EXAMINATIONS

21 September 2004 (pm)

Subject 103 — Stochastic Modelling

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available Actuarial Tables and your own electronic calculator.

1 A sequence $x_1, x_2, \ldots, x_n$ of observations is provided, each of which takes one of a finite set, S, of possible values. A researcher wishes to fit a discrete-time Markov chain model to these data.

(i) Write down estimates of the transition probabilities of the Markov chain. [1]

(ii) Outline the purpose of model validation. [1]

(iii) Describe the role of simulation in sensitivity analysis in the given context. [3]
[Total 5]

2 A person who catches a particular virus for the first time immediately falls ill with Disease A. The illness lasts for a random length of time, whose mean depends on the age of the person. Having recovered from Disease A, the person will not fall ill with it again. However, if the person catches the same virus again after the age of 18, the result will be Disease B, which will affect the person for an average of three months regardless of age. After suffering from Disease B the person will not catch the virus again.

It is proposed that a continuous-time Markov model can be used to model the person's medical history, with states:

0: not currently ill with Disease A or Disease B
1: currently ill with Disease A
2: currently ill with Disease B

(i) Explain why the proposed state space is inadequate and suggest an enlarged state space which can support a Markov model. [2]

(ii) Draw a diagram to illustrate the possible transitions. [2]

(iii) State, with reasons, whether a time-homogeneous or a time-inhomogeneous model is more suitable to model the medical history of a single person. [1]

(iv) Explain why a national medical service might find that a time-homogeneous model was just as good as a time-inhomogeneous one for planning the provision of treatment for Diseases A and B. [1]
[Total 6]

3 You are given a stream of standard identically and independently distributed uniform [0,1] random variables U_1, U_2, U_3, …..

Describe how to use this random variable stream to generate random variates with the following distributions:

(i) The discrete distribution with possible values A, B and C with respective probabilities 1/5, 1/4 and 11/20. [2]

(ii) The continuous distribution with probability density function

$$f(x) = \begin{cases} (10 - x)/18 & \text{for } 4 < x < 10 \\ 0 & \text{otherwise} \end{cases}$$ [3]

(iii) The continuous distribution function with probability density function

$$f(x) = \begin{cases} \dfrac{2}{\pi}\sin^2 x & \text{for } 0 < x < \pi \\ 0 & \text{otherwise} \end{cases}$$ [3]

[Hint: Use the Acceptance-Rejection method.]

[Total 8]

4 (i) Define a standard Brownian motion $\{B_t : t \geq 0\}$. [2]

(ii) Assume that S_t, the price of a share at time t, satisfies the stochastic differential equation

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t, \quad t \geq 0.$$

Solve the equation for S_t in terms of S_0 and B_t. [3]

(iii) For the values $\mu = 20\%$ and $\sigma = 15\%$ on an annual basis, calculate the probability that the price of the share will exceed 120 in three months' time if the current price of the share is 96. [4]

[Total 9]

5 A bag contains N balls, all green or red. At each stage a ball is taken out of the bag at random and is replaced by a ball of the other colour. Let X_n denote the number of green balls in the bag after n stages.

(i) Explain why X_n, $n \geq 0$ is a discrete time Markov chain with state space $S = \{0, 1, 2, \ldots.., N\}$ and with transition probabilities

$$p_{i,i+1} = \frac{N-i}{N} \text{ and } p_{i,i-1} = \frac{i}{N}$$ [2]

(ii) Show that the Markov chain X_n, $n \geq 0$ is irreducible and periodic and state its period. [3]

(iii) Show that for this process the stationary distribution $\pi = (\pi_0, \pi_1, \pi_2, \ldots\ldots.., \pi_N)$ is given by

$$\pi_i = \binom{N}{i} \frac{1}{2^N} \text{ for } i = 0, 1, 2, \ldots., N.$$ [4]

(iv) State, with a reason, whether it is the case that

$$\lim_{n\to\infty} P(X_n = j \mid X_0 = 0) = \pi_j.$$ [1]

[Total 10]

6 A model frequently used for interest rates is the Cox-Ingersoll-Ross process, which is an Itô process X_t satisfying the stochastic differential equation

$$dX_t = \alpha(b - X_t)dt + \sigma\sqrt{X_t}dB_t,$$

where B_t is a standard Brownian motion.

(i) Define $m_1(t) = \mathbf{E}(X_t \mid X_0 = x)$.

(a) Verify that m_1 satisfies the ordinary differential equation

$$\frac{dm_1}{dt} = \alpha(b - m_1)$$

(b) Solve the equation to determine $m_1(t)$ for all $t \geq 0$. [3]

(ii) Define $Y_t = X_t^2$ and $m_2(t) = E[X_t^2 \mid X_0 = x]$.

(a) Use Itô's Lemma to derive a stochastic differential equation satisfied by Y_t.

(b) Deduce an ordinary differential equation satisfied by $m_2(t)$.

(c) Assuming that $\dfrac{dm_2}{dt} \to 0$ as $t \to \infty$, show that

$$\lim_{t\to\infty} \text{Var}[X_t \mid X_0 = x] = \frac{b\sigma^2}{2\alpha}.$$

[7]
[Total 10]

7 A stationary autoregressive process X_t is defined by the recursive relationship

$$X_t = \mu + \alpha_1(X_{t-1} - \mu) + \alpha_2(X_{t-2} - \mu) + \cdots + \alpha_p(X_{t-p} - \mu) + e_t,$$

where $\{e_t : t \geq 1\}$ is a sequence of independent, zero-mean Normal variables, each with variance σ^2.

(i) Derive the Yule-Walker equations

$$\gamma_k = \alpha_1\gamma_{|k-1|} + \alpha_2\gamma_{|k-2|} + \cdots + \alpha_p\gamma_{|k-p|} + \sigma^2 1_{k=0}$$

for $0 \leq k \leq p$, where $\gamma_k = \text{Cov}(X_t, X_{t-k})$. [2]

(ii) Describe a diagnostic procedure based on a sequence of observations from a time series for testing whether the underlying time series can be modelled as a second order autoregressive process. [2]

(iii) Consider the second order autoregressive process

$$X_t = 0.6X_{t-1} + 0.3X_{t-2} + e_t$$

(a) Determine whether the process can be stationary.

(b) State, with a reason, whether the process possesses the Markov property.

(c) Assuming that $\sigma = 1$, calculate the values of $\gamma_0, \gamma_1, \gamma_2$. [7]
[Total 11]

8 (i) Write down the defining equation of an ARMA(1,1) process, identifying the parameters of the process. [2]

(ii) Explain what it means to say that a time series is stationary and state (but do not prove) a condition needed to ensure that an ARMA(1,1) process can be stationary. [2]

(iii) Outline the method of moments parameter estimation technique as it would be applied to estimate the parameters of an ARMA(1,1) process. [3]

(iv) Suppose an individual has fitted the following model to a dataset

$$x_t = 9.12 + 0.71x_{t-1} + e_t + 0.17e_{t-1}$$

The most recently observed value in the series is $x_{25} = 14.82$, with estimated residual $\hat{e}_{25} = -1.98$.

(a) Obtain estimates $\hat{x}_{25}(1)$ and $\hat{x}_{25}(2)$ for x_{26} and x_{27}.

(b) The simplest form of exponential smoothing used at time 24 gave a forecast for x_{25} of 12.97. Assuming the smoothing parameter is equal to 0.3, find the forecast for x_{26}.

[4]

(v) Discuss when the method of exponential smoothing might in practice be preferred to a method based on the Box-Jenkins technique. [2]

[Total 13]

9 Vehicles in a certain country are required to be assessed every year for road-worthiness. At one vehicle assessment centre, drivers wait for an average of 15 minutes before the road-worthiness assessment of their vehicle commences. The assessment takes on average 20 minutes to complete. Following the assessment, 80% of vehicles are passed as road-worthy allowing the driver to drive home. A further 15% of vehicles are categorised as a “minor fail”; these vehicles require on average 30 minutes of repair work before the driver is allowed to drive home. The remaining 5% of vehicles are categorised as a “significant fail”; these vehicles require on average three hours of repair work before the driver can go home.

A continuous-time Markov model is to be used to model the operation of the vehicle assessment centre, with states W (waiting for assessment), A (assessment taking place), M (minor repair taking place), S (significant repair taking place) and H (travelling home).

(i) Explain what assumption must be made about the distribution of the time spent in each state. [1]

(ii) Write down the generator matrix for this process. [2]

(a) Use Kolmogorov's Forward Equations to write down differential equations satisfied by $p_{WM}(t)$ and by $p_{WA}(t)$.

(b) Verify that $p_{WA}(t) = 4e^{-t/20} - 4e^{-t/15}$ for $t \geq 0$, where t is measured in minutes.

(c) Derive an expression for $p_{WM}(t)$ for $t \geq 0$.

[7]

(iii) Let T_i be the expected length of time (in minutes) until the vehicle can be driven home given that the assessment process is currently in state i.

(a) Explain why $T_W = 15 + T_A$.

(b) Derive corresponding equations for T_A, T_M and T_S.

(c) Calculate T_W.

[4]
[Total 14]

10 Assume that the spot rate of interest at time t, $S(t)$, can be modelled by $S(t) = e^{-2\mu W(t)}$, where $W(t)$ is a Wiener process with drift coefficient μ and diffusion coefficient 1 such that $W(0) = 0$.

(i) Write down an expression for $W(t)$ in terms of a standard Brownian motion $B(t)$. [1]

(ii) Show that $\{S(t) : t > 0\}$ is a continuous-time martingale. [4]

(iii) Let $T_a = \inf\{t : S(t) = a\}$ for some $0 < a < 1$.

(a) Prove that $P[S(t) < a] \to 1$ as $t \to \infty$.

(b) Deduce that $P[T_a < \infty] = 1$ and that $E[S(T_a)] = a$.

(c) Explain why the fact that $E[S(T_a)] \neq S(0)$ does not contradict the optional stopping theorem.

[5]

(iv) Now suppose instead that $a > 1$ and define T_a as before.

(a) Explain why $E[S(T_a)] = aP[T_a < \infty]$.

(b) Apply the optional stopping theorem to find $P\left[T_a < \infty\right]$.

[4]
[Total 14]

END OF PAPER