c). [6]

[Total 13]