Faculty of Actuaries Institute of Actuaries

EXAMINATIONS

April 2000

Subject 103 — Stochastic Modelling

EXAMINERS' REPORT

1 $\mathbf{E}[M_{n+1} \mid \mathbf{F}_n] = \mathbf{E}[e^{-\lambda(n+1)+\gamma X_{n+1}} \mid \mathbf{F}_n]$

$$= e^{-\lambda(n+1)}\ \mathbf{E}[e^{\gamma(X_n+Y_{n+1})} \mid \mathbf{F}_n] = e^{-\lambda(n+1)}\ e^{\gamma X_n}\ \mathbf{E}[e^{\gamma Y_{n+1}} \mid \mathbf{F}_n]$$

$$= e^{-\lambda}\ M_n \mathbf{E}[e^{\gamma Y_{n+1}}] = e^{-\lambda} M_n (pe^{\gamma} + (1-p)\, e^{-\gamma}).$$

Hence the condition for martingale:

$$pe^{\gamma} + (1-p)e^{-\gamma} = e^{\lambda}.$$

Multiply by e^{γ} and solve quadratic equation in unknown e^{γ}:

$$e^{\gamma} = \frac{e^{\lambda} \pm \sqrt{e^{2\lambda} - 4p(1-p)}}{2p}.$$

2 Assume $\mathbf{E}[X \mid Y] = \alpha + \beta Y$ and determine α, β using:

(i) orthogonality condition $\mathbf{E}\{(X - \mathbf{E}[X \mid Y])Y\} = 0$.

(ii) $\mathbf{E}\{\mathbf{E}[X \mid Y]\} = \mathbf{E}[X]$.

(i) gives $\mathbf{E}[XY] - \alpha\mathbf{E}[Y] - \beta\mathbf{E}[Y^2] = 0$;

Since the correlation coefficient ρ is

$$\rho = \frac{\mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y]}{\sigma_X \sigma_Y},$$

we have $\mathbf{E}[XY] = \rho\sigma_X\sigma_Y + \mu_X\mu_Y$ and (i) yields

$$\alpha\mu_Y + \beta(\sigma_Y^2 + \mu_Y^2) = \rho\sigma_X\sigma_Y + \mu_X\mu_Y.$$

(ii) gives $\alpha + \beta\mu_Y = \mu_X$.

Solve the two simultaneous equations to get

$$\beta = \frac{\rho\sigma_X}{\sigma_Y}, \qquad \alpha = \mu_X - \frac{\rho\sigma_X}{\sigma_Y}\mu_Y$$

3 $-0 \overset{\lambda(t)}{\longrightarrow} 1 \overset{\lambda(t)}{\longrightarrow} 2 \overset{\lambda(t)}{\longrightarrow} 3 \;\cdots.$

$$A(t) = \begin{pmatrix} -\lambda(t) & \lambda(t) & & \\ & -\lambda(t) & \lambda(t) & \\ & & -\lambda(t) & \ddots \\ & & & \ddots \end{pmatrix}.$$

Forward equations:

$$\frac{\partial}{\partial t} P(s, t) = P(s, t)\, A(t), \qquad t \ge s.$$

$\frac{\partial}{\partial t} P_{00}(s, t) = -\lambda(t)\, P_{00}(s, t)$ and $P_{00}(s, s) = 1$ imply that $P_{00}(s, t) = \exp(-\int_s^t \lambda(u)du) = e^{-m(s,t)}$.

For $j > 0$, we have $\frac{\partial}{\partial t} P_{0j}(s, t) = \lambda(t)\, P_{0,\,j-1}(s, t) - \lambda(t)\, P_{0j}(s, t)$ with initial condition $P_{0j}(s, s) = 0$.

Verify that the form of $P_{0j}(s, t)$ given in the question satisfies this equation:

$$\text{LHS} = (jm(s, t)^{j-1} - m(s, t)^j)\, \frac{e^{-m(s,t)}}{j!} \frac{\partial}{\partial t} m(s, t),$$

$$\text{RHS} = \lambda(t) \frac{m(s,t)^{j-1}\, e^{-m(s,t)}}{(j-1)!} - \lambda(t) \frac{m(s,t)^j\, e^{-m(s,t)}}{j!}$$

The observation that $\frac{\partial}{\partial t} m(s, t) = \lambda(t)$ is sufficient to finish the verification.

4 (a) Z is stationary, i.e. $I(0)$, as it is a first-order autoregression; X is not stationary but ∇X is just a linear combination of Z and e_1, so is stationary; this implies that X is $I(1)$. The same goes for Y.

(b) Z satisfies the Markov property on its own; X and Y do not, since they depend on values of Z.

(c) (X, Y, Z) *is* Markov; indeed, it is a vector autoregression.

(d) X and Y are not cointegrated. Although both are $I(1)$, any linear combination $W = aX + bY$ satisfies $W_n = W_{n-1} + \theta_W Z_{n-1} + e_{3,n}$ which does not define a stationary process.

5 (i) $\mathbf{E}[B_t^2|\mathrm{F}_s] = \mathbf{E}[(B_t - B_s + B_s)^2 \,|\, \mathrm{F}_s]$

$= \mathbf{E}[(B_t - B_s)^2 + 2(B_t - B_s)\, B_s + B_s^2 \,|\, \mathrm{F}_s]$

$= \mathbf{E}[(B_t - B_s)^2 \,|\, \mathrm{F}_s] + 2B_s\mathbf{E}[B_t - B_s \,|\, \mathrm{F}_s] + B_s^2,$

by the property of conditional expectations which allows one to "take out what is known". Moreover, by independence of the increments, the above is $\mathbf{E}[(B_t - B_s)^2] + B_s^2 = t - s + B_s^2$.

Similarly,

$\mathbf{E}[B_t^4 \,|\, \mathrm{F}_s] = \mathbf{E}[(B_t - B_s + B_s)^4 \,|\, \mathrm{F}_s]$

$= \mathbf{E}[(B_t - B_s)^4 + 4(B_t - B_s)^3 + 6(B_t - B_s)^2 B_s^2 + 4(B_t - B_s)^{B_s}\, B_s^3 + B_s^4 \,|\, \mathrm{F}_s]$

$= \mathbf{E}[(B_t - B_s)^4] + 6B_s^2\ \mathbf{E}[(B_t - B_s)^2] + B_s^4\,,$

where we used the independence of increments property as well as the fact that moments of odd order of $N(0, \sigma^2)$ vanish. Finally

$$\mathbf{E}[B_t^4 \,|\, \mathrm{F}_s] = B_s^4 + 6(t - s)\, B_s^2 + 3(t - s)^2.$$

(ii) From above

$\mathbf{E}[B_t^4 - 6tB_t^2 | \mathrm{F}_s] = B_s^4 + 6(t - s)\ B_s^2 + 3(t - s)^2 - 6t(t - s + B_s^2)$

$= B_s^4 + 6sB_s^2 + 3(t - s)^2 - 6t(t - s) = B_s^4 - 6sB_s^2 + 3(s^2 - t^2)$

$\therefore\ B_t^4 - 6tB_t^2 + 3t^2$ is a martingale.

6 (i) $u = F_1(x) = \dfrac{x}{1 + x}$ is solved by $x = F_1^{-1}(u) = \dfrac{u}{1 - u}$.

For the symmetrised version, the simplest thing is to multiply x by a variable y which takes ± 1 depending on whether another pseudo-random uniform number v is in the range (0, 0.5) or (0.5, 1).

(ii) By symmetry we only need consider $x > 0$, so we find $\max_{x>0} \dfrac{2\theta(1 + x)^2}{\pi(\theta^2 + x^2)}$.

Differentiating the logarithm of this fraction and setting equal to 0, we get $\dfrac{2}{1 + x} = \dfrac{2x}{\theta^2 + x^2}$, with solution $x = \theta^2$. Substituting this value in, we obtain the required value of C.

Let $g(x) = \dfrac{f(x\,|\,\theta)}{Cf_2(x)}$, which we observe is less than or equal to 1 everywhere.
The method of Acceptance-Rejection sampling goes as follows: use (i) to generate a variable y from density f_2. We accept y as a valid observation from $f(x\,|\,\theta)$ with probability $g(y)$, otherwise reject it. (Generate a uniform variable u, and reject if $u > g(y)$.) If we reject it, go back and generate another y from f_2, and continue to do the same until eventual acceptance.

7 (i) (a) $\gamma_0 = \text{Var}(e_t + \beta_1 e_{t-1}) = (1+\beta_1^2)\sigma_e^2$ and $\gamma_1 = \text{Cov}(e_t + \beta_1 e_{t-1}, e_{t-1} + \beta_1 e_{t-2}) = \beta_1\sigma_e^2$, with $\gamma_k = 0$ for $k > 1$.

This gives $\rho_0 = 1$, $\rho_1 = \beta_1 / (1 + \beta_1^2)$, $\rho_k = 0$ otherwise.

(b) Invertibility requires that $|\beta_1| < 1$, so that the sum $X_t - \beta_1 X_{t-1} + \beta_1^2 X_{t-2} + \ldots$ converges. μ and σ_e are irrelevant.

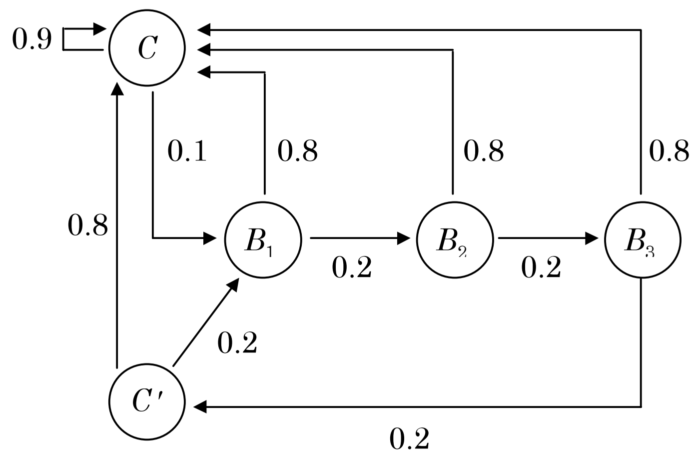
(ii) We need to solve $(1+\beta_1^2)\sigma_e^2 = 14.5$, $\beta_1\sigma_e^2 = 5.0$. Eliminating σ_e^2, we have $1+\beta_1^2 = 2.9\beta_1$, or $\beta_1 = ½(2.9 \pm \sqrt{2.9^2 - 4}) = 2.5$ or 0.4.
$\beta_1 = 2.5$ corresponds to $\sigma_e^2 = 2$, whereas $\beta_1 = 0.4$ corresponds to $\sigma_e^2 = 12.5$.

For invertibility, solve $1 + \beta_1 z = 0$. In the first case, $z = -0.4$ (no good); in the second, $z = -2.5$ (OK).

8 (i) (a) States:

C: healthy contributor
C': contributor but ill
B_1, B_2, B_3: beneficiary, with index giving duration of illness

(b) Transition graph:



(c) Transition matrix (states ordered C, C', B_1, B_2, B_3):

$$P = \begin{pmatrix} 0.9 & 0 & 0.1 & 0 & 0 \\ 0.8 & 0 & 0.2 & 0 & 0 \\ 0.8 & 0 & 0 & 0.2 & 0 \\ 0.8 & 0 & 0 & 0 & 0.2 \\ 0.8 & 0.2 & 0 & 0 & 0 \end{pmatrix}$$

(ii) The chain is irreducible by inspection: every state is accessible from every other state. State C is clearly aperiodic because of the one-step loop from C to C; because of irreducibility, every other state must be aperiodic too.

(iii) (a) $\pi = \pi P$ reads

$$\pi_c = 0.9\pi_c + 0.8(\pi_{c'} + \pi_1 + \pi_2 + \pi_3)$$

$$\pi_{c'} = 0.2\pi_3$$

$$\pi_1 = 0.1\pi_c + 0.2\pi_{c'}$$

$$\pi_2 = 0.2\pi_1$$

$$\pi_3 = 0.2\pi_2 \; .$$

Discard first equation and choose $\pi_{c'}$ as working variable:

$$\pi_3 = \frac{1}{0.2}\pi_{c'} = 5\pi_{c'}$$

$$\pi_2 = \frac{1}{0.2}\pi_3 = 5\pi_3 = 25\pi_{c'}$$

$$\therefore \; \pi = \pi_{c'}(1248, \; 1, \; 125, \; 25, \; 5)$$

$$\pi_1 = \frac{1}{0.2}\pi_2 = 5\pi_2 = 125\pi_{c'}$$

$$\pi_c = \frac{1}{0.1}\pi_1 - \frac{0.2}{0.1}\pi_{c'} = 10\pi_1 - 2\pi_{c'} = 1248\pi_{c'} \; .$$

Find $\pi_{c'}$ by normalisation: $\pi_{c'}(1248 + 1 + 125 + 25 + 5) = 1$,

$$\therefore \; \pi_{c'} = \frac{1}{1404} \, .$$

(b) Proportion of beneficiaries: $\dfrac{125 + 25 + 5}{1404} = 11.04\%$.

(iv) (a) Average profit per period per member in stationary régime is

$$Z = (f-c)\left(\frac{1248+1}{1404}\right) + (f-b)\left(\frac{125+25+5}{1404}\right)$$

$$= f - c\,\frac{1249}{1404} - b\frac{155}{1404}.$$

For $Z > 0$ you need $f > c\frac{1249}{1404} + b\,\frac{155}{1404}$.

(b) With the given data

$$Z = 300 - \frac{150\times 1249 + 600\times 155}{1404} = 100.32.$$

9 (i) (a) $\gamma_1 = \text{Cov}(X_t, X_{t-1}) = \text{Cov}(\alpha_1 X_{t-1} + \alpha_2 X_{t-2} + e_t, X_{t-1}) = \alpha_1\gamma_0 + \alpha_2\gamma_1 + 0$, since e_t is independent of X_{t-1}.

(b) Similarly $\gamma_2 = \alpha_1\gamma_1 + \alpha_2\gamma_0$ and $\gamma_0 = \alpha_1\gamma_1 + \alpha_2\gamma_2 + \text{Cov}(X_t, e_t)$. A further application of the same technique gives $\text{Cov}(X_t, e_t) = \sigma_e^2$.

Thus $\gamma_1 = \dfrac{\alpha_1}{1-\alpha_2}\gamma_0$ and $\gamma_2 = \left(\alpha_2 + \dfrac{\alpha_1^2}{1-\alpha_2}\right)\gamma_0$.

(c) ρ_k is found by the relation $\rho_k = \gamma_k / \gamma_0$.

(ii) We have $\hat{\alpha}_1 = r_1(1-\hat{\alpha}_2)$ and $\hat{\alpha}_2 + \dfrac{\hat{\alpha}_1^2}{1-\hat{\alpha}_2} = r_2$, which are solved by

$$\hat{\alpha}_1 = \frac{r_1(1-r_2)}{1-r_1^2}, \qquad \hat{\alpha}_2 = \frac{r_2 - r_1^2}{1-r_1^2}.$$

10 (i) (a) B_t defined by following properties:

- Independent increments: $B_t - B_s$ independent of B_a, $0 \le a \le s$ whenever $s \le t$.
- Stationary Gaussian increments: $B_t - B_s \sim N(0, t-s)$.
- Continuous sample paths: $t \to B_t$ continuous.

Transition density to go from x at time s to y at time t:

$$g_{t-s}(y-x) = \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-(y-x)^2/2(t-s)}.$$

(b) $\{W_s = x,\ W_t = y\} = \{\sigma B_s + \mu s = x,\ \sigma B_t + \mu t = y\} = \{B_s = \dfrac{x-\mu s}{\sigma}, B_t = \dfrac{y-\mu t}{\sigma}\}$. Hence transition density of W is

$$\frac{1}{\sigma}\, g_{t-s}\left(\frac{y-x-\mu(t-s)}{\sigma}\right).$$

(ii) By Itô's lemma

$$d(\log S_t) = \frac{1}{S_t}\, dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(dS_t)^2$$

$$= \mu dt + \sigma dB_t - \frac{\sigma^2}{2}\, dt.$$

Hence

$$\log S_t = \log S_0 + \left(\mu - \frac{\sigma^2}{2}\right)\, t + \sigma B_t\,,$$

and finally

$$S_t = S_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma B_t}.$$

(iii) $$\mathbf{P}[S_t > b \mid S_0 = a] = \mathbf{P}\left[\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right)t > \log\frac{b}{a}\right]$$

$$= \mathbf{P}\left[B_t > \frac{1}{\sigma}\left(\log\frac{b}{a} - \left(\mu - \frac{\sigma^2}{2}\right)t\right)\right]$$

$$= 1 - G\left(\frac{\log\frac{b}{a} - \left(\mu - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right).$$

Here $a = 38$, $b = 45$, $\mu = 0.25$, $\sigma = 0.2$, $t = \frac{1}{3}$ year.

So, above quantity is

$$1 - G(0.800) = 1 - 0.7881 = 0.2119.$$

(iv) $$\mathbf{P}\left[\max_{0\le s\le t} S_s \ge b \,\middle|\, S_0 = a\right] = \mathbf{P}\left[\max_{0\le s\le t}\left(B_s + \left(\mu - \frac{\sigma^2}{2}\right)\frac{s}{\sigma}\right) \ge \frac{1}{\sigma}\log\frac{b}{a}\right]$$

$$= G\left(\frac{\left(\mu - \frac{\sigma^2}{2}\right)t - \log\frac{b}{a}}{\sigma\sqrt{t}}\right) + \exp\left\{\frac{2\mu - \sigma^2}{\sigma^2}\log\frac{b}{a}\right\} G\left(-\frac{\left(\mu - \frac{\sigma^2}{2}\right)t + \log\frac{b}{a}}{\sigma\sqrt{t}}\right).$$

The first term is 0.2119 by (iii).

The second term is the product of $\left(\frac{b}{a}\right)^{\frac{2\mu-\sigma^2}{\sigma^2}} = 6.9893$ with $G(-2.128)$ $= 1 - G(2.128) = 1 - 0.9833$.

So the result is finally 0.2119 x 0.0167 = 0.3286.

11 (i) The generator matrix of the process would be

$$\begin{pmatrix} -1 & 0.4 & 0.1 & 0.5 & 0 \\ 0 & -\frac{1}{3} & \frac{1}{12} & \frac{1}{12} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{60} & 0 & \frac{1}{60} \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(ii) The probability of ever visiting state I is $\frac{1}{10} + \frac{4}{10} \times \frac{1}{4} = \frac{1}{5}$.

(iii) (a) $\frac{d}{dt}p_{AA}(t) = -p_{AA}(t)$, which has solution $p_{AA}(t) = e^{-t}$.

(b) Similarly, $\frac{d}{dt}p_{AF}(t) = -\frac{1}{3}\,p_{AF}(t) + 0.4p_{AA}(t)$, so that

$\frac{d}{dt}\{e^{t/3}\,p_{AF}(t)\} = 0.4e^{t/3}\,p_{AA}(t) = 0.4e^{-2t/3}$,

giving $p_{AF}(t) = e^{-t/3} \times 0.6(1 - e^{-2t/3})$.

(iv) (a) The equation arises as follows: when the process arrives in state i the subsequent holding time has mean λ_i^{-1}, after which the process jumps to a different state, choosing state j with probability $p_{ij} = \sigma_{ij} / \lambda_i$ (independent of the length of the holding time). The total time to reach state D is therefore the time until the first jump plus the time from arriving in the new state until hitting D (unless the new state is D).

(b) We have $m_I = 60$, $m_O = 2$, $m_F = 3 + \frac{1}{4} \times 60 + \frac{1}{4} \times 2 = 18.5$,

$m_A = 1 + 0.1 \times 60 + 0.5 \times 2 + 0.4 \times 18.5 = 15.4$ hours.

(v) The time-homogeneous Markov model has exponential holding times, so the distribution is completely determined by the expectation.

(vi) A simple check on whether the Markov model fits the data is therefore to verify that the distributions of holding times are at least roughly exponential, and a simple way of doing that is to compare sample standard deviations with sample means. More detailed comparisons might be possible, depending on the size of the data set.

(vii) (a) Calculations required in the first case would include working out the expected duration of stay if the change were implemented, which involves solving the equations in (iv) again. For the second situation, just replace m_O in the original calculation. New parameter values will need to be guessed. Whichever model comes out better should be compared with the initial situation, to determine whether the improvement was worth the additional resources.

(b) Model suitability: on the one hand the required decision is couched in terms of expectations, which lend themselves well to Markov process treatment. On the other, the fundamental problem in the system is queue length, which can never be successfully modelled by a process which tracks only a single individual at a time. (A network of queuing processes would be a much better model.)