Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

September 2001

Subject 103 — Stochastic Modelling

EXAMINERS' REPORT

1 (i) (a) Applying the equation iteratively for t, $t-1$, … , 1, we obtain
$i_t - 5\% = e_t + 0.9e_{t-1} + 0.9^2\, e_{t-2} + \ldots + 0.9^{t-1}\, e_1 + 0.9^t\,(i_0 - 5\%)$.

(b) The RHS has expectation $0.9^t \times 3\%$, since each e_u has expectation zero, and has variance $\sigma^2(1 + 0.9^2 + 0.9^4 + \ldots + 0.9^{2t-2}) = \sigma^2\ \frac{(1-0.9^{2t})}{0.19}$.

(ii) We have the expression

$$e_t = i_t - 5\% - 0.9(i_{t-1} - 5\%),$$

so that a sensible estimator for σ^2 would be

$$\frac{1}{n}\sum_{t=2}^{n}(i_t - 5\% - 0.9(i_{t-1} - 5\%))^2\ .$$

(This is both least squares estimator and maximum likelihood estimator.)

2 (i) $EB_1(t) = tEB(t^{-1}) = 0;$

$\text{Var}\ B_1(t) = t^2\ \text{Var}\ B(t^{-1}) = t;$

$\text{Cov}\ (B_1(s), B_1(t)) = st\text{Cov}\ (B(s^{-1}), B(t^{-1})) = stt^{-1} = s$ for $s < t$.

These are identical to the corresponding quantities for B.

(ii) (a) $$P[B(t) < ct \text{ for all } t \geq 1] = P\left[B\left(\frac{1}{u}\right) < \frac{c}{u} \text{ for all } \frac{1}{u} \geq 1\right]$$

$$= P[B_1(u) < c \text{ for all } u \leq 1]$$

(b) We know that the density of M_1 is $2\phi(y)$ for $y > 0$, where ϕ is the standard Normal density.

It follows that the required probability is

$$P[M_1 < c] = 2\int_0^c \phi(y)dy = 2\Phi(c) - 1.$$

3 (i) $\gamma_1 = 0.8\gamma_0 - 0.4\gamma_1 + 0$ and $\gamma_2 = 0.8\gamma_1 - 0.4\gamma_0 + 0$

Hence $\gamma_1 = \frac{4}{7}\gamma_0$ and $\gamma_2 = \frac{2}{35}\gamma_0$, implying that $\rho_1 = \frac{4}{7}$ and $\rho_2 = \frac{2}{35}$.

$$\phi_1 = \rho_1 = \tfrac{4}{7},\ \phi_2 = \tfrac{\rho_2 - \rho_1^2}{1-\rho_1^2} = -0.4.$$

(ii) The ACF will reduce to zero as k increases;
the PACF, however, will be equal to zero for all $k > 2$.

4 (i) The generator is

$$A = \begin{pmatrix} -4\alpha & 4\alpha & 0 \\ \alpha & -4\alpha & 3\alpha \\ 0 & 0 & 0 \end{pmatrix} = \alpha \begin{pmatrix} -4 & 4 & 0 \\ 1 & -4 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

The forward equation is $\frac{d}{dt}P_t = P_t A$.

Calculate the right hand side

$$P_t A = \alpha \begin{pmatrix} e^{-2\alpha t}(-2+1) + e^{-6\alpha t}(-2-1) & e^{-2\alpha t}(2-4) + e^{-6\alpha t}(2+4) & 3e^{-2\alpha t} - 3e^{-6\alpha t} \\ e^{-2\alpha t}(-1+\frac{1}{2}) + e^{-6\alpha t}(1+\frac{1}{2}) & e^{-2\alpha t}(1-2) + e^{-6\alpha t}(-1-2) & \frac{3}{2}e^{-2\alpha t} + \frac{3}{2}e^{-6\alpha t} \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\alpha e^{-2\alpha t} - 3\alpha e^{-6\alpha t} & -2\alpha e^{-2\alpha t} + 6\alpha e^{-6\alpha t} & 3\alpha e^{-2\alpha t} - 3\alpha e^{-6\alpha t} \\ \frac{-\alpha}{2} e^{-2\alpha t} + \frac{3}{2}\alpha e^{-6\alpha t} & -\alpha e^{-2\alpha t} - 3\alpha e^{-6\alpha t} & \frac{3\alpha}{2} e^{-2\alpha t} + \frac{3}{2}\alpha e^{-6\alpha t} \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \tfrac{d}{dt}P_t$$

(ii) $P_{AA}(t) \le P_{AD}(t) \quad t \ge \tau$ implies

$$1 - \frac{3}{2}e^{-2\alpha\tau} = \frac{1}{2}e^{-2\alpha\tau}.$$

Hence,

$$\tau = \frac{\log 2}{2\alpha}.$$

5 (i) Use the inverse distribution function technique.

We have $F(x) = 1 - e^{-2x}$, and we require $U = F(X) = 1 - e^{-2X}$, so that $X = -\frac{1}{2}\log(1-U)$.

(ii) We need two Poisson pseudo-random variables, Y_1 and Y_2, each with mean $\mu = \frac{1}{2}X$.

There are two possible methods for generating Poisson(μ):

- Keep generating exponential variables with mean 1 until the cumulative sum exceeds μ, then set Y equal to one less than the number of variables generated.

- Draw up a table of the cumulative distribution function F_μ of Poisson (μ), use a single uniform U and let Y be the first y such that $F_\mu(y) > U$.

(iii) Carry out the above procedures a large number of times, independently. Let N_0 be the number of times no claims were made in the first six months, $N_{0,2+}$ the number of times no claims were made in the first six months but two or more in the next six. The required estimate is $N_{0,2+} / N_0$.

6 (i) Let $A = \{T_K < T_0\}$, $U = \{S_1 - S_0 = 1\}$, $D = \{S_1 - S_0 = -1\}$. Conditioning after one step, we find:

$$p_k = P[A \mid S_0 = k] = pP[A \mid S_0 = k \cap U] + qP[A \mid S_0 = k \cap D] = p\, p_{k+1} + q\, p_{k-1}.$$

(Together with $p_K = 1$, $p_0 = 0$ this could be used to solve p_k …)

(ii) (a) Let $F_n = \sigma\{X_1, X_2, \ldots, X_n\}$. To show S_n is a martingale we need to show that the conditional expectation of the increments of S_n is 0.

$$E[S_{n+1} - S_n \mid F_n] = E[X_{n+1} \mid F_n] = E[X_{n+1}] = 0.$$

(b) By the optional stopping theorem,

$$E_k(S_T) = S_0 = k$$

On the other hand

$$E_k(S_T) = K\, p_k + 0\,(1 - p_k) = k,$$

which gives

$$p_k = \frac{k}{K}.$$

(iii) The exponential process $Y_n = \theta^{S_n}$ is a martingale if and only if the expectation of the factors is 1, i.e. if and only if $E\theta^{X_1} = (p\theta + q\theta^{-1}) = 1$. This equation has two roots $\theta = 1$, $\theta = \frac{q}{p}$.

Applying the optional stopping theorem to the martingale yielded by the second root gives: $\theta^k = \theta^K p_k + \theta^0 (1 - p_k)$ and $p_k = \frac{\theta^k - 1}{\theta^K - 1}$.

7 (i) $P\{T_u > t\} = e^{-\lambda_u t}$, $P\{T_d > t\} = e^{-\lambda_d t}$, $P\{T > t\} = P\{T_u > t\}\, P\{T_d > t\} = e^{-(\lambda_u + \lambda_d)t}$

(ii) $P\{I = 1\} = P\{T_d > T_u\} = \int_0^\infty \lambda_u e^{-\lambda_u t}\, P\{T_d > t\}\, dt = \frac{\lambda_u}{\lambda_u + \lambda_d}$

(iii) $P\{T > t \cap I = 1\} = \int_0^\infty \lambda_u e^{-\lambda_u t}\, P\{T_d > t\}\, dt = \frac{\lambda_u}{\lambda_u + \lambda_d} e^{-(\lambda_u + \lambda_d)t}$

This is the product of $P\{T > t\}$ and $P\{I = 1\}$.

(iv) $Er_t = r_0 + (j_u \lambda_u + j_d \lambda_d)\, t$ and $\mathrm{Var}\ r_t = j_u^2 \lambda_u t + j_d^2 \lambda_d t$

(v) Take $0 < s < t$. Then $r_{t+s} - r_t = j_u (N_u(t+s) - N_u(t)) + j_d (N_d(t+s) - N_d(t))$. Now $N_u(t+s) - N_u(t)$ has a Poisson($\lambda_u s$) distribution, not depending on t and independent of $\{N_u(r) : 0 \le r \le s\}$, and $N_d(t+s) - N_d(t)$ has a Poisson($\lambda_d s$) distribution, also independent of t and of $\{N_d(r) : 0 \le r \le s\}$, so $r_{t+s} - r_t$ has the same distribution as $r_s - r_0$ and is independent of $r_s - r_0$.

8 (i) Fluctuations in sales tend to be proportional to sales in the sense that $\mathrm{Var}(S_{t+1} - S_t) \propto S_t^2$, company growth tends to be exponential rather than linear. (Either of these explanations is sufficient.)

(ii) $\mu + \beta t$ represents a deterministic linear trend in the main sales volume; β is related to the average annual percentage increase in sales, whereas μ is related to the initial value.

The θ_q refer to predictable seasonal fluctuations above and below the average from one quarter to another due to the weather, timing of Christmas, etc.

We assume that $\Sigma\theta_q = 0$ because any non-zero value could be subsumed in μ; estimation procedures report an indeterminacy if we do not make this assumption.

(ii) We need to ensure that each θ_q has a coefficient of $\frac{1}{4}$ (assuming that the filter coefficients add to 1).

The filter $\left(\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right)$ will do.

(iv) We have

$$Y_t = \mu + \beta t + \tfrac{1}{8} X_{t-2} + \tfrac{1}{4} X_{t-1} + \tfrac{1}{4} X_t + \tfrac{1}{4} X_{t+1} + \tfrac{1}{8} X_{t+2} .$$

Hence

$$\nabla Y_t = Y_t - Y_{t-1} = \beta + \tfrac{1}{8} (X_{t+2} + X_{t+1} - X_{t-2} - X_{t-3}) .$$

Clearly Y is not stationary, if only because it has a trend in the mean.

∇Y, however, does look stationary, so it is reasonable to claim that Y is $I(1)$.

9 (i)

$$dU_t = d(e^{at} r_t) = ae^{at} r_t \, dt + e^{at} \, dr_t$$

$$= e^{at}[ar_t \, dt + ab \, dt - ar_t \, dt + \sigma dB_t]$$

$$= e^{at}(ab \, dt + \sigma dB_t) .$$

Hence

$$U_t = U_0 + ab \int_0^t e^{as} \, ds + \sigma \int_0^t e^{as} \, dB_s$$

$$= r_0 + b(e^{at} - 1) + \sigma \int_0^t e^{as} \, dB_s$$

Thus

$$r_t = e^{-at} \, U_t = b + (r_0 - b) \, e^{-at} + \sigma \int_0^t e^{a(s-t)} \, dB_s .$$

(ii) From above, r_t is Gaussian
with mean $b + (r_0 - b) \, e^{-at}$ and variance

$$\sigma^2 \int_0^t e^{2a(s-t)} \, ds = \sigma^2 \frac{1 - e^{-2at}}{2a} .$$

As $t \to \infty$, the distribution of the spot rate is $N(b, \sigma^2 / 2a)$.

(iii) $$r_t = b + (r_0 - b)\, e^{-at} + \sigma e^{-at} \int_0^t e^{au}\, dB_u.$$

Hence, by the martingale property of Itô integrals

$$E[r_t \mid \mathcal{F}_s] = b + (r_0 - b)\, e^{-at} + \sigma e^{-at}\, E\left[\int_0^t e^{au}\, dB_u \,\middle|\, \mathcal{F}_s \right]$$

$$= b + (r_0 - b)\, e^{-at} + \sigma e^{-at} \int_0^s e^{au}\, dB_u$$

$$= e^{a(s-t)}\, r_s + b(1 - e^{a(s-t)}).$$

10 (i) There is an explicit dependence on the past behaviour of Y_j, $j \le n$ in the probability distribution of Y_{n+1} ; hence the Markov property does not hold.

On the other hand

$$P[X_{n+1} = j \mid X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}, X_n = i]$$

$$= P[Y_{n+1} = j - i \mid Y_1 = i_1, Y_2 = i_2 - i_1, \ldots, Y_{n-1} = i_{n-1} - i_{n-2}, Y_n = i - i_{n-1}]$$

$$= \begin{cases} pe^{-\lambda i} & \text{if } j - i = 0, \\ 1 - pe^{-\lambda i} & \text{if } j - i = 1. \end{cases}$$

This is independent of $i_1, i_2, \ldots, i_{n-1}$.

(ii) Transition graph

$$\begin{tikzcd}
0 \arrow[r, "1-p"] \arrow[loop below, "p"] & 1 \arrow[r, "1-pe^{-\lambda}"] \arrow[loop below, "pe^{-\lambda}"] & 2 \arrow[r, "1-pe^{-2\lambda}"] \arrow[loop below, "pe^{-2\lambda}"] & 3 & \cdots
\end{tikzcd}$$

Transition matrix:

$$\begin{pmatrix} p & 1-p & & \\ & pe^{-\lambda} & 1-pe^{-\lambda} & 0 \\ & & pe^{-2\lambda} & 1-pe^{-2\lambda} \\ 0 & & \ddots & \ddots \end{pmatrix}.$$

(iii) (a) Chain is time-homogeneous since transition probabilities calculated in (i) do not depend on time n.

(b) It is not irreducible since the number of accidents can never go down.

(c) There are no recurrent states, hence there can be no stationary distribution. Alternatively, a stationary distribution, π, if it exists, must obey

$$\pi_0 p = \pi_0$$

$$\pi_0 (1 - p) + \pi_1 pe^{-\lambda} = \pi_1$$

$$\pi_1 (1 - pe^{-\lambda}) + \pi_2 pe^{-2\lambda} = \pi_2 .$$

$$\vdots$$

Since $p < 1$ we have $\pi_0 = 0$, and then $\pi_1 = 0$ etc. Hence no stationary probability distribution exists.

(iv) No new accident:

$$(pe^{-j\lambda})^n = p^n e^{-nj\lambda}$$

(v) (a) Maximum likelihood would be very easy in this case: choose λ and p to maximise $\Pi\{(pe^{-\lambda x_k})^{1-y_k}(1 - pe^{-\lambda x_k})^{y_k}\}$.

(b) Change the model to:

$$P[Y_{n+1} = 0 \mid Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n] = pe^{-\lambda(x_n, n)},$$

then test the hypothesis that $\lambda(x, n) \equiv \lambda(x)$ for all n.