EXAMINATIONS

12 April 2001 (pm)

Subject 104 — Survival Models

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Write your surname in full, the initials of your other names and your Candidate's Number on the front of the answer booklet.*
2. *Mark allocations are shown in brackets.*
3. *Attempt all 10 questions, beginning your answer to each question on a separate sheet.*

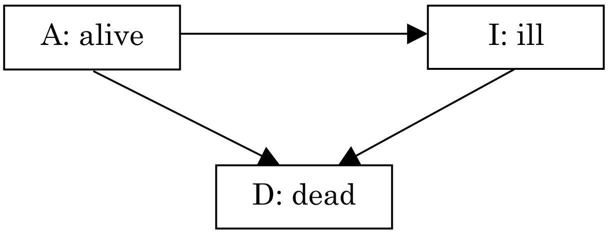
Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet and this question paper.

In addition to this paper you should have available Actuarial Tables and an electronic calculator.

1 The following Markov Model is to be used to model the progression of an illness for which there is no known cure.



(a) Explain in general the difference between the events which have probabilities

$${}_t p_x^{II} \text{ and } {}_t p_x^{\overline{II}}$$

(b) State the relationship between the numerical values of these probabilities in this model.

[3]

2 A special endowment assurance with a term of n years is issued to lives aged x. It is special because the death benefit at time t (where $0 < t < n$) is restricted to the net premium policy value at that time ${}_tV$, payable immediately on death. The net premium is P p.a., payable continuously. The survival benefit at time n is S.

(i) Write down Thiele's differential equation for this policy. [1]

(ii) Show that the solution to the equation in (i) is ${}_tV = P\bar{s}_{\overline{t}|}$ at the appropriate rate of interest. [5]

(iii) Suppose that $x = 45$, $n = 20$, the rate of interest at all times is 4% p.a., and the survival benefit S is sufficient to purchase an annuity, payable annually in arrear, of £20,000 p.a. Find P.

Mortality basis: a(55) Ultimate Mortality Table (males). [3]

[Total 9]

3 Consider a "limited-premium" whole-life assurance, i.e. a whole-life assurance for which premiums are payable until death but for at most m years. The sum assured is £1, payable at the end of the year of death, the age at entry is x, and the premiums are payable annually in advance.

(i) Give expressions, in terms of standard actuarial functions, for:

(a) the net premium
(b) the prospective net premium policy value, at (integer) time $t > m$
(c) the retrospective net premium policy value, at (integer) time $t > m$

[4]

(ii) Hence show that, if all three of the expressions in (i) are calculated on the same mortality and interest basis, the prospective and retrospective net premium policy values are equal. [4]

[Total 8]

4 A life insurance company has investigated the recent mortality experience of its male term assurance policy holders by estimating the initial rate of mortality q_x. The crude estimates $\hat{q}_x$, of these rates will be graduated by reference to a standard mortality table for male permanent assurance policy holders with rates q_x^s so that the graduated rates $\mathring{q}_x$ are given by

$$\mathring{q}_x = a + bq_x^s \qquad \text{(A)}$$

where a and b are constants. The estimates of a and b will be determined by minimising

$$\sum_x w_x \left(\hat{q}_x - a - bq_x^s\right)^2 \qquad \text{(B)}$$

where w_x is a suitably chosen weighting function.

(i) Describe how the suitability of the formula (A) could be investigated. [1]

(ii) Explain why it is important to use the weighting function, w_x, in formula (B). Determine an expression for a suitable choice of function for w_x. [4]

(iii) Explain how the smoothness of the graduated rates, $\mathring{q}_x$, is ensured. [2]
[Total 7]

5 A special deferred annuity provides the following benefits for a life aged 45.

- on survival to age 65 an annuity of £5,000 p.a. payable weekly in advance for five years certain and for life thereafter
- on death before age 55, £10,000 payable immediately on death
- on death between ages 55 and 65, £25,000 payable immediately on death

Annual premiums are payable in advance until age 65 or earlier death.

Determine the level annual premium payable.

Basis: A67–70 Ultimate Mortality Table before age 65
a(55) Ultimate Mortality Tables for males after age 65
4% p.a. throughout [8]

6 A mortality investigation has been carried out over the three calendar years; 1997, 1998 and 1999.

Deaths during the Period of Investigation, θ_x have been classified by age x at the date of death, where

x = calendar year of death – calendar year of birth

(i) State the rate year implied by this classification, and give the age range of the lives at the beginning of the rate year. [2]

(ii) Censuses of the numbers alive on 1 July 1997, 1 July 1998 and 1 July 1999 have been tabulated and denoted by $P_x(½)$, $P_x(1½)$ and $P_x(2½)$ respectively, where x is the age determined at the date of each census.

The force of mortality at age $x + f$ is to be estimated using the formula

$$\hat{\mu}_{x+f} = \frac{\theta_x}{P_x(½) + P_x(1½) + P_x(2½)}$$

(a) Determine the age definition x in $P_x(t)$, t = ½, 1½, 2½ if this formula is correct.

(b) Determine the value of f stating clearly all the assumptions you have made. [6]

[Total 8]

7 A life insurance company wishes to test if the recent mortality experience of its annuitants is consistent with the mortality basis used in the calculation of the annuity rates.

If $\mathring{q}_x$ and q_x^p denote the mortality rates for the experience and the premium basis respectively:

(i) State the null hypothesis for the investigation. [1]

(ii) For each of

- the Chi-Squared Test
- the Grouping of Signs Test

(a) State the test statistic.

(b) Derive the sampling distribution of the test statistic, when the null hypothesis is true.

(c) Indicate the observed values of the test statistic that would lead to the rejection of the null hypothesis.

Define clearly all the symbols that you use. [10]

[Total 11]

8 (i) If K_x is a random variable representing the curtate future lifetime of a life now aged x, and W is a random variable which represents the present value of the benefits payable from a temporary immediate annuity with a term of 20 years sold to a life aged 40, write down an expression for W in terms of K_x. [2]

(ii) Using as a basis the A67–70 Ultimate Mortality Table and an effective rate of interest of 5% p.a. calculate

(a) $P[W < E[W]]$

(b) Using suitable diagrams or otherwise, explain why the solution to (a) is not 0.5. [6]

(iii) Show that the variance of W is

$$441\left(A'_{40:\overline{21}|} - \left(A_{40:\overline{21}|}\right)^2\right)$$

State the rates of interest that should be used to evaluate A' and A respectively. [7]

[Total 15]

9 (i) (a) Show that at age x if $0 \le a < b \le 1$ then

$$_{b-a}q_{x+a} = 1 - \frac{_b p_x}{_a p_x}$$

(b) Hence, or otherwise, show that if deaths are uniformly distributed over the year of age $(x, x + 1)$ then

$$_{b-a}q_{x+a} = \frac{(b-a)\,q_x}{1-aq_x}$$ [3]

(ii) In a mortality investigation, the following data have been recorded for six independent lives observed between exact age 70 and exact age 71.

a_i the time in years after exact age 70 when the ith life came under observation;

b_i the time in years after exact age 70 when ith life was censored;

d_i = 1 if the ith life died before $x + b_i$;
= 0 if the ith life survived to $x + b_i$;

t_i if $d_i = 1$, then $x + t_i$ is the age at which the ith life died.

Life i	a_i	b_i	d_i	t_i
1	0	1	0	–
2	0.3	0.9	0	–
3	0.5	1	1	0.9
4	0	0.4	0	–
5	0	0.9	1	0.7
6	0	1	1	0.8

(a) Using the Binomial Model of Mortality write down the likelihood of these observations.

If deaths are assumed to be uniformly distributed over (70, 71) express this likelihood as a function of q_{70}.

(b) Using the Poisson Model of Mortality and assuming a constant force of mortality, $\bar{\mu}_{70}$, over (70, 71) write down the likelihood of these observations.

Calculate the maximum likelihood estimate of $\bar{\mu}_{70}$ and hence obtain the maximum likelihood estimate of q_{70}. [11]
[Total 14]

10 The table gives the data for a small sample of employees in a factory. It shows the time in months until the first absence from work. Observations marked + show the time of leaving for those employees who left employment without being absent from work.

Male employees	6+	11	13+	15	16+	19+	20	
Female employees	2+	4	7	8+	10+	12+	17	21+

A Cox Proportional Hazards Model

$$\lambda(t \mid x) = \lambda_0(t) \exp(\beta x)$$

is to be fitted to these data where t is the time until the first absence from work, $\lambda_0(t)$ is the baseline hazard and $x = 0$ for males, $= 1$ for females.

(i) Show that the partial log-likelihood for these data can be written

$$l(\beta) = 3\beta - 4\log_e(1 + e^{\beta}) - 2\log_e(2 + e^{\beta}) + c$$

where c is a constant that does not depend on β. [7]

(ii) Calculate the maximum partial likelihood estimate of β. [5]

(iii) Calculate the asymptotic standard error of this estimate. [3]

(iv) Test the hypothesis that female employees experience higher first absence rates than male employees. Explain the steps in your argument and state your conclusions. [2]

[Total 17]