Faculty of Actuaries Institute of Actuaries

EXAMINATIONS

12 April 2000 (pm)

Subject 104 — Survival Models

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Write your surname in full, the initials of your other names and your Candidate's Number on the front of the answer booklet.*
2. *Mark allocations are shown in brackets.*
3. *Attempt all 12 questions, beginning your answer to each question on a separate sheet.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet and this question paper.

In addition to this paper you should have available Actuarial Tables and an electronic calculator.

1 Describe how smoothness is ensured when mortality rates are graduated using each of the following methods:

(a) fitting a parametric formula to the crude estimates; and
(b) graduation by reference to a standard mortality table. [2]

2 Explain why crude estimates of mortality rates should be smoothed before they can be used in financial calculations. [3]

3 Determine an approximate value of $\bar{A}_{50:\overline{15}|}$ using the A1967–70 Mortality Table and 6% per annum interest. [3]

4 (i) The random variable Z represents the present value of the benefits of a deferred life annuity-due with an annual payment of 1 issued to a life aged x. The annuity has a deferred period of n years.

Sketch and label a diagram to show the probability function of Z. [2]

(ii) Calculate the net level annual premium payable for a deferred annuity-due of £10,000 per annum, issued to a man currently aged 38 exact, with a deferred period of 25 years. The premium is payable throughout the deferred period only.

Basis: A1967–70 Ultimate Mortality Table. 4% per annum interest. [3]
[Total 5]

5 Thiele's equation for the policy value at duration t, ${}_t\bar{V}$, of an immediate life annuity payable continuously at a rate of £1 per annum from age x is:

$$\frac{\partial}{\partial t}\,{}_t\bar{V} = \mu_{x+t}\;.\;{}_t\bar{V} - 1 + \delta\;.\;{}_t\bar{V}$$

(i) Derive this result algebraically showing all the steps in your argument. [5]

(ii) Explain this result by general reasoning. [3]
[Total 8]

6 In a mortality investigation the actual number of deaths at age x last birthday is d_x. The goodness of fit between the data and the force of mortality, $\mu_{x+½}$, over the age range $x_1, x_1+1, x_1+2 \ldots, x_1+m-1$ can be tested using the statistic

$$\sum_{x=x_1}^{x=x_1+m-1} \frac{\left(d_x - E_x^c\,\mu_{x+½}\right)^2}{E_x^c\,\mu_{x+½}}$$

where E_x^c is the central exposed to risk which corresponds to d_x.

For each of the following cases state the null hypothesis being tested, and give the sampling distribution of this statistic as precisely as you can from the given information:

(a) if $\mu_{x+½}$ are taken from a standard mortality table;

(b) if $\mu_{x+½}$ are the graduated rates obtained from the crude estimates using the Gompertz-Makeham formula

$$g(x) = a_0 + \exp\{b_0 + b_1x + b_2x^2\};$$

(c) if $\mu_{x+½}$ are the graduated rates obtained from the crude estimates using the formula

$$g(x) = \mu_x^s + k$$

where μ_x^s are taken from a standard mortality table and k is a constant; and

(d) if $\mu_{x+½}$ are the graduated rates obtained from a graphical graduation of the crude estimates d_x / E_x^c. [8]

7 Let X be a random variable representing the present value of the benefits of a whole of life assurance, and Y be a random variable representing the present value of the benefits of a temporary assurance with a term of n years. Both assurances have a sum assured of 1 payable at the end of the year of death and were issued to the same life aged x.

(i) Describe the benefits provided by the contract which has a present value represented by the random variable $X - Y$. [1]

(ii) Show that

$$\text{Cov}(X, Y) = {}^2A^{1}_{x:\overline{n}|} - A_x\ A^{1}_{x:\overline{n}|}$$

and hence or otherwise that

$$\text{Var}(X - Y) = {}^2A_x - ({}_{n|}A_x)^2 - {}^2A^{1}_{x:\overline{n}|}$$

where the functions A are determined using an interest rate of i, and functions ${}^2A$ are determined using an interest rate of $i^2 + 2i$. [7]

[Total 8]

8 X is a random variable which measures the duration from the date of a kidney transplant until death.

(i) Express the hazard rate and the integrated hazard function at duration x, in terms of probabilities. [3]

(ii) If the hazard rate at duration x is

$$h(x) = \alpha\lambda\, x^{\alpha-1}$$

derive an expression for the integrated hazard, $H(x)$. [2]

(iii) The hazard rate $h(x)$, as defined in part (i), varies between transplant patients in such a way that

$$\alpha = \alpha_0 + \alpha_1 z_1$$

$$\lambda = \lambda_1 z_1 + \lambda_2 z_2$$

where α_0, α_1, λ_1, λ_2 are constants; z_1 is the age of the patient at the date of the transplant and z_2 is the patient's sex where $z_2 = 0$ = female, $z_2 = 1$ = male.

Show that these hazards are not in general proportional, but that if $\alpha_1 = 0$ the hazards are proportional. [4]

[Total 9]

9 A special endowment assurance issued to a man aged 40 exact has a term of 25 years. A sum assured of £10,000 is payable immediately on death, and a sum assured of £15,000 is payable on survival to the end of the term.

The policy is secured by annual premiums of P during the first five years, $2P$ during the second five years and $3P$ during the remainder of the term. Premiums are payable annually in advance for 25 years or until death, if earlier.

(i) Show that the initial net annual premium for this policy is £164.23. [6]

(ii) Determine the prospective net premium policy value at the end of the twelfth policy year, immediately before the payment of the thirteenth premium. [4]

Basis: A1967–70 Ultimate Mortality Table. 4% per annum interest.

[Total 10]

10 The population of elderly people in a prison is observed during the period 1 January 1994 to 31 December 1996. The duration of residence (measured to the nearest number of months) is recorded for those who die during the period, for those who are released from the prison during the period and for those who are still in residence on 31 December 1996.

The recorded data measured in months are

$6^{\dagger}$	6	6	6	7	$9^{\dagger}$
$10^{\dagger}$	10	$11^{\dagger}$	13	16	$17^{\dagger}$
20	$23^{\dagger}$				

where † indicates those who were released from the prison during the period or who were still in residence on 31 December 1996.

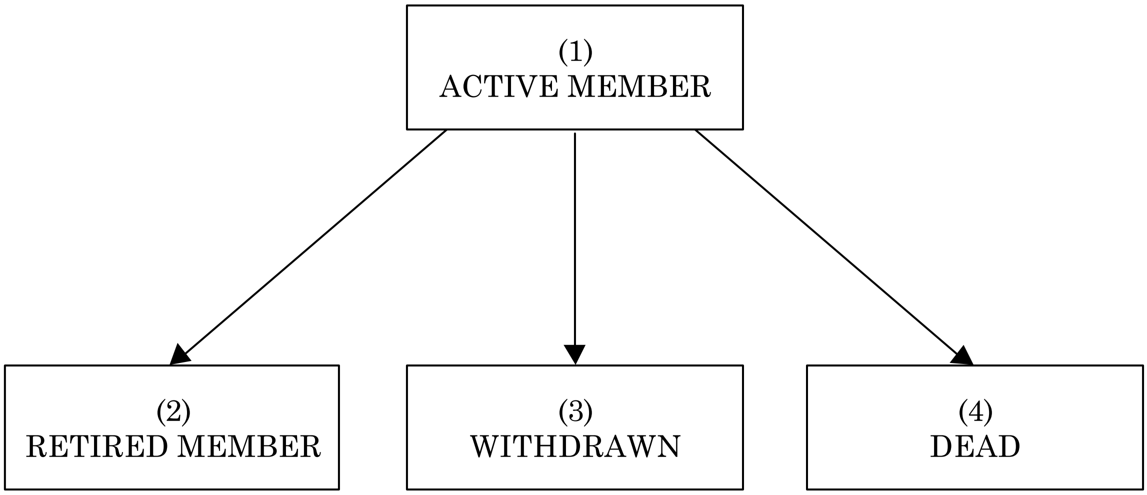
(i) State the type(s) of censoring present in these data. [2]

(ii) Calculate the product-limit (Kaplan-Meier) estimate of the survival function, $S(t)$, where t is the duration of residence in the prison. [7]

(iii) State the assumptions underlying the estimate in (ii), and explain how each of these assumptions would apply to these data. [4]

[Total 13]

11 A pension scheme is to be modelled using the following four-state model:



The following functions are defined for integer x:

μ_{x+t}^{1j} = force of transition from state 1 to state j at exact age $x + t$, ($j = 2, 3, 4$), for $0 \le t < 1$.

${}_tp_x^{1j}$ = probability that a life in state 1 at exact age x is in state j at exact age $x + t$, ($j = 1, 2, 3, 4$), for $t > 0$.

(i) Derive from first principles a differential equation for ${}_tp_x^{11}$, and show that the solution to this equation is:

$${}_tp_x^{11} = \exp\left(-\int_0^t \sum_{j=2}^{4} \mu_{x+s}^{1j} \, . \, ds\right)$$

stating all assumptions made. [6]

(ii) State, or otherwise obtain, the differential equation for ${}_tp_x^{1j}$ ($j \neq 1$), and show that the solution to this equation is:

$${}_tp_x^{1j} = \int_0^t {}_rp_x^{11} \, . \, \mu_{x+r}^{1j} dr; \quad \text{for } j \neq 1.$$ [2]

(iii) Assuming that the forces of transition μ_{x+t}^{1j} have a constant value $\mu_{x+½}^{1j}$ for $0 \le t < 1$, show that:

$${}_1p_x^{1j} = \frac{\mu_{x+½}^{1j}}{\sum_{k=2}^{4} \mu_{x+½}^{1k}} \left(1 - \exp\left[-\sum_{k=2}^{4} \mu_{x+½}^{1k}\right]\right)$$ [3]

(iv) The values of $\mu^{1j}_{x+½}$ are:

x	$\mu^{12}_{x+½}$	$\mu^{13}_{x+½}$	$\mu^{14}_{x+½}$
60	.25	.08	.012
61	.10	.04	.014

Calculate ${}_2p^{12}_{60}$. [3]

[Total 14]

12 A life insurance company issues only annual premium life assurance policies. The company keeps records of its life assurance policies in two files: an in-force file and a claims file.

For each policy paying premiums the in-force file includes the following information:

- age last birthday at the date of policy issue
- smoking status (smoker or non-smoker)
- sex
- type of policy (temporary assurance, whole life assurance or endowment assurance)

On 1 January each year the company tabulates summary statistics from this file. For each age x where:

$x =$ age last birthday at date of policy issue
+ number of annual premiums paid since policy issue

a count of the number of policies sub-divided by smoking status, sex and type of policy is tabulated.

For each policy for which a death claim has been paid the claims file includes the same information as the in-force file.

On 1 January each year the company tabulates summary statistics from this file for death claims paid in the previous calendar year. For each age x where:

$x =$ age last birthday at date of policy issue
+ number of annual premiums paid up to the date of death

a count of the number of policies divided by smoking status, sex and type of policy is tabulated.

The company wishes to investigate the recent mortality experience of its life assurance policies.

(i) (a) Explain why the subdivisions of data by smoking status, sex and type of policy are important for the company's mortality analysis.

(b) Describe the statistical problems that can arise when data are sub-divided in this way. [4]

(ii) Defining suitable symbols, derive a formula for the estimation of the force of mortality using the data for an age group x according to the definitions given. State any assumptions that are required for your formula. State, with reasons, the age to which your estimate of the force of mortality applies. [7]

(iii) The company proposes to graduate the crude estimates obtained from its investigations before using them for determining premiums. Describe an appropriate method of graduation indicating how the suitability of the graduated rates would be assessed. Detailed descriptions of graduation tests are **not** required. [6]

[Total 17]