Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

September 2001

Subject 105 — Actuarial Mathematics 1

EXAMINERS' REPORT

Examiners' Comments

Overall the standard of attempts was lower than the examiners would have expected. There was evidence that many candidates spent too much time on the earlier questions, with consequent time problems later on.

Questions 1,2,3,4,9 and 11 were well answered. In question 5, career average salary was not dealt with well. Candidates had difficulty with year-end decrements in question 7 and with the duration in question 8. Question 9(i) was poorly answered, although it was a standard question.

Question 10 was the most poorly attempted, with few candidates scoring more than half marks. There was an ambiguity in this question: the benefit payable on the second death could have been interpreted as £200,000 or £300,000. Candidates were given credit for either approach. Many candidates did not give sufficient detail in their answers to question 12(i). Question 13(ii) was poorly attempted and the answers to question 14 were not as strong as one would have expected for a fairly standard question.

1 $s_x = 52.18\int_0^1 {}_tp_x \bar{z}_{x+t}dt$

$$z_x = \frac{52.18\int_0^1 l_{x+t}\bar{z}_{x+t}dt}{\int_0^1 l_{x+t}dt} = \frac{s_x * l_x}{\int_0^1 l_{x+t}dt} = \frac{s_x}{\int_0^1 {}_tp_x dt} = \frac{5}{0.9} = 5.56$$

2 The required formula is:

$$P_{21}(2003) = P_{20}(2002)(1 - q_{20\frac{1}{2}}(2002)) + M_{21}(2003)$$

$q_{20\frac{1}{2}}(2002)$ is the probability that a life aged 20 last birthday at mid-year 2002 dies between mid-year 2002 and mid-year 2003, assuming those aged 20 last birthday at mid-year 2002 have birthdays uniformly distributed over the calendar year.

$M_{21}(2003)$ denotes the number of migrants entering the population during mid-year 2002 and mid-year 2003 who survive to be aged 21 last birthday at mid-year 2003.

The formula is applied separately to males and females.

3 To allow for the fact that benefit cannot be paid for at least one year, the sickness benefit could be valued using the factor

$100\dfrac{K_{31} - K_x}{D_{30}}$, where x is the ceasing age for benefits.

However, the factor K_{31} is not accurate as it takes into account sickness of all durations, whereas a new policyholder aged 30 cannot experience sickness of all durations from age 31. For this reason and because the numerical effect is not significant, I would use the factor K_{30} rather than K_{31} in the above formula.

4 The required expression is

$$25{,}000\int_0^{45} e^{-\delta t}\{2p_{20,t}^{HH}.\mu_{20+t} + 3p_{20,t}^{HS}.\nu_{20+t}\}dt$$

5 The value of the benefits is

$$\frac{40{,}000}{60}\frac{{}^{s}\bar{R}^{ra}_{40}}{s_{39\frac{1}{2}}D_{40}} = \frac{40{,}000}{60}\frac{1{,}758{,}471}{\frac{1}{2}(3.48+3.58).5{,}204} = £63{,}816.35$$

6 The required single premium is given by

$$10000\,{}_{10}p^{f}_{60}v^{10}\left(\ddot{a}^{(12)}_{70^m|70^f}\right) = \frac{D^f_{70}}{D_{60}}\left(a_{70^f} - a_{70^m:70^f}\right)$$

$$= 10000\frac{3571.2}{8858.7}(7.308 - 5.106)$$

$$= £8{,}876.90$$

7 Construct a multiple decrement table.

Age	*No. alive*	*No. deaths*	*No. withdrawals over year*	*No. withdrawals at year end*
20	100000	97.50	4997.5	4745.25
21	90159.75	87.9058		

At age 20, no. of deaths = 100000*0.001(1-0.5*0.05) = 97.50

no. of withdrawals over year = 100000*0.05*(1-0.5*.001) = 4997.5

no. of withdrawals at year end = 100000*(1-0.05)*(1-0.001)* 0.05 = 4745.25

Required probability = 87.9058/100000 = 0.00087906.

8 Define a census taken at time *t* after the start of the period of investigation (1.1.98), $P'_{x,d}(t)$, of those lives having a policy in force at time *t,* who were *x* nearest birthday at entry and will be duration *d* on the policy anniversary next following time *t*.

The central exposed to risk is then given by

$$E^c_{x,d} = \int_{t=0}^{t=3.5} P'_{x,d}(t)\,dt$$

Assuming that $P'_{x,d}(t)$ varies linearly between the census dates the integral can be approximated by

$$\tfrac{1}{2} * \tfrac{1}{2}\left\{P'_{x,d}(0) + P'_{x,d}(\tfrac{1}{2})\right\} + \tfrac{1}{2} * 2\left\{P'_{x,d}(\tfrac{1}{2}) + P'_{x,d}(2\tfrac{1}{2})\right\} + \tfrac{1}{2} * 1\left\{P'_{x,d}(2\tfrac{1}{2}) + P'_{x,d}(3\tfrac{1}{2})\right\}$$

However, the census data have been recorded according to age x nearest birthday at entry and curtate duration d at time t. The following formula may be written:

$P'_{x,d}(t) = P_{x,d-1}(t)$.

Substituting this into the equation above gives

$$E^c_{x,d} = \tfrac{1}{2} * \tfrac{1}{2}\left\{P_{x,d-1}(0) + P_{x,d-1}(\tfrac{1}{2})\right\} + \tfrac{1}{2} * 2\left\{P_{x,d-1}(\tfrac{1}{2}) + P_{x,d-1}(2\tfrac{1}{2})\right\}$$
$$+ \tfrac{1}{2} * 1\left\{P_{x,d-1}(2\tfrac{1}{2}) + P_{x,d-1}(3\tfrac{1}{2})\right\}$$

$\mathring{m}_{x,d} = \dfrac{\theta_{x,d}}{E^c_{x,d}}$ estimates $m_{[x]+d-1}$ because the average age at entry is x assuming birthdays are uniformly distributed over the policy year and the exact duration at the start of the rate year of death is $d-1$ for all lives (no assumptions are necessary).

9 (i) Gross premium retrospective and prospective reserves will be equal if:

- The mortality and interest rate basis is the same for the retrospective and prospective reserves and is the same as that used to determine the gross premium at the date of issue of the policy.

- The same expenses (excluding the initial expenses) are valued in the retrospective and prospective reserves and also the expenses valued in the retrospective reserves are the same as those used to determine the original gross premium.

- The gross premium valued in the retrospective and prospective reserves is that determined on the original basis using the equivalence principle.

(ii) The prospective reserves at time t are given by

$$S\bar{A}_{x+t} + e\ddot{a}^{(m)}_{x+t} + f\bar{A}_{x+t} - G\ddot{a}^{(m)}_{x+t} \qquad \text{............ (a)}$$

where S is the sum assured
e is the annual rate of renewal expenses
f is the claim expense
G is the annual rate of gross premium

The retrospective reserve at time t is given by

$$\frac{D_x}{D_{x+t}}\left\{G\ddot{a}_{x:\overline{t}|} - S\bar{A}^{1}_{x:\overline{t}|} - I - e\ddot{a}^{(m)}_{x:\overline{t}|} - fA^{1}_{x:\overline{t}|}\right\} \qquad \text{...............(b)}$$

where I is the additional initial expense.

The original gross premium is given by

$$G\ddot{a}^{(m)}_x - S\bar{A}_x - I - e\ddot{a}^{(m)}_x - f\bar{A}_x = 0 \qquad \text{.............. (c)}$$

Add $\dfrac{D_x}{D_{x+t}}\left\{G\ddot{a}^{(m)}_x - S\bar{A}_x - I - e\ddot{a}^{(m)}_x - f\bar{A}_x\right\}$, which is identically 0, to (a).

Combining terms, e.g. $\dfrac{D_x}{D_{x+t}}G\ddot{a}^{(m)}_x - G\ddot{a}^{(m)}_{x+t} = \dfrac{D_x}{D_{x+t}}G\ddot{a}^{(m)}_{x:\overline{t}|}$ gives (b), the expression for the retrospective reserve.

10 The value of benefit (a) and (b) is

$$100000\left(2\bar{A}^{1}_{50:\overline{10}|} + \bar{A}^{\;\;1}_{\underbrace{\overline{50:50}}:\overline{10}|}\right)$$

$$\bar{A}^{\;\;1}_{\underbrace{\overline{50:50}}:\overline{10}|} \approx (1.04)^{½}\left(1 - d\ddot{a}_{\overline{50:50}:\overline{10}|} - 2\frac{D_{60}}{D_{50}} + \frac{D_{60:60}}{D_{50:50}}\right)$$

$$\ddot{a}_{\overline{50:50}:\overline{10}|} = 2\ddot{a}_{50:\overline{10}|} - \ddot{a}_{50:50:\overline{10}|}$$

$$\ddot{a}_{50:50:\overline{10}|} = \frac{N_{50:50} - N_{60:60}}{D_{50:50}} = \frac{59513.103 - 24729.51}{4354.5857} = 7.98781$$

$$\ddot{a}_{50:\overline{10}|} = 8.207$$

$$\frac{D_{60}}{D_{50}} = \frac{2855.5942}{4597.0607} = 0.621178$$

$$\frac{D_{60:60}}{D_{50:50}} = \frac{2487.2117}{4354.5857} = 0.57117$$

$$\bar{A}^{1}_{50:\overline{10}|} = 1.04^{½}\frac{M_{50} - M_{60}}{D_{50}} = 1.04^{½}\frac{1767.5555 - 1477.0842}{4597.0607} = 0.064438$$

$\therefore$ the required premium is

$$100000\left(2*0.064438+(1.04)^{1/2}\left(1-0.038462\left(2*8.207-7.98781\right)-2*0.621178+0.57117\right)\right)$$

$= £13,369.55$

Alternative solution

The benefits payable may be regarded as a sum of £200,000 payable on either death, less a sum of £100,000 payable on the first death.

$\therefore$ the value of the benefits is:

$$200000*2\bar{A}^{1}_{50:\overline{10|}}-100000\bar{A}^{\;\;1}_{\overline{50:50}:\overline{10|}}$$

$$\bar{A}^{1}_{50:\overline{10|}}=0.064438$$

$$\bar{A}^{\;\;1}_{\overline{50:50}:\overline{10|}}=1.04^{1/2}\left(1-d\ddot{a}_{50:50:\overline{10|}}-\frac{D_{60:60}}{D_{50:50}}\right)$$

$$=1.04^{1/2}\left(1-0.038462*7.98781-0.57117\right)=0.124011$$

$\therefore$ the required premium is

200000*2*0.064438-100000*0.124011

=£13,374.10

(The difference in the two answers is due to rounding)

11 (i) The directly standardised rate (*DSR*) is given by

$$DSR = \frac{\sum_x E^{cs}_{x,t} m_{x,t}}{\sum_x E^{cs}_{x,t}}$$

The standardised mortality ratio (*SMR*) is given by

$$SMR = \frac{\sum_x E^{c}_{x,t} m_{x,t}}{\sum_x E^{c}_{x,t} m^{s}_{x,t}}$$

For 10-year policies:

$$DSR = \frac{6.991 * 0.86 + \ldots}{6.991 +} = 13.56053 \qquad SMR = \frac{6.013 * 0.86 + \ldots}{6.013 * 1.08 + \ldots} = 0.920149$$

For 20-year policies:

$$DSR = \frac{6.991 * 2.12 + \ldots}{6.991 +} = 21.06187 \qquad SMR = \frac{0.978 * 2.12 + \ldots}{0.978 * 1.08 + \ldots} = 1.424669$$

Note: In each of the above the *DSR* is expressed as the number of deaths per 1,000.

(ii) I would favour the standardised mortality ratio. The directly standardised mortality rate requires $m_{x,t}$ to be recorded for each age group, for the 10-year and 20-year policies separately. The data may not be readily available. The SMR requires the number of deaths in each age and policy group only to be recorded: these data should be easily recorded.

12 (i) In pricing the mortality option using the conventional method, the actuary pricing the option assumes:

- that all lives eligible to take up the option will do so, and
- that the mortality experience of those who take up the option will be the Ultimate experience which corresponds to the Select experience that would have been used as a basis if underwriting had been completed as normal when the option was exercised

The mortality basis used is not usually assumed to change over time, so that the only data required are the Select and Ultimate mortality tables used in the original pricing basis.

In pricing the mortality option, the actuary values the premium income assuming that the premium payable at the end of the ten years is calculated using Select rates according to the original premium basis and values the premiums assuming Select rates apply only from the date of issue of the original policy. The actuary values the liabilities similarly. The difference in the present value of the premium income and benefit liability per policy originally issued gives the additional option single premium, per policy issued.

(ii) Whole life premium payable = $100000P_{[40]}$

Whole life premium which should be paid according to the actuary's basis = $100000P_{40}$

Option premium = present value of the difference in premiums =

$$100000\left\{\left(P_{40}-P_{[40]}\right)v^{10}\frac{l_{40}}{l_{[30]}}\ddot{a}_{40}\right\}$$

$$=100000\left\{(0.01063-0.01058)\frac{1}{1.06^{10}}\frac{33542.311}{33828.764}*14.874\right\}=£41.18$$

(iii) I would require the following data:

- an estimate of the probability of those reaching age 40 as policyholders, who exercise the option
- a multiple decrement table to describe the mortality and other relevant decrements (such as surrender) of those who exercise the option, commencing at age 40
- the basis on which the whole life premium payable is to be calculated: this would normally be assumed to based on the 1967-70 Select mortality, similar to the premium basis set out

I would calculate the present value of the additional liability, using the multiple decrement table from age 40 and allowing for the probability of exercise of the option and A1967-70 Select mortality before age 40.

I would then calculate the whole life premium payable and also the present value of the whole life premiums payable, similarly to the method used to calculate the additional liability.

The difference between the two values, per term assurance policy issued, would be the option premium.

(iv) The more accurate method is the North American method. However, I would favour the conventional method for the following reasons.

- There may not be sufficient data available to apply the North American method.
- If the policy were being sold in a market where the conventional method was generally used for pricing, then there would be adequate experience of the use of the method in the market.
- Even if sufficient data were available in respect of the North American method, they might not be appropriate for pricing the portfolio concerned, particularly if the pricing were being done when the business was first issued.

13 (i) Present value of annuity payments:

$1.01923/1.06 = 1.04 \Rightarrow$ annuity payments are valued at 4% p.a.

$$\text{Value} = 100000\frac{D_{60}^{6\%}}{D_{50}}\ddot{a}_{60}^{4\%} = 100000\frac{1}{1.06^{10}}\frac{l_{60}}{l_{50}}(11.625+1)$$

$$= 10000\frac{1}{1.06^{10}}\frac{30039.787}{32669.855}12.625 = 64{,}821.99$$

Present value of death benefits:

Present value of death benefit at age $50+t$

$= P\ddot{s}_{\overline{t|}}^{1.923\%}v_{6\%}^{t}$, where P is the annual premium.

$$= P\frac{1.01923}{0.01923}\left(1.01923^t - 1\right)\frac{1}{1.06^t} = \frac{1.01923}{0.01923}P\left(v_{4\%}^t - v_{6\%}^t\right)$$

$\therefore$ the present value of the death benefits is

$$\frac{1.01923}{0.01923}P\left(A_{\overset{1}{50}:\overline{10|}}^{4\%} - A_{\overset{1}{50}:\overline{10|}}^{6\%}\right)$$

$$A_{\overset{1}{50}:\overline{10|}}^{4\%} = A_{50:\overline{10|}} - \frac{D_{60}}{D_{50}} = 0.68436 - \frac{2855.5942}{4597.0607} = 0.063182$$

$$A_{\overset{1}{50}:\overline{10|}}^{6\%} = A_{50} - \frac{1}{1.06^{10}}\frac{l_{60}}{l_{50}}A_{60} = 0.25736 - 0.55839 * \frac{30039.787}{32669.855} * 0.39136$$

$$=0.056421$$

$\therefore$ the present value of the death benefits is

$$\frac{1.01923}{0.01923}(0.063182 - 0.056421)P = 0.358347P$$

Present value of premiums less expenses:

$$= 0.95P\ddot{a}_{50:\overline{10|}} - 0.05P - 100A^{1}_{50:\overline{10|}}$$

$$= 0.95P * 7.599 - 0.05P - 100 * 0.056421$$

$$= 7.16905P - 5.6421$$

$$\therefore 7.16905P - 5.6421 = 64821.99 + 0.358347P$$

$P = £9,518.49$

(ii) At the date of alteration:

Present value of annuity payments before alteration

$$= 10000\frac{D^{6\%}_{60}}{D_{55}}\ddot{a}^{4\%}_{60} = 10000\frac{1}{1.06^5}\frac{l_{60}}{l_{55}}(11.625 + 1) = 10000 * 0.708453 * 12.625$$

$$= 89442.19$$

Present value of death benefit before alteration

$$P\ddot{s}^{1.923\%}_{\overline{5|}}A^{1\ 4\%}_{55:\overline{5|}} + \frac{1.01923}{0.01923}P\left(A^{1\ 4\%}_{55:\overline{5|}} - A^{1\ 6\%}_{55:\overline{5|}}\right)$$

$$= P * 5.295953 * 0.045886 + \frac{1.01923}{0.01923}P(0.045886 - 0.043249)$$

$$= 0.382777P$$

$$= 3643.459$$

Present value of annuity payments after alteration

=89442.19+3643.459-100 = 92985.649

Present value of annuity payments before alteration, based on a rate of interest of 6% after age 60

$$= \frac{10.813}{12.625} * 89442.19 = 76605.02$$

Estimated interest rate underlying annuity payments after alteration

$$0.04 - \frac{92985.649 - 89442.19}{89442.19 - 76605.02} * 0.02 = 0.034479$$

Revised rate of escalation

$$= \frac{1.06}{1.034479} - 1 = 2.467\%$$

14 **Multiple decrement table**

Age	q_x^d	$(al)_x^d$	$(ad)_x^d$	$(ad)_x^s$
60	0.014432	100000	1443.246	19711.35
61	0.016014	78845.4	1262.596	15516.56
62	0.01775	62066.25	1101.658	12192.92

Probabilities of survival

Age	${}_tp_x$
60	1
61	0.788454
62	0.620662

Unit fund (ignoring actuarial funding)

Year	*1*	*2*	*3*
Fund brought forward	0	4970.97	5100.215
Premiums allocated to CU	4845	0	0
Interest	387.6	397.6776	408.0172
Management charge	261.63	268.4324	275.4116
Fund carried forward	4970.97	5100.215	5232.821
Fund brought forward	0	0	5180.274
Premiums allocated to AU	0	4845	4845
Interest	0	387.6	802.0219
Management charge	0	52.326	108.273
Fund carried forward	0	5180.274	10719.02
Surrender values	3976.776	9260.446	15951.84

Unit fund (with actuarial funding)

Year	*1*	*2*	*3*
Actuarial funding factor	0.890605	0.925148	0.961538
Fund brought forward	0	4598.885	4904.053
Premiums allocated to CU	4314.979	0	0
Interest	345.1983	367.9108	392.3242
Management charge	233.0089	248.3398	264.8189

Fund carried forward	4427.169	4718.456	5031.558

Sterling fund

Year	*1*	*2*	*3*
Unallocated premium	685.021	155	155
Expenses	400	80	80
Interest	14.251	3.75	3.75
Management charge	233.0089	300.6658	373.0918
Mortality charge	109.2946	33.64881	0
Surrender profit	88.7785	125.6126	0
Additional allocation	135.3906	146.0999	201.2624
Fund at year end	376.3742	325.2797	250.5794

Present value of profit = 623.4689

Present value of premiums = 10774.61

Profit margin = 6.19%