Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

April 2004

Subject 104 — Survival Models

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

J Curtis
Chairman of the Board of Examiners

22 June 2004

EXAMINERS' COMMENTS

As in previous years, the Examiners aimed to set questions covering all the aspects of survival modelling: life contingencies including its stochastic treatment, graduation including its statistical aspects and the determination of exposures. The Examiners aim to strike a balance between questions requiring numerical solutions and those requiring verbal and algebraic answers, as well as between those with and without a statistical theme.

Comments on solutions presented to individual questions for the April 2004 paper are given below:

Question 1: *On previous occasions, questions on this recursive relationship have been poorly answered. However, this question was well answered, with many candidates scoring full marks.*

Question 2: *This was reasonably well answered. Some candidates did not understand left censoring and some lost marks for unclear reasons given.*

Question 3: *Part (i) was well answered, although the working was not always clear.*
Part (ii) was not well answered. Few candidates made sufficient points to score highly and a number of candidates ignored the circumstances of the question and debated the merits of the Binomial and Poisson models.

Question 4: *This was well answered, particularly part (i).*
Marks were lost by candidates who failed to show how their estimate was reached in (i) or who did not produce a sensible argument in (ii).

Question 5: *This was very poorly answered.*
In part (i), few candidates remembered the correct formula for h(t), and fewer could explain what it meant.
In part (ii) many candidates struggled to produce a full, coherent explanation.
In part (ii) very few candidates made any real progress. Many candidates showed a lack of understanding by attempting to show that the ratio of the survival functions (not the hazard functions) for two individuals was constant over time.

Question 6: *Overall this was reasonably well answered.*
In part (i), many candidates missed the value of L if death occurred after age 67. Candidates lost marks if they went straight to the calculation of E(L) and failed to show the values of L as requested. Some candidates did not understand that L was a present value, not an expected present value, and attempted to include survival or death probabilities in its calculation.
Part (iii) was less well answered.

Question 7: *Part (i) was well answered.*
Solutions to part (ii)(a) were particularly poor. Candidates should note that the question asked them to "derive a formula" and therefore most of the marks were available for working logically through the derivation rather than the final answer. Very few candidates produced a coherent argument here.
For part (ii)(b), many candidates produced the correct answer; those who did not often lost marks as little or no working was shown.
Many candidates did not attempt part (iii). Again, those who produced incorrect answers often lost marks as little or no working was shown

Question 8: *Part (i) was well answered by the majority of the candidates. Candidates who lost marks generally did so for not giving the integrated hazard (F(t) and S(t) were often quoted), or not giving the correct ranges for time.*
Part (ii) was intended to demonstrate if candidates knew how to use the hazard function. Many did not, with ${}_t p_x = \exp(-\Lambda_t \cdot t)$ rather than ${}_t p_x = \exp(-\Lambda_t)$ being a common error. Other common errors were not making allowance for continuous payments, ignoring interest and not providing food for the first year of the dog's life.
A surprising number of candidates did not attempt part (iii), although this was reasonably straightforward and most of the marks could be gained without having an answer for part (ii). Those who did answer, did so reasonably well. The most common errors were in the value of the legacies, being the use of A_{70} rather than $\bar{A}_{70}$; incorrect interpretation of the effect of payments being one year after death; and an inability to estimate A_{70}, which is not given in the tables.

Question 9: *This was reasonably well answered overall.*
Part (i) was well answered. Some candidates lost marks in (a) by not defining all the terms used.
In part (ii) marks were most commonly lost because of lack of relevant detail in carrying out the tests.
Part (iii) was less well answered. Many candidates just re-stated the conclusions of their tests, rather than explaining what this meant in respect of this investigation.
Part (iv) was fairly standard bookwork and was reasonably well answered. Candidates lost marks by not giving sufficient explanation of the points made.

1 We can use the recursive formula:

$$V_5(1 + i) = p_{65} \,.\, V_6 + q_{65} \,.\, 3{,}000$$

So that

$$V_6 = (V_5(1 + i) - q_{65} \,.\, 3{,}000) \times \frac{1}{p_{65}}$$

$$= (1200(1.06) - 0.02 \times 3{,}000) \times \frac{1}{(1-0.02)}$$

$$= \frac{1{,}212}{0.98}$$

$$= 1{,}236.73$$

$$= £1{,}237 \text{ to nearest } £.$$

2 (a) *Left-censoring*
Not present.
We know for all the toys in the observation exactly when they were turned on.

(b) *Interval censoring*
Present.
We only know the hour during which the toy ceased to function.

(c) *Type I censoring*
Present
Those fluffy toys still running after 24 hours are type I censored.

(d) *Non-informative censoring*
Present
Those not turned on, or failing due to mechanical problems, or censored after 24 hours tell us nothing about battery life.

3 (i) The death data will be classified as age = calendar year of death – calendar year of birth. This is the same as age at the birthday in the calendar year of death.

This is a calendar year rate interval, and the range of exact ages for a life labelled x at the start of the rate interval is (x – 1, x).

So the average age at the start of the rate interval is x – 0.5 and because we are using a binomial model, this is the age to which the calculated rate would apply.

(ii) Some of the difficulties will be:

- low volume of death data
- deaths will cover maybe 200+ years, during which time mortality patterns will have changed greatly
- there could be selection issues — what about people too poor to have a tombstone?
- it will be almost impossible to estimate a corresponding exposed to risk — due to lack of data, migration etc.
- some tombstones could be more weathered than others (and so illegible)
- people may have been buried in the graveyard who did not live in the village. It may not be possible to identify this and eliminate them from the analysis
- For the older tombstones in particular, the data might be incorrect (e.g. the correct date of birth might not have been known)

Credit was given for other valid comments concerning the particular circumstances

4 (i) (a) The assumption of a uniform distribution of deaths implies that

$$_t q_x = t q_x \qquad (\text{for } 0 \le t \le 1).$$

Since q_{80} = 1 – (20,010/22,933) = 0.12746,

$$_{0.5}q_{80} = 0.5 \times 0.12746 = 0.06373$$

and $_{0.5}p_{80} = 1 - {}_{0.5}q_{80} = 1 - 0.06373 = 0.93627$.

(b) Let the constant force of mortality be μ. Then

$$\mu = -\log p_{80} = -\log \frac{20,010}{22,933} = 0.13634$$

and ${}_{0.5}p_{80} = \exp(-0.5\mu) = \exp(-0.5 \times 0.13634) = 0.93410$.

(ii) The two estimates in (i) are different because they make different assumptions about the distribution of the force of mortality within the year of age from 80 to 81 years.

The uniform distribution of deaths (UDD) assumption implies an increasing force of mortality within this age range.

Since it is likely that the true force of mortality is increasing with age in this age range, the UDD estimate is to be preferred.

5 (i) $h(t) = \lim\limits_{dt \to 0} (1/dt) \times \Pr[X \le t + dt \mid X > t]$

For a small interval of time *dt*, after duration *t*, the probability that the event will occur is *dt* multiplied by the hazard.

(ii) In a proportional hazards model, the hazard factorises into two parts, algebraically,

$$h(t) = h_0(t)\, f(Z_1, Z_2)\,.$$

One part (the baseline hazard) is the same for all individuals and depends only on duration *t*, and
the other depends on the values of the covariates (in this case Z_1 and Z_2), which vary among individuals.

A feature of this model is that the ratio between the hazards for two individuals with different values of Z_1 and Z_2 does not depend on duration *t*.

(iii) We have

$$S(t) = \exp\,[-(\lambda t)^{\alpha}].$$

Therefore

$$\log S(t) = -(\lambda t)^{\alpha}.$$

Since the hazard $h(t) = d[-\log S(t)]/dt$,

$$h(t) = d[(\lambda t)^{\alpha}]/\mathrm{dt} = \lambda^{\alpha}\alpha t^{\alpha-1}.$$

Letting λ be a function of Z_1 and Z_2, so that

$$\lambda = g(Z_1, Z_2)\,,$$

and substituting into the expression above for $h(t)$

$$h(t) = [g(Z_1, Z_2)]^{\alpha}\ \alpha t^{\alpha-1},$$

which is of the form

$$h(t) = h_0(t)\ .\ \ f(Z_1, Z_2),$$

where

$$h_0(t) = \alpha t^{\alpha-1} \text{ and } f(Z_1, Z_2) = [g(Z_1, Z_2)]^{\alpha}.$$

Therefore the hazard factorises into two parts, one depending only on t and one depending only on Z_1 and Z_2, so the model is a proportional hazards model.

6 (i) The relevant calculations are:

Age at death	*Probability*	*PV Benefits*	*PV Premiums*	*L*	*L* x *Prob*	L^2 x *Prob*
65	0.01424	£9,708.74	£400.00	£9,308.74	£132.58	£1,234,193
66	0.01571	£14,138.94	£788.35	£13,350.59	£209.78	£2,800,651
67	0.01729	£27,454.25	£1,165.39	£26,288.86	£454.54	£11,949,237
>67	0.95275	£0.00	£1,165.39	–£1,165.39	–£1,110.33	£1,293,963
					–£313.43	**£17,278,044**

Therefore $E(L) = –£313$

and
$\text{Var}(L) = 17{,}278{,}044 - 313.43^2$
$= 17{,}179{,}806$
so the standard deviation of L is £4,145.

(ii) The relevant calculations are:

Age at death	*Probability*	*PV Benefits*	*PV Premiums*	*L**	*(L*)* x *Prob*	$(L^*)^2$ x *Prob*
65	0.01424	£9,708.74	£9,500.00	£208.74	£2.97	£621
66	0.01571	£14,138.94	£18,723.30	–£4,584.36	–£72.03	£330,230
67	0.01729	£27,454.25	£27,677.96	–£223.71	–£3.87	£865
>67	0.95275	£27,454.25	£27,677.96	–£223.71	–£213.14	£47,683
					–£286.07	**£379,399**

So $E(L^*) = –£286$

and
$\text{Var}(L^*) = 379{,}399 - 286.07^2$
$= 297{,}563$
so the standard deviation of L^* is £545.

(iii) Although the means are more or less unchanged, the introduction of the endowment element has greatly reduced the variance of the loss. This is because there is much more certainty over the amount of the benefit that will be paid — only the timing remains uncertain.

7 (i) Suppose there are N voluntary leavers, M involuntary leavers and D deaths. Then if we let $W = \sum_i W_i$ be the total waiting time in the "working" state, the likelihood function can be written as:

$$L = K \times \exp(-W(\lambda + \mu + \sigma)) \times \lambda^N \times \mu^D \times \sigma^M$$

where K is some constant. The log-likelihood function is therefore

$$l = \log L = K' + N \log \lambda - W\lambda$$

where K' is another constant, independent of λ. Differentiating with respect to λ gives

$$\frac{dl}{d\lambda} = \frac{N}{\lambda} - W$$

and equating this expression to 0 gives

$$\frac{N}{\hat{\lambda}} - W = 0 \text{ i.e. } \hat{\lambda} = \frac{N}{W}$$

To check we have a maximum, note that

$$\frac{dl^2}{d\lambda^2} = -\frac{N}{\lambda^2} < 0 .$$

The formula for the variance of the estimator is

$$Var(\hat{\lambda}) = \frac{\lambda}{E(W)} .$$

(ii) (a) The age classification of the exit data is "age last birthday on employment anniversary prior to exit". By the principle of correspondence, we must estimate the exposure on the same basis. If we knew $P'(30, t)$ which is defined as the population aged 30 last birthday at employment anniversary prior to t then the exposure would be:

$$E_{30}^{c} = \int_{0}^{1} P'(30,t)\,dt$$

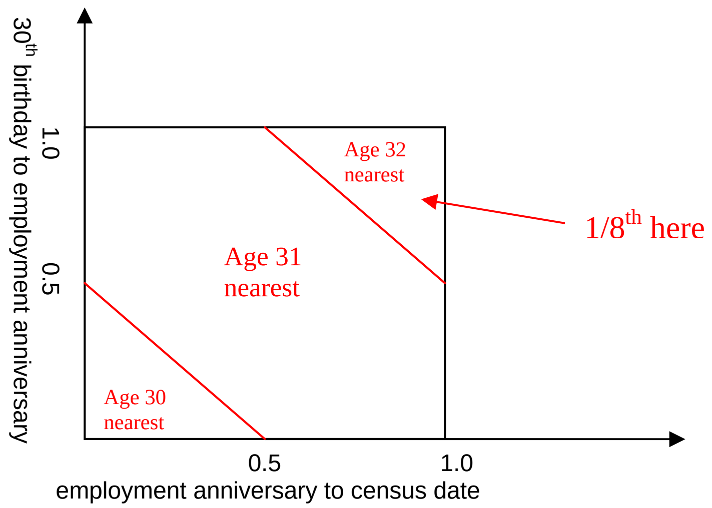
and assuming the population varies approximately linearly over the year, we could approximate this by
$E_{30}^{c} = 0.5 \times (P'(30,0) + P'(30,1))$.

The range of exact ages that could apply to a life aged 30 last birthday on employment anniversary prior to leaving is (30, 32).

The uniformity assumptions mean that the time elapsing between the previous employment anniversary and the date of death is uniformly distributed, as is the time between the previous employment anniversary and the birthday before that. Therefore the time between the birthday before the previous employment anniversary and the date of death is distributed as the sum of two independent identical uniform distributions and

$$P'(30,t) = \frac{1}{8} \times (P(30,t) + 6P(31,t) + P(32,t))$$

A diagram may be used to demonstrate this:



and substituting into the approximation above, we have

$$E_{30}^{c} \approx \frac{1}{2}\left(\begin{array}{l} \frac{1}{8}\times\left(P(30,0)+6P(31,0)+P(32,0)\right) \\ +\frac{1}{8}\times\left(P(30,1)+6P(31,1)+P(32,1)\right) \end{array} \right)$$

ie,

$$E_{30}^{c} \approx \frac{1}{16}\left(P(30,0)+6P(31,0)+P(32,0)+P(30,1)+6P(31,1)+P(32,1)\right)$$

Credit was also given for the following approximation (although it should be noted that the majority of the marks for this part of the question were available for the derivation rather than the final answer):

Assuming employment anniversaries are uniformly distributed across calendar years, then the "average" anniversary prior to 1 January 2002 will be 1 July 2001. A life aged 30 last birthday on 1 July 2001 is aged between 30 and 31 on that date and so will be aged between 30.5 and 31.5 on 1 January 2002. So, we can approximate

$P'(30,t) \approx$ on average, the number of employees on 1 January 2002 aged between 30.5 and 31.5

$\approx$ the number of employees on 1 January 2002 aged 31 nearest

$\approx P\left(31,0\right)$

and substituting into the approximation above, we have

$$E_{30}^{c} \approx \frac{1}{2}\times\left(P(31,0)+P(31,1)\right)$$

(b) If we denote the number of voluntary exits in 2002 by employees aged 30 last birthday on last employment anniversary by N_{30} then the estimator is

$$\hat{\lambda}_{30} = N_{30} / E_{30}^{c}.$$

This is a central rate, applying to ages at the middle of the rate interval. Since the age range at the start of the rate interval (crossing an employment anniversary) is (30, 31) the average age at the start of the rate interval is 30.5 and the average age in the middle is 31.

(iii) In this case, the age definition is the same as "age last birthday" so that the census formula would have to be adapted (indeed simplified) accordingly.

If we define $P''(30,t)$ as the population aged 30 last birthday at time t then the exposure would be:
$$P''(30,t) = 0.5 \times \left(P(30,t) + P(31,t)\right)$$

And, using (as before)
$$E_{30}^{c} = 0.5 \times \left(P''(30,0) + P''(30,1)\right)$$

The relevant formula is then
$$E_{30}^{c} \approx 0.25(P(30,0) + P(31,0) + P(30,1) + P(31,1))$$

with the calculated rate applying at age 30.5.

8 (i) The relevant calculations are:

Time	*Number of deaths*	*Number of other exits*	*Lives exposed*		
t_j	d_j	c_j	n_j	λ_j	$\Sigma\lambda_j$
0	0	2	100	0	0
0.5	1	8	98	1/98	0.0102
4	2	1	89	2/89	0.03268
6	1	4	86	1/86	0.0443

Thus the Nelson-Aalen estimate of the integrated hazard is:

t	Λ_t
$0 \le t < 0.5$	0
$0.5 \le t < 4$	0.0102
$4 \le t < 6$	0.03268
$6 \le t$	0.0443

(ii) The values given for the integrated hazard mean that the survival function is given by ${}_t p_x = \exp\{-\Lambda_t\}$

$${}_t p_0 = 1 \qquad 0 < t < 0.5$$
$${}_t p_0 = 0.9899 \qquad 0.5 \le t < 4$$
$${}_t p_0 = 0.9678 \qquad 4 \le t < 6$$
$${}_t p_0 = 0.9567 \qquad 6 \le t$$

And so the cost of training is given by

$$V = £3,000 \times \int_0^1 {}_t p_0 \exp(-\delta t)dt$$

$$= £3,000 \times \left(\bar{a}_{\overline{0.5}|} + 0.9899 v^{0.5} \bar{a}_{\overline{0.5}|}\right)$$

$$= £3,000 \times (0.49513 + 0.9899 \times 0.98058 \times 0.49513)$$

$$= £2,927$$

Similarly, the cost of the food etc is calculated as:

$$F = £2,000 \times (\bar{a}_{\overline{0.5}|} + v^{0.5} \times 0.9899 \times \bar{a}_{\overline{3.5}|} + 0.9678 \times v^4 \times \bar{a}_{\overline{2}|} + 0.9567 \times v^6 \times \bar{a}_{\overline{4}|})$$

$$= £2,000 \times (0.49513 + 0.9899 \times 0.98058 \times 3.27040 + 0.9678 \times 0.85480$$

$$\times 1.92357 + 0.9567 \times 0.79031 \times 3.70202)$$

$$= £16,120$$

and finally, the value of the retirement benefit is

$$£300 \times {}_{10} p_0 \times v^{10} = £300 \times 0.9567 \times 0.67556 = £194$$

so the capital cost of providing a guide dog is

£194 + £16,120 + £2,927 = £19,241.

(iii) The value can be calculated in 2 parts. Firstly, annual commitments are worth:

$$£125,000 \times \ddot{a}_{50:\overline{10}|}$$

Now $\ddot{a}_{50:\overline{10}|} = \ddot{a}_{50} - {}_{10} p_{50} v^{10} \ddot{a}_{60} = 19.539 - \dfrac{9848.431}{9952.697} \times 1.04^{-10} \times 16.652 = 8.407$

so the annual commitments are worth £125,000 × 8.407 = £1,050,875.

The values of the legacies are

$$£900,000 \times v \times \bar{A}_{70} \approx £900,000 \times v \times 1.04^{½} \times A_{70}$$

and we can calculate the assurances via premium conversion:

$$A_{70} = 1 - d\ddot{a}_{70} = 1 - 0.04/1.04 \times (12.934) = 0.5025$$

So the value of the legacies is £900,000 × $1.04^{-½}$ × 0.5025 = £443,468.

The total value of the benefactions is

£1,050,875 + £443,468 = £1,494,343

and the number of guide dogs that can be provided is

1,494,343 / 19,241 = 77.

9 (i) (a) If the observed number of deaths at age x is θ_x, the exposed-to-risk at age x is E_x, and the mortality rate in English Life Table 15 (Males) at age x is q_x, then the standardised deviation in age-group x, z_x, is given by the formula

$$z_x = \frac{\theta_x - E_x q_x}{\sqrt{E_x q_x}},$$

using the approximation $E_x q_x \simeq E_x q_x (1 - q_x)$,

Alternatively, if the observed number of deaths at age x is θ_x, the central exposed-to-risk at age x is E_x^c, and the force of mortality in English Life Table 15 (Males) at age x is $\mu_{x+0.5}$, then the standardised deviation in age-group x, z_x, is given by the formula

$$z_x = \frac{\theta_x - E_x^c \mu_{x+0.5}}{\sqrt{E_x^c \mu_{x+0.5}}}.$$

(b) The null hypothesis is that the underlying mortality of the lives in the investigation is that of English Life Table 15 (Males).

(c) The test statistic is $\sum_x z_x^2 \sim \chi_m^2$,

where m is the number of age groups (m = 10 in our case), because we are comparing an experience with a standard table.

(d) The calculations are shown in the table below

Age x	*E*	θ_x	q_x	$E_x\ q_x$	z_x	z_x^2
18	34,000	40	0.00087	29.58	1.9159	3.6707
19	33,000	35	0.00083	27.39	1.4541	2.1144
20	29,500	27	0.00084	24.78	0.4460	0.1989
21	30,000	26	0.00086	25.80	0.0394	0.0016
22	25,500	22	0.00089	22.70	–0.1469	0.0216
23	24,000	19	0.00089	21.36	–0.5106	0.2607
24	17,000	13	0.00088	14.96	–0.5067	0.2567
25	23,500	20	0.00086	20.21	–0.0467	0.0022
26	18,000	12	0.00085	15.30	–0.8437	0.7118
27	14,000	11	0.00085	11.90	–0.2609	0.0681
Sums				**213.98**	**1.5399**	**7.3067**

Using the data in the table above, $\sum_x z_x^2 = 7.3067$.

The critical value of the χ^2_{10} distribution at the 5% level is 18.31, which is much greater than the calculated value.

So we accept the null hypothesis.

On the basis of this test, we conclude that the underlying mortality of the lives in our investigation is represented by English Life Table 15 (Males), and that the suggestion seems to be true.

(ii) (a) (1) There could be a few large deviations offset by a lot of very small deviations. The chi-squared test will be satisfied, but the data do not satisfy the distributional assumptions which underlie it.

(2) The graduation might be biased above or below the data by a smallish amount at all ages.

(3) Even if the graduation is not biased as a whole, there could be significant groups of consecutive ages (runs or clumps) over which it is biased up or down.

Full marks were available for any two of these.

(b) *The appropriate tests are as follows.*
For difference (1) (few large deviations): standardised deviations test.
For difference (2) (overall bias): signs test; cumulative deviations test over whole age range.
For difference (3) (runs or clumps): grouping of signs (Stevens') test; serial correlation test.
Candidates were expected to perform one test for each of the difference they identified in part (a).

STANDARDISED DEVIATIONS TEST

This tests for the possibility that there are a small number of age groups with large differences between the mortality rates in the investigation and the standard table.

The z_xs comprise m independent samples from a Normal (0,1) distribution. We can compare the expected and actual number of z_x in the following intervals

Interval	(-∞, -3)	(-3, -2)	(-2, -1)	(-1, 0)	(0, 1)	(1, 2)	(2, 3)	(3, ∞)
Expected number	0	0.2	1.4	3.4	3.4	1.4	0.2	0
Actual number	0	0	0	6	2	2	0	0

Therefore under the null hypothesis we should expect fewer than 1 in 20 to be > 2 in absolute magnitude.

In this case none of the z_xs exceeds 2 in absolute value.

So we accept the null hypothesis.

SIGNS TEST

This tests for the possibility of the mortality rates in the investigation being systematically lower or higher than those in the standard table.

Let P be the number of z_xs that are positive.

Then under the null hypothesis, P ~ Binomial(10, 0.5).

We have 4 positive signs. The probability of getting 4 or fewer positive signs if the null hypothesis is true is

$$\binom{10}{4}\frac{1}{2^{10}}+\binom{10}{3}\frac{1}{2^{10}}+\binom{10}{2}\frac{1}{2^{10}}+\binom{10}{1}\frac{1}{2^{10}}+\binom{10}{0}\frac{1}{2^{10}}$$
$$=\frac{10!}{4!6!}\frac{1}{2^{10}}+\frac{10!}{3!7!}\frac{1}{2^{10}}+\frac{10!}{2!8!}\frac{1}{2^{10}}+\frac{10!}{1!9!}\frac{1}{2^{10}}+\frac{1}{2^{10}}$$
$$= (210+120+45+10+1)0.0009765625$$
$$= 0.37695$$

[Alternatively, candidates could just evaluate the probability of getting exactly 4 positive deviations, which is 0.205].

This is greater than 0.025 (2-tailed test)

We accept the null hypothesis.

CUMULATIVE DEVIATIONS TEST

When using the whole age range, this tests for the possibility of the mortality rates in the investigation being systematically lower or higher than those in the standard table.

Under the null hypothesis,

$$\frac{\sum_{x=1}^{m}(\theta_x - E_x q_x^s)}{\sqrt{\sum_{x=1}^{m} E_x q_x^s}} \sim \text{Normal}(0,1)$$

Using the data in the table, we have

$$\frac{\sum_{x=1}^{m}(\theta_x - E_x q_x^s)}{\sqrt{\sum_{x=1}^{m} E_x q_x^s}} = \frac{11.025}{\sqrt{213.98}} = 0.75369$$

Since both positive and negative cumulative deviations are of interest we use a two-tailed test.

Since $|0.75369|<1.96$,

we accept the null hypothesis.

GROUPING OF SIGNS TEST

This tests for runs of deviations of the same sign, that is for subsections of the age range for which the mortality rates of lives in the investigation are systematically lower or higher than the rates in the standard table.

Let G be the number of groups of positive z_xs, n_1 be the number of positive z_xs and n_2 be the number of negative z_xs.

In our case $G = 1$, $n_1 = 4$ and $n_2 = 6$.

Then the probability of getting 1 group of positive signs is

$$\frac{\binom{3}{0}\binom{7}{1}}{\binom{10}{4}} = \frac{\frac{7!}{1!6!}}{\frac{10!}{4!6!}} = \frac{7}{210} = 0.0333$$

Using a one-tailed test, since only small values of G are of interest, we find that $0.0333 < 0.05$.

We reject the null hypothesis.

SERIAL CORRELATIONS TEST

This tests for runs of deviations of the same sign, that is for subsections of the age range for which the mortality rates of lives in the investigation are systematically lower or higher than the rates in the standard table.

The correlation coefficient at lag 1 is

$$r_1 = \frac{\sum_{x-1}^{m-1}(z_x - \overline{z^*})(z_{x+1} - \overline{z^{**}})}{\sqrt{\sum_{x=1}^{m-1}(z_x - \overline{z^*})\sum_{x=1}^{m-1}(z_{x+1} - \overline{z^{**}})}}.$$

The calculations are shown in the table below.

Age x	z_x	$z_x - \overline{z^*}$	$z_x - \overline{z^{**}}$
18	1.9159	1.7158	1.4959
19	1.4541	1.2540	0.4878
20	0.4460	0.2459	0.0812
21	0.0394	–0.1607	-0.1051
22	–0.1469	–0.3470	-0.4688
23	–0.5106	–0.7107	-0.4649
24	–0.5067	–0.7068	-0.0049
25	–0.0467	–0.2468	-0.8019
26	–0.8437	–1.0438	-0.2191
27	–0.2609	–0.4610	

$\overline{z^*} = 0.2001$

$\overline{z^{**}} = -0.0418$

Age x	$(z_x - \overline{z^*})^2$	$(z_{x+1} - \overline{z^{**}})^2$	$(z_x - \overline{z^*})(z_{x+1} - \overline{z^{**}})$
18	2.9440	2.2377	2.5666
19	1.5725	0.2379	0.6117
20	0.0605	0.0066	0.0200
21	0.0258	0.0111	0.0169
22	0.1204	0.2198	0.1627
23	0.5051	0.2162	0.3304
24	0.4996	0.0000	0.0035
25	0.0609	0.6431	0.1979
26	1.0895	0.0480	0.2287
Total	6.8783	3.6203	4.1384

Hence

$$r_1 = \frac{\sum_{x-1}^{m-1}(z_x - \overline{z^*})(z_{x+1} - \overline{z^{**}})}{\sqrt{\sum_{x=1}^{m-1}(z_x - \overline{z^*})\sum_{x=1}^{m-1}(z_{x+1} - \overline{z^{**}})}} = \frac{4.1383}{\sqrt{(6.8783)(3.6203)}} = 0.8293.$$

Now $r_1\sqrt{m}$ ~ Normal(0, 1).

Since $m = 10$, we have $r_1\sqrt{m} = 3.1623 \text{ x } 0.8293 = 2.6225$.

Using a one-tailed test (since we are only interested in positive serial correlations), the probability of getting a value as high as 2.6225 is (1 – 0.9957) = 0.0043

Therefore we have evidence to reject the null hypothesis at the 5% level.

(iii) The results of the tests suggest that the underlying mortality of the lives in the investigation is, overall, not significantly different from ELT 15 (Males).

There are no individual ages with suspiciously large deviations.

Neither does there appear to be any overall bias.

However, although, overall, the experience fits ELT 15 well, there is a problem with the shape of the mortality curve. At younger ages the observed mortality is systematically higher than that of ELT 15 (Males), whereas at all ages above 21 years it is lower.

The serial correlations and grouping of signs (Stevens') tests, which are designed to pick up this kind of difference, thus lead us to reject the null hypothesis that the underlying mortality is the same as ELT 15 (Males).

It seems that the observed experience has a much more pronounced "accident hump" in it than ELT 15 (Males).

Therefore, ELT 15 (Males) is probably not a good standard table to use for graduating this experience.

In fact, ELT 14 (Males), which is based on 1980-82 data, might be better, since the "accident hump" is more obvious in ELT 14 than ELT 15.

(iv) Choose an appropriate standard table.

Seek a simple function relating the observed experience to that of the standard table.

If, for example, $\overset{o}{q}_x$ and $\overset{o}{\mu}_x$ refer to the observed mortality at age x, and q_x^s and μ_x^s refer to the corresponding mortality in the standard table, we might choose

$$\overset{o}{q}_x = a + bq_x^s,$$

or

$$\overset{o}{\mu}_x = \mu_x^s + k$$

where a, b and k are parameters.

To find a suitable function we can plot $\overset{o}{q}_x$ against q_x^s to check for a linear relationship in the q_xs , or $-\log(1-\overset{o}{q}_x)$ against $-\log(1-q_x^s)$ to check for a linear relationship in the μ_xs.

If a simple function cannot be found, then a different standard table should be chosen and the procedure repeated.

Once a possible relationship has been identified, the best-fitting parameters may be found using maximum likelihood or weighted least squares methods.

In weighted least squares, natural weights would be the exposed-to-risk at each age, as we wish to give more emphasis to those ages where the $\overset{o}{q}_x$s or $\overset{o}{\mu}_x$s are based on abundant data.

The resulting graduation needs to be tested for goodness-of-fit, but not for smoothness, since the standard table should already be smooth.

If the graduation fails the goodness-of-fit tests, then either a new function or a new standard table should be sought and the graduation repeated.

END OF REPORT