EXAMINATIONS

September 2004

Subject 104 — Survival Models

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M Flaherty
Chairman of the Board of Examiners

23 November 2004

EXAMINERS' COMMENTS

As in previous years, the Examiners aimed to set questions covering all the aspects of Survival Modelling: life contingencies including its stochastic treatment, graduation including its statistical aspects and the determination of exposures. The Examiners aim to strike a balance between questions requiring numerical solutions and those requiring verbal and algebraic answers, as well as between those with and without a statistical theme.

Comments on solutions presented to individual questions for this September 2004 paper are given below:

Question 1 — *This was very poorly answered. In particular, few candidates realised that if the life survives to age x+n, no benefit is payable.*

Question 2 — *Almost all candidates wrote down the correct equation in part (i).*
In part (ii), many candidates were able to write down the prospective and retrospective policy values. Disappointingly few, however, were able to make much headway in showing that these are equal if evaluated on the same basis. Of the few who did make a serious attempt at this, most used commutation functions, as in ALTERNATIVE 3.

Question 3 — *No comments.*

Question 4 — *In part (i), most candidates showed that var(Z) was equal to* $\frac{1}{d^2}\text{var}\left(v^{K_x+1}\right)$, *but only a minority made any attempt to go on to show that* $\frac{1}{d^2}\text{var}\left(v^{K_x+1}\right)$ *is equal to* $\frac{1}{d^2}\left[{}^2A_x-(A_x)^2\right]$. *Most just wrote the last quantity down, either from memory or from the Gold Book.*
Part (ii) was well answered.

Question 5 — *Those candidates who attempted this question generally answered it well. A surprisingly high proportion of candidates, however, failed to make a serious attempt.*

Question 6 — *This question was very well answered.*

Question 7 — *Parts (a) and (b) were tackled fairly well by most candidates.*
Part (c), however, clearly caused more difficulty: very few candidates spotted that the age range at the start of the rate interval was two years wide (x-2, x). There is still a tendency for candidates to adopt a 'scattergun' approach to stating assumptions in this type of question. The examiners were looking for statements of assumptions that were specific to each example, and penalised general lists of (sometimes irrelevant) assumptions.

Question 8 — *There were some good attempts at this question. The most common mistakes were to omit the end-of-year survival benefit R in the death strain at risk, and to miss out the death benefits completely when calculating the reserve.*

Question 9 — *Part (i) of this question was very well answered.*
By contrast, the standard of attempts at part (ii) was very poor. Very few candidates correctly identified the function to be summed, and most seemed to have little idea of what was required.

Question 10 — *In parts (i) and (iii), many candidates failed to relate their answers to the specific context of the question ('a medium-sized UK pension scheme'). Instead many produced vague general comments, some of which were of little relevance in the specific context, and failed to emphasise other issues which were especially important in the context of a medium-sized pension scheme (for example, the problem of scanty data at older ages).*
Part (ii) was well answered, though a surprisingly high number of candidates tested for the wrong problems (bias over part of the age range, or individual ages with large deviations) by performing the Grouping of Signs test or the Individual Standardised Deviations test.

Question 11 — *This question was poorly answered.*
There were few attempts at part (ii), and most of these made the (theoretically incorrect) assumption that a uniform distribution of deaths between ages x and x+1 implies that the complete expectation of life changes linearly between these ages. In fact, this assumption turns out to be quite close to reality in the particular case of ages 39-40 in the AM92 tables, and so the assumption could have been justified, but few candidates made the attempt to justify it on these grounds.
In part (iii) there were many incorrect approximations used in calculating the premium for the continuous annuity.
Part (iv) was very poorly answered. The only point which was made by an appreciable number of candidates was the need to take into account future mortality improvements.

1

This is a benefit of 1 p.a. payable continuously between the death of the life and the date that would have been the life's $x + n$ th birthday.
If the life survives to age $x + n$, then no benefit is payable.

2

(i) $P\ddot{a}_{45:\overline{20}|} = 20,000A^{1}_{45:\overline{20}|}$

(ii) *ALTERNATIVE 1*

The prospective policy value at duration t (when the life is aged 45 + t) is

$$ {}_tV_{\text{pro}} = 20,000A^{1}_{45+t:\overline{20-t}|} - P\ddot{a}_{45+t:\overline{20-t}|}. $$

The retrospective policy value is

$$ P\left[\frac{\ddot{a}_{45:\overline{t}|}}{v^t\ {}_tp_{45}}\right] - \left[\frac{20,000A^{1}_{45:\overline{t}|}}{v^t\ {}_tp_{45}}\right]. $$

To show that they are equal, note that

$$ \ddot{a}_{45:\overline{20}|} = \ddot{a}_{45:\overline{t}|} + v^t\ {}_tp_{45}\ddot{a}_{45+t:\overline{20-t}|}, $$

so that

$$ \ddot{a}_{45+t:\overline{20-t}|} = \frac{\ddot{a}_{45:\overline{20}|} - \ddot{a}_{45:\overline{t}|}}{v^t\ {}_tp_{45}}. $$

Substituting in the above expression for ${}_tV_{\text{pro}}$ yields

$$ {}_tV_{\text{pro}} = 20,000A^{1}_{45+t:\overline{20-t}|} - \frac{P}{v^t\ {}_tp_{45}}(\ddot{a}_{45:\overline{20}|} - \ddot{a}_{45:\overline{t}|}) $$

But from the solution to (i) above,

$$ \ddot{a}_{45:\overline{20}|} = \frac{20,000A^{1}_{45:\overline{20}|}}{P}, $$

so that

$$ {}_tV_{\text{pro}} = 20,000A^{1}_{45+t:\overline{20-t}|} - \frac{P}{v^t\ {}_tp_{45}}\left(\frac{20,000A^{1}_{45:\overline{20}|}}{P} - \ddot{a}_{45:\overline{t}|}\right). $$

Since

$$ A^{1}_{45:\overline{20}|} = A^{1}_{45:\overline{t}|} + v^t\ {}_tp_{45}A^{1}_{45+t:\overline{20-t}|}, $$

then

$$
{}_tV_{\text{pro}} = 20,000A^{1}_{45+t:\overline{20-t}|} - \frac{P}{v^t\;{}_tp_{45}}\left(\frac{20,000A^{1}_{45:\overline{t}|} + 20,000v^t\;{}_tp_{45}A^{1}_{45+t:\overline{20-t}|}}{P} - \ddot{a}_{45:\overline{t}|}\right)
$$

$$
= \frac{P\ddot{a}_{45:\overline{t}|}}{v^t\;{}_tp_{45}} - \frac{20,000A^{1}_{45:\overline{t}|}}{v^t\;{}_tp_{45}}
$$

which is the retrospective policy value.

ALTERNATIVE 2

The prospective policy value at duration t (when the life is aged $45 + t$) is

$$
{}_tV_{\text{pro}} = 20,000A^{1}_{45+t:\overline{20-t}|} - P\ddot{a}_{45+t:\overline{20-t}|}.
$$

The retrospective policy value may be written

$$
\frac{D_{45}}{D_{45+t}}\left[P\ddot{a}_{45:\overline{t}|} - 20,000A^{1}_{45:\overline{t}|}\right].
$$

From part (i) we have

$$
P\ddot{a}_{45:\overline{20}|} - 20,000A^{1}_{45:\overline{20}|} = 0. \quad (*)
$$

Multiplying both sides of equation (*) by $\dfrac{D_{45}}{D_{45+t}}$ we have

$$
\frac{D_{45}}{D_{45+t}}\left[P\ddot{a}_{45:\overline{20}|} - 20,000A^{1}_{45:\overline{20}|}\right] = 0.
$$

Adding the left-hand side of this equation to the prospective policy value produces

$$
{}_tV_{\text{pro}} = 20,000A^{1}_{45+t:\overline{20-t}|} - P\ddot{a}_{45+t:\overline{20-t}|} + \frac{D_{45}}{D_{45+t}}\left[P\ddot{a}_{45:\overline{20}|} - 20,000A^{1}_{45:\overline{20}|}\right]
$$

so that

$$
{}_tV_{\text{pro}} = \frac{D_{45}}{D_{45+t}}\left[20,000A^{1}_{45+t:\overline{20-t}|}\frac{D_{45+t}}{D_{45}} - 20,000A^{1}_{45:\overline{20}|} + P\ddot{a}_{45:\overline{20}|} - P\ddot{a}_{45+t:\overline{20-t}|}\frac{D_{45+t}}{D_{45}}\right]
$$

$$
= \frac{D_{45}}{D_{45+t}}\left[P\ddot{a}_{45:\overline{t}|} - 20,000A^{1}_{45:\overline{t}|}\right]
$$

which is the same as the retrospective policy value.

ALTERNATIVE 3

Using commutation functions, we have

$$ {}_tV_{\text{pro}} = 20,000\left[\frac{M_{45+t} - M_{65}}{D_{45+t}}\right] - P\left[\frac{N_{45+t} - N_{65}}{D_{45+t}}\right] $$

and

$$ \begin{aligned} {}_tV_{\text{retro}} &= P\frac{D_{45}}{D_{45+t}}\left[\frac{N_{45} - N_{45+t}}{D_{45}}\right] - 20000\frac{D_{45}}{D_{45+t}}\left[\frac{M_{45} - M_{45+t}}{D_{45}}\right] \\ &= P\left[\frac{N_{45} - N_{45+t}}{D_{45+t}}\right] - 20000\left[\frac{M_{45} - M_{45+t}}{D_{45+t}}\right] \end{aligned} . $$

From part (i) we have

$P\ddot{a}_{45:\overline{20}|} - 20,000A^{1}_{45:\overline{20}|} = 0$,

which, using commutation functions, may be written as

$P(N_{45} - N_{65}) - 20,000(M_{45} - M_{65}) = 0$.

Therefore, dividing this equation by D_{45+t} produces

$$ P\left[\frac{N_{45} - N_{65}}{D_{45+t}}\right] + -20,000\left[\frac{M_{45} - M_{65}}{D_{45+t}}\right] = 0 . $$

Subtracting this equation from the retrospective policy value produces

$$ \begin{aligned} {}_tV_{\text{retro}} &= P\left[\frac{N_{45} - N_{45+t}}{D_{45+t}}\right] - P\left[\frac{N_{45} - N_{65}}{D_{45+t}}\right] \\ &\quad - 20000\left[\frac{M_{45} - M_{45+t}}{D_{45+t}}\right] + 20000\left[\frac{M_{45} - M_{65}}{D_{45+t}}\right] \end{aligned} $$

which may be simplified to

$$ {}_tV_{\text{retro}} = 20000\left[\frac{M_{45+t} - M_{65}}{D_{45+t}}\right] - P\left[\frac{N_{45+t} - N_{65}}{D_{45+t}}\right] $$

which is the same as the prospective policy value.

3

(i) Females who received the existing treatment at the time of diagnosis.

(ii) (a) $h_i(t) = h_0(t)\ \exp\left\{\left(0.5\times\frac{6}{12}\right) + (0.01\times1) + (-0.05\times1)\right\}$

$$= h_0(t)\ e^{0.21}$$

(b) $$S(t) = \exp\left\{-\int_0^t h(s)\ ds\right\}$$

$$= \exp\left\{-\int_0^t h_0(s)\ e^{+0.21}\ ds\right\}$$

$$= \exp\left\{-e^{+0.21}\int_0^t h_0(s)\ ds\right\}$$

$$= \exp\left\{-\int_0^t h_0(s)\ ds\right\}^{e^{+0.21}}$$

(iii) *ALTERNATIVE 1*

For the female life:

$$h_i(t) = h_0(t)\ \exp\{(0.5\times 0) + (0.01\times 1) + (-0.05\times 0)\} = h_0(t)\ e^{0.01}$$

$$S(5) = \exp\left\{-\int_0^5 h_0(s)\ ds\right\}^{e^{0.01}} = 0.75$$

$$\Rightarrow \exp\left\{-\int_0^5 h_0(s)\ ds\right\} = (0.75)^{e^{-0.01}}$$

So, for the male life:

$$S(5) = \exp\left\{-\int_0^5 h_0(s)\ ds\right\}^{e^{0.21}}$$

$$= \left[(0.75)^{e^{-0.01}}\right]^{e^{0.21}}$$

$$= 0.7037$$

ALTERNATIVE 2

Defining the hazard for the male h_1 and the hazard for the female h_2, the ratio of hazards is

$$\begin{aligned}
\frac{h_1}{h_2} &= \frac{h_0(5)\ e^{0.21}}{h_0(5)\ e^{0.01}} \\
&= e^{0.20} \\
S_1 &= (S_2)^{e^{0.20}} \\
&= (0.75)^{e^{0.20}} \\
&= 0.7037
\end{aligned}$$

4 (i) $Var(Z) = Var\left(a_{\overline{K_x}|}\right)$, where K_x is the curtate future lifetime

$$= Var\left(\ddot{a}_{\overline{K_x+1}|} - 1\right)$$

$$= Var\left(\ddot{a}_{\overline{K_x+1}|}\right)$$

$$= Var\left(\frac{1-v^{K_x+1}}{d}\right)$$

$$= \frac{1}{d^2} Var\left(v^{K_x+1}\right)$$

$$= \frac{1}{d^2}\left\{\sum_{k=0}^{\infty}\left(v^{k+1}\right)^2 \cdot {}_{k|}q_x - \left(\sum_{k=0}^{\infty}\left(v^{k+1}\right)\cdot {}_{k|}q_x\right)^2\right\}$$

$$= \frac{1}{d^2}\left\{\sum_{k=0}^{\infty}\left(v^2\right)^{k+1} \cdot {}_{k|}q_x - (A_x)^2\right\}$$

$$= \frac{1}{d^2}\left\{{}^2A_x - (A_x)^2\right\}$$

where ${}^2A_x$ is calculated using a rate of interest $(1+i)^2 - 1$

(ii) $Var(Z) = \dfrac{100^2}{d^2}\left\{{}^2A_{60} - (A_{60})^2\right\}$

$$= \frac{100^2}{0.038462^2}\left\{0.23723 - (0.45640)^2\right\}\text{, from tables}$$

$$= 195{,}555.617$$

So, standard deviation = $(195{,}555.617)^{0.5}$ = £442.22.

5 (i) $$\mathring{e}_x = \int_0^\infty t \cdot f_x(t)\,dt$$

$$= \int_0^\infty t \cdot \left({}_t p_x \cdot \mu_{x+t}\right) dt$$

$$= \int_0^\infty t \cdot \left(-\frac{\partial}{\partial t}\,{}_t p_x\right) dt$$

$$= -\left[t \cdot {}_t p_x\right]_0^\infty + \int_0^\infty {}_t p_x\, dt$$

$$= \int_0^\infty {}_t p_x\, dt$$

(ii) $${}_t p_x = \exp\left\{-\int_0^t \mu_{x+s}\, ds\right\}$$

$$= \exp\left\{-\int_0^t \mu\, ds\right\} = \exp\left\{-[\mu s]_0^t\right\} = \exp\{-\mu t\}$$

So,

$$\mathring{e}_{25} = \int_0^\infty {}_t p_{25}\, dt = \int_0^\infty e^{-\mu t}\, dt$$

$$= \left[\frac{e^{-\mu t}}{-\mu}\right]_0^\infty$$

$$= \frac{1}{\mu} = \frac{1}{0.02} = 50 \text{ years}$$

6 (i) $L \propto e^{-(\mu+\sigma)625} e^{-(\nu+\rho)35} \mu^6 \sigma^{15} \rho^5 \nu^1$

Taking logs gives:

$$\log L = -(\mu+\sigma)625 - (\nu+\rho)35 + 6\ln\mu + 15\ln\sigma + 5\ln\rho + \ln\nu$$

$$= -625\sigma + 15\ln\sigma + K$$

Differentiate with respect to σ:

$$\frac{\partial \ln L}{\partial \sigma} = -625 + \frac{15}{\sigma}$$

Set to zero to find maximum:

$$0 = -625 + \frac{15}{\hat{\sigma}} \Rightarrow \hat{\sigma} = \frac{15}{625} = 0.024$$

Check that this is a maximum:

$$\frac{\partial^2 \ln L}{\partial \sigma^2} = -\frac{15}{\sigma^2} < 0$$

(ii) (a) $\tilde{\sigma} \sim N\left(\sigma, \dfrac{\sigma}{E[V]}\right)$, where V is the waiting time in state 1

Note that the question asks for the asymptotic distribution of the maximum likelihood estimator. Inserting numbers or $\hat{\sigma}$ *gives the formula we use, in practice, to estimate this distribution, rather than the distribution itself.*

(b) *ALTERNATIVE 1*

$$Var(\tilde{\sigma}) = \frac{\sigma}{E[V]}$$

Estimate σ and $E[V]$ by the observed values of $\hat{\sigma}$ and v respectively.

$$Var(\tilde{\sigma}) \approx \frac{15}{625^2}$$

So, estimate of sd of $\tilde{\sigma} = \dfrac{\sqrt{15}}{625} = 0.00620$

ALTERNATIVE 2

$$Var(\tilde{\sigma}) \approx \left(-\frac{\partial^2 L}{\partial \sigma^2}\right)^{-1} = \frac{\sigma^2}{15}$$

Estimate σ by the observed value of $\hat{\sigma}$.

So, estimate of sd of $\tilde{\sigma} = \dfrac{\hat{\sigma}}{\sqrt{15}} = \dfrac{\sqrt{15}}{625} = 0.00620$

7 $x + f$ is the average age of the lives half-way through the rate interval.

(a) Age changes on 1 January each year, so calendar year rate interval starting on 1 January.

The age range at the start of the interval is (x - 1, x).

Assuming birthdays are uniformly distributed throughout the year, the average age at the start of the interval is $x - \frac{1}{2}$

So, average age half-way through interval is $x - \frac{1}{2} + \frac{1}{2} = x$

So, $f = 0$

(b) Age x = [Age last birthday at entry] + [curtate duration]
Age changes on a policy anniversary, so policy year rate interval.

The age range at the start of the interval is (x, x + 1).

Assuming birthdays are uniformly distributed throughout the policy year, the average age at the start of the interval is $x + \frac{1}{2}$

So, average age half-way through interval is $x + \frac{1}{2} + \frac{1}{2} = x + 1$

So, $f = 1$

(c) Age x = Age next birthday at anniversary in calendar year of death
Age changes at start of calendar year, so calendar year rate interval starting on 1 January.

The age range at the start of the interval is (x - 2, x).

Assuming birthdays and policy anniversaries are uniformly distributed throughout the calendar year, the average age at the start of the interval is $x - 1$

So, average age half-way through interval is $x - 1 + \frac{1}{2} = x - \frac{1}{2}$

So, $f = -\dfrac{1}{2}$

8 We need to calculate the reserve V needed for each life who survives to age 76. This is just the value of the future benefits.

The value of the survival benefits is

$$\begin{aligned} £10,000 \times a_{76} &= £10,000 \times (9.049 - 1) \\ &= £80,490 \end{aligned}$$

The value of the death benefits can be calculated via premium conversion:

$$\begin{aligned} £2,000 \times A_{76} &= £2,000(1 - d\ddot{a}_{76}) \\ &= £2,000(1 - 0.04/1.04 \times 9.049) \\ &= £1,304 \end{aligned}$$

The total reserve required is therefore £80,490 + £1,304 = £81,794.

The death strain at risk is equal to

S - (V_t + R)
£2,000 - (£81,794 + £10,000) = -£89,794

The expected number of deaths during 2002 is 100 x q_{75}

= 100 x 0.028121 = 2.8121

The mortality profit is therefore equal to

Expected death strain - Actual death strain
= (Expected deaths - actual deaths) x DSAR
= (2.8121 - 7) x (-£89,794)
= £376,048.

An alternative method provided by a number of candidates used
Mortality profit = reserve at beginning of year plus interest
- actual payments on death and in pensions
- end of year reserves
This method was given full credit.

9 (i)

t_j	n_j	d_j	c_j	$\lambda_j = d_j/n_j$	$1 - \lambda_j$
0	100	0	1	0	1
2	99	3	5	1/33	32/33
6	91	2	2	2/91	89/91
8	87	2	5	2/87	85/87
17	80	1	79	1/80	79/80

t	$S(t) = \prod(1 - \lambda_j)$
$0 <= t < 2$	1
$2 <= t < 6$	0.9697
$6 <= t < 8$	0.9484
$8 <= t < 17$	0.9266
$t >= 17$	0.9150

(ii) The expected present value of the cost over 2 years of treating a patient who contracts the disease is given by

$$X = £1{,}000\ .\ \sum_{0}^{2} S(t).h(t).v^t$$

For a patient who has not been vaccinated, this is

$$X_1 = £1{,}000 \text{ x } (1.0 \text{ x } 0.08 \text{ x } 1.04^{-0.41666} + 0.92 \text{ x } 0.02717 \text{ x } 1.04^{-0.6667} + 0.895 \text{ x } 0.02793 \text{ x } 1.04^{-1})$$

$= £1{,}000$ x 0.12708

$= £127.08$

And, similarly for a patient who has been vaccinated

$$X_2 = £1{,}000\left(1.0\times\frac{1}{33}\times 1.04^{-0.1666} + 0.9697\times\frac{2}{91}\times 1.04^{-0.5} + 0.9484\times\frac{2}{87}\times 1.04^{-0.6667} + 0.9266\times\frac{1}{80}\times 1.04^{-1.4166}\right)$$

$= £1{,}000$ x 0.08320

$= £83.20$

So, the total expected present value of the cost of treating a vaccinated patient is £83.20 + £20(cost of vaccine) = £103.20.

This is less than the expected cost of treating a patient without the vaccine and the expected saving is £23.88.

10 (i) Reasons why crude rates will require graduation:

- low data volumes at older ages — graduation means data at nearby ages can be used to improve estimates

- overall low data volumes mean that crude rates are likely to be subject to relatively large sampling errors and therefore will not progress smoothly with age, as we assume that the underlying rates do

Volume of data will be too small to attempt a direct graduation, ruling out use of parametric formula.
Also, large sampling errors would make graphical graduation imprecise, and in any case computationally inefficient. So suggest we graduate via some simple relationship to a standard table, many of which exist based on large volumes of data relating to similar lives. This approach also allows us to compare our population with the population underlying the standard table.

(ii)

H_0: The crude rates come from a population in which the true underlying rates of mortality are the graduated rates.

(a) **Test for overall goodness of fit** (Chi-squared)

The test statistic is $X = \sum_i z_i^2$,

which under H_0 has a χ^2 distribution.

The degrees of freedom will be the number of age groups less some allowance for the method of graduation, so at most 9 in this case.

The observed value of the test statistic $\sum z_i^2$ is 5.11.

This is a one-sided test, so we wish to compare this to the upper 5% point of the χ^2 distribution.

But 5.11 is below the upper 5% point for all degrees of freedom except 1.

Hence we conclude there is insufficient evidence to reject H_0.

(b) **Test for Bias**

ALTERNATIVE 1

Test statistic is P = number of age groups for which the graduated rate is below the crude rate = number of positive standardised deviations.

Under H_0 $P \sim Bin(9, 0.5)$

The observed value of P is 7.

$$\Pr(P >= 7) = 0.5^9(1 + 9 + 9 * 8 / 2) = 0.0898$$

This is a 2-sided test, so we would reject H_0 if this probability is below 0.025.

Since it is not, we conclude that we do not have significant evidence of bias in the graduation.

ALTERNATIVE 2 (cumulative deviations)

The test statistic is $Z = \dfrac{\sum d_x - E_x^c \, \mathring{\mu}_{x+½}}{\sqrt{\sum E_x^c \, \mathring{\mu}_{x+½}}}$

Under $H_{0,}$ $Z \sim N(0, 1)$

Observed value of Z is $\dfrac{11.7192}{\sqrt{149.2808}} = 0.9592$

This is a two-sided test so we would reject H_0 if $|Z| > 1.96$
Since it is not, we accept H_0 and conclude that we do not have significant evidence of bias.

(iii) **Use of graduated rates to value benefits**

Areas for consideration:

- Do we expect the experience of 2000–2002 to be typical for the scheme as a whole?
  Is there any reason why rates might have been higher / lower (e.g. harsh winter?)
- These rates would be used to value benefits payable many years into the future.
  Mortality rates have fallen consistently in the past, so we should make

some allowance for future changes also. If we do not there is a danger that we will undervalue the benefits.

11 (i) Sex
Smoker/non-smoker status
Occupation
Known impairments
Geographical location

(ii) *ALTERNATIVE 1*

Using AM92, we find that $\overset{\circ}{e}_{39} > 39$ and $\overset{\circ}{e}_{40} < 40$

so 39 < a < 40.

If $\overset{\circ}{e}_a = a$, then

$$a = \frac{\int_a^{\omega} l_x dx}{l_a}. \quad (*)$$

Assuming a uniform distribution of deaths (UDD) between ages 39 and 40, then

$$l_a = l_{39} - (a-39)d_{39}$$

So that, substituting into (*) we obtain

$$a = \frac{\int_a^{\omega} l_x dx}{l_{39} - (a-39)d_{39}} \qquad (**)$$

To evaluate $\int_a^{\omega} l_x dx$, note that

$$\int_a^{\omega} l_x dx = \int_a^{40} l_x dx + \int_{40}^{\omega} l_x dx = \int_a^{40} l_x dx + \overset{\circ}{e}_{40} l_{40},$$

and that, assuming UDD,

$$\int_a^{40} l_x dx = \frac{40-a}{2}(l_a + l_{40}) + \overset{\circ}{e}_{40} l_{40}$$

Therefore, substituting for $\int_a^{w} l_x \, dx$ in (**) we obtain

$$a = \frac{\frac{40-a}{2}(l_{39} - (a-39)d_{39} + l_{40}) + \mathring{e}_{40} l_{40}}{l_{39} - (a-39)d_{39}}$$

From the tables, e_{40} = 39.064, so using the approximation given $\mathring{e}_{40}$ = 39.064 + 0.5 = 39.564. Inserting this, the tabulated values for l_{39}, l_{40} and d_{39} and the value 39.778 for a into the equation, the right hand side becomes:

$$\frac{\frac{40-a}{2}(9{,}864.8688 - (8.5824)(0.778) + 9{,}856.2863) + (39.564)(9{,}856.2863)}{9{,}864.8688 - (0.778)(8.5824)}$$

$$= \frac{\frac{0.222}{2}(9{,}858.1917 + 9{,}856.2863) + 389{,}954.1112}{9{,}858.1917}$$

= 39.778 (to 3 decimal places)

ALTERNATIVE 2

Using AM92, we find that $\mathring{e}_{39} > 39$ and $\mathring{e}_{40} < 40$

so 39 < a < 40.

To evaluate $\mathring{e}_{x+t}$ $(0 \le t \le 1)$ assuming UDD between exact ages x and x+1 consider that $\mathring{e}_{x+t}$ is a weighted average of the complete expectations of life of those who survive to exact age x+1 and those who die between x+t and exact age x+1.

The proportion of those alive at age x+t who survive to exact age x+1 is ${}_{1-t}p_{x+t}$.

These lives will have a complete expectation of life equal to $1 - t + \mathring{e}_{x+1}$.

The proportion of those alive at age x+t who die before exact age x+1 is 1-${}_{1-t}p_{x+t}$.

These lives will live, on average, $\frac{1}{2}(1-t)$ more years.

Therefore

$$\mathring{e}_{x+t} = \frac{1}{2}(1-t)(1 - {}_{1-t}p_{x+t}) + \left(1 - t + \mathring{e}_{x+1}\right){}_{1-t}p_{x+t}. \quad (*)$$

In our case, we have t = 0.778, and x = 39. From the tables, e_{40} = 39.064, so using the approximation given, $\mathring{e}_{40}$ = 39.064 + 0.5 = 39.564.

Note also that under UDD, ${}_{1-t}p_{x+t} = 1 - {}_{1-t}q_{x+t} = 1 - \left[\frac{(1-t)q_x}{1-tq_x}\right] = \frac{1-q_x}{1-tq_x}$.

From the tables, q_{39} = 0.00087. Therefore

$${}_{1-t}p_{x+t} = \frac{1-0.00087}{1-(0.00087)(0.778)} = 0.99981$$

Substituting into (*) above, therefore, produces

$$\overset{o}{e}_{39.778} = \frac{1}{2}\big(1-0.778\big)(1-0.99981) + \big(1-0.778+39.564\big)(0.99981)$$

$$= \frac{1}{2}(0.222)(0.00019) + (39.786)(0.99981)$$

= 39.778 (to 3 d.p.)

Therefore we have shown that a = 0.778.

(iii) In this case, $2a$ = 79.556, so the birthday nearest to this age is the 80th.

An annuity paid weekly can be treated as being paid continuously, so the premium for the annuity, P_a, is given by the equation

$$P_a = 10,000(\bar{a}_{40:\overline{40}|} - \bar{a}_{40:\overline{25}|})$$

Evaluation proceeds as follows: for the first term we have

$$\bar{a}_{40:\overline{40}|} = \ddot{a}_{40:\overline{40}|} - 0.5(1 - v^{40}\ {}_{40}p_{40})$$

$$= \ddot{a}_{40} - \frac{D_{80}}{D_{40}}\ddot{a}_{80} - 0.5(1 - v^{40}\frac{l_{80}}{l_{40}})$$

and, from the AM92 tables this is evaluated as

$$20.005 - \frac{228.48}{2052.96}(6.818) - 0.5\left[1 - (0.20829)\frac{5266.4604}{9856.2863}\right]$$

$$= 20.005 - (0.111293)(6.818) - 0.5[1 - (0.20829)(0.534325)]$$

= 20.005 – 0.75880 – 0.44435

= 18.80185.

For the second term we have

$$\bar{a}_{40:\overline{25}|} = \ddot{a}_{40:\overline{25}|} - 0.5(1 - v^{25}\frac{l_{65}}{l_{40}}).$$

From tables, this is evaluated as

$$15.884 - 0.5(1 - 0.37512 \times \frac{8821.2612}{9856.2863})$$
$$= 15.884 - 0.5[1 - (0.37512)(0.89499)]$$
$$= 15.884 - 0.33214$$
$$= 15.5519$$

Therefore

$$P_a = 10,000(18.80185 - 15.5519) = £32,500.$$

For the pure endowment, the premium, P_e, is given by

$$P_e = 100,000 v^{40}\ {}_{40}p_{40}$$

which, using the figures in the previous evaluation, is

$$100,000 \text{ x } (0.20829)\frac{5266.4604}{9856.2863} = £11,129.$$

So the total single premium payable is

£11,129 + £32,500 = £43,629.

(iv) When pricing these annuities and pure endowments, companies will need to consider the likely risk of future mortality improvements.

If mortality improves, then the expected outgo will increase. Premiums should be set to take this into account, for otherwise the office might become insolvent.

In addition, offices might wish to use a mortality basis that takes into account factors other than those listed in the solution to part (i), such as the policy size, the level of underwriting and the sales channel.

If an office does not take these factors into account, and if mortality is related to them, then the premiums charged will be too high for the low risks (who will take their business elsewhere to other offices who do take these factors into account) and too low for the high risks (resulting in the possibility of insolvency).

END OF EXAMINERS' REPORT