EXAMINATIONS

14 September 2001 (pm)

Subject 104 — Survival Models

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Write your surname in full, the initials of your other names and your Candidate's Number on the front of the answer booklet.*

2. *Mark allocations are shown in brackets.*

3. *Attempt all 9 questions, beginning your answer to each question on a separate sheet.*

Graph paper is required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet and this question paper.

In addition to this paper you should have available Actuarial Tables and an electronic calculator.

1 Describe how smoothness is checked when mortality rates are graduated using a graphical method. [2]

2 A total of N independent lives are observed during a finite period of observation. Between the ages of x and $x + 1$, $x + a_i$ is the age at which observation of the ith life starts, and $x + t_i$ is the age at which observation of the ith life ceases. For each life, an indicator variable $\mathbf{D}_i$ indicates whether life i is observed to die.

$$\mathbf{D}_i = \begin{cases} 1 & \text{if life } i \text{ dies at } x + t_i \qquad i = 1, 2, \ldots N \\ 0 & \text{otherwise} \end{cases}$$

(i) If the force of mortality at age $x + t$, $\mu_{x+t} = \mu$ $(0 \le t < 1)$, derive the maximum likelihood estimate of μ. [6]

(ii) State the asymptotic sampling distribution of the maximum likelihood estimator, $\hat{\mu}$. [2]

[Total 8]

3 (i) If T_x and K_x are random variables measuring the complete and curtate length of life for a life aged x, define $\bar{a}_{x:\overline{n}|}$ and $\ddot{a}_{x:\overline{n}|}$ as expected values of functions of T_x and K_x respectively. [2]

(ii) Using the definition in (i) derive the following result

$$\bar{a}_{x:\overline{n}|} = \int_0^n v^t \,{}_t p_x \, dt ,$$

and state the corresponding result for $\ddot{a}_{x:\overline{n}|}$. [4]

(iii) Hence or otherwise show that $\bar{a}_{x:\overline{n}|}$ can be approximated using the formula

$$\bar{a}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - ½\left(1 - v^n \,{}_n p_x\right). \qquad [3]$$

(iv) Hence or otherwise find a simple approximation for $\bar{a}_x$. [1]

[Total 10]

4 A cash-flow is payable continuously at a rate of $\rho(t)$ per annum at time t provided a life who is aged x at time 0 is still alive. T_x is a random variable which measures the complete future lifetime in years of a life aged x.

(i) Write down an expression, in terms of T_x, for the present value at time 0 of this cash-flow, at a constant force of interest δ p.a. [1]

(ii) Hence or otherwise show that the expected present value at time 0 of this cash-flow is equal to:

$$\int_0^\infty e^{-\delta s}\,\rho(s)\;{}_s p_x\,ds. \qquad [3]$$

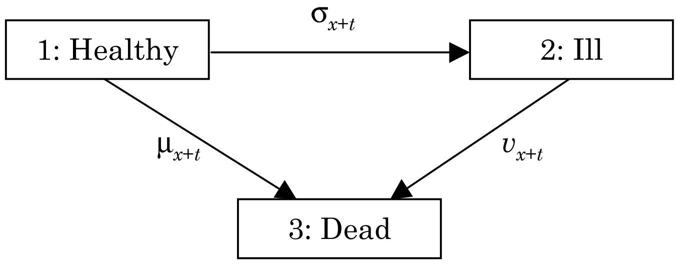
(iii) An annuity is payable continuously during the lifetime of a life now aged 30, but for at most 10 years. The rate of payment at all times t during the first 5 years is £5,000 p.a., and thereafter it is £10,000 p.a. The force of mortality to which this life is subject is assumed to be 0.01 p.a. at all ages between 30 and 35, and 0.02 p.a. between 35 and 40.

Find the expected present value of this annuity at a force of interest of 0.05 p.a. [5]

(iv) If the mortality and interest assumptions are as in (iii), find the expected present value of the benefits of a term assurance, issued to the life in (iii), which pays £40,000 immediately on death within 10 years. [3]

[Total 12]

5 The following diagram represents a 3-state continuous-time Markov Model, in which states 1 and 2 represent the two stages of a terminal disease and transitions from state 2 to state 1 are not possible.



The symbols σ_{x+t}, μ_{x+t} and υ_{x+t} represent the forces of transition. The probability ${}_t p_x^{ab}$ is defined as

$$ {}_t p_x^{ab} = P(S(t) = b \mid S(0) = a), $$

where $S(t)$ is the state of the life at time t, and x is the age at time 0.

(i) For a = 1, 2 and 3, write down expressions for ${}_t p_x^{aa}$ in terms of the forces of transition. [2]

(ii) Determine an expression for ${}_t p_x^{23}$ in terms of the forces of transition. [1]

(iii) Derive, from first principles, the Kolmogorov forward differential equation for ${}_t p_x^{12}$, and state the relevant initial condition, i.e. the value of ${}_0 p_x^{12}$. [4]

(iv) If the forces of transition are assumed constant, show that:

$$ {}_t p_x^{12} = \frac{\sigma}{\upsilon - \sigma - \mu}\left(e^{-(\sigma+\mu)t} - e^{-\upsilon t}\right) \qquad [5] $$

[Total 12]

6 You have been asked by a government department to help them calculate childhood mortality rates. They have provided you with the following data collected from a recent investigation.

Age category *(t)*	*Age range*	*Exposure* *(in years)* E_t^c	*Deaths* (d_t)
1	age < 1 month	1,536.2	396
2	1 month ≤ age < 3 months	3,041.9	139
3	3 months ≤ age < 6 months	4,498.1	144
4	6 months ≤ age < 12 months	8,792.8	219
5	1 year ≤ age < 2 years	16,999.2	176
6	2 years ≤ age < 3 years	16,440.2	79
	Total	51,308.4	1,153

Exposure is defined as "the number of lives who survived through the interval multiplied by the length of the interval in years, plus the fraction of the year lived in the interval by those lives that died in the interval". Exposure is measured in years.

Estimate each of the following probabilities:

(i) the probability of dying in the first month of life [3]

(ii) (a) q_0

(b) ${}_3q_0$ [10]

In each case state any assumptions you have made in calculating your estimates.
[Total 13]

7 On 1 January 1995 an office issued a number of level annual premium policies to independent lives, each of whom was aged exactly 40. The sum assured for each policy was £10,000. There were two types:

- whole-life assurances, with the sum assured being payable at the end of the year of death
- pure endowments with a term of 20 years, under which the only benefit was the payment of the sum assured on survival to the maturity date

On 1 January 2000 there were 600 of the whole-life assurances and 400 of the pure endowments in force. During the calendar year 2000, 2 whole-life policyholders and 1 pure endowment policyholder died. There were no exits other than by death.

The premium and policy value bases used were A1967–70 ultimate mortality, 4% interest p.a. and expenses can be ignored.

(i) Show that the annual premiums payable were approximately £144.65 for a whole-life policy and £296.96 for a pure endowment policy. [3]

(ii) Find the total policy value in respect of each class of policy, at the beginning and at the end of the calendar year 2000. [5]

(iii) Find the mortality profit or loss for the calendar year 2000. [3]

(iv) If on 1 January 2000 the office held funds equal to the total of the policy values of all in force policies, what rate of interest would the office have had to earn on its funds during the calendar year 2000 for there to be neither a profit nor a loss on this business in that year? [2]

[Total 13]

8 (i) Explain why graduated rates, rather than crude estimates of mortality rates are used in the construction of standard mortality tables. [3]

(ii) A graduation of the mortality experience of the male population of a region of the United Kingdom has been carried out using a graphical method. The following is an extract from the results.

Age x	*Actual number of deaths,* θ_x	*Graduated mortality rate,* $\mathring{q}_x$	*Initial exposed to risk,* E_x	$E_x \mathring{q}_x$
14	3	0.00038	12,800	4.86
15	8	0.00043	15,300	6.58
16	5	0.00048	12,500	6.00
17	14	0.00053	15,000	7.95
18	17	0.00059	16,500	9.74
19	9	0.00066	10,100	6.67
20	15	0.00074	12,800	9.47
21	10	0.00083	13,700	11.37
22	10	0.00093	11,900	11.07
Total	91			73.71

Use the Chi-squared test to test the adherence of the graduated rates to the data. State clearly the null hypothesis you are testing and comment on the result. [4]

(iii) Perform two other tests which detect different aspects of the adherence of the graduation to the data. For each test state clearly the features of the graduation which the test is able to detect, and comment on your results. [8]

[Total 15]

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9 A life insurance company has carried out a mortality investigation. It followed a sample of independent policyholders aged between 40 and 45 years. Policyholders were followed from their 40th birthday until either they died, or they withdrew from the investigation while still alive, or they celebrated their 45th birthday (whichever of these events occurred first).

(i) Describe the types of censoring that are present in this investigation. [2]

(ii) An extract from the data for 20 policyholders is shown in the table below. Use these data to calculate the Kaplan-Meier estimate of the survival function. Determine an approximate 95% confidence interval for your estimate.

Person number	*Last age at which person was observed (years and months)*	*Outcome*
1	40 6	Died
2	40 6	Withdrew
3	41 0	Died
4	41 0	Died
5	41 6	Withdrew
6	42 3	Died
7	42 3	Withdrew
8	42 3	Died
9	42 6	Withdrew
10	43 0	Withdrew
11	43 3	Died
12	43 3	Withdrew
13	44 3	Withdrew
14	44 6	Withdrew
15	44 9	Died
16	45 0	Survived
17	45 0	Survived
18	45 0	Survived
19	45 0	Survived
20	45 0	Survived

[10]

(iii) Plot clearly on a suitably labelled graph the Kaplan-Meier estimate of the survival function and its associated 95% confidence band. [3]

[Total 15]