Faculty of Actuaries **Institute of Actuaries**

REPORT OF THE BOARD OF EXAMINERS ON THE EXAMINATIONS HELD IN

April 2002

Subject 104 — Survival Models

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

K Forman
Chairman of the Board of Examiners

11 June 2002

EXAMINERS' COMMENTS

The overall standard achieved by candidates was an improvement on that achieved in recent examinations.

By its nature Survival Models is a statistical subject. The paper will contain questions that require the use of the skills taught in Subject 101. The Examiners aim to strike a balance between questions requiring numerical solutions and those requiring verbal and algebraic answers, as well as between those with and without a statistical theme. The questions set will always cover Survival Modelling in all its aspects, Life Contingencies including its stochastic treatment, Graduation including its statistical aspects and the determination of exposures.

The solutions presented to Questions 1, 2, 6 and 10 were, on the whole, good. Those presented to Questions 4 and 5 were slightly poorer than expected, but the general standard in Questions 3, 7, 8 and 9 was well below what was expected.

Questions 3, 7, 8 and 9 covered Stochastic Life Contingencies, Proportional Hazards Models and general ideas about exposure calculations. Many of the basic skills needed for these topics come from Subject 101, but it is often clear from the solutions presented that concepts such as a random variable, its expected value, its probability distribution, parameters and their estimates and hypothesis testing are misunderstood.

1 (i) Using the notation given in the question the likelihood L is

$$L = Ke^{-(\mu+\rho)n}e^{-(\nu+\sigma)m}\mu^a\rho^b\nu^c\sigma^d,$$

where K is a constant which does not depend on μ, ν, ρ or σ.

Assumptions are not required as instruction is "write down". An expression omitting K but with a proportionality sign is also acceptable.

(ii) Taking logarithms of the likelihood in (i) produces

$$\ln L = \ln K - (\mu + \rho)n - (\nu + \sigma)m + a \ln \mu + b \ln \rho + c \ln \nu + d \ln \sigma.$$

Differentiating the log-likelihood with respect to μ gives

$$\frac{d \ln L}{d\mu} = -n + \frac{a}{\mu}.$$

Setting this equal to zero and solving for μ gives the maximum likelihood estimate which is

$$\hat{\mu} = \frac{a}{n}.$$

Since $\frac{d^2 \ln L}{d\mu^2} = \frac{-a}{\mu^2}$, which is negative, then the likelihood is, indeed, maximised.

2 (a) H_0: The observed rates are a sample from a population in which the graduated rates are the true rates.

(b) If the null hypothesis is true then the observed number of positive deviations (where deviation = observed number of deaths − expected number of deaths if H_0 is true), P will be such that

$$P \sim \text{Binomial } (97, ½)$$

(c) Using normal approx. to Binomial because binomial "n" parameter is large enough to use the Central Limit Theorem.

Observed value of test statistic $\dfrac{57 - 97 \times ½}{\sqrt{97 \times ½ \times ½}}$

$$= \frac{8.5}{4.92}$$

$$= 1.73$$

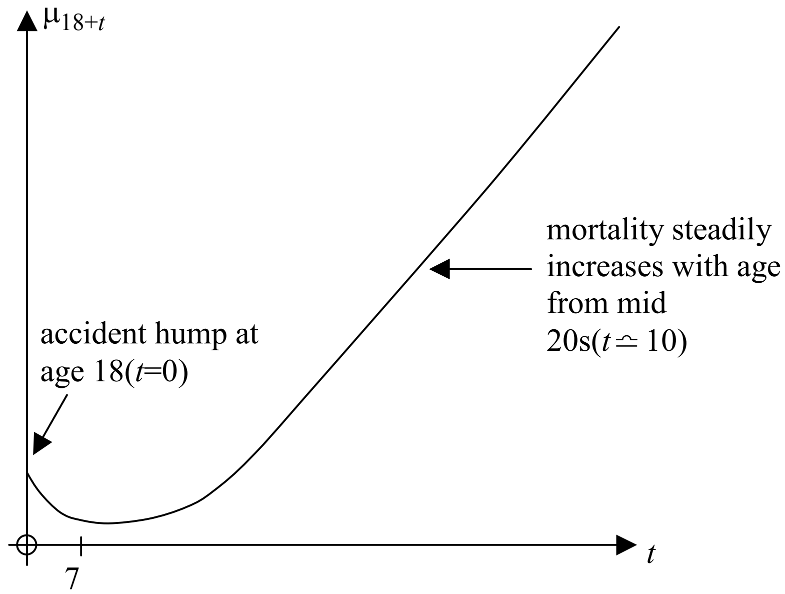
which will be $N(0, 1)$ if H_0 is true.

Now $P[-1.73 < Z < 1.73] < 0.95$ or $P[\,|Z|>1.73] > 0.05$

so no reason to reject null hypothesis and graduation appears acceptable.

A solution using the continuity correction is also acceptable.

3 (i)

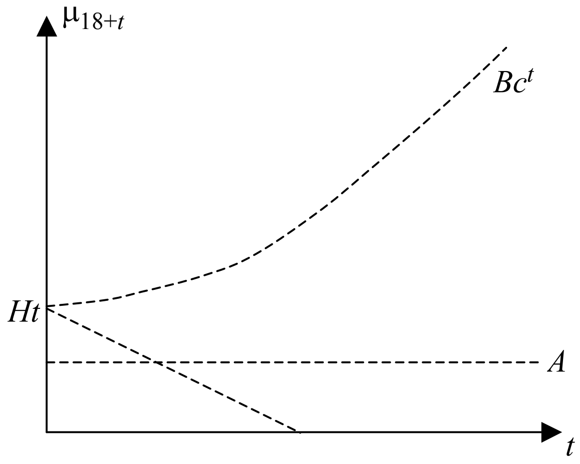


(ii) $\text{GM}(2, 2) = \alpha_1 + \alpha_2 t + \exp(\alpha_3 + \alpha_4 t)$

$= A + Ht + Bc^t$ relabelling constants

This curve has 3 components:

A	independent of age
Ht	varying linearly with age
Bc^t	varying geometrically or exponentially with age



which add together to give required shape.

A	gives "general level" to curve
Ht	a linearly decreasing function gives "accident hump"
Bc^t	reflects the rapid increase in mortality with increasing age

4 (i) Age x = [age last at 6th April prior start of policy] + [5th April's passed]
= age x last on 6th April prior to death
= age $(x + 1)$ next on 6th April prior to death
= age on birthday in calendar year (6th April to 5th April) of death

Age changes on 6 April each year, so calendar year rate interval starting on 6 April.

Age range at start of calendar year = x to $x + 1$

Derivation is not required as the question says "state".

(ii) (a) Principle of correspondence: a life alive at time t should be included in E_x^c if and only if, were that life to die **immediately**, the life would be included in the death data θ_x.

(b) E_x^c = total number of years for which lives were exposed to the risk of dying whilst aged x last birthday on the immediately preceding 6th April.

Or total number of years for which lives were exposed to the risk of dying whilst aged $x + 1$ next birthday on the immediately preceding 6th April during the Period of the Investigation.

Or $\int_0^T P_x(t).dt$ where $P_x(t)$ is a census at time t after the start of the Period of Investigation of those lives aged x last birthday on 6th April immediately prior to t; the period of investigation is $(0, T)$.

(iii) $x + f$ is the mean age of lives half way through the rate interval assuming force of mortality is constant over the calendar year rate interval.

Assuming birthdays are uniformly distributed over the calendar year, the average age at the start of the rate interval is $x + ½$ and halfway through the interval is $x + 1$.

So, $x + f = x + 1$.

In (ii)(b) a correctly specified census formula received partial credit.

5 (i) The Principle of Equivalence says that

$$\begin{matrix}\text{Expected Present Value} \\ \text{of Income to a Policy}\end{matrix} \equiv \begin{matrix}\text{Expected Present Value} \\ \text{of Outgo from a Policy}\end{matrix}$$

(ii) Value of premiums

$$= P\ddot{a}_{44:\overline{10|}} = P \times \left(\ddot{a}_{44} - v^{10} \times \frac{l_{54}}{l_{44}} \times \ddot{a}_{54} \right)$$

$$= P\left(15.859 - (1.05^{-10}) \times \frac{31,926.430}{33,309.271} \times 13.357 \right)$$

$$= P(15.859 - 0.5884265 \times 13.357)$$

$$= 7.99939P$$

Value of annuity

$$= 5,000 \times \frac{D_{54}}{D_{44}} \times a_{54:\overline{10|}}$$

$$= 5,000\left(v^{10} \times \frac{l_{54}}{l_{44}} \right)\{(1.05^{-1}) \times p_{54} \times \ddot{a}_{55:\overline{10|}}\}$$

$$= 5,000 \times 0.5884265 \times \{(1.05^{-1})\,(1 - 0.00755572) \times 7.741\}$$

$= 5,000 \times 0.5884265 \times 7.316677$

$= 21,526.6341$

Equation of value:

$7.99939P = 21,526.6341$

$\Rightarrow \quad P = £2,691.03$ per annum

(iii) Policy value at end of 2004

$_{15}V = 5,000a_{59:\overline{5}|} - P \times 0$

(after deferred period, so no further premiums)

$= 5,000 \ \{(1.05^{-1}) \times p_{59} \times \ddot{a}_{60:\overline{5}|}\}$

$= 5,000 \ \{(1.05^{-1}) \ (1 - 0.01299373) \times 4.409\}$

$= 5,000 \times 4.1445$

$= 20,722.4316$

Expected Death Strain for year 2004

$= -q_{58} \ (5,000 + {}_{15}V)$

$= -0.01168566 \ (5,000 + 20,722.4316)$

$= -£300.58$ i.e. a release of reserves

(iv) If the annuitant died, the actual death strain is

$= -(5,000 + {}_{15}V)$

$= -£25,722.43$

Other methods of evaluation are possible particularly in (ii) and (iii).

6 (i) H_0: the true underlying mortality of the annuitants is that of the standard table.

Age x	*Exposed to Risk* E_x	*Observed Deaths* θ_x	q_x *(from tables)*	$E_x q_x$	$z_x = \dfrac{\theta_x - E_x q_x}{\sqrt{E_x q_x (1-q_x)}}$	z_x^2
70	600	23	0.03776	22.656	0.073676	0.00543
71	750	31	0.04170	31.275	−0.050232	0.00252
72	725	33	0.04602	33.3645	−0.064608	0.00417
73	650	29	0.05075	32.9875	−0.712583	0.50778
74	700	35	0.05595	39.165	−0.684965	0.46918
75	675	39	0.06164	41.607	−0.417228	0.17408
					Total	1.1632

Degrees of freedom for χ^2 test = number of ages = 6.

One tailed test as large values of $\sum z_x^2$ indicate excessive deviations.

$$\chi^2_{6,0.95} = 12.59$$

$\sum z_x^2 < 12.59$, so, there is no evidence to reject H_0.

All deviations but one are negative, which could indicate the true mortality is lighter than a(55). This is not detected by the Chi-squared test as the statistic is based on squared deviations.

(ii) If the true mortality is lighter than a(55), the company will charge inadequate premiums and will suffer a loss on the policies.

(iii) For testing adherence to data, the test statistic is unchanged, but the number of degrees of freedom is reduced.

In fitting the relationship two parameters have been estimated so the number of degrees of freedom will be reduced from 6 to 4, with a further deduction of 2 or 3 degrees of freedom for the constraints imposed by the choice of the standard table.

A solution to (i) using $E_x q_x$ as an estimate of the variance is also acceptable, provided the assumption $1 - q_x \simeq 1$ is stated. Candidates who modified their answers because the data was an extract from the whole experience received credit.

7 (i) $h(x, t) = h_0(t)\exp(\beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k)$
where $h(x, t)$ is the hazard at duration t, $h_0(t)$ is some unspecified baseline hazard, $x_1 \ldots x_k$ are covariates and $\beta_1 \ldots \beta_k$ are their associated parameters.

(ii) Men who were "unable to care for themselves" at the time of diagnosis, who were given the "existing" treatment, and whose tumours were of the "large" type.

(iii) The value of the parameter associated with the new treatment is 0.25. This implies that the ratio of the hazards of death for two otherwise identical patients, one of whom is given the new treatment and the other the existing treatment is exp(0.25) = 1.28. Thus the new treatment appears to increase the risk of death.

However, the standard error associated with the parameter is 0.25. The approximate 95% confidence interval is therefore $0.25 \pm 1.96(0.25) = (-0.24, 0.74)$, which includes 0. Therefore, the value of the parameter is not significantly different from zero at the 5% level, so it is not possible to say with the available data whether the new treatment affects the risk of death.

(iv) The hazard for men with "adeno" type tumours who were "able to care for themselves" at the time of diagnosis is $h_0(t)\exp(-0.60 + 0.75)$.

The hazard for men with "large" type tumours who were "unable to care for themselves" at the time of diagnosis is $h_0(t)$, since this is the baseline category.

The ratio is thus

$$\frac{h_0(t)\exp(-0.60+0.75)}{h_0(t)} = \exp(0.15) = 1.16$$

so the risk of death is 16% greater for men with "adeno" type tumours who were "able to care for themselves" at the time of diagnosis.

8 (i) Let the time individual i enters observation be a_i and the time that individual i leaves observation be b_i. Define an indicator variable d_i such that $d_i = 0$ if individual i is not observed to die and $d_i = 1$ if individual i dies. Then the traditional actuarial estimate of q_{60} is

$$\hat{q}_{60} = \frac{\sum_{i=1}^{10} d_i}{\sum_{i=1}^{10} (1 - a_i - [(1-d_i)(1-b_i)])}.$$

The calculations are shown in the table below.

Person	a_i	b_i	d_i	$1 - a_i$	$1 - b_i$	$1 - a_i - (1 - d_i)(1 - b_i)$
1	0	6/12	0	1	6/12	6/12
2	1/12	1	0	11/12	0	11/12
3	1/12	3/12	1	11/12	9/12	11/12
4	2/12	1	0	10/12	0	10/12
5	3/12	9/12	1	9/12	3/12	9/12
6	4/12	1	0	8/12	0	8/12
7	5/12	11/12	1	7/12	1/12	7/12
8	7/12	1	0	5/12	0	5/12
9	8/12	10/12	1	4/12	2/12	4/12
10	9/12	1	0	3/12	0	3/12
Totals			4			74/12

Therefore $\hat{q}_{60} = \dfrac{4}{74/12} = 0.6486.$

ALTERNATIVELY

The data allow the exact calculation of the initial exposed to risk. The approximate calculation using

$$E_x^c + ½\theta_x = \frac{59}{12} + \frac{1}{2} \times 4 = \frac{83}{12}$$

makes an unnecessary assumption, giving $\hat{q}_{60} = 0.5783.$

The assumption is that the dates of death are uniformly distributed over the life year (60, 61).

(ii) In the two-state model we estimate $\hat{\mu}_{60} = \dfrac{\sum_{i=1}^{10} d_i}{\sum_{i=1}^{10} (b_i - a_i)}$. The necessary working is shown in the table below.

Person	a_i	b_i	$b_i - a_i$
1	0	6/12	6/12
2	1/12	1	11/12
3	1/12	3/12	2/12
4	2/12	1	10/12
5	3/12	9/12	6/12
6	4/12	1	8/12
7	5/12	11/12	6/12
8	7/12	1	5/12
9	8/12	10/12	2/12
10	9/12	1	3/12
Totals			59/12

So $\hat{\mu}_{60} = \dfrac{4}{59/12} = 0.81355$

and assuming that the force of mortality is constant over (60, 61)

$\hat{q}_{60} = 1 - \exp(-\mu) = 0.5567.$

(iii) The estimates of q_{60} differ, the actuarial estimate being higher than the estimate produced by the two-state model.

The two-state model uses all the information we are given, including that of times of death, whereas the actuarial estimate places relatively more weight on the number of deaths.

In this case, μ is large, so most of the information is in the times of death rather than the number of deaths, and the actuarial estimate does not produce acceptable results. The estimate from the two-state model is to be preferred.

Explicit statements of formulae are not necessary, provided they are implicit in the calculations presented. The alternative solution to (i) received partial credit.

9 (i) $P_{K_{70}}(t) = {}_tp_{70}\, q_{70+t} = \dfrac{d_{70+t}}{\ell_{70}}$ $\quad t = 0, 1, 2, 3, \ldots$

Using tables we obtain $\ell_{70} = 54{,}806$

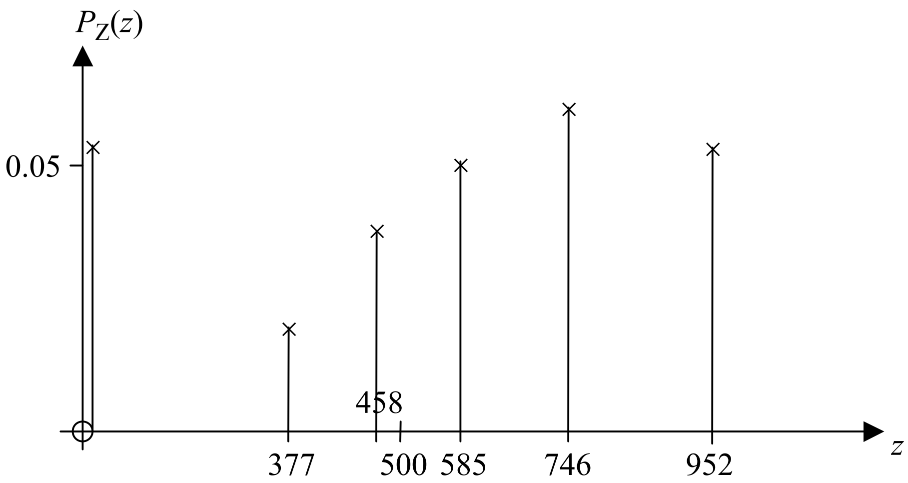
t	*0*	*5*	*10*	*15*	*20*
d_{70+t}	3,051	3,282	2,923	1,897.4	779.9
Value	0.05567	0.05988	0.05333	0.03462	0.01422

Explicit statements of formulae are not needed provided they are implicit in the calculations presented.

(ii) (a) $Z = \begin{cases} 1{,}000(1.05)^{-t-1} & t = 0, 1, 2, \ldots, 19 \\ 0 & t \geq 20 \end{cases}$

So values for density function are

t	*0*	*5*	*10*	*15*	*19*	*≥ 20*
z	952	746	585	458	377	0
Height of probability function	0.056	0.060	0.053	0.035	0.018	$\dfrac{\ell_{90}}{\ell_{70}} = 0.056$



The graph should be labelled to explain its shape and the "spike" at Z = 0.

(b) From the results in (ii)(a) we can see that $z = 500$ will occur between $t = 10$ and $t = 15$.

Expanding the tabulation in (ii)(a) gives

t	*11*	*12*	*13*	*14*
z	557	530	505	481

ALTERNATIVELY

We can find the "cross-over" point by solving

$$1{,}000(1.05)^{-t-1} = 500$$

$$(1.05)^{t+1} = \frac{1{,}000}{500} = 2$$

$$t = \frac{\log_e 2}{\log_e 1.05} - 1 = 13.21 \text{ years}$$

So $t = 13$ gives $Z > 500$ and $t = 14$ gives $Z < 500$.

So $Z > 500$ corresponds to $K_{70} \leq 13$ that the life dies in the interval (70, 84). We can write

$$P[Z > 500] = P[K_{70} \leq 13]$$

$$= 1 - \frac{\ell_{84}}{\ell_{70}}$$

$$= 1 - \frac{12{,}306}{54{,}806}$$

$$= 1 - 0.22454$$

$$= 0.775$$

10 (i) $$\ddot{a}_{x:\overline{n|}} = E\left[\ddot{a}_{\overline{\min(K_x+1,n)|}}\right]$$

$$= E\left[\frac{1-v^{\min(K_x+1,n)}}{d}\right]$$

$$= \frac{1}{d} - \frac{1}{d}E\left[v^{\min(K_x+1,n)}\right]$$

$$= \frac{1}{d} - \frac{1}{d}A_{x:\overline{n|}}$$

So $A_{x:\overline{n|}} = 1 - d\ddot{a}_{x:\overline{n|}}$

ALTERNATIVELY

Let $$C_x = v^{x+1}d_x$$

$$= v^{x+1}(\ell_x - \ell_{x+1})$$

$$= v.v^x\ell_x - v^{x+1}\ell_{x+1}$$

$$= vD_x - D_{x+1}$$

Then $$\sum_{t=0}^{t=\infty} C_{x+t} = M_x$$

$$= v\sum_{t=0}^{t=\infty} D_{x+t} - \sum_{t=0}^{t=\infty} D_{x+t+1}$$

$$= vN_x - N_{x+1}$$

So $$A_{x:\overline{n|}} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

$$= \frac{vN_x - N_{x+1} - vN_{x+n} + N_{x+n+1} + D_{x+n}}{D_x}$$

since $N_{x+n+1} + D_{x+n} - N_{x+1}$

$$= N_{x+n} - (N_x - D_x)$$

$$= N_{x+n} - N_x + D_x$$

then $$A_{x:\overline{n}|} = \frac{(1-d)(N_x - N_{x+n}) - (N_x - N_{x+n}) + D_x}{D_x}$$

$$= 1 - d\ddot{a}_{x:\overline{n}|}$$

ALTERNATIVELY

$$A_{x:\overline{n}|} = \sum_{k=0}^{k=n-1} v^{k+1} \cdot {}_{k|}q_x + v^n {}_np_x$$

Now ${}_{k|}q_x = {}_kp_x - {}_{k+1}p_x$

Substitute

$$= \sum_{k=0}^{k=n-1} v^{k+1} {}_kp_x - \sum_{k=0}^{k=n-1} v^{k+1} {}_{k+1}p_x + v^n {}_np_x$$

Then EITHER

$$= v\ddot{a}_{x:\overline{n}|} - (\ddot{a}_{x:\overline{n+1}|} - 1) + v^n {}_np_x$$

$$= v\ddot{a}_{x:\overline{n}|} - \ddot{a}_{x:\overline{n}|} + 1$$

$$= 1 - (1-v)\ddot{a}_{x:\overline{n}|}$$

$$= 1 - d\ddot{a}_{x:\overline{n}|}$$

OR

$$= v + v^2p_x \ldots + v^n {}_{n-1}p_x - (vp_x + v^2 {}_2p_x \ldots + v^{n-1} {}_{n-1}p_x + v^n {}_np_x) + v^n p_x$$

$$= 1 - (1 - v)(1 + vp_x \ldots + v^{n-1} {}_{n-1}p_x)$$

$$= 1 - d\ddot{a}_{x:\overline{n}|}$$

(ii) (a) $$ {}_t p'_x = \exp\left\{-\int_{u=0}^{u=t} \mu'_{x+u}.du\right\}$$

$$= \exp\left\{-\int_{u=0}^{u=t} \mu_{x+u} + k.du\right\}$$

$$= \exp\left\{-\int_{u=0}^{u=t} \mu_{x+u}.du\right\} \exp\left\{-\int_{u=0}^{u=t} k.du\right\}$$

$$= {}_t p_x.\exp(-kt)$$

(b) $$\ddot{a}'_{x:\overline{n|}} = \sum_{t=0}^{t=n-1} {}_t p'_x.v^t$$

$$= \sum_{t=0}^{t=n-1} {}_t p_x.e^{-kt}.v^t$$

$$= \sum_{t=0}^{t=n-1} {}_t p_x(v.e^{-k})^t$$

$$= \sum_{t=0}^{t=n-1} {}_t p_x v_j^t$$

$$= \ddot{a}_{x:\overline{n|}j}$$

where $\dfrac{1}{1+j} = \dfrac{1}{(1+i)}.\dfrac{1}{e^k}$

i.e. $j = (1 + i)\, e^k - 1$

(iii) Use (i): $A'_{60:\overline{20|}6\%} = 1 - d_{6\%}\ddot{a}'_{60:\overline{20|}6\%}$

Use (ii): $\ddot{a}'_{60:\overline{20|}6\%} = \ddot{a}_{60:\overline{20|}j}$

where $j = (1.06)e^{0.018692} - 1 = 0.08$ i.e. 8%

Also $A'^{\;\;1}_{60:\overline{20}|6\%} = \dfrac{D'_{80}}{D'_{60}} = v^{20}_{6\%}\ {}_{20}p'_{60}$

$= (1.06)^{-20} \cdot {}_{20}p_{60} \cdot e^{-20\times 0.018692}$

$= (1.06)^{-20}\ \dfrac{363,991}{859,916}\, e^{-0.37384}$

$= 0.31180 \times 0.42329 \times 0.68809$

$= 0.09082$

Alternatively $\dfrac{D_{80}}{D_{60}}$ @ 8% $= \dfrac{771.2}{8,492.4} = 0.09081$

$\ddot{a}'_{60:\overline{20}|6\%} = \ddot{a}_{60:\overline{20}|8\%}$

$= \ddot{a}_{60} - \dfrac{D_{80}}{D_{60}}\ddot{a}_{80}$

$= (8.432 + 1) - 0.09082 \times (3.989 + 1)$

$= 8.97890$ (8.97895 with 0.09081)

Then $A'_{60:\overline{20}|6\%} = 1 - \dfrac{0.06}{1.06} \times 8.97890 = 0.49176$

Then Premium $= \dfrac{10,000(0.49176 - 0.09082)}{8.97890}$

$= 446.5362$ £446.54 p.a.