REPORT OF THE BOARD OF EXAMINERS

September 2003

Subject 105 — Actuarial Mathematics 1

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

J Curtis
Chairman of the Board of Examiners

11 November 2003

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EXAMINATIONS

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Overall Comments

The standard of answering overall was at a lower level than the examiners expected. Candidates found particular difficulty with questions 4, 8, 9, 13 and 14. Attempts at questions 9 and 13 in particular were generally unsatisfactory. In relation to the other questions many candidates performed well.

Individual comments follow after each question and we hope that these will be of assistance to students.

1 $p_{[40]} = 0.999212$

$p_{[40]+1} = 0.999038$

The reserve required per policy in force at the end of year 2, ${}_2V = \dfrac{150}{1.05} = 142.857$

The cost of this, at the end of year 2, per policy in force at the start of year 2 = $p_{[40]+1} * {}_2V = 142.720$

The adjusted value of $(NUCF)_2 = 100 - 142.720 = -42.720$

The reserve required per policy in force at the end of year 1, ${}_1V = \dfrac{42.720}{1.05} = 40.686$

The cost of this, at the end of year 1, per policy in force at the start of year 1 = $p_{[40]} * {}_1V = 40.654$

The adjusted value of $(NUCF)_1 = 100 - 40.654 = 59.35$

The question was well answered in general. A number of candidates used incorrect mortality rates.

2 **Return of the member's contributions**

Under this option, the total of the member's contributions are returned, with or without interest. This option is available normally only after a short period of service. There is likely to be a tax charge on the sum paid to the member.

A deferred pension payable from normal pension age

This option provides for the member to receive, from the scheme the member is leaving, a pension payable from normal pension age. The pension is normally based on the number of years' service to the date of leaving and final pensionable salary at the date of leaving. The basic amount of the deferred pension is increased each year, from the date of leaving to normal pension age, by a revaluation rate.

An immediate pension from the date of leaving

This option provides an immediate pension payable from the scheme, from the date of leaving. This option is normally restricted to members close to normal pension age. The pension can be calculated in a number of ways: a common method is to determine the pension amount as that which is actuarially equivalent to the deferred pension the member would otherwise have received.

A transfer cash equivalent

The transfer cash equivalent is an amount determined by the scheme actuary as a fair assessment of the present value of the deferred pension and other benefits given up by the member leaving the scheme. The transfer cash equivalent may be paid to a new scheme that the member is joining, or to a special individual policy that a member can effect for this purpose with a life insurance company.

This question was well answered in general. Some candidates just listed the benefit options, whereas use of the word "Describe" required a fuller treatment.

3 Let P be the annual premium.

P is given by

$$P\left(\bar{a}_{45:\overline{20}|}-\bar{a}^{HS(1/all)}_{45:\overline{20}|}\right)=10000*1.015*\bar{a}^{HS(1/all)}_{45:\overline{20}|}+0.6P+50\bar{a}_{45:\overline{20}|}$$

$$\therefore P(11.299-0.242488)=10150*0.242488+0.6P+50*11.299$$

$$\therefore P=£289.41$$

Overall this question was answered well. Some candidate had difficulty with valuing the waiver benefit.

4 The retrospective policy value is determined, using a basis that reflects the experience of the policy and takes account of the cost of surrender. The formula for the policy value is as follows:

$$\frac{D_x}{D_{x+t}}\left\{G\ddot{a}^{(12)}_{x:\overline{t}|}-S\bar{A}^{1}_{x:\overline{t}|}-I-e\ddot{a}^{(12)}_{x:\overline{t}|}-f\bar{A}^{1}_{x:\overline{t}|}\right\}-C\text{, where}$$

x is the age of policyholder at inception

t is the policy duration at which the surrender value is being calculated
G is the annual office premium
S is the sum assured
I are the initial expenses, in excess of the regular expenses occurring each year
e are the regular annual expenses
f are the additional expenses that occur when the contract terminates
C are the surrender expenses

The prospective policy value is calculated using a basis that reflects the future expected investment earnings, future expected expenses and future expected mortality experience of the surrendering policyholders, less the cost of surrender. The formula is as follows:

$$S\bar{A}_{x+t:\overline{n-t|}} + e\ddot{a}^{(12)}_{x+t:\overline{n-t|}} + f\bar{A}_{x+t:\overline{n-t|}} - G\ddot{a}^{(12)}_{x+t:\overline{n-t|}} - C$$

Additional definition: n is the original term of the policy.

A table of surrender values by policy duration is produced. The surrender value at a particular duration is usually a blend of the retrospective and prospective policy values, subject to a minimum of zero. Generally, the retrospective policy value is given a greater weighting at earlier durations and the prospective value is given a greater weighting at later durations. Other considerations, such as the asset share and marketing influences, are also generally taken into account. Where possible, the surrender value should be less than the asset share. Marketing considerations may mean adjusting surrender values upwards.

Most candidates did not answer this question well. The examiners' view was that this was a standard theoretical question and well-prepared candidates should have scored reasonably. Very few candidates mentioned both prospective and retrospective reserves; most formulae given were not fully correct; and very few candidates dealt with the considerations set out in the final part of the solution.

5 Let P be the annual premium.

P is given by

$$P\ddot{a}^{(12)}_{[55]:\overline{10|}} = 95000\bar{A}^{1}_{[55]:\overline{10|}} + 5000(I\bar{A})^{1}_{[55]:\overline{10|}}$$

$$\ddot{a}^{(12)}_{[55]:\overline{10|}} = \ddot{a}_{[55]:\overline{10|}} - 0.458\left(1 - \frac{D_{65}}{D_{[55]}}\right) = 8.228 - 0.458\left(1 - \frac{689.23}{1104.05}\right) = 8.056$$

$$\bar{A}^{1}_{[55]\overline{10|}} = 1.04^{0.5}\left(A_{[55]\overline{10|}} - \frac{D_{65}}{D_{[55]}}\right)$$

$$= 1.04^{0.5}\left(0.68354 - 0.624274\right) = 0.060439$$

$$(I\bar{A})^{1}_{[55]:\overline{10|}} = 1.04^{0.5} * \frac{R_{[55]} - R_{65} - 10 * M_{65}}{D_{[55]}}$$

$$= 1.04^{0.5} * \frac{9482.75 - 5441.07 - 10 * 363.82}{1104.05} = 0.372692$$

$$P = \frac{95000 * 0.060439 + 5000 * 0.372692}{8.056} = £944.04$$

Candidates attempted this question well in general. There were some minor errors in the formulae and numerical calculations.

6 (i) Salary at age 50 exact $\Rightarrow$ salary earned between age 49.5 and 50.5, assuming that the salary increase was given at age 49.5.

$$s_{49.5} = 0.5*(9.031+9.165) = 9.098$$

Value of future contributions

$$= 0.05*50000*\frac{{}^{s}\bar{N}_{50}}{s_{49.5}*D_{50}} = 2500*\frac{163638}{9.098*1796} = £25,036.40.$$

(ii) Value of future retirement benefits

$$= \frac{50000}{60}*\frac{{}^{z}\bar{R}^{ra}_{50} + {}^{z}\bar{R}^{ia}_{50}}{s_{49.5}D_{50}} = \frac{50000}{60}*\frac{1604000+363963}{9.098*1796} = £100,365.26.$$

The solution given is based on the assumption that Member A's salary was increased 6 months before the valuation date. The examiners gave full credit for any other sensible assumption so long as the assumption was stated. For example, assuming that the salary had just been increased, s_{50} *would be used in place of* $s_{49.5}$. *Candidates answered the question well, in general.*

7 The remaining transition probabilities are:

$$p^{HH}_{50+t} = 0.85 \quad p^{SS}_{50+t} = 0.05$$

Probability of being sick at $t = 1$

= 0.1

Probability of being sick at $t = 2$

$$= p^{HH}_{50}\,p^{HS}_{51} + p^{HS}_{50}\,p^{SS}_{51} = 0.85*0.1+0.1*0.05 = 0.09$$

Probability of being sick at $t = 3$

$$= p^{HH}_{50}\,p^{HH}_{51}\,p^{HS}_{52} + p^{HH}_{50}\,p^{HS}_{51}\,p^{SS}_{52} + p^{HS}_{50}\,p^{SS}_{51}\,p^{SS}_{52} + p^{HS}_{50}\,p^{SH}_{51}\,p^{HS}_{52}$$

= 0.85*0.85*0.1+0.85*0.1*0.05+0.1*0.05*0.05+0.1*0.8*0.1 = 0.08475

P is given by

$$0.95P = 10000 * \left(0.1v + 0.09v^2 + 0.08475v^3\right)$$

$P = £2,585.23$

Most candidates scored well on this question, with many getting full marks.

8 (i) The actuarial funding factor is given by

$$A_{53:\overline{7}|}$$

at a rate of interest of 3% and mortality given by $q_x = 0.001 \;\; 53 \le x \le 60$.

$$A_{53:\overline{7}|} = 1 - d\ddot{a}_{53:\overline{7}|}$$

$$\ddot{a}_{53:\overline{7}|} = 1 + 0.999v + (0.999v)^2 + ... + (0.999v)^6$$

$$= \frac{1-(0.999v)^7}{1-0.999v} = 6.39873$$

$$A_{53:\overline{7}|} = 1 - \frac{0.03}{1.03} * 6.39873 = 0.81363$$

(ii) In assessing the maximum rate of interest, I would make a prudent estimate of the level of the company's future renewal expenses (including renewal commissions) and express this as a regular percentage of the projected bid values of the funded capital and accumulation units, say *i*%.

I would use discounted cash flow techniques to calculate *i*.

Conventionally, the rate *i*% tends to be the management charge used for accumulation units, 1% in this case. In practice, we might tend to increase the 3% interest rate to 4% (5%-1%).

Mathematically, however, the maximum rate of interest is $(5\% - i\%)/(100\% - 5\%)$. In this case, assuming $i = 1\%$, this would give a maximum theoretical rate of 4.21%.

In assessing whether this would be prudent to use, I would compare the funded value of capital units at the end of the third year using the revised actuarial funding factor with the surrender value of capital units at that time. The funded value should not be less than the surrender value. A further check should be made to ensure that this remains the case at all subsequent policy durations.

I would also consider whether the mortality assumption was appropriate for calculating the actuarial funding factor. The assumed level of mortality should not be lighter than that prudently expected for the group of policyholders. Otherwise the company would be anticipating future management charges it might not receive.

Part (i) was not well answered. Many candidates did not show that the actuarial funding factor as the present value of an endowment benefit.

Credit was given for variations from the solution set out: if a candidate assumed that the amount of the management charge being pre-funded was 3% per annum and used a rate of interest of $\frac{0.03}{0.95}$ *for the present value of the endowment benefit, credit was given; if a candidate assumed that the death benefit was payable immediately on death rather than at the end of the year of death in the calculation of the present value of the endowment benefit, credit was also given.*

Part (ii) caused particular difficulties. Few candidates mentioned the use of discounted cash flow techniques or the considerations set out in the final two paragraphs of the solution.

9 Let t be the future lifetime of the joint status. For the payments to be exactly 95% likely to be sufficient, since the lives are independent with respect to mortality, the value of t is given by

$${}_t p_{\overline{60:60}} = 0.05$$

$$\Rightarrow 2\,{}_t p_{60} - {}_t p_{60:60} = 0.05$$

$$\Rightarrow ({}_t p_{60})^2 - 2\,{}_t p_{60} + 0.05 = 0$$

$$\Rightarrow {}_t p_{60} = \frac{2 \pm \sqrt{4 - 4*0.05}}{2} = 0.02532 \text{ or } 1.9747$$

$$\Rightarrow {}_t p_{60} = 0.02532$$

$$\Rightarrow \frac{l_{60+t}}{l_{60}} = 0.02532 \Rightarrow 39 < t < 40$$

Therefore, for the payments to be **at least** 95% likely to be sufficient, there must be at least 40 payments.

<u>Alternative derivation that there must be at least 40 payments</u>

For payments to be at least 95% likely to be sufficient, t is given by

$${}_t q_{\overline{60:60}} \geq 0.95$$

$$\Rightarrow \left({}_t q_{60}\right)^2 \geq 0.95$$

$$\Rightarrow {}_t q_{60} \geq 0.97468$$

$$\Rightarrow {}_t p_{60} \leq 0.02532$$

$$l_{60} = 9826.131 \Rightarrow l_{60+t} \leq 248.798$$

$$\Rightarrow t \geq 40$$

I is given by

$$1000000 = I\ddot{a}_{\overline{40}|} \quad i = \frac{1.06}{1.05} - 1 = 0.9524\%$$

$$\ddot{a}_{\overline{40}|} = 33.44892$$

$$I = £29,896$$

This was the most poorly answered of all the questions, with few candidates gaining many marks. The question was based on a practical application of standard joint life mortality and the examiners would have expected candidates to have performed much better.

10 (i) Under the conventional method, the premiums that should be charged and the premiums that will be charged for the new policy or policies that the policyholder can opt to take are determined. The present value of the differences between the premiums is then calculated and this is the present value of the cost of the option. Where there is more than one option, the present value of one option only is taken into account: the option chosen is the one that gives the highest present value of the differences in premiums.

In carrying out the calculations, the following assumptions are made:

all lives eligible to take up the option will do so;

the mortality experience of those who take up the option will be the Ultimate experience which corresponds to the Select experience that would have been used as a basis if underwriting had been completed as normal when the option had been exercised.

The mortality basis used is not usually assumed to change over time, so the only data required are the Select and Ultimate mortality rates used in the original pricing basis.

(ii) The present value of the differences in premiums are as follows:

Option exercised at the fifth anniversary

$$\text{Present value} = 200000\frac{D_{50}}{D_{[45]}}\left(\frac{\bar{A}_{50}}{\ddot{a}_{50}} - \frac{\bar{A}_{[50]}}{\ddot{a}_{[50]}}\right)\ddot{a}_{50}$$

$$= 200000*\frac{1366.61}{1677.42}*1.04^{0.5}\left(\frac{0.32907}{17.444} - \frac{0.32868}{17.454}\right)*17.444$$

$$= 96.10$$

Option exercised at the tenth anniversary

$$\text{Present value} = 200000\frac{D_{55}}{D_{[45]}}\left(\frac{\bar{A}_{55}}{\ddot{a}_{55}} - \frac{\bar{A}_{[55]}}{\ddot{a}_{[55]}}\right)\ddot{a}_{55}$$

$$= 200000*\frac{1105.41}{1677.42}*1.04^{0.5}\left(\frac{0.38950}{15.873} - \frac{0.38879}{15.891}\right)*15.873$$

$$= 154.62$$

The cost of the option is the greater value, i.e., £154.62

The basic single premium is given by

$$P = 200000*1.04^{0.5}\,A^{1}_{[45]:\overline{10}|} = 200000*1.04^{0.5}*\frac{M_{[45]} - M_{55}}{D_{[45]}}$$

$$= 200000*1.04^{0.5}\,\frac{462.68 - 430.55}{1677.42} = £3,906.75$$

$\therefore$ The total single premium = £3,906.75 + £154.62 = £4,061.37.

Candidates performed well on this question in general. In part (ii) there is a subtle point that if the 5 year option is taken then a release of the Term Assurance reserve would take place. The Examiners did not expect students to cover this and the solution is based on this assumption. A few candidates did point this out and due credit was allowed within the total marks in these cases.

11 (i) The original gross premium is given by

$$0.95P\ddot{a}_{50:\overline{10}|} = 300 + 100000A_{50:\overline{10}|}$$

$$\ddot{a}_{50:\overline{10}|} = 8.314$$

$$A_{50:\overline{10}|} = 0.68024$$

$$P = £8,650.47$$

The gross premium reserve

$$= 100000A_{53:\overline{7}|} - 0.95 * 8650.47 * \ddot{a}_{53:\overline{7}|}$$

$$\ddot{a}_{53:\overline{7}|} = 6.166$$

$$A_{53:\overline{7}|} = 0.76286$$

Gross premium reserve = £25,614.14

(ii) Net premium reserve with Zillmer adjustment

$$= 100000\left(1 - \frac{\ddot{a}_{53:\overline{7}|}}{\ddot{a}_{50:\overline{10}|}}\right) - 300 * \frac{\ddot{a}_{53:\overline{7}|}}{\ddot{a}_{50:\overline{10}|}}$$

= 25,835.94 – 222.49 = £25,613.45

£222.49 is the Zillmer adjustment.

(iii) The net premium reserve with Zillmer adjustment equals the gross premium reserve calculated in part (i) (subject to rounding errors). If the insurance company actuary is satisfied that there are sufficient margins in the gross premium reserve then the net premium reserve with Zillmer adjustment would be adequate. In addition, the use of the net premium reserve with Zillmer adjustment compared with the use of the reserve without adjustment would reduce the company's funding requirements.

(iv) If the life insurance company's actuary decided that the gross premium reserve using 4% interest was no longer adequate given the fall in market interest rates and that 3.5% interest should be used, this would give a higher value for the gross premium reserve. The net premium reserve calculated in part (ii) was equal to the gross premium reserve using 4% interest and this net premium reserve would not be adequate.

Many of the well prepared performed well on this question. A surprising number of candidates showed a lack of understanding of a Zillmer adjustment.

12 The multiple decrement table is as follows.

Age (x)	$(al)_x$	$(ad)_x^d$	$(ad)_x^w$
50	100000	192.17	4995.07
51	94812.76	252.55	4734.16
52	89826.04		

Values for the multiple decrement table are calculated from formulas of the following type:

$$(aq)_x^d = q_x^d\left(1-\tfrac{1}{2}q_x^w\right)$$

$$(ad)_x^d = (al)_x * (ad)_x^d$$

$$(al)_{x+1} = (al)_x - (ad)_x^d - (ad)_x^w$$

The profit test is set out as follows.

Year	*1*	*2*
Premium	3000	3000
Expenses	150	
Interest	142.5	150
Death benefit	19.217	26.637
Withdrawal benefit	112.389	224.692
Survival benefit		4737.02
Cash flow	2860.894	−1838.349
Probability in force	1	0.94813
Discounted cash flow	2487.734	−1317.954
Net present value	£1,169.78	

Candidates performed well on this question in general. Where errors occurred, they were mostly in respect of the multiple decrement table. A number of candidates did not use a cash flow approach which is what the Examiners were expecting.

13 (i) With $i = 0.06$ and payments increasing at the rate of 1.9231% per annum, we can value at 4%, but we must make the initial payment = 10000/1.019231.

age	l_x *(male)*	l_x *(female)*	*k*	${}_kp_{xy}$	$Pr(K_{xy} = k)$
60	9826.131	9848.431	0	1	0.004504
61	9802.048	9828.163	1	0.995496	0.005364
62	9773.083	9804.173	2	0.990132	0.006361
63	9738.388	9775.888	3	0.98377	0.007514
64	9696.99	9742.64	≥4	0.976257	0.976257

k	$a_{\overline{\min(k,4)}|}$	$a_{\overline{\min(k,4)}|}^2$	*E[x]*	*E[x²]*
0	0	0	0	0
1	0.961538	0.924556	0.005158	0.00496
2	1.886095	3.557353	0.011998	0.02263
3	2.775091	7.70113	0.020851	0.057863
≥4	3.629895	13.17614	3.543709	12.86329
			3.581717	12.94875

Variance = 12.94875 – $(3.581717)^2$ = 0.120052

Std Dev: $(0.120052)^{0.5}$ = 0.346485

$\therefore$ Std Dev for this annuity is (10000/1.019231)*0.346485=3399.48

Alternative solution

With $i = 0.06$ and payments increasing at the rate of 1.9231% per annum, we can value at 4%, but we must make the initial payment = 10000/1.019231.

We require

$$Var(a_{\overline{\min(K_{xy},4)}|}) = Var(\ddot{a}_{\overline{\min(K_{xy}+1,5)}|} - 1) = Var\left(\frac{1 - v^{\min(K_{xy}+1,5)}}{d}\right)$$

$$= \frac{1}{d^2}\left({}^2A_{60:60:\overline{5}|} - \left(A_{60:60:\overline{5}|}\right)^2\right)$$

age	l_x *(male)*	l_x *(female)*	*k*	$Pr(K_{xy} = k)$
60	9826.131	9848.431	0	0.004504
61	9802.048	9828.163	1	0.005364
62	9773.083	9804.173	2	0.006361
63	9738.388	9775.888	3	0.007514

64	9696.99	9742.64	≥4	0.976257

k	$v_{4\%}^{k+1} * \Pr(K_{xy} = k)$	$v_{8.16\%}^{k+1} * \Pr(K_{xy} = k)$
0	0.0043308	0.0041642
1	0.0049593	0.0045852
2	0.0056549	0.0050272
3	0.0064230	0.0054904
≥4	0.8024121	0.6595242
	0.8237801	0.6787912

$$\left(A_{60:60:\overline{5|}}\right)^2 = 0.82378^2 = 0.6786136$$

Variance = $\dfrac{1}{d_{4\%}^2}\left(0.6787912 - 0.6786136\right) = 0.12005$

Std Dev: $(0.12005)^{0.5} = 0.34648$

$\therefore$ Std Dev for this annuity is (10000/1.019231)*0.34648=3399.43

(ii) If the annuity were a last survivor annuity, the standard deviation would be smaller. The chances of both lives dying during the 4 years would be much lower, so more annuities would be payable for 4 years, with a consequent reduction in the deviation from the average present value of the annuity payments.

This question was very poorly answered in general. Many candidates were unable to make any reasonable attempt. The examiners had expected the question to be challenging, but not to the extent experienced. 2 alternative solutions are given which the Examiners hope will assist.

14 (i) With no recovery to the healthy state, premiums are payable only until the first claim.

$$ {}_tp_x^{00} = {}_tp_x^{\overline{00}} = (0.87)^t $$

$\therefore$ EPV premiums $P\{1 + 0.87v + (0.87v)^2 + (0.87v)^3 + ...\} = 5.578947P$

Valuing the benefit from the point when the first claim arises, we get the following probabilities:

the first claim payment will be at level 1;

the second claim payment will be at level 1 with probability 0.6 and level 2 with probability 0.3;

the third claim payment will be at level 1 with probability $0.6^2 = 0.36$ and at level 2 with probability $0.6*0.3+0.3*0.6 = 0.36$;

the fourth claim payment will be at level 1 with probability $0.6^3 = 0.216$ and at level 2 with probability $0.6*0.3*0.6+0.6*0.6*0.3+0.3*0.6*0.6 = 0.324$.

If the first claim is in n years time, the expected present value will be $50000*0.6*1.06^n*v^n$. With v at 6%, this is 30,000 for all n. Similarly the present value of any level 2 claim will be 50,000, so we can ignore interest in valuing claims.

The EPV of all claims at the point of the first claim payment arising is therefore:

$$ 30,000*(1+0.6+0.36+0.216)+50,000*(0+0.3+0.36+0.324) = 114,480 $$

Finally the probability that the first claim occurs at the end of year 1 is 0.1, at the end of year 2 is (0.87)*(0.1), at the end year 3 is $(0.87)^2*(0.1)$ and in general at the end of year n is $(0.87)^{n-1}*(0.1)$.

The probability of a claim is therefore

$$ 0.1*(1+0.87+0.87^2+...) = \frac{0.1}{0.13} = 0.76923 $$

The EPV of all claims = (0.76923)*(114,480) = 88,061.45

The equation of value is:

$$(1-0.075)*5.578947*P = 88{,}061.45 \;\Rightarrow\; P = £17{,}064.43$$

(a) If the third instalment is at level 1, then the fourth claim will be at level 1 with probability 0.6, or at level 2 with probability 0.3.

However, interest and claim inflation no longer cancel, so the reserve immediately after paying the third claim is:

$$V = 42{,}000*\left(\frac{1.07}{1.05}\right)*(0.6)+70{,}000*\left(\frac{1.07}{1.05}\right)*(0.3) = £47{,}080$$

(b) If the third instalment is at level 2, then the fourth can only be at level 2, and will occur with probability 0.6.

This gives the following reserve value

$$V = 70{,}000*\left(\frac{1.07}{1.05}\right)*(0.6) = £42{,}800$$

This question was also not answered well. Many candidates valued the policies as four-year policies only and many also failed to appreciate that interest could be ignored in valuing claims in part (i) after which the question became much easier to complete. Few candidates made reasonable attempts at part (ii).