EXAMINATIONS

10 April 2001 (pm)

Subject 103 — Stochastic Modelling

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Write your surname in full, the initials of your other names and your Candidate's Number on the front of the answer booklet.*

2. *Mark allocations are shown in brackets.*

3. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet and this question paper.

In addition to this paper you should have available Actuarial Tables and an electronic calculator.

1 $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ and $\{\mathcal{F}_t : t \geq 0\}$ is the filtration associated with N.

(i) Write down the conditional distribution of $N(t + s) - N(t)$ given $\mathcal{F}_t$, where $s > 0$ and use your answer to find $E(\theta^{N(t+s)} \mid \mathcal{F}_t)$. [3]

(ii) Find a process of the form $M(t) = \eta(t)\theta^{N(t)}$ which is a martingale. [2]

[Total 5]

2 An insurance company wishes to test the assumption that claims of a particular type arrive according to a Poisson process model. The times of arrival of the next 20 incoming claims of this type are to be recorded, giving a sequence $T_1, \ldots, T_{20}$.

(i) Give reasons why tests for the goodness of fit should be based on the inter-arrival times $X_i = T_i - T_{i-1}$ rather than on the arrival times T_i. [1]

(ii) Write down the distribution of the inter-arrival times if the Poisson process model is correct and state **one** statistical test which could be applied to determine whether this distribution is realised in practice. [2]

(iii) State the relationship between successive values of the inter-arrival times if the Poisson process model is correct and state **one** method which could be applied to determine whether this relationship holds in practice. [2]

[Total 5]

3 (i) Give a definition of the spectral density of a stationary time series, expressed in terms of the autocovariance function $\{\gamma_k : k \in Z\}$ of the process. Use this definition to derive the spectral density of a first-order moving average process and of a first-order autoregression. [5]

(ii) Suppose the "inverse" of a time series model with spectral density $f(\omega)$ is defined to be the model with spectral density $\dfrac{1}{f(\omega)}$. Using part (i), state the form of the inverse of a first-order moving average and state the way in which the inverse of an invertible MA(1) differs from the inverse of a non-invertible MA(1). [2]

[Total 7]

4 Let X_n denote an autoregressive high frequency time series modelled by:

$$X_{n+1} = (1 - \alpha) X_n + \theta + \tau e_n ,$$

where $e_n = \pm 1$ with equal probabilities and α, θ, τ are constant parameters. An analyst wishes to investigate whether this series may be approximated by some continuous time diffusion, i.e. $X_n \approx Y_{nh}$, where Y_t satisfies a stochastic differential equation

$$dY_t = \mu(Y_t)\, dt + dB_t$$

and B_t denotes standard Brownian motion.

(i) State the expectation and variance of $dY_t = Y_{t+h} - Y_t$, the increment of the process Y_t over a small interval of size h, conditional on $Y_t = y$. [2]

(ii) Calculate the expectation and variance of the increment $X_{n+1} - X_n$ of the autoregression, conditional on $X_n = y$. [2]

(iii) Find, by equating the first and second moments of the increments in (i) and (ii) above, an expression for the drift $\mu(y)$ of the approximating diffusion in a form which does not involve the time increment h. [2]

(iv) State a condition under which the approximating process in (iii) is a Brownian motion with drift. [1]

(v) State a condition under which the approximating process in (iii) is an Ornstein-Uhlenbeck process. [1]

[Total 8]

5 (i) Derive expressions for ρ_1 and ρ_2, the autocorrelation function of X at lags 1 and 2, in the case that X is a stationary process satisfying the recursion:

$$X_t = \alpha X_{t-1} + e_t + \beta e_{t-1} ,$$

where $\{e_t : t = 1, 2, \ldots\}$ is a sequence of uncorrelated random variables with mean 0, variance σ^2. [5]

(ii) A company's monthly sales figures, corrected for trend and seasonal factors, exhibit sample autocorrelation function at lags 1 and 2 of $r_1 = 0.5$, $r_2 = 0.4$. Find method of moments estimators of α and β for the model in (i). [3]

[Total 8]

6 A motor insurance company has 80,000 policy holders, paying an average annual premium of £400. The company receives claims at a rate of 2000 per month, the sizes of the claims having mean £1,200, standard deviation σ = £200.

Let $S(t)$ denote the company's total surplus at time t, with $S(0)$ equal to the initial reserve, set at £20,000,000.

(i) Calculate the expectation of the total amount paid out in claims in a given month and the safety loading employed by the company. [2]

(ii) State, with reasons, whether it would be appropriate to use a diffusion approximation to calculate the probability of ruin, that is the probability that the process $\{S(t) : t \geq 0\}$ ever hits 0. [3]

(iii) Describe a simulation-based method for estimating the probability of ruin. Indicate why it would be important to use a method of generating pseudo-random variables which gives rise to reproducible sequences. [4]

[Total 9]

7 The evolution of a stock price S_t is modelled by

$$S_t = e^{\mu t + \sigma B_t},$$

where B_t represents a standard Brownian motion, μ and σ are fixed parameters and the initial value of the stock is $S_0 = 1$.

(i) Derive an expression for $P\{S_t \leq x\}$. [2]

(ii) Derive expressions for the median of S_t and the expectation of S_t. [4]

(iii) (a) Determine an expression for the conditional expectation $E(S_t \mid F_s)$, where $s < t$ and where $\{F_s : s \geq 0\}$ denotes the filtration associated with the process S.

(b) Find conditions on μ and σ under which the process $\{S_t : t \geq 0\}$ is a martingale.

(c) State, with reasons, whether or not the stock would be a good long term investment in this case.

[5]

[Total 11]

8 A company assesses the credit-worthiness of various firms every quarter; the ratings are, in order of decreasing merit, *A, B, C* and *D* (default). Historical data support the view that the credit rating of a typical firm evolves as a Markov chain with transition matrix

$$P = \begin{pmatrix} 1-\alpha-\alpha^2 & \alpha & \alpha^2 & 0 \\ \alpha & 1-2\alpha-\alpha^2 & \alpha & \alpha^2 \\ \alpha^2 & \alpha & 1-2\alpha-\alpha^2 & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for some parameter α

(i) Draw the transition graph of the chain. [2]

(ii) Determine the range of values of α for which the matrix P is a valid transition matrix. [2]

(iii) State, with reasons, whether the chain is irreducible and aperiodic. [2]

(iv) Derive a stationary probability distribution for the chain and establish whether it is unique. [4]

(v) For the value $\alpha = 0.1$, calculate the probability that the company's rating in the third quarter, X_3, is in the default state D:

(a) in the case where the company's rating in the first quarter, X_1, is equal to A

(b) in the case $X_1 = \text{B}$

(c) in the case $X_1 = \text{C}$

(d) in the case $X_1 = \text{D}$

[3]
[Total 13]

9 A continuous-time Markov sickness and death model has four states: H (healthy), S (sick), T (terminally ill) and D (dead). From a healthy state transitions are possible to states S and D, each at rate 0.05 per year. A sick person recovers his health at rate 1.0 per year; other possible transitions are to D and T, each with rate 0.1 per year. Only one transition is possible from the terminally ill state, and that is to state D with transition rate 0.4 per year.

(i) Draw the transition graph for this process. [2]

(ii) Define $P(t) = \{p_{ij}(t) : i, j \in H, S, T, D\}$ where $p_{ij}(t)$ denotes the probability of being in state j at time t given that the individual was in state i at time 0. State the Kolmogorov forward equation satisfied by the matrix $P(t)$, making sure that you specify the entries of the matrix A which appears. [3]

(iii) Calculate the probability of being healthy for at least 10 uninterrupted years given that you are healthy now. [1]

(iv) Let d_j denote the probability that a life which is currently in state j will never suffer a terminal illness. By considering the first transition from state H, show that $d_H = \frac{1}{2} + \frac{1}{2} d_S$ and deduce similarly that $d_S = \frac{1}{12} + \frac{5}{6} d_H$. Hence evaluate d_H and d_S. [5]

(v) Write down the expected duration of a terminal illness, starting from the moment of the first transition into state T. Use the result of (iv) to deduce the expectation of the future time spent terminally ill by an individual who is currently healthy. [4]
[Total 15]

10 A family agrees an expenditure target, Y_n, for year n, in such a way that the annual increase in the expenditure target is proportional to the increase in the family income over the previous year. The actual expenditure during the year, X_n, is assumed to be related to the expenditure target, but incorporating an element of randomness and a factor accounting for the family's propensity to overspend. The family income, I_n, is assumed to grow at a constant annual rate, before randomness is taken into account.

The head of the household believes that the following three equations form an appropriate representation of the above information:

$$Y_n = Y_{n-1} + \beta(I_{n-1} - I_{n-2})$$

$$X_n = (1 + \pi)\, Y_n + e_n^{(1)}$$

$$I_n = (1 + \alpha)\, I_{n-1} + e_n^{(2)}$$

where $\{(e_n^{(1)}, e_n^{(2)}) : n = 1, 2, \ldots\}$ is a sequence of zero-mean bivariate Normal random variables and α, β and π are positive parameters (with $\beta < 1$).

(i) Express the first of the equations in terms of the backshift operator, B, and deduce that a linear relationship exists between Y_n and I_{n-1}. [3]

(ii) Show that the process $\mathbf{Z}_n = (X_n, I_n)$ is a first-order multivariate autoregressive process. [2]

(iii) State, with reasons, whether $\{I_n : n \geq 1\}$ is a stationary time series, and hence determine whether $\{\mathbf{Z}_n : n \geq 1\}$ is $I(0)$, $I(1)$ or neither. [3]

(iv) Find an estimator for the parameter α by minimising the quantity $\Sigma_{t=2}^{n}(e_t^{(2)})^2$. [3]

(v) The head of the household wishes to perform a simulation to investigate whether the propensity to overspend will result in negative net savings. It is assumed that $\text{Var}(e_n^{(1)}) = \sigma_1^2, \text{Var}(e_n^{(2)}) = \sigma_2^2$ and $\text{Cov}(e_n^{(1)}, e_n^{(2)}) = \rho\sigma_1\sigma_2$, where $-1 < \rho < 1$.

(a) Describe a method of simulating an observation of the pair $(e_n^{(1)}, e_n^{(2)})$ starting from two uniformly distributed pseudo-random variables U_1, U_2.

(b) Describe the role of sensitivity analysis in drawing conclusions from the simulation. [6]

(vi) An alternative model is proposed, involving the logarithms of the quantities I_n, X_n and Y_n:

$$\ln Y_n = \ln Y_{n-1} + \ln I_{n-1} - \ln I_{n-2}$$

$$\ln X_n = \theta + \ln Y_n + e_n^{(1)}$$

$$\ln I_n = \phi + \ln I_{n-1} + e_n^{(2)}$$

Discuss whether this model is more suitable than the original model. [2]

[Total 19]