EXAMINATIONS

12 September 2001 (pm)

Subject 103 — Stochastic Modelling

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Write your surname in full, the initials of your other names and your Candidate's Number on the front of the answer booklet.*
2. *Mark allocations are shown in brackets.*
3. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet and this question paper.

In addition to this paper you should have available Actuarial Tables and an electronic calculator.

1 A stochastic interest rate model postulates that the base lending rate in month t, i_t, follows the model:

$$i_t = 5\% + 0.9\,(i_{t-1} - 5\%) + e_t\,,$$

where e_t, e_2, ...is a sequence of independent Normal random variables with mean 0, variance σ^2.

(i) (a) Determine an expression for i_t in terms of e_t, e_{t-1}, e_{t-2}, ..., e_1 and i_0.

(b) Calculate the conditional mean and variance of the base lending rate in month t given that $i_0 = 8\%$.

[4]

(ii) Derive an estimator for σ^2 based on observations i_1, i_2, ..., i_n of the base lending rate in n successive months. [2]

[Total 6]

2 Let $\{B(t) : t \geq 0\}$ be a standard Brownian motion and define

$$B_1(t) = tB\left(\frac{1}{t}\right) \text{ for } t > 0, \text{ with } B_1(0) = 0.$$

(i) Calculate $EB_1(t)$, $\mathrm{Var}(B_1(t))$ and $\mathrm{Cov}\{B_1(s), B_1(t))$ for $s < t$. Deduce that B_1 is a standard Brownian motion. [3]

(ii) (a) Show that the two probabilities

$$P[B(t) < ct \text{ for all } t \geq 1]$$

and

$$P[B(t) < c \text{ for all } 0 \leq t \leq 1]$$

are equal to one another where $c > 0$ is a constant.

(b) Find an expression for the value of these probabilities by stating the probability density function of $M_1 = \max_{0 \leq t \leq 1} B(t)$. [4]

[Total 7]

3 A stationary stochastic process $\{Y_t : t = 0, 1, \ldots\}$ satisfies the relationship

$$Y_t = \mu + 0.8(Y_{t-1} - \mu) - 0.4(Y_{t-2} - \mu) + e_t \, ,$$

where $\{e_t : t = 0, 1, \ldots\}$ is a sequence of independent, zero-mean Normal random variables with common variance σ^2.

(i) Calculate the autocorrelation function, ρ_k, and the partial autocorrelation function, ϕ_k, of Y for $k = 1$ and 2. [5]

(ii) State, without performing additional calculations, what you would expect to find if you were to calculate ρ_k and ϕ_k for larger values of k. [2]

[Total 7]

4 Consider the simplified model of credit rating of companies in continuous time described below. There are three ratings which a company can have, A, B and D (default) and the possible transitions are as follows:

- from A to B with rate 4α
- from B to A with rate α
- from B to D with rate 3α

(i) Write down the matrix form of Kolmogorov's forward equations as it applies to this model and verify that the transition matrix $P(t) = P(0, t)$ given below is a solution:

$$P(t) = \begin{pmatrix} \frac{1}{2}e^{-2\alpha t} + \frac{1}{2}e^{-6\alpha t} & e^{-2\alpha t} - e^{-6\alpha t} & 1 - \frac{3}{2}e^{-2\alpha t} + \frac{1}{2}e^{-6\alpha t} \\ \frac{1}{4}e^{-2\alpha t} - \frac{1}{4}e^{-6\alpha t} & \frac{1}{2}e^{-2\alpha t} + \frac{1}{2}e^{-6\alpha t} & 1 - \frac{3}{4}e^{-2\alpha t} - \frac{1}{4}e^{-6\alpha t} \\ 0 & 0 & 1 \end{pmatrix}. \quad [5]$$

(ii) Find the time τ after which a company starting in state A is more likely to be in state D than in state A. [2]

[Total 7]

5 A motor insurance company assumes that a holder of a provisional driver's licence will make claims according to a Poisson process with rate X per year, where X is not fixed but is determined randomly for each driver according to the density function

$$f(x) = 2e^{-2x} \qquad (x > 0).$$

(i) Describe how to simulate an observation X from the density f using a single pseudo-random variable U assumed uniformly distributed on [0, 1]. [3]

(ii) Explain how, given the value X generated in (i), you would use a sequence $U_1, U_2, \ldots$ of uniform pseudo-random variables to simulate the number of claims made in two six-month periods by a provisional driver with mean claim rate X per year. [4]

(iii) Describe a simulation-based method for estimating the conditional probability that a provisional driver makes 2 or more claims in the second six months of driving given that no claim was made in the first six months. [Here the value X is to be assumed **unknown**.] [2]

[Total 9]

6 The evolution of a stock price is modelled as a discrete time process $S_n = \Sigma_{i=1}^{n} X_i$, where $X_1, X_2, \ldots$ are independent, identically distributed random variables with $P\{X_i = 1\} = p$ and $P\{X_i = -1\} = q = 1 - p$. The investment will be liquidated at either the bankruptcy time T_0 (the first time n when the price S_n hits 0) or the first time T_K when the price attains a fixed target K, whichever occurs first.

Let $T = \min(T_0, T_K)$ denote the liquidation time (the exit time from $[0, K]$). Let $A = \{T_K < T_0\}$ denote the event that the target is met before bankruptcy and let $p_k = P[A \mid S_0 = k]$ denote the probability of this event, given that the initial price is $S_0 = k$, where $k \in \{1, \ldots, K-1\}$.

(i) By conditioning on the price of the stock at time 1, determine a difference equation satisfied by p_k, $k = 1, \ldots, K-1$. [2]

(ii) Assume that $p = q = \frac{1}{2}$.

(a) Show that S_n is a martingale.

(b) Derive an expression for p_k by applying the optional stopping theorem to this martingale stopped at T. [5]

(iii) Assume now that $p \neq q$.

(a) Determine a value $\theta \neq 1$ such that $Y_n = \theta^{S_n}$ is a martingale.

(b) Derive an expression for p_k in this case. [4]

[Total 11]

7 According to an interest rate model which operates in continuous time, the interest rate $r(t)$ may change only by upward jumps of fixed size j_u or by downward jumps of fixed size j_d (where $j_d < 0$), occurring independently according to Poisson processes $N_u(t)$ (with rate λ_u) and $N_d(t)$ (with rate λ_d).

Let T_u denote the time of the first up jump in the interest rate, T_d the time of the first down jump, $T = \min(T_u, T_d)$ the time of the first jump. Further, let I be defined as an indicator taking the value 1 if the first jump is an up jump or 0 otherwise.

(i) Determine expressions for the probabilities $P\{T_u > t\}$, $P\{T_d > t\}$ and $P\{T > t\}$ [2]

(ii) Determine the distribution of I. [2]

(iii) Show, by evaluating $P\{T > t \text{ and } I = 1\}$, that I and T are independent random variables. [3]

(iv) Calculate the expectation and variance of the interest rate at time t given the current rate $r(0)$.

Hint: $r(t) = r(0) + j_u N_u(t) + j_d N_d(t)$. [2]

(v) Show that $\{r(t) : t \geq 0\}$ is a process with stationary, independent increments. [3]

[Total 12]

8 A company keeps records of quarterly sales figures, $\{S_t : t = 1, 2, \ldots n\}$ for the most recent n quarters. It wishes to analyse the records with the aim of predicting the sales figures in the near future.

The model suggested by the company is:

$$\log S_t = \mu + \beta t + \theta_{Q(t)} + X_t,$$

where $\{X_t : t = 1, 2, \ldots, n\}$ is a stationary time series, $Q(t)$ takes the value 1, 2, 3 or 4 depending on whether the tth quarter is the first, second, third or fourth quarter of the financial year, and $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 0$.

(i) Explain why the company has suggested a linear model for $\log S_t$ rather than a linear model for S_t. [1]

(ii) Explain the significance of the parameters μ, β and $\{\theta_q : 1 \le q \le 4\}$ and give a reason for the assumption that $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 0$. [4]

(iii) Derive a linear filter $Y_t = \Sigma_{k=-2}^{+2} a_k \log S_{t+k}$ which has the property that the filtered series $\{Y_t\}$ does not depend on $\{\theta_q : 1 \le q \le 4\}$. [3]

(iv) (a) Using the filtered series $\{Y_t\}$ obtained in (iii), derive an expression for ∇Y_t in terms of μ, β, X_t, X_{t-1}, …

(b) State, with reasons, whether $\{Y_t\}$ is $I(0)$, $I(1)$ or neither.

[4]
[Total 12]

9 In the Vasicek model, the spot rate of interest is governed by the stochastic differential equation

$$dr_t = a(b - r_t)\, dt + \sigma dB_t$$

where B_t is a standard Brownian motion and a, $b > 0$.

(i) A stochastic process $\{U_t : t \ge 0\}$ is defined by $U_t = e^{at} r_t$.

(a) Derive an equation for dU_t.

(b) Solve the equation to find U_t.

(c) Show that

$$r_t = b + (r_0 - b)\, e^{-at} + \sigma \int_0^t e^{a(s-t)}\, dB_s \qquad [5]$$

(ii) State the probability distribution of r_t and its limit for large t. [4]

(iii) Derive, in the case $s < t$, the conditional expectation $E[r_t \mid \mathcal{F}_s]$, where $\{\mathcal{F}_s : s \geq 0\}$ is the filtration generated by the Brownian motion B. [3]

[Total 12]

10 For a given driver, any period j is either accident free ($Y_j = 0$) or gives rise to one accident ($Y_j = 1$). The probability of having **no** accident during the next period is estimated using the driver's past record as follows (all values y_j are either 0 or 1):

$$P[Y_{n+1} = 0 \mid Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n] = pe^{-\lambda(y_1 + y_2 + \ldots + y_n)},$$

where $0 < p < 1$, $\lambda \geq 0$. The cumulative number of accidents suffered by the driver over the time from period 1 up to period n is

$$X_n = \sum_{j=1}^{n} Y_j.$$

(i) Verify that the Markov property holds for the sequence $X_1, X_2, \ldots, X_n, \ldots$ and explain why the sequence $Y_1, Y_2, \ldots, Y_n, \ldots$ does not form a Markov chain. [3]

(ii) Draw the transition graph of the Markov chain X and write down its transition matrix. [4]

(iii) Determine, being careful to explain your reasons in each case:

(a) whether the Markov chain X is time-homogeneous
(b) whether it is irreducible
(c) whether it admits a stationary probability distribution [4]

(iv) Starting from the state $X_t = j$, calculate the probability of suffering no further accident for the next n successive periods. [2]

(v) Suppose you are provided with full claims records for a number of a company's policy holders.

(a) Describe a method for estimating the parameters λ and p.

(b) Explain how to test the assumption that the probability of an accident depends only on the cumulative number of accidents, X_n, and does not have a direct dependence on n. [4]

[Total 17]