EXAMINATIONS

10 April 2002 (pm)

Subject 103 — Stochastic Modelling

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available Actuarial Tables and your own electronic calculator.

1 A stock price $\{S_t: t = 1, 2, \ldots\}$ is modelled as

$$S_t = S_0 \exp\left(\sum_{j=1}^{t} X_j\right),$$

where $X_1, X_2, \ldots$ is a sequence of independent Normal random variables, with $E(X_j) = \mu_j$, $\mathrm{Var}(X_j) = \sigma_j^2$.

In order to perform a martingale analysis it is necessary to find a sequence of constants $\alpha_1, \alpha_2, \ldots$ such that $Y_t = \alpha_t S_t$ is a martingale.

(i) Derive a recurrence equation satisfied by the constants $\{\alpha_t: t = 1, 2, \ldots\}$. [You may use the fact that the moment generating function of $N(\mu, \sigma^2)$ is $M(s) = \exp(\mu s + \frac{1}{2}\sigma^2 s^2)$.] [4]

(ii) State, with a brief explanation, whether the same answer would be obtained if X_j had a non-normal distribution with mean μ_j, variance σ_j^2. [1]

[Total 5]

2 An analyst wishes to use a model which is based on Brownian motion, but which does not become too large and positive for large t. The model proposed is

$$X_t = B_t e^{-cB_t},$$

where B_t is standard Brownian motion, and c is a positive constant.

(i) Verify that there is an upper bound which X never exceeds. [2]

(ii) Use Itô's Lemma to find dX_t. [3]

(iii) State, with a brief explanation, whether the suggested model is appropriate for a process which is asymptotically stationary. [1]

[Total 6]

3 (i) Explain briefly what is meant by a *linear trend* and by *seasonal variation* in respect of a sequence of observed values $\{x_1, x_2, \ldots, x_n\}$ forming a time series. [2]

(ii) Describe an operation which can be applied to the data in order to remove additive seasonal variation of period 2. [2]

(iii) Derive an appropriate operation to perform on the process

$$X_t = \exp(a + bt + Z_t),$$

where Z is $I(1)$, in order to obtain a stationary process Y_t. [2]

[Total 6]

4 Consider the second-order autoregressive process

$$Y_t = -2\alpha Y_{t-1} + \alpha^2 Y_{t-2} + Z_t$$

where $\{Z_t\}$ is a zero-mean white noise process with $\mathrm{Var}(Z_t) = \sigma^2$.

(i) Determine the range of values of α for which the process Y can be stationary. [3]

(ii) Derive the autocovariances γ_1 and γ_2 of Y in terms of α and σ. [6]

[Total 9]

5 The ratio of claims to premium income is calculated annually by a division of a large insurance company over a period of 30 years. The observed values are plotted against time in Figure 1a, with the sample autocorrelation function (ACF) plotted in Figure 1b; the dotted lines indicate the cutoff points for significance at the 5% level.

Figure 1a

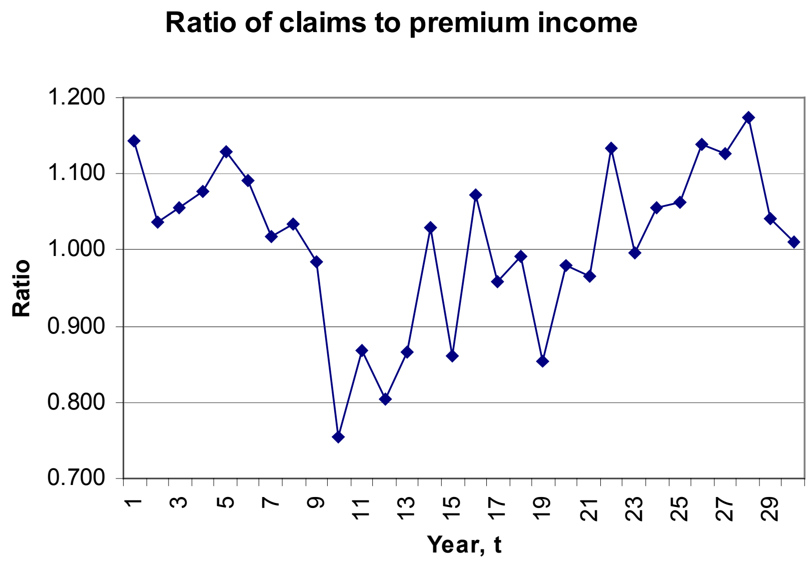
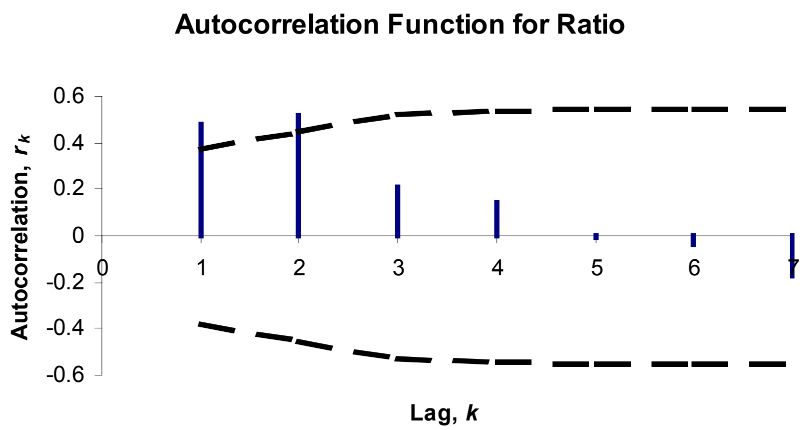


Figure 1b

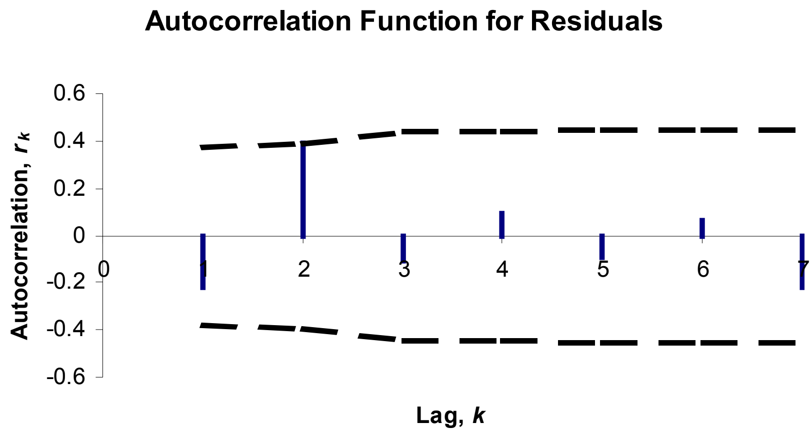


(i) Explain which feature of the Figures indicates that differencing is not required in order to obtain a stationary series. [1]

(ii) On the basis of the sample ACF, r_k, the company's analyst decides to fit a first-order autoregressive model to the data. State, with reasons, whether you consider this to be a reasonable decision and indicate what additional plot you would require in order to make a firmer recommendation. [3]

(iii) The model is fitted and the residuals calculated. The sample ACF of the residuals is shown in Figure 1c. State what conclusions you would draw from the plot. [1]

Figure 1c



(iv) The fitted model is

$$X_t = 0.49541 + 0.5118X_{t-1} + e_t .$$

(a) Derive forecasts $\hat{x}_{30}(1)$ and $\hat{x}_{30}(2)$ for the next two values in the series.

(b) Comment on the reliability of the forecasts. [4]

[Total 9]

6 A disability benefit scheme is modelled in continuous time by a Markov jump process with states A (active), T (temporarily disabled), P (permanently disabled) and D (dead). The transition rates are as follows:

$A \to T$: 3λ	$T \to A$: 5λ	$P \to D$: 2α
$A \to P$: λ	$T \to P$: 2λ	
$A \to D$: α	$T \to D$: α	

(i) Write down the generator matrix of the process. [2]

(ii) Calculate the probability that, having started in state A, the process has visited neither T nor P by time t. [3]

(iii) Write down the matrix form of the Kolmogorov backward differential equations and use this to derive a differential equation for $p_{PD}(t)$, the probability that a scheme member in state P at time 0 will be in state D at time t. [2]

(iv) Solve the equation for $p_{PD}(t)$, the probability that a policyholder, initially permanently disabled, is dead by t. [2]

[Total 9]

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7 (i) Demonstrate that the random variable

$$X = -\tfrac{1}{4}\log(U)$$

has an exponential distribution with mean ¼ when U is uniformly distributed over the range (0, 1). [2]

(ii) (a) Explain how the above can be used to simulate a path of the Poisson process with intensity $\lambda = 4$.

(b) Derive a method for simulating a Poisson random variable with mean 4. [4]

(iii) Describe an alternative method for generating a Poisson random variable with mean 4 based on the cumulative distribution function. [2]

(iv) State, giving reasons, which of the two methods in (ii) and (iii) would be more efficient if a simulation calls for a large number of Poisson random variables with mean 4. [2]

[Total 10]

8 A company is studying the health records of its longest serving employee in order to improve its provision for health insurance. Let $X(t) = H$ if the employee is healthy at time t, $X(t) = S$ otherwise. The available information includes the value of $X(t)$ for all $0 \le t \le T$.

(i) Explain the principal stages in the formulation and verification of a stochastic model for this process. [3]

(ii) The company chooses to model X as a two-state time-homogeneous Markov jump process with transition rates $\sigma_{HS} = \sigma$, $\sigma_{SH} = \rho$.

(a) State the distribution of a typical holding time in state H and of a typical holding time in state S assuming the model is valid.

(b) Write down estimates for the parameters σ and ρ of the model in terms of quantities which may be derived from the available data.

(c) Indicate one test which could be used to determine whether the data support the assumption that the Markov jump process model is suitable. [4]

(iii) Having fitted the model the company discovers that the observed distribution of holding times in state H does not fit the predictions of the time-homogeneous Markov model; in particular, the mean holding time in state H between visits to state S appears to be decreasing with t.

(a) Describe the principal difference between a time-inhomogeneous model and a time-homogeneous one and indicate whether a time-inhomogeneous model might provide a better fit to the observations.

(b) Explain why the original model could still be used if the company is large and has a roughly constant age profile. [4]

[Total 11]

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9 A motor insurer operates a no claims discount system that has five levels. The percentage of the basic premium paid by the insured in each level is as follows:

Level	*% premium charged*
5	100
4	90
3	80
2	70
1	60

Insured motorists move between levels depending on the number of claims in the previous year. For each policyholder, the number of claims per year follows a Poisson distribution with mean 0.25.

For those in Levels 2, 3, 4 and 5 at the start of the previous year:

- if no claims are made during the previous year, the insured moves down one level (e.g. from Level 4 to Level 3)
- if one claim is made during the previous year, the insured moves up one level (except those in Level 5 at the start of the previous year, who will remain in Level 5)
- if two claims are made during the previous year, the insured moves up two levels (except those in Level 5 at the start of the previous year, who will remain in Level 5 and those in Level 4, who will move to Level 5)
- if three or more claims are made during the previous year, the insured moves to Level 5

For those in Level 1 at the start of the start of the previous year, a no claims discount protection policy applies whereby they remain in Level 1 if they make one claim. If they make two claims, they move to Level 2. If they make three or more claims, they move to Level 5. If they make no claims, they remain in Level 1.

(i) Determine the transition matrix for the no claims discount system (assuming that all motorists continue their policies). [3]

(ii) A policyholder is in Level 3 for the first year of the policy. Assuming that the policy is maintained, calculate the probability that at the start of the third year the policyholder will be (a) in Level 1, (b) in Level 3. [3]

(iii) (a) State conditions under which the probability of being in a particular state after n years converges as $n \to \infty$ to some limit which is independent of the initial state.

(b) Verify that the conditions are satisfied in this instance.

(c) Determine the ultimate probability that the insured will be in Level 1.
[8]

(iv) The insurer suspects that the model used for its calculations may be too simplistic. Given annual data listing numbers of claims per policy, broken down by discount level, state which test would be most appropriate to test the assumption that the distribution of the number of claims per policy per year is Poisson with mean 0.25. [1]

[Total 15]

10 (i) State the Lévy decomposition theorem which describes the constituent parts of a Lévy process. [3]

(ii) Let $M_t = \exp(-2ab + 2bB_t - 2b^2 t)$, where B_t is standard Brownian motion and where a and b are positive constants. Define T as the first time that $M_t = 1$, with the definition $T = \infty$ if M never hits the point 1. You may assume that $M_t \to 0$ as $t \to \infty$, so that

$$M_T = \begin{cases} 1 & \text{if } M \text{ hits } 1 \\ 0 & \text{if } M \text{ never hits } 1 \end{cases}$$

(a) Show that M is a martingale and that $0 \le M_t \le 1$ for all $0 \le t \le T$.

(b) State the optional stopping theorem and use it to prove that the probability that the Brownian motion B_t ever hits the line $a + bt$ is e^{-2ab}. [6]

(iii) An insurance company earns premium income at a constant rate c per unit time. Claims arrive according to a Poisson process with rate λ; each claim may be assumed to be of fixed size k. Let X_t denote the total value of all claims received up until time t and denote by S_t the company's surplus at time t,

$$S_t = s_0 + ct - X_t,$$

where s_0 is a positive constant. Show that $\{S_t : t \ge 0\}$ is a Lévy process and identify the components of the Lévy decomposition of S. [2]

(iv) Calculate the expectation and variance of S_t. [2]

(v) The company is interested in the probability of ruin, defined as the probability that S_t ever goes below 0. An investigator proposes using a Brownian model

$$S_t^* = s_0 + \mu t + \sigma B_t,$$

where B_t is a standard Brownian motion, as an approximation to the surplus process S_t.

(a) Calculate the appropriate values of μ and σ.

(b) State the significance of the condition $c > k\lambda$ in this situation.

(c) Write down the probability that the approximating process S^* ever hits 0 assuming that $c > k\lambda$.

(d) Outline the principal difference between S and S^* and state whether you consider that the probability obtained in (c) would be an acceptable approximation to the probability of ruin. [7]

[Total 20]