Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

September 2002

Subject 105 — Actuarial Mathematics 1

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners
12 November 2002

EXAMINERS' COMMENTS

The overall standard of attempts by candidates was high. A number of questions were answered very well. The more challenging questions were less well answered, such as Questions 2,6,7, 12 and 13, with evidence of lack of proper preparation. A common mistake, which was also a feature of previous examinations, was to misread some of the questions.

Detailed comments are given after the solution to each question.

1 A logistic model for projecting the size of a population is a model under which an initial rate of growth for the population is assumed to decrease over time in proportion to the size of the population.

The model may be expressed in the form

$$\frac{1}{P(t)}\frac{dP(t)}{dt} = \rho - kP(t)\text{, where}$$

$P(t)$ is the size of the population at time t
ρ is the rate of growth, a constant, >0
k is a constant, >0.

Full credit was given for the solution set out. However, a fuller treatment would be to assume that the initial rate of growth, ρ, may also be negative, in which case ρ may be assumed to either increase or decrease over time in proportion to the size of the population. Few candidates gave this fuller treatment.

The question was answered well in general. A number of candidates were vague in the definition, omitting the point that the initial rate of growth decreased over time in proportion to the size of the population.

2 A guaranteed annuity rate corresponds to a call option on the bonds that would be necessary to ensure that the guarantee was met, i.e. at an exercise price that generated the required fixed rate of return. Alternatively, it can be modelled by an option to swap floating rate returns at the option date for fixed rate returns sufficient to meet the guaranteed option.

It is difficult to ensure that the whole investment fund corresponds to a single option traded in the market. An approximation is possible by using options written on indices.

At the date of policy issue, all guarantees will be out of the money, i.e. they will have no intrinsic value because current market rates are more than sufficient to meet the guarantees, but will have a time value that is the result of the views of many investors ("the market") of the present value of the likely future costs of the option.

Thus the market price of a suitable option produces a way of pricing the guaranteed annuity rates.

This question was poorly answered. Many candidates described other techniques rather than option pricing techniques.

3 (a) Class selection is the process whereby lives are divided into separate groups, within which mortality or morbidity is homogenous, where each group is specified by a category or class of a particular characteristic of the population. An example in life assurance business is the use of individual rating factors which produce mortality differences, e.g. smoking status.

(b) Spurious selection is the process whereby lives are divided into separate groups, within which mortality or morbidity is homogenous, where the differences in mortality or morbidity are due to factors other than those used to form the groups. An example is a change in underwriting over time leading to mortality improvements, where such improvements are ascribed to the passage of time.

(c) Adverse selection is the process whereby lives are divided into groups that tend to act against a controlled selection process imposed on the groups, in respect of mortality or morbidity. An example is where smokers will tend to select policies from a life office that does not use smoking status as a rating factor.

This question was generally well answered. Credit was given for all reasonable examples. Some candidates confused self-selection with adverse selection in part (c).

4 (a) The asset share for a with profit contract is the accumulation of premiums less deductions associated with the contract plus an allocation of profits on non-profits business, all accumulated at the actual rate of return earned on investments. The deductions include expenses, claims, cost of capital and transfer to shareholder funds, if relevant and are based on actual experience.

(b) The retrospective valuation reserve is the expected accumulation of past premiums received, less expected expenses and benefits including any reversionary or interim and terminal bonuses included in past claims.

This question was well answered. Some candidates omitted the allocation from non-profit business in part (a).

5 The net future loss random variable is given by

$$S\left(1+b\right)^{K_{60}+1} v^{T_{60}} - P\ddot{a}_{\overline{\min(K_{60}+1,25)}|}.$$

b is the annual rate of future bonus

K_{60}, T_{60} are the curtate and complete future lifetimes of a life aged 60

P is the annual premium

This question was well answered. A common error was the inclusion of K_{60} *rather than* $K_{60}+1$.

6 The required single premium is given by

$$10000a^{(4)}_{\overline{60:60:10|}} = 10000\left(2a^{(4)}_{\overline{60:10|}} - a^{(4)}_{\overline{60:60:10|}}\right)$$

$$a^{(4)}_{\overline{60:10|}} = a_{60} + \tfrac{3}{8} - \frac{D_{70}}{D_{60}}\left(a_{70} + \tfrac{3}{8}\right)$$

$$= 11.551 + \tfrac{3}{8} - \frac{1516.9972}{2855.5942}\left(7.957 + \tfrac{3}{8}\right)$$

$$= 7.500$$

$$a^{(4)}_{\overline{60:60:10|}} = a_{60:60} + \tfrac{3}{8} - \left(\frac{D_{70:70}}{D_{60:60}}\right)\left(a_{70:70} + \tfrac{3}{8}\right)$$

$$= 8.943 + \tfrac{3}{8} - \left(\frac{1039.0172}{2487.2117}\right)\left(5.498 + \tfrac{3}{8}\right)$$

$$= 6.865$$

$\therefore$ the single premium is $10000 * (2 * 7.500 - 6.865) = £81{,}350$

This question was relatively poorly answered. Many candidates struggled with the correct evaluation of the annuity factors.

7 The required probability is

$$\int_0^\infty {}_t p_y \mu_{y+t} \, {}_{t+10}p_x dt, \text{ where } {}_t p_x = \exp\left(-\int_0^t 0.01 ds\right)$$

$$= \int_0^\infty e^{-0.01t} * 0.01 * e^{-0.1-0.01t} dt$$

$$= 0.01 * e^{-0.1} * \left[-\frac{e^{-0.02t}}{0.02}\right]_0^\infty$$

$$= 0.5 * e^{-0.1}$$

$$= 0.45242$$

This question was answered less well than the examiners had expected, with many candidates setting out the initial integral expression incorrectly.

8 $A^{\;\;2}_{[20]:[20]} = \left(A_{[20]} - A^{\;\;1}_{[20]:[20]} \right)$

$= \left(A_{[20]} - \tfrac{1}{2} A_{[20]:[20]} \right)$

$A_{[20]} = 0.13312$

$A_{[20]:[20]} = 1 - d\ddot{a}_{[20]:[20]}$

$= 1 - 0.038462*21.509$

$= 0.17272$

$\therefore A^{\;\;2}_{[20]:[20]} = 0.13312 - 0.5*0.17272$

$= 0.04676$

This question was well answered by most candidates.

9 Independent rates of withdrawal:

Age attained	*Rate*
20	0.05
21	0.0525
22	0.055125

Probability of survival to age 22 =

$$0.95*0.9475*\frac{l_{22}}{l_{20}} = 0.95*0.9475*\frac{34029.283}{34088.257} = 0.898568$$

$$(aq)^w_{22} = q^w_{22}\left(1 - \tfrac{1}{2} q^d_{22}\right) = 0.055125*\left(1 - \tfrac{1}{2}*0.00079739\right) = 0.055103$$

Required probability = 0.898568*0.055103 = 0.049514

This question was well answered. Some candidates used q^w_{22} in place of $(aq)^w_{22}$.

10 (i) The Standardised Mortality Ratio is the ratio of the actual deaths in a population compared with the expected deaths, based on standard mortality rates.

The formula is

$$\frac{\sum_x E^c_{x,t} m_{x,t}}{\sum_x E^c_{x,t}\, {}^s m_{x,t}}, \text{ where}$$

$E^c_{x,t}$ is the central exposed to risk in the population between ages x and $x + t$

$m_{x,t}$ is the central rate of mortality for the population between ages x and $x + t$

${}^s m_{x,t}$ is the central rate of mortality for a standard population between ages x and $x + t$

(ii) The Ratio may be written in the form

$$\frac{\sum_x E^c_{x,t}\,{}^s m_{x,t} \dfrac{m_{x,t}}{{}^s m_{x,t}}}{\sum_x E^c_{x,t}\,{}^s m_{x,t}}$$

which is the weighted average of the age-specific mortality differentials between the population being studied and the standard population.

i.e. $\dfrac{m_{x,t}}{{}^s m_{x,t}}$

weighted by the expected deaths in the population being studied based on standard mortality.

i.e. $E^c_{x,t}\,{}^s m_{x,t}$

Part (i) was well answered in general. A common error was not basing expected deaths on the mortality of a standard population.

Part (ii) was very poorly answered, with few candidates obtaining full marks.

11 (a) The net rate of interest is 4% per annum in deferment and payment.

The value of the deferred pension in payment is

$$10000 * \frac{D_{65}}{D_{55}} * \left(\ddot{a}^{(12)}_{\overline{5}|} + \frac{D_{70}}{D_{65}} * \ddot{a}^{(12)}_{70} \right)$$

$$\frac{D_{65}}{D_{55}} = \frac{2144.1713}{3664.5684} = 0.585109$$

$$\ddot{a}_{\overline{5|}}^{(12)} = \frac{i}{d^{(12)}} * a_{\overline{5|}} = 1.021537 * 4.4518 = 4.5477$$

$$\frac{D_{70}}{D_{65}} = \frac{1516.9972}{2144.1713} = 0.707498$$

$$\ddot{a}_{70}^{(12)} = 8.957 - 0.458 = 8.499$$

Required value = 10000*0.585109*(4.5477+(0.707498*8.499)) = 61792

(b) The value of the pension on death in retirement is

$$0.9 * 0.5 * 10000 * \frac{D_{65}}{D_{55}} * \left(\frac{D_{70}}{D_{65}} * \left(\frac{l_{70}}{l_{65}} * \ddot{a}_{70|70}^{(12)} + \left(1 - \frac{l_{70}}{l_{65}}\right) * \ddot{a}_{70}^{(12)} \right) \right)$$

$$\ddot{a}_{70|70}^{(12)} = a_{70|70} = a_{70} - a_{70:70} = 7.957 - 5.498 = 2.459$$

$$\frac{l_{70}}{l_{65}} = \frac{23622.102}{27442.681} = 0.8607797$$

Required value =

$$0.9 * 0.5 * 10000 * 0.585109 * \left(0.707498 * \left(0.8607797 * 2.459 + (1 - 0.8607797) * 8.499\right)\right)$$
$$= 6147$$

Total value = £67,939

Attempts at valuing the benefits in part (a) were reasonable in general. Attempts at valuing the benefits in part (b) were very poor.

12 (i) The approach uses two double decrement tables. One table relates to healthy policyholders and decrements of falling sick and dying. Recovery and subsequent rates of sickness are allowed for in the table. The table is used to calculate probabilities of surviving to be a healthy policyholder at age 30+t, $\frac{(al)_{30+t}}{(al)_{30}}$, $0 \le t \le 35$. The table is also used to calculate the dependent initial rate of falling sick at age 30+t, $(aq)_{30+t}$, $0 \le t \le 35$. The rate $(aq)_{30+t}$ is called the inception rate for disability.

The second table relates to policyholders receiving disability benefits and has decrements of recovery from disability and dying (while disabled). The survival probabilities from this double decrement table are used, together with an appropriate interest rate, to determine the present value at the date of becoming disabled of a disability annuity of £20,000 per annum, increasing in payment continuously at the rate of 3% per annum compound and payable according to the policy conditions until the policyholder dies while disabled, recovers or reaches age 65.

The probabilities of surviving as a healthy policyholder to age 30+*t*, the inception rates for disability at age 30+*t* and the disability annuity payable from age 30+*t* are calculated for each value of *t* and are integrated or summed over the range $0 \le t \le 35$.

The data required are those set out in the two multiple decrement tables above, for ages from 30 to 65.

(ii) The value of a disability benefit of £1 p.a. payable weekly to a healthy life now aged 30 exact is

$$\int_{t=0}^{t=35} \frac{(al)_{30+t}}{(al)_{30}} (a\mu)^i_{30+t}\, v^t \left(20000 * \bar{a}^{i'}_{30+t}\right) dt \text{ , where}$$

$(al)_{30+t}, (a\mu)^i_{30+t}$ are based on a double decrement table for healthy policyholders described in part (i) above and gives the number surviving to age 30 + *t* while healthy and the force of inception of disability at age 30 + *t*.

$\bar{a}^{i'}_{30+t}$ is a continuous annuity based on the second double decrement table described above, evaluated at rate of interest i', where

$$i' = \frac{1+i}{1.03} - 1$$

Assume that lives becoming disabled in $(30+t, 30+t+1)$ do so on average at age $30+t+\frac{1}{2}$ and the integral is approximated by

$$\sum_{t=0}^{t=35} \frac{(ad)^i_{30+t}\, v^{30+t+\frac{1}{2}} \left(20000 * \bar{a}^{i'}_{30+t+\frac{1}{2}}\right)}{(al)_{30}\, v^{30}} \text{, where}$$

$(ad)^i_{30+t}$ is the number of lives becoming disabled at age 30 + *t* last birthday in the first decrement table described in part (i) above.

Define commutation functions as follows

$$C'^{ia}_{30+t} = (ad)^{i}_{30+t}\, v^{30\frac{1}{2}}\, \bar{a}^{i'}_{30+t+\frac{1}{2}}$$

$$D_{30} = (al)_{30}\, v^{30}$$

$$M'^{ia}_{30} = \sum_{t=0}^{t=\infty} C'^{ia}_{30+t}$$

So the value is approximated by

$$20000 * \frac{M'^{ia}_{30}}{D_{30}}$$

Poorly answered in general. The evidence was that many candidates were not well prepared for this topic, with some candidates not attempting the question and others giving very poor answers. For a well-prepared candidate, the question should have been relatively straightforward and a small number of candidates did achieve high marks.

13 (i) The expected present value of benefits at outset is given by

$$\sum_{t=1}^{\infty}\left\{\frac{D^{4\%}_{50+t}}{D_{50}}\left[\sum_{c=1}^{2}\left(I^{c}_{50+t}L^{c}\right)\right]\right\} + P\sum_{t=1}^{\infty}\left\{\frac{D^{6\%}_{50+t}}{D_{50}}\left[\sum_{c=1}^{2} I^{c}_{50+t}\right]\right\},$$

using A1967-70 Ultimate

I^{c}_{50+t} is the proportion of policyholders needing care at exact age 50+t, at benefit level $c = 1, 2$

Benefit levels: $c = 1$ is the benefit level at 50% of maximum
$c = 2$ is the benefit level at 100% of maximum

$L^{1} = £25,000,\ \ L^{2} = £50,000$

P is the annual premium

(ii)

Exact age	*Proportion needing care at 50% of maximum*	*Proportion needing care at 100% of maximum*	*Total proportion*	*Expected payment per current life*
51-70	0.01	0.01	0.02	750
71-85	0.04	0.06	0.10	4000
86+	0.08	0.10	0.18	7000

Present Value of Long Term Care Benefits (all calculated at 4% interest):

$D_{50} = 4597.0607$

Exact age (x to x+ t)	*Expected payment per current life* A	$(N_x - N_{x+t+1})/D_{50}$ B	*Value* A*B
51-70	750	12.37730	9282.98
71-85	4000	2.44004	9760.16
86+	7000	0.18574	1300.18
		Total Present Value	20343.32

Present value of Waiver Payments (all calculated at 6% interest)

Present value is = $P^*\{(0.02^*(D_{51}+D_{52}+\ldots.+D_{70})+0.1^*(D_{71}+D_{72}+\ldots.+D_{85})$
$+0.18^*(D_{86}+D_{87}+\ldots\ldots)\}/D_{50}$

This can be rewritten as:

$$\text{PV} = P^*\{0.02^*a_{50}+0.08^*v^{20*}(l_{70}/l_{50})^*a_{70}+0.08^*v^{35*}(l_{85}/l_{50})^*a_{85}\}$$

$$= P^*(0.02^*12.120+0.08^*0.31180^*0.723055^*7.018$$
$$+0.08^*0.13011^*0.20712^*3.297)$$

$$= 0.376084^*P$$

So the final premium equation allowing for expenses is:

$$0.9^*P^*\ddot{a}_{50} \text{ (at 6\%)} = 20343.32+0.376084^*P$$

i.e. $P = 20343.32/(0.9^*13.120-0.376084)$

i.e. $P =$ £1,779.52

The examiners expected this to be a challenging question. It required the application of basic actuarial techniques to pricing a product that was probably relatively unfamiliar to most candidates. A number of candidates performed very well, achieving full, or near to full, *marks. However, the majority of candidates performed poorly, with part (i) being better answered than part (ii).*

14 (i)

Multiple decrement table

x	q_x^d	q_x^s	$(al)_x$	$(ad)_x^d$	$(ad)_x^s$
63	0.01965464	0.1	100000	1965.464	9803.454
64	0.0217431	0	88231.08	1918.417	0
65			86312.67		

Unit Fund

Year, t	*1*	*2*
Value of Capital units at start	0	5965.164
Premium to Capital units	5814	0
Interest on Capital units	465.120	477.213
Management charge on CUs	313.956	322.119
Value of Capital units at end	5965.164	6120.258
Value of Accumulation units at start	0	0
Premium to Accumulation units	0	5814
Interest on Accumulation units	0	465.120
Management charge on Aus	0	62.791
Value of Accumulation Units at end	0	6216.329
Total value of units	5965.164	12336.587
Surrender value	5189.693	0

Non-unit Fund

Unallocated premium	186	186
Expenses	500	100
Interest	-12.56	3.44
MC on Capital units	313.956	322.119
MC on Accumulation units	0	62.791
Surrender profit	76.023	0
Extra death benefit	79.303	0
End of year cash flow	-15.884	474.350
Probability in force	1	0.882311
Profit signature	-15.884	418.52
Discount factor	0.869565	0.756144
Expected present value	-13.812	316.465
Net present value	**302.65**	

(ii)

Unit Fund

Year, t	1	2
Actuarial funding factor	0.92528	0.96154
Value of Capital units at start	0	5735.744
Premium to Capital units	5379.578	0
Interest on Capital units	430.366	458.860
Management charge on CUs	290.497	309.730
Value of Capital units at end	5519.447	5884.873
Total value of units	5519.447	12101.202*
Surrender value	5189.693	0

*including accumulation units

Non-unit Fund

Unallocated premium	620.422	186
Expenses	500	100
Interest	4.817	3.440
Surrender profit	32.327	0
Extra death benefit	88.064	5.118
MC on Capital units less cost of additional allocation	99.656	79.463
MC on Accumulation units	0	62.791
End of year cash flow	169.158	226.576
Probability in force	1	0.882311
Profit signature	169.158	199.91
Discount factor	0.869565	0.756144
Expected present value	147.094	151.161
Net present value	**£298.25**	

(iii) If A1967-70 Select mortality were used in the profit tests instead of A1967-70 Ultimate mortality, the cost of the extra death benefit would decrease and, separately, the profit signature would increase. The effect of these two factors would be to increase the net present value of profit in part (i) and part (ii).

Part (i) was generally well answered. A surprising number of candidates calculated the probability of being in force for year 2 incorrectly.

A number of candidates achieved full marks for part (ii). However, in general this part was less well answered than part (i). Common errors were the incorrect calculation of the surrender profit, extra death benefit and management charge on capital units less the charge of additional allocation.

Part (iii) was well answered. A number of candidates described the effect of changing the mortality basis from Ultimate to Select in the actuarial funding factors, in addition to the two factors given in the solution. Full credit was also given for this approach.