EXAMINATIONS

17 September 2003 (pm)

Subject 104 — Survival Models

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 11 questions, beginning your answer to each question on a separate sheet.*

Graph paper is required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available Actuarial Tables and your own electronic calculator.

1 State the reasons why crude mortality rates are graduated before they are used in financial calculations. [3]

2 An investigation into mortality by an insurer follows N independent and identical lives over the year of age from x to $x + 1$. Each life holds one or more term assurance policies. The proportion of the N lives holding i policies is π_i for $i = 1, 2, \ldots$

Let C denote the total number of claims made on all policies, and let D be the number of deaths among the N policyholders. Show that

$$\frac{\mathrm{Var}(C)}{\mathrm{Var}(D)} = \frac{\sum_i i^2 \pi_i}{\sum_i \pi_i}. \qquad [4]$$

3 (i) List the data required for the exact calculation of the central exposed to risk of lives aged x last birthday in a mortality investigation over the two year period from 1 January 2001 to 1 January 2003. [2]

(ii) In an investigation of mortality during the period 1 January 2001 to 1 January 2003, data are available on the number of lives under observation, aged x last birthday, on 1 January 2001, 1 July 2001 and 1 January 2003.

Derive an approximation for the central exposed to risk at age x last birthday over the period in terms of the populations recorded on each of these three dates. [3]

[Total 5]

4 During a 2 year trial for a new medical treatment, 50 patients were observed after receiving the new treatment on 1 July 2000. For those patients who died or who left the trial before 30 June 2002, a record was kept of their time spent under observation. The details are shown below:

Period under observation (in months) for patients who:

Died	*Left trial*
4	2
6	5
6	7
8	9
11	10
16	13
22	15
	18

(i) Calculate the Kaplan-Meier estimate of the survival function, $S(t)$. [5]

(ii) Sketch the hazard, $h(t)$, implied by the Kaplan-Meier estimate of $S(t)$. [3]

[Total 8]

5 A life insurance company has carried out an investigation, over N years, of the mortality of its term assurance policyholders. Premiums for these contracts are based on the policyholder's age last birthday at the date of issue.

The following data are available from the investigation:

d_x = number of deaths during the investigation aged x

$P_{x,t}$ = number of lives under observation aged x at time t ($t = 0, 1, \ldots N$)

where x is the policyholder's age last birthday at date of issue plus the number of policy anniversaries passed.

(i) (a) State the type of rate interval.

(b) Write down an expression that may be used to evaluate the central exposed to risk using the available data for $P_{x,t}$. State any assumptions made. [3]

(ii) The following is an extract from the data:

x	d_x	$P_{x,t}$	$P_{x,t+1}$
59	75	6,276	6,824
60	67	6,551	6,340
61	80	6,689	6,750

Calculate an estimate of μ_{60}, stating any further assumptions made. [3]

(iii) Give an example of why the assumptions you have made may not be appropriate in this particular investigation. [1]

[Total 7]

6 (i) T_x denotes the future lifetime of a life currently aged x. Write down the probability density function of T_x. [1]

(ii) Using your answer to (i), show that:

(a) $$\frac{\partial}{\partial s}\log {}_s p_x = -\mu_{x+s}\text{, and}$$

(b) $${}_t p_x = \exp\left\{-\int_0^t \mu_{x+s}\ ds\right\}.$$

[4]

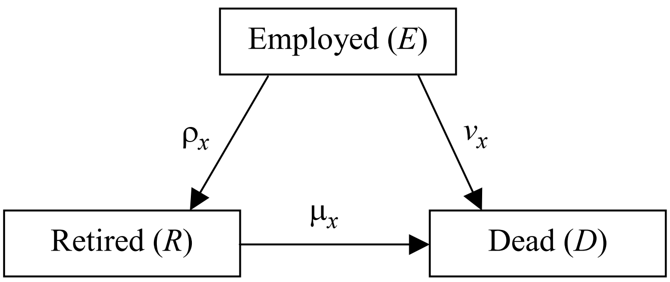
(iii) In a certain population, the force of mortality is given by:

	μ_x
$60 < x \le 70$	0.01
$70 < x \le 80$	0.015
$x > 80$	0.025

Calculate the probability that a life aged exactly 65 will die between exact ages 80 and 83. [3]

[Total 8]

7 A very large employer is considering the simple three state Markov model of its pension scheme shown below:



(i) Define ${}_t p_x^{RD}$. [1]

(ii) Suppose we observe a large number N of employees aged x. The ith life is observed from age $x + a_i$ to age $x + b_i$ where $0 \le a_i < b_i \le 1$. Suppose that of the employees observed, K retire and M die.

(a) Assuming the transition intensities are constant between ages x and $x + 1$, write down the likelihood function for this data.

(b) Derive an expression for the maximum likelihood estimate $\hat{\rho}_x$ of ρ_x.

[5]

(iii) The employer has observed that retired people are living longer than anticipated and therefore increasing the cost of the pension scheme. Annual records are kept of numbers of retired people at each age, and a summary is given below.

Age last birthday	*Number of retired people at 1 January 2001*	*Number of retired people at 1 January 2002*	*Number of deaths in 2001*
70	8,143	8,741	103
71	7,592	8,062	76
72	6,811	7,493	107
73	7,249	6,693	94

Use this data to estimate q_{71} and q_{72} stating clearly any assumptions you make. [5]

[Total 11]

8 An insurance company offers different annuity rates to smokers and non-smokers. The premium rates are calculated using the following forces of mortality:

non-smokers: $\mu_{x+t}^{NS} = \mu_{x+t}$

smokers: $\mu_{x+t}^{S} = \mu_{x+t} + 0.019048$

μ_{x+t} is from the standard mortality table AM92 ultimate.

An annuity is issued to a life aged exactly 50. The annuity is deferred for 15 years and is then payable continuously for life. A single premium of £20,000 is paid at the date the policy is issued.

(i) Calculate the level annual annuity that would be paid assuming the policyholder is:

(a) a non-smoker
(b) a smoker

[8]

(ii) A policyholder fails to declare the fact that he is a smoker and is sold a non-smoker policy. Calculate the expected present value of the profit that the insurance company will make. [2]

Basis: 4% per annum interest throughout
Expenses are ignored

[Total 10]

9 On 1 June 2001 an insurer issued a 20 year level temporary annuity to a life then aged exactly 60. The single premium was paid on 1 June 2001.

Let V_t denote the prospective policy value held after t years ($t = 0, 1, \ldots, 20$) calculated using a constant force of interest δ and assuming the member is subject to force of mortality μ_{60+t}.

(i) If the benefit is £1 paid annually in arrears (so the first payment was made on 31 May 2002), write down and explain a recurrence relation between V_t and V_{t+1} for $t = 0, 1, \ldots, 19$. [3]

(ii) Suppose the benefit is £h payable in arrears every h years, where $h < 1$. Write down a recurrence relation between V_t and V_{t+h} for $t = 0, h, 2h, 3h\ldots$ [1]

(iii) By considering the limit as $h \to 0$ show that Thiele's equation for the policy value if the benefit is paid continuously is

$$\frac{\partial V_t}{\partial t} = (\mu_{60+t} + \delta)\, V_t - 1$$

and state the boundary condition for V_{20}. [5]

(iv) Assuming that μ_{60+t} is a constant μ, solve Thiele's equation for the policy value in (iii). [Hint: use an integrating factor.] [4]

Basis: Expenses are ignored [Total 13]

10 On 1 September 2003 a man aged exactly 60 takes out a 5 year temporary annuity of £1,000 per year payable annually in arrears. The annuity is deferred for 5 years. The annuity is purchased by a single premium paid on 1 September 2003.

(i) Give the dates of the first and last annuity payments to be made (assuming the man is still alive). [1]

(ii) Let X represent the present value of the benefits payable under this annuity and K_x the man's curtate future lifetime. Calculate and plot the value of X for $K_x = 0, 1, \ldots\ldots$ [5]

(iii) $P[X = x]$ denotes the probability that the random variable X has a value of x. Calculate $P[X = x]$ for all possible values of x and plot a graph of $P[X = x]$ against x.

[5]

(iv) Calculate the premium. [4]

Basis: Mortality: AM92 ultimate
5% per annum interest throughout
Expenses are ignored [Total 15]

11 A large investigation has been carried out into mortality among people of working age. It is suggested that the underlying mortality rates among the lives in the investigation are the same as those tabulated in a well-known standard table.

(i) Describe three possible features of the mortality data in the investigation that might lead you to reject the suggestion. [3]

(ii) The actual number of deaths of people in the investigation, together with the exposed to risk, are tabulated below. The expected number of deaths which would arise if we applied the rates in the standard table to the exposed to risk in the investigation, and the standardised deviations, are also shown below.

Age	*Exposed to risk* E_x	*Actual deaths* d_x	*Expected deaths from rates in standard table* $E_x q_x^s$	*Standardised deviations* z_x
20–24	35,000	35	34	0.17150
25–29	33,000	30	29	0.18569
30–34	30,000	31	35	−0.67612
35–39	30,000	45	52	−0.97072
40–44	31,000	84	80	0.44721
45–49	28,000	138	130	0.70165
50–54	25,000	229	213	1.09630
55–59	23,000	360	348	0.64327
60–64	20,000	522	505	0.75649

Using 3 appropriate statistical tests, compare the observed mortality rates with those in the standard table.

For each test you perform, state the possible differences between the data and the standard table that it is designed to detect, perform the test and clearly state your conclusions.

Note: the standardised deviations, z_x, are given by the formula:

$$z_x = \frac{d_x - E_x q_x^s}{\sqrt{E_x q_x^s}}$$

using the approximation $E_x q_x^s \simeq E_x q_x^s (1 - q_x^s)$. You should use this approximation in your calculations. [10]

(iii) Comment on your results. Include in your comments an observation on the consequences for a life office which used the standard table mortality rates when calculating premiums for a class of lives that experienced the mortality rates underlying those observed in the investigation. [3]

[Total 16]