REPORT OF THE BOARD OF EXAMINERS

April 2003

Subject 104 — Survival Models

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

J Curtis
Chairman of the Board of Examiners

3 June 2003

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EXAMINERS' COMMENTS

As in previous years, the Examiners aimed to set questions covering all the aspects of Survival Modelling: Life contingencies including its stochastic treatment, Graduation including its statistical aspects and the determination of exposures. The Examiners aim to strike a balance between questions requiring numerical solutions and those requiring verbal and algebraic answers, as well as between those with and without a statistical theme.

Comments on solutions presented to individual questions for this April 2003 paper are given below:

Question 1: *This question was intended to be a relatively straightforward first question, but many candidates either attempted it towards the end of their script, or did not attempt it at all. It was generally not well answered, with many candidates making insufficient points.*

Question 2: *This was generally well answered, although some candidates struggled to express clearly the answer to part (i).*

Question 3: *This was generally well answered.*

Question 4: *Many of the solutions presented lacked sufficient detail.*

Question 5: *Many candidates gave correct answers to (ii). However, in (i) the majority failed to give an expression for the particular model in the question, giving instead a general expression for the hazard function. Very few candidates scored well in part (iii).*

Question 6: *Was generally poorly answered. Most candidates made reasonable attempts at part (ii), but many did not attempt part (iii).*

Question 7: *This exposed to risk question was, as in previous exams, generally poorly answered. Few candidates gave a clear derivation of their answer, which was required for full credit in part (i), although many went on to score well in part (ii).*

Question 8: *This was reasonably well answered on the whole. Most candidates stated the 3 defects, although many failed to give reasons why these were not detected. Marks were generally lost in part (ii) by giving insufficient detail.*

Question 9: *Many candidates correctly calculated the variances, although some number seemed unaware of availability of the formula and the* ${}^{2}A_x$ *functions tabulated in the Gold book. Few candidates gave sufficient explanation in part (ii).*

Question 10: *This was generally well answered. Some candidates lost marks in part (ii) for not showing all the steps required.*

Question 11: *In part (i), many candidates lost marks for not indicating the different mortality rates pre and post age 65. Some credit was given to the many candidates who failed to show the correct result, but who had shown some* *correct, clear working. In part (iii), many candidates incorrectly used the present value of the premiums and/or the annuity payments.*

Question 12: *This was generally well answered. Many candidates stated the types of censoring in part (i), although this was not required. In part (iii), some candidates recalculated S(t) from scratch rather than using the results from part (ii) as required.*

Question 13: *Many candidates gave insufficient detail in their answer to part (ii)(b). A large number of candidates used an incorrect approximation for* $\bar{A}_{x:\overline{n}|}$*, applying the acceleration term* $(1+i)^{1/2}$ *to the whole of the endowment instead of just the term assurance part.*

1 If data on exact times of death are available, the two-state model uses all the information available (ie the times of death), whereas the Binomial model does not (it uses the fact that death has occurred).

The Binomial model requires estimation of q and some assumption about the distribution of deaths with age in order to calculate μ; the two-state model does not.

The two-state model is extended very simply to processes with more than one decrement (i.e. to a multiple-state model), and to processes with increments and decrements. The Binomial model is not.

2 (i) The assumption of a uniform distribution of deaths between integer ages x and $x + 1$ means that, for $0 \leqslant t \leqslant 1$, the function ${}_tp_x \mu_{x+t}$ is a constant.

The assumption implies that, for $0 \leqslant t \leqslant 1$, ${}_tq_x = tq_x$.

Alternatively,

The assumption of a uniform distribution of deaths (UDD) between integer ages x and $x + 1$ means that the exact ages at death of persons dying within this age range are evenly spaced along the age axis.

This, for example, if there are j deaths between exact ages x and $x + 1$, UDD would be achieved if the exact ages at death are $x + 1/(j + 1)$, $x + 2/(j + 1)$, …, $x + j/(j + 1)$.

(ii) ${}_{0.25}p_{75} = 1 - {}_{0.25}q_{75} = 1 - 0.25 \times q_{75}$ under the assumption of a uniform distribution of deaths (UDD) between ages 75 and 76.

From ELT 15, $q_{75} = 0.06197$, so

$$ {}_{0.25}p_{75} = 1 - 0.25 \times 0.06197 = 0.98451 $$

Either

Under UDD we have, for $0 \leqslant s < t \leqslant 1$, ${}_{t-s}q_{x+s} = \dfrac{(t-s)q_x}{1 - sq_x}$.

Putting $t = 0.75$, $s = 0.5$ and $x = 75$, therefore,

$$ {}_{0.75-0.5}q_{75+0.5} = \frac{0.25q_{75}}{1 - 0.5q_{75}}, \text{ and so} $$

$$_{0.25}p_{75.5} = 1 - \frac{0.25q_{75}}{1 - 0.5q_{75}}.$$

Using ELT15, this is evaluated as

$$1 - \frac{0.25(0.06197)}{1 - 0.5(0.06197)} = 1 - \frac{0.0154925}{0.969015} = 1 - 0.0159879 = 0.98401$$

Or

Using $_{t}p_{x} \quad = {}_{s}p_{x} \cdot {}_{t-s}p_{x+s},$

$$_{0.75}p_{75} \quad = {}_{0.5}p_{75} \cdot {}_{0.25}p_{75.5}$$

Using an assumption of UDD between 75 and 76, we have

$$_{0.5}p_{75} \quad = 1 - 0.5 \times 0.06197 = 0.969015$$

$$_{0.75}p_{75} \quad = 1 - 0.75 \times 0.06197 = 0.9535225$$

So, $\quad {}_{0.25}p_{75.5} \quad = \dfrac{_{0.75}p_{75}}{_{0.5}p_{75}} = \dfrac{0.9535225}{0.969015} = 0.98401$

3 (i) A population under investigation in a mortality study is homogeneous if all the lives have similar mortality characteristics.

The problem with lack of homogeneity is that the investigation will reveal only the average experience of the group.

This means that the insurer will mis-price the policy — for some individuals it will be too expensive, for others too cheap.

This will be a problem if other insurers do take account of the factor causing the lack of homogeneity and price correctly. The insurer will be too expensive for the good risks (and so lose their business) and too cheap for the bad risks (and so write unprofitable business).

(ii) Factors usually considered:

- sex
- age
- type of policy
- smoker/non-smoker status
- level of underwriting
- duration <u>in force</u>
- other factors such as sales channel, policy size, occupation and known impairments are acceptable

Valid alternative suggestions were credited, but note the word 'often' in the question. Factors which are very rarely used did not receive credit.

4 (i) It is generally assumed that the true mortality of the population progresses smoothly with age.

The crude rates at each age will be estimated independently of the data at other ages. The data will not be spread evenly across ages and at some ages may be sparse. For these reasons the rates will not necessarily progress smoothly.

Graduation allows us to use the information from adjacent ages to smooth the rates.

Insurance companies prefer to use smooth rates for its premium calculations — any irregularities in the premium rates will be hard to justify to customers or would leave the company open to the risk of anti-selection or lapse/re-entry.

(ii) (a) The first step is to select a suitable standard table, based on a similar group of lives.

We then plot the crude rates q_x against q_x^s from the standard table to identify any simple relationship (logs may be plotted rather than q_x).

Having selected a suitable relationship, we find the best-fit parameters, for example by maximum likelihood or least squares estimates.

We then test the graduation for goodness of fit using statistical tests; if the fit is not adequate the process is repeated.

(b) The rates will be used for annuity premiums. It is important for the company not to overestimate mortality (as the premiums charged would then be inadequate).

In addition, as the company is using the rates as an estimate of future mortality, it should take into account any trends — particularly any likely reduction in future mortality rates.

The company should also consider premium rates charged by competitors — if rates are out of line either too little business or too much (unprofitable) business may be attracted.

5 (i) $$\lambda(t;\underline{z}_i) = \lambda_0(t)\exp(\underline{\beta}z_i^T)$$

where $\lambda(t;\underline{z}_i)$ is the hazard at duration t,

$$\underline{\beta} = (0.4, 0.3, -0.1, 0.2, -0.3, 0.4)$$

and

$$\underline{z}_i = (z_1, z_2, z_3, z_4, z_5, z_6)$$

where

$$z_1 = \begin{cases} 1 & : \text{ small employer} \\ 0 & : \text{ otherwise} \end{cases}$$

$$z_2 = \begin{cases} 1 & : \text{ no degree} \\ 0 & : \text{ otherwise} \end{cases}$$

$$z_3 = \begin{cases} 1 & : \text{ Science degree} \\ 0 & : \text{ otherwise} \end{cases}$$

$$z_4 = \begin{cases} 1 & : \text{ Arts degree} \\ 0 & : \text{ otherwise} \end{cases}$$

$$z_5 = \begin{cases} 1 & : \text{ other UK location} \\ 0 & : \text{ otherwise} \end{cases}$$

$$z_6 = \begin{cases} 1 & : \text{ overseas location} \\ 0 & : \text{ otherwise} \end{cases}$$

$\lambda_0(t)$ = baseline hazard at duration t

Alternatively

$$\lambda(t;\underline{z}_i) = \lambda_0(t)\exp(0.4z_1 + 0.3z_2 - 0.1z_3 + 0.2z_4 - 0.3z_5 + 0.4z_6)$$

where $\lambda(t;\underline{z}_i)$ is the hazard at duration t,

$\lambda_0(t)$ = baseline hazard at duration t

and

where

$$z_1 = \begin{cases} 1 & : \text{ small employer} \\ 0 & : \text{ otherwise} \end{cases}$$

$$z_2 = \begin{cases} 1 & : \text{ no degree} \\ 0 & : \text{ otherwise} \end{cases}$$

$$z_3 = \begin{cases} 1 & : \text{ Science degree} \\ 0 & : \text{ otherwise} \end{cases}$$

$$z_4 = \begin{cases} 1 & : \text{ Arts degree} \\ 0 & : \text{ otherwise} \end{cases}$$

$$z_5 = \begin{cases} 1 & : \text{ other UK location} \\ 0 & : \text{ otherwise} \end{cases}$$

$$z_6 = \begin{cases} 1 & : \text{ overseas location} \\ 0 & : \text{ otherwise} \end{cases}$$

Alternative solutions for β and z were acceptable, but the important point was that the vector β is a constant. One possible alternative given was:

β = (0.4, 0.1, 0.1)
z1 = 0 (large); 1(small)
z2 = 3 (none); -1 (science); 2 (arts); 0 (other)
z3 = -3 (other UK); 0 (London); 4 (overseas)

(ii) Most likely corresponds to lowest hazard, so large employer, Science degree, other UK location.

Least likely is highest hazard, so small employer, no degree, overseas location.

(iii) We have

$$P_L = e^{-\int_1^3 \lambda_L(t;\underline{z}_i)dt}$$

where $\lambda_L(t;\underline{z}_i)$ is the hazard rate for a large employer

We know that $\dfrac{\lambda_L(t;\underline{z}_i)}{\lambda_s(t;\underline{z}_i)} = \dfrac{e^0}{e^{0.4}} = 0.6703$

so, $P_L = e^{-\int_1^3 \lambda_L(t;z_i)dt}$

$$= e^{-\int_1^3 0.6703\lambda_s(t;\underline{z}_i)dt}$$

$$= \left(e^{-\int_1^3 \lambda_s(t;\underline{z}_i)dt}\right)^{0.6703}$$

$$= (P_S)^{0.6703}.$$

6 (i) Gompertz' Law is a suitable model for human mortality for middle to older ages, say 35 and over.

There is evidence that the Gompertz' Law breaks down at very advanced ages and therefore 35 to 90 years is acceptable.

(ii) Since ${}_t p_x = \exp\left[-\int_0^t \mu_{x+s} ds \right]$,

putting $\mu_x = Bc^x$ produces

$${}_t p_x = \exp\left[-\int_0^t Bc^{x+s} ds \right].$$

Evaluating the integral, we obtain

$${}_t p_x = \exp\left(-\left[\frac{Bc^x c^s}{\log c} \right]_0^t \right),$$

$$= \exp\left(-\left[\frac{Bc^x c^t - Bc^x}{\log c} \right] \right)$$

$$= \exp\left(\frac{-B}{\log c} \right)^{c^x(c^t - 1)}.$$

(iii) Suppose that a Poisson model is used in the investigation.

Then the likelihood for the age interval x to $x + 1$ is

$$K\left(\mu_{x+\frac{1}{2}} \right)^{d_x} \exp\left(-\mu_{x+\frac{1}{2}} E_x^c \right),$$

where K is a constant, $\mu_{x+\frac{1}{2}}$ is the force of mortality between exact ages x and $x + 1$, d_x is the number of deaths between exact ages x and $x + 1$ and E_x^c is the central exposed to risk at age x.

Gompertz' Law implies that $\mu_{x+\frac{1}{2}} = Bc^{x+\frac{1}{2}}$.

Substituting this expression into the likelihood produces

$$K\left(Bc^{x+\frac{1}{2}}\right)^{d_x}\ \exp\left(-Bc^{x+\frac{1}{2}}E_x^c\right).$$

Over all ages in the investigation the likelihood is then proportional to

$$\prod_x\left(Bc^{x+\frac{1}{2}}\right)^{d_x}\ \exp\left(-Bc^{x+\frac{1}{2}}E_x^c\right),$$

and B and c can be obtained by numerical maximisation of this expression.

Alternatively,

If, between exact ages x and $x + 1$, the force of mortality is $\mu_{x+\frac{1}{2}}$, then Gompertz' Law implies that

$$\mu_{x+\frac{1}{2}} = Bc^{x+\frac{1}{2}}.$$

Taking logarithms of this equation produces

$$\log\mu_{x+\frac{1}{2}} = \log B + (x + \tfrac{1}{2})\log c\ ,$$

and B and c can be estimated from the crude estimates of $\mu_{x+\frac{1}{2}}$ by a linear regression of $\log\ \mu_{x+\frac{1}{2}}$ against $x + ½$.

Alternative solutions such as these using weighted squares were also given credit.

Calculate the crude mortality rates $\hat{q}_x$

Calculate $\sum w_x\left(\hat{q}_x - q_x\right)^2 = S^2$

where $w_x = E_x$ *or* $w_x \propto \dfrac{1}{Var\left(E_x\right)}$

and $q_x = 1 - \left[\exp\left(\dfrac{-b}{\log c}\right)\right]^{c^x(c-1)}$

Choose B and c to minimise S^2

Or similarly, but using the crude values $\hat{\mu}_x$ *and* $\mu_x = Bc^x$

7 (i) Since the age label will change at the end of the calendar year, we have a calendar year rate interval.

Given the data we have, a person dying in calendar year t and born in year $t - x$ will be classified as aged x at death, OR age x next birthday at previous or coincident 1 January. 1

This person will be aged between exact ages $x - 1$ and x at the start of the year.

So, at the end of the year the person would (had he or she not died during the year) have been aged between exact ages x and $x + 1$.

$P_{x,t}$ is the number of persons aged x last birthday on 1 January in year t.

We want the central exposed to risk at age x during year t corresponding to the definition of the deaths data (principle of correspondence).

Assuming $P_{x,t}$ varies linearly over the calendar year, this consists of the average of the number of persons aged $x - 1$ last birthday at the start of year t and the number of persons aged x last birthday at the start of year $t + 1$, ie $0.5\,(P_{x-1,t} + P_{x,t+1})$.

Summing this over the whole period of the investigation produces

$$\sum_{K}^{K+N} 0.5(P_{x-1,t} + P_{x,t+1}).$$

(ii) At the start of the rate interval, ages range from $x - 1$ to x exact.

Thus, assuming birthdays are distributed evenly over the calendar year, the average age at the start of the rate interval is $x - ½$,

so q estimates $q_{x-\frac{1}{2}}$

Thus $f = -½$.

8 (i) A few large deviations can be offset by a lot of small deviations. This is not detected as the information is summarised into just one figure.

The true mortality may be consistently slightly lighter or heavier than the standard table. As the test statistic is based on squared deviations, the test will miss this. (Note that large differences should be detected.)

Even if the true mortality is not biased as a whole, there could be significant runs or clumps of ages for which it is biased. Again because of the use of squared deviations and a single figure, this will not be detected.

(ii) **Standardised deviations test**

This is used to test for a small number of large deviations.

Calculate the standardised deviations at each age x, $z_x = \dfrac{d_x - E_x q_x^s}{\sqrt{E_x q_x^s (1 - q_x^s)}}$

where

d_x = observed number of deaths at age x
E_x = exposed to risk at age x
q_x^s = mortality rate for age x from standard table
Under the null hypothesis, the z_x are independent samples from $N(0, 1)$ distribution.

We can calculate the expected number of z_x in various intervals (for example we expect fewer than 1 in 20 to be greater than 2 or less than −2) and compare with the observed number in each interval.

The comparison can be formalised by using a χ^2-statistic equal to

$$\sum_{\text{all intervals}} \frac{(\text{Actual number } - \text{ Expected number})^2}{\text{Expected number}}$$

provided the number of intervals is large (such that the expected number in each interval is not less that five (as a rule of thumb).

Signs test

This tests for the true mortality being lighter or heavier than the standard table.

Either

Calculate the standardised deviations at each age x, $z_x = \dfrac{d_x - E_x q_x^s}{\sqrt{E_x q_x^s (1 - q_x^s)}}$

where

d_x = observed number of deaths at age x
E_x = exposed to risk at age x
q_x^s = mortality rate for age x from standard table

Or

Calculate the sign of the deviation at each age x, $d_x - E_x q_x^s$.

where

d_x = observed number of deaths at age x
E_x = exposed to risk at age x
q_x^s = mortality rate for age x from standard table

Then

The test statistic is P, the number of positive deviations.

Under the null hypothesis $P \sim \text{Bin}(m, \tfrac{1}{2})$ where m is the number of ages.

This is a two-tailed test as too many positive or negative deviations is a defect.

Then either

Find k^*, the smallest k such that

$$\sum_{j=0}^{k} \binom{m}{j} \left(\frac{1}{2}\right)^m \geq 0.025 \text{ (available in tables).}$$

At the 5% level, we would accept the null hypothesis if $k^* < P < m - k^*$.

Or

Find the p-value for the test statistic P.

If this p-value is greater than 0.025, we would accept the null hypothesis (at the 5% level.

Or

If the number of age groups is large, use the approximation

$P \sim \text{Normal}(m/2, m/4)$.

Cumulative deviations

This tests for the true mortality being lighter or heavier than the standard table either over the whole age range or over subsections.

Calculate the deviations at each age x, $d_x - E_x q_x^s$.

where

d_x = observed number of deaths at age x
E_x = exposed to risk at age x
q_x^s = mortality rate for age x from standard table

Sum these over all ages and standardise, to give our test statistic:

$$\frac{\sum_{\text{all } x} d_x - E_x q_x^s}{\sqrt{\sum_{\text{all } x} \left(E_x q_x^s \left(1 - q_x^s\right)\right)}}$$

Under the null hypothesis this is distributed $N(0, 1)$.

We use a two-tailed test as both positive and negative cumulative deviations are of interest.

At the 5% level, we will accept the null hypothesis if the absolute value of the statistic is less than 1.96.

Grouping of signs

This tests for runs of deviations of the same sign.

Either

Calculate the standardised deviations at each age x, $z_x = \dfrac{d_x - E_x q_x^s}{\sqrt{E_x q_x^s (1 - q_x^s)}}$

where

d_x = observed number of deaths at age x
E_x = exposed to risk at age x
q_x^s = mortality rate for age x from standard table

Or

Calculate the sign of the deviation at each age x, $d_x - E_x q_x^s$.

where

d_x = observed number of deaths at age x
E_x = exposed to risk at age x
q_x^s = mortality rate for age x from standard table

Then

Calculate n_1 = number of positive deviations
n_2 = number of negative deviations
G = number of groups of positive deviations

Under the null hypothesis

$$P(G = t) = \frac{\binom{n_1 - 1}{t-1}\binom{n_2 + 1}{t}}{\binom{n_1 + n_2}{n_1}}.$$

This is a one-tailed test as we are only interested in small values of G.

Then either

Find k^*, the smallest k such that

$$\sum_{t=1}^{k} \frac{\binom{n_1 - 1}{t-1}\binom{n_2 + 1}{t}}{\binom{n_1 + n_2}{n_1}} \geq .05$$

At the 5% level, we accept the null hypothesis if $G > k^*$.

Or

Calculate

$$\Pr[\text{exactly } G \text{ positive groups}] = \frac{\binom{n_1 - 1}{G-1}\binom{n_2 + 1}{G}}{\binom{n_1 + n_2}{n_1}}$$

and if this is greater than 0.05, accept the null hypothesis.

Or

If the number of age groups is large (> about 20)

use a normal approximation as follows:

$$G \sim \text{Normal}\left(\frac{n_1(n_2+1)}{n_1+n_2}, \frac{(n_1 n_2)^2}{(n_1+n_2)^3}\right).$$

Serial correlations

This tests for runs of deviations of the same sign.

Calculate the standardised deviations at each age x, $z_x = \dfrac{d_x - E_x q_x^s}{\sqrt{E_x q_x^s (1 - q_x^s)}}$

where

d_x = observed number of deaths at age x
E_x = exposed to risk at age x
q_x^s = mortality rate for age x from standard table

Calculate the correlation coefficient of the jth lagged sequence, r_j using the formula in the Gold Book.

Under the null hypothesis, r_j is distributed $N(0, 1/m)$.

This is a one-tailed test as we are only interested in high values of r_j (which indicates a tendency for z_x to cluster).

Calculate $\sqrt{m} \times r_j$

At the 5% level, we will accept the null hypothesis if $\sqrt{m} \times r_j$ is less than 1.645.

9 (i) From the formulae in the gold tables, we have

$$\text{Var}(X) = {}^2A^{\,1}_{45:\overline{20|}} - \left(A^{\,1}_{45:\overline{20|}}\right)^2$$

and $${}^2A^{\,1}_{45:\overline{20|}} = {}^2A_{45} - 1.04^{-40} \times \frac{l_{65}}{l_{45}} \times {}^2A_{65}$$

$$= 0.09458 - 1.04^{-40} \times \frac{8821.2612}{9801.3123} \times 0.30855$$

$$= 0.0367386$$

Then either

$$A^{\,1}_{45:\overline{20|}} = A_{45} - 1.04^{-20} \times \frac{l_{65}}{l_{45}} \times A_{65}$$

$$= 0.27605 - 1.04^{-20} \times \frac{8821.2612}{9801.3123} \times 0.52786$$

$$= 0.059230$$

so that

$$\text{Var}(X) = 0.0367386 - 0.059230^2$$

$$= 0.033230$$

Or

$$A^{\,1}_{45:\overline{20|}} = \frac{M_{45} - M_{65}}{D_{45}}$$

$$= \frac{463.20 - 363.82}{1677.97}$$

$$= 0.059226$$

so that

$$\text{Var}(X) = 0.0367386 - 0.059226^2$$

$$= 0.033231$$

Then

Similarly

$$\text{Var}(Y) = {}^2A_{\overset{1}{45}:\overline{20}|} - \left(A_{\overset{1}{45}:\overline{20}|}\right)^2$$

$$
\begin{aligned}
{}^2A_{\overset{1}{45}:\overline{20}|} &= 1.04^{-40} \times \frac{l_{65}}{l_{45}} \\
&= 1.04^{-40} \times \frac{8821.2612}{9801.3123} \\
&= 0.187462
\end{aligned}
$$

and

$$
\begin{aligned}
A_{\overset{1}{45}:\overline{20}|} &= 1.04^{-20} \times \frac{l_{65}}{l_{45}} \\
&= 1.04^{-20} \times \frac{8821.2612}{9801.3123} \\
&= 0.410752
\end{aligned}
$$

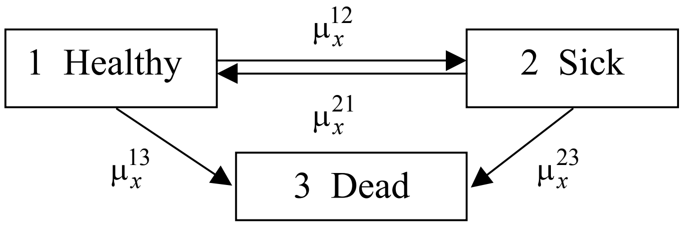
so that

$$\text{Var}(Y) = 0.187462 - 0.410752^2 = 0.0187448$$

(ii) For Z the endowment and the term assurance are issued to the same life. Therefore, the only uncertainty for Z is when the payment will be made — the benefit is certain to be paid at some point. Therefore the lowest value taken by Z is v^{20}.

On the other hand $X + Y$ represents two policies issued to independent lives. Therefore, there is a possibility that no benefits could be paid at all, or that a benefit of 2 could be paid in total. The results are therefore more variable.

10 (i)



(ii) By the Markov assumption, consider the survival probability ${}_{t+dt}p_x^{12}$ and condition on the state occupied at t.

We have

$$ {}_{t+dt}p_x^{12} = {}_tp_x^{11}\,{}_{dt}p_{x+t}^{12} + {}_tp_x^{12}\,{}_{dt}p_{x+t}^{22} + {}_tp_x^{13}\,{}_{dt}p_{x+t}^{32}. \quad (*)$$

But the last term in this equation is zero, since ${}_{dt}p_{x+t}^{32} = 0$.

By the law of total probability,

$$ {}_{dt}p_{x+t}^{22} = 1 - {}_{dt}p_{x+t}^{21} - {}_{dt}p_{x+t}^{23}, $$

and, substituting in (*), this produces

$$ {}_{t+dt}p_x^{12} = {}_tp_x^{11}\,{}_{dt}p_{x+t}^{12} + {}_tp_x^{12}(1 - {}_{dt}p_{x+t}^{21} - {}_{dt}p_{x+t}^{23}). $$

We now assume that, for small dt,

$$ {}_{dt}p_{x+t}^{12} = \mu_{x+t}^{12}dt + o(dt), $$

$$ {}_{dt}p_{x+t}^{21} = \mu_{x+t}^{21}dt + o(dt), \text{ and} $$

$$ {}_{dt}p_{x+t}^{23} = \mu_{x+t}^{23}dt + o(dt), $$

where $o(dt)$ is the probability that a life makes two or more transitions in the time interval dt, and

$$ \lim_{dt \to 0} \frac{o(dt)}{dt} = 0. $$

Substituting for ${}_{dt}p_{x+t}^{12}$, ${}_{dt}p_{x+t}^{21}$ and ${}_{dt}p_{x+t}^{23}$ gives

$$ {}_{t+dt}p_x^{12} = {}_tp_x^{11}\mu_{x+t}^{12}dt + {}_tp_x^{12}\left(1 - \mu_{x+t}^{21}dt - \mu_{x+t}^{23}dt\right) + o(dt) $$

Thus

$$ {}_{t+dt}p_x^{12} - {}_t p_x^{12} = {}_t p_x^{11} \mu_{x+t}^{12} dt - {}_t p_x^{12} \mu_{x+t}^{21} dt - {}_t p_x^{12} \mu_{x+t}^{23} dt + o(dt) $$

and

$$ \frac{\partial}{\partial t} {}_t p_x^{12} = \lim_{dt \to 0^+} \frac{{}_{t+dt}p_x^{12} - {}_t p_x^{12}}{dt} = {}_t p_x^{11} \mu_{x+t}^{12} - {}_t p_x^{12} \mu_{x+t}^{21} - {}_t p_x^{12} \mu_{x+t}^{23} $$

11 (i) The equations of value are, for the standard annuity:

Either

$$X \times a'_{65} \times v^{20}{}_{20}p_{45} = 1500\ddot{a}_{45:\overline{20}|} + 3000 \times \left(v^5{}_5p_{45} + v^{10}{}_{10}p_{45}\right)$$

Or

$$X \times a'_{65} \times \frac{D_{65}}{D_{45}} = 1500 \times \frac{(N_{45} - N_{65})}{D_{45}} + 3000\left(\frac{D_{50}}{D_{45}} + \frac{D_{55}}{D_{45}}\right)$$

and for the guaranteed annuity

Either

$$X^*\left(a_{\overline{10}|} + v^{10} \times {}_{10}p'_{65} \times a'_{75}\right) \times v^{20}{}_{20}p_{45} = 1500\ddot{a}_{45:\overline{20}|} + 3000 \times \left(v^5{}_5p_{45} + v^{10}{}_{10}p_{45}\right)$$

Or

$$X^* \times a'_{\overline{65:10}|} \times v^{20}{}_{20}p_{45} = 1500\ddot{a}_{45:\overline{20}|} + 3000 \times \left(v^5{}_5p_{45} + v^{10}{}_{10}p_{45}\right)$$

where p' and a' denote PMA92C20 mortality.

(ii) Present value of premiums is given by

$$1500\ddot{a}_{45:\overline{20}|} + 3000 \times \left(\frac{D_{50}}{D_{45}} + \frac{D_{55}}{D_{45}}\right) @ 4\%$$

$$= 1500 \times 13.780 + 3000 \times \left(\frac{1366.61 + 1105.41}{1677.97}\right) \text{ (from tables)}$$

$$= 25089.66$$

The present value of the standard annuity is

$$\frac{D_{65}}{D_{45}} \times a'_{65} = \frac{689.23}{1677.97} \times (13.666 - 1) \text{ (from tables)}$$

$$= 5.2026$$

and the standard annual annuity is

$$\frac{25089.66}{5.2026} = 4822.52$$

The present value of the guaranteed annuity is

$$\frac{689.23}{1677.97}\left(8.1109 + \frac{8405.16}{9647.797} \times 1.04^{-10} \times (9.456 - 1)\right) \text{ (from tables)}$$

$$= \quad 5.3758$$

and the guaranteed annuity is therefore

$$\frac{25089.66}{5.3758} = 4667.15$$

so the difference is

$$4822.52 - 4667.15 = 155.37, \text{ i.e. £155 to nearest £}$$

(iii) We must solve

$$4822.52 \times n = 1500 \times 20 + 3000 \times 2$$

The solution is $n = 7.46$. So the purchaser must survive 8 years. The probability of doing so is

$${}_8p'_{65} = \frac{l'_{73}}{l'_{65}}$$

$$= \frac{8803.265}{9647.797} \text{ (using PMA92C20 mortality)}$$

$$= 0.9125$$

12 (i) We only know during which three month period the chip broke down, not the actual date of breakdown.

The investigation is cut short at 1 January 2002.

We don't have any information on the chips where observation ceased for some other reason.

(ii) *Either*

Assuming that the chips break down or are censored at the dates given in the question, and measuring time in months, we have

t_j	n_j	c_j	d_j	$\dfrac{d_j}{n_j}$	$\Lambda_t = \sum \dfrac{d_j}{n_j}$
0	20	0	0	0	0
3	20	0	1	1/20	0.05
6	19	3	1	1/19	0.1026
12	15	6	2	2/15	0.2360
33	7	2	1	1/7	0.3788
45	4	0	1	1/4	0.6288
48	3	2	1	1/3	0.9622

Or

Assuming that the chips break down or are censored mid-way between the three-monthly checks (i.e. that a chip reported as breaking down or being censored on 1 April 1997 actually broke down or was censored mid-way between 1 January 1997 – when it was known to be working – and 1 April 1997), we have

t_j	n_j	c_j	d_j	$\dfrac{d_j}{n_j}$	$\Lambda_t = \sum \dfrac{d_j}{n_j}$
0	20	0	0	0	0
1.5	20	0	1	1/20	0.05
4.5	19	3	1	1/19	0.1026
10.5	15	6	2	2/15	0.2360
31.5	7	2	1	1/7	0.3788
43.5	4	0	1	1/4	0.6288
46.5	3	2	1	1/3	0.9622

(iii) $S(t) = \exp(-\Lambda_t)$,

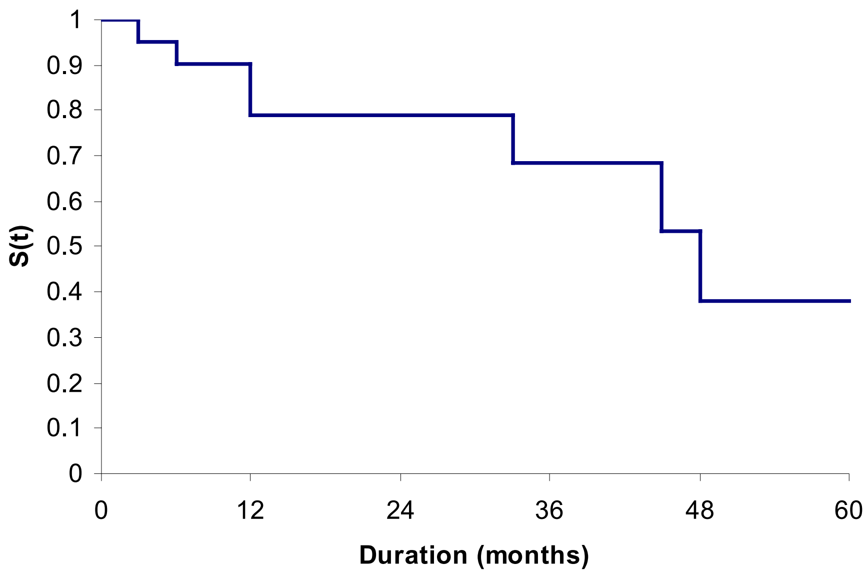
so, depending on which assumption is made about the exact dates of breakdowns and censorings

Either

t	$S(t)$
$0 \leq t < 3$	1
$3 \leq t < 6$	0.9512
$6 \leq t < 12$	0.9025
$12 \leq t < 33$	0.7898
$33 \leq t < 45$	0.6847
$45 \leq t < 48$	0.5332
$48 \leq t$	0.3821

or

t	$S(t)$
$0 \leq t < 1.5$	1
$1.5 \leq t < 4.5$	0.9512
$4.5 \leq t < 10.5$	0.9025
$10.5 \leq t < 31.5$	0.7898
$31.5 \leq t < 43.5$	0.6847
$43.5 \leq t < 46.5$	0.5332
$46.5 \leq t$	0.3821



(iv) Two similarities:

High initial levels of failure, similar to high infant mortality in humans.

Increasing failure for older chips, similar to rising mortality in old age for humans.

Note that the diagram shown above in the solution to part (iii) assumes that breakdowns and censorings take place at the dates given in the question. The alternative assumption implies that the steps in the step function should all be shifted 1.5 months to the left.

The two alternative assumptions given about when breakdowns and censorings take place are the only plausible ones. Other assumptions (for example, that breakdowns and censorings are uniformly distributed within each interval) are wrong and did not receive credit

13 (i) For this policy, the expected cost of paying benefits increases over the term, but the premiums paid are level.

This means that the premiums paid in the early years will be more than the expected cost of the benefits, but those in later years will be less.

It is prudent for the insurance company to set aside (or reserve) the premiums not required in the early years to cover the shortfall in later years. If the company spent all the premiums received in the early years it may not be able to find the money required to pay the benefit later in the contract — ultimately the company could become insolvent.

(ii) (a) Prospective policy value = Expected present value of future outgo less Expected present value of future income

Retrospective policy value = Accumulated value of premiums received to date, allowing for interest and survivorship less Accumulated value of benefits (and expenses) paid to date, allowing for interest and survivorship

Alternatively

At time t:

Prospective policy value $= 10000\bar{A}_{40+t:\overline{20-t}|} - 300\times \ddot{a}_{40+t:\overline{20-t}|}$

Retrospective policy value $= 300 \times \ddot{s}_{40:\overline{t}|} - 10000\times\dfrac{(1.06)^t}{{}_t p_{40}}\times \bar{A}^{1}_{40:\overline{t}|}$

(b) The two values will be equal if:

1 they are calculated on the same basis and

2 the same basis was used to calculate the premiums in the policy value calculations

The assumptions used to calculate the retrospective value will be based on the experienced conditions to date.

For the prospective calculation, the assumptions will be those considered appropriate for the future remaining term.

These two sets of assumptions will not generally be the same.

Further the assumptions used to calculate the premiums, whilst appropriate at the time, may not be considered appropriate for either policy value calculation.

(iii) The prospective policy value at the end of 2003 is given by the formula:

$${}_tV_{x:\overline{t}|} = 10000 \times \bar{A}_{x+t:\overline{n-t}|} - P \times \ddot{a}_{x+t:\overline{n-t}|}$$

So, $${}_{13}V_{40:\overline{20}|} = 10000 \times \bar{A}_{53:\overline{7}|} - P \times \ddot{a}_{53:\overline{7}|}$$

$$= 10000 \times \bar{A}_{53:\overline{7}|} - 300 \times 5.847$$

But, $$\bar{A}_{53:\overline{7}|} = \bar{A}^{1}_{53:\overline{7}|} + A_{53:\overline{7}|}^{\ \ \ \,1}$$

$$= (1.06)^{1/2} A^{1}_{53:\overline{7}|} + A_{53:\overline{7}|}^{\ \ \ \,1} = (1.06)^{1/2} \times \left(A_{53:\overline{7}|} - A_{53:\overline{7}|}^{\ \ \ \,1} \right) + A_{53:\overline{7}|}^{\ \ \ \,1}$$

$$A_{53:\overline{7}|}^{\ \ \ \,1} = \frac{v^{60} l_{60}}{v^{53} l_{53}} = \frac{1.06^{-60} \times 9287.2164}{1.06^{-53} \times 9630.0522} = 0.64138$$

$$A_{53:\overline{7}|} = 0.66904 \text{(from tables)}$$

So, $$\bar{A}_{53:\overline{7}|} = 1.06^{1/2} \times (0.66904 - 0.64138) + 0.64138 = 0.66986$$

So, $${}_{13}V_{40:\overline{20}|} = 10{,}000 \times 0.66986 - 300 \times 5.847 = £4{,}944.50$$

(iv) Mortality profit = Expected Death Strain – Actual Death Strain

Expected Death Strain $= q_{x+t}(S - {}_{t+1}V) = q_{52}(10000 - {}_{13}V)$
$= 0.003152\ (10000 - 4944.50)$
$= £15.93$

Actual Death Strain $= 1 \times (S - {}_{t+1}V) = 10000 - {}_{13}V$
$= 10000 - 4944.50$
$= £5{,}055.50$

Mortality Profit $= 15.93 - 5055.50 = -£5{,}039.57$ (i.e. a loss)

This answer assumes the death benefit is paid at the end of year of death. The alternative, assuming payment on average half-way through the year was given full credit.