EXAMINATIONS

April 2001

Subject 105 — Actuarial Mathematics 1

EXAMINERS' REPORT

Examiners' Comments

The overall standard of scripts was somewhat disappointing, especially with regard to answers offered for Questions 10, 11 and 12 where candidates were required to apply principles from the syllabus to a problem that they may not have seen before. It is also obvious that many candidates did not read the question asked, or at least did not address the specifics of the question in their solution. Comments on the individual questions follow.

Question 3

A surprisingly high number of candidates used the half-year approximation to work out the integral without any justification, and a number omitted the factor of 52.18

Question 4

Many students used q_{55} and not q_{54} to calculate the death cost

Question 7

The majority of candidates could not quote any of the acceptable integrals and a further number did not know how to obtain an expression for the force of the decrement from the available data.

Question 8

Common errors included adding rather than subtracting the net number of emigrants and not applying a survival factor to these emigrants. Clearly students did not read the question correctly.

Question 9

Generally answered well, although a number of candidates seemed unaware of selective decrements.

Question 10

This question was answered very poorly. Far too many candidates averaged the interest rate as a first step and calculated the cost of annuity benefits at the expected interest rate (6%). A further number then overlooked the option when finalising their solutions.

Question 11

This question was answered poorly by many candidates. Most candidates who made progress favoured the alternative answer shown below. While a number of students identified the reversionary element correctly, few were able to correctly handle the 5-year element.

Question 12

While many candidates had a general understanding of the construction of select mortality rates, most had trouble applying it with the data given. Only a minority of candidates appreciated that the in force data would have to be manipulated in order to get a central exposed to risk which corresponded with the data for deaths.

Question 13

Part (ii) was generally well done. The commonest errors were to limit the benefit term to 25 years and to ignore that it was paid immediately on death. Part (i) was not answered well as students seemed to confuse random variables and their expected values. Part(iii) proved the most difficult. Many students calculated a gross premium reserve, others correctly omitted allowance for expenses and future bonuses but used the gross premium from (ii) and finally most of those who could correctly compute a net premium did so using select mortality.

Question 14

This was generally answered quite well. A significant minority failed to recognise it as a profit test question, and others constructed the reserves using the gross premium. In part (ii), most students identified the correct effect on the NPV of the profit (a reduction) but wrongly assumed that the premium must also reduce.

1 The office can more closely match income and outgo. The initial strain caused by high initial expenses is reduced by capitalising the higher management charges from the capital unit fund.

2 When a member leaves a pension scheme with an entitlement to deferred benefits, they may elect in lieu to have a cash payment made by the scheme to either a new scheme or an individual pension policy.

3 $s_x = 52.18 \int_0^1 {}_t p_x \, \bar{z}_{x+t} \, dt$

$$= 52.18 \int_0^1 (1 - .05t^2)\,(0.1)\,dt = 5.218 \left[t - \frac{.05t^3}{3} \right]_0^1$$

$$= (5.218) \left[1 - \frac{.05}{3} \right] = 5.131$$

4 In general: $({}_tV + P - E)\,(1 + i) = (q_{x+t})\,(S) + (p_{x+t})\,({}_{t+1}V)$

Here $(12{,}500 + 1{,}150 - 75)\,(1.055) = (q_{54})\,(50{,}000) + (p_{54})\,({}_{t+1}V)$

$q_{54} = .00755572$

$\Rightarrow p_{54} = .99244428$

$$\Rightarrow {}_{t+1}V = \frac{1}{.99244428} \{(13{,}575)(1.055) - (.00755572)(50000)\}$$

$$= \frac{1}{.99244428} \{14{,}321.62 - 377.79\} = 14{,}050$$

5 Company A: $_{15}V_{40:\overline{25|}} = 1 - \dfrac{\ddot{a}_{55:\overline{10|}}}{\ddot{a}_{40:\overline{25|}}} = 1 - \dfrac{8.371}{17.169} = 0.5124$

$$\text{PUP SA} = \frac{_{15}V_{40:\overline{25|}}}{A_{55:\overline{10|}}} = \frac{0.5124}{0.75619} = 0.6777$$

Company B: $t = 15 \quad n = 25 \quad \Rightarrow \text{PUP SA} = \dfrac{t}{n} = \dfrac{15}{25} = \dfrac{3}{5} = 0.6$

Therefore Company A provides the higher PUP SA.

6 (a) Uncertain $\rightarrow$ healthy to sick transition rate increase causes reduced premium income and higher claims but increased sickness death rate leads to reduced claims.
Depends on interaction of both effects.

(b) Definite increase $\rightarrow$ Both effects mean people are sick for longer. Therefore higher premiums will be needed to meet higher claims.

7 $(aq)_x^\alpha = \int_0^1 {}_t(ap)_x \, (a\mu)_{x+t}^\alpha \, dt = \int_0^1 ({}_tp_x^\alpha)({}_tp_x^\beta)(\mu_{x+t}^\alpha)\, dt$

$$= \int_0^1 ({}_tp_x^\alpha \, \mu_{x+t}^\alpha)\, {}_tp_x^\beta dt$$

But ${}_tp_x^\alpha \, \mu_{x+t}^\alpha = -\dfrac{d}{dt}({}_tp_x^\alpha)$

$$\Rightarrow {}_tp_x^\alpha \mu_{x+t}^\alpha = -\frac{d}{dt}\left(\frac{x}{x+t}\right)^2 = (-1)\,(-2)\,(x^2)\,(x+t)^{-3} = \frac{2x^2}{(x+t)^3}$$

$$\therefore (aq)_x^\alpha = \int_0^1 \frac{2x^2}{(x+t)^3} \cdot \frac{x^3}{(x+t)^3}\, dt = 2x^5 \int_0^1 \frac{1}{(x+t)^6}\, dt$$

$$= 2x^5 \left[-\frac{1}{5} \frac{1}{(x+t)^5} \Bigg|_0^1 \right] = \frac{2}{5}\left[1 - \frac{x^5}{(x+1)^5}\right]$$

8 We need $P(57, 2)$ so we need to project $P(55, 0)$ for 2 years, allowing appropriately for emigrants.

2001 $P(55, 0)$ = 700369 all have their 56th birthday during 2001 $\Rightarrow$ use $q(56, 0)$

$P(56, 1)$ = (700,369) (1 – .0152) – (35,868) [1 – (½) (.0152)] assuming net emigration is spread uniformly across year

= 689723.4 – 35,595.4 = 654,128.0

2002 $P(57, 2)$ = (654,128) (1 – .0161) – (28,994) [1 – (½) (.0161)] using ($q_{57,\,1}$)

= 643,596.5 –28,760.6 = 614,835.9 = 614,836

9 (a) Class selection: Refers to a factor affecting relative mortality which is a permanent feature, e.g. age, sex, smoking status etc.

(b) Selective decrement: When lives grouped by one decrement experience different levels of another decrement, e.g. ill health retirers usually experience heavier mortality than other scheme members or retired members of similar age/sex

or

marriage/mortality.

(c) Spurious selection: An investigation wrongly suggests that a certain selection is present when it is not.

It usually results from unrecognised heterogeneity in the data, with perhaps changing proportions of lives subject to different underlying mortality rates, e.g. occupational differences being the underlying cause of "regional" mortality effects

or

changing sex mix leading to wrongly attributing mortality rate progressions to temporary initial selection.

10 $\ddot{a}_{60}^{(12)} = a_{60} + \dfrac{13}{24} \Rightarrow$ @ 4% = .542 + 13.294 = 13.836

@ 6% = .542 + 10.996 = 11.538

@ 8% = .542 + 9.294 = 9.836

$\Rightarrow$ annuity per annum available is @ 4%: (200,000 ÷ 13.836) = 14,455 p.a.
@ 6%: (200,000 ÷ 11.538) = 17,334 p.a.
@ 8%: (200,000 ÷ 9.836) = 20,333 p.a.

so the guaranteed minimum annuity option will only be chosen if i = .04

$\Rightarrow$ if i = .06 or .08 (occurs with probability 0.75), the office will only make a loss if assets at age 60 < £200,000.

The guarantee has value at vesting of (15,000) $\ddot{a}_{60}^{(12)}$

= (15,000) (13.836) = 207,540

so the insurer makes a loss if i = .04 (with prob = .25) if assets at age 60 < 207,540

$\Rightarrow$ Probability of loss = (.75) [Prob (Assets < 200,000)] + (.25) [Prob(Assets < 207,540)]

$$= (.75)\left[P\left(\frac{A-250{,}000}{50{,}000} < \frac{200{,}000-250{,}000}{50{,}000}\right)\right] \text{ where } A \sim N(250{,}000,\ 50{,}000^2)$$

$$+ (.25)\left[P\left(\frac{A-250{,}000}{50{,}000} < \frac{207{,}540-250{,}000}{50{,}000}\right)\right]$$

= (.75) [$P(z < -1)$] + (.25) [$P(z < -.8492)$]

= (.75) [1 – .84134] + (.25) [1 – .80234] = (.75) (.15866) + (.25) (.19766)

= .118995 + .049415 = .16841 = 0.17

11 From considering **only** the second condition, the first five payments are certain to occur:

$$\Rightarrow \qquad \text{EPV} = 40{,}000\sum_{t=1}^{5} v^t = 40{,}000 a_{\overline{5|}}$$

Payments thereafter will be made if either life was alive five years earlier, with probability ${}_{t-5}p_{\overline{40:40}}$

$$\Rightarrow \qquad \text{EPV} = 40{,}000\sum_{t=6}^{\infty} v^t \, {}_{t-5}p_{\overline{40:40}}$$

Setting $n = t - 5$ ($\therefore\ t = n + 5$) we get

$$\text{EPV} = 40{,}000\sum_{n=1}^{\infty} v^{n+5} \, {}_n p_{\overline{40:40}} = 40{,}000 v^5 \sum_{n=1}^{\infty} v^n \, {}_n p_{\overline{40:40}}$$

$$= 40{,}000\, v^5\, a_{\overline{40:40}}$$

However, the first condition means that no payment occurs on any date when both lives are still alive, which occurs with probability ${}_t p_{40:40}$.

The previous values are too high by $40{,}000\sum_{t=1}^{\infty} v^t \, {}_n p_{40:40} = 40{,}000\ a_{40:40}$.

Therefore the total expected present value is given by:

$$40{,}000[a_{\overline{5|}} + v^5(a_{\overline{40:40}}) - a_{40:40}]$$

$$= 40{,}000[a_{\overline{5|}} + v^5(a_{40} + a_{40} - a_{40:40}) - a_{40:40}]$$

$$= 40{,}000\left[a_{\overline{5|}} + v^5\left\{\frac{2N_{41}}{D_{40}} - \frac{N_{41:41}}{D_{40:40}}\right\} - \frac{N_{41:41}}{D_{40:40}}\right]$$

$$= 40{,}000\left[4.45182 + (.82193)\left\{\frac{(2)(125{,}015.43)}{6{,}986.4959} - \frac{109{,}071.05}{6{,}794.7238}\right\} - \frac{109{,}071.05}{6{,}794.7238}\right]$$

$= 40{,}000[4.45182 + (.82193)\ \{(2)(17.89387) - (16.05232)\} - 16.05232]$

$= 40{,}000[4.45182 + (.82193)\ (19.73542) - 16.05232]$

$= (40{,}000)(4.62063) = 184{,}825$

Alternative solution:

The first condition is the sum of two reversionary annuities:

$(40{,}000)(2)(a_{40|40}) = (80{,}000)(a_{40} - a_{40:40})$

or

is the difference between a last survivor annuity and joint life annuity:

$(40{,}000)(a_{\overline{40:40}} - a_{40:40}) = (80{,}000)(a_{40} - a_{40:40})$

The second condition pays a 5 year annuity certain, with the first payment at the end of the year of death of the survivor:

$(40{,}000)(\ddot{a}_{\overline{5}|})\,A_{\overline{40:40}}$

Using $A_{\overline{40:40}} = 1 - d\ddot{a}_{\overline{40:40}} = 1 - d(2\,\ddot{a}_{40} - \ddot{a}_{40:40})$

evaluating and summing the 2 elements leads to the same solution as above.

12 (i) Force of mortality ⇒ Deaths ÷ Corresponding central exposed to risk

In this case let $\theta_{y,r}$ = number of deaths where

y = age last birthday at previous policy anniversary

r = duration at previous policy anniversary

⇒ both are policy year rate intervals

$$\frac{\theta_{y,r}}{E^c_{y,r}} \text{ estimates } \mu_{[y+\frac{1}{2}-r]+r+\frac{1}{2}}$$

At the start of the policy year rate interval lives are aged y last birthday, with duration in force of exactly r years. Assuming birthdays are spread uniformly over the policy year, this gives an average exact age at the start of the rate interval of $y + \frac{1}{2}$, or $y + \frac{1}{2} - r$ at the start of the policy.

The duration in force half-way through the rate interval (appropriate for force of mortality) is clearly $r + \frac{1}{2}$.

For the central exposed to risk, initially we define

${}_nP_{x,t}$ = number of lives in force on 1.1.n where

x = age next birthday at issue

t = calendar year of issue

$\Rightarrow \quad x + n - t =$ age next birthday at following policy anniversary

$\Rightarrow \quad x + n - t - 1 =$ age last birthday at following policy anniversary

$\Rightarrow \quad x + n - t - 2 =$ age last birthday at previous policy anniversary

and also $\quad n - t - 1 =$ duration at previous policy anniversary

For correspondence with deaths we need

$${}_nP_{y+2+t-n,\, n-r-1} \text{ and } {}_{n+1}P_{y+1+t-n,\, n-r}$$

and the appropriate central exposed to risk for calendar year n is:

$${}_nE^c_{y,r} = \tfrac{1}{2}\left({}_nP_{y+2+t-n,\, n-r-1} + {}_{n+1}P_{y+1+t-n,\, n-r}\right)$$

assuming all movements (new business, deaths, lapses etc.) are spread evenly throughout the calendar year.

Then $\quad \mu_{[y+\frac{1}{2}-r]+r+\frac{1}{2}} = \dfrac{\theta_{y,r}}{\sum_n {}_nE^c_{y,r}}$

summing the central exposed to risk over the years of the study.

Alternative method for (i)

It is probably easier to actually restructure in force as follows:

At each census date, calculate r = census year – calendar year of issue

Let $y = x + r$

$\Rightarrow$ age $y - 2$ last birthday at preceding policy anniversary

$\Rightarrow$ duration $r - 1$ at preceding policy anniversary

Redefine in force as ${}_nP_{y,r} =$ no. of lives in force on 1.1.n where

$y =$ age next birthday at following policy anniversary

$r =$ duration at following policy anniversary

$\Rightarrow$ Appropriate central exposed to risk for calendar year n is

$${}_nE^c_{y,r} = ½({}_nP_{y+2,\,r+1} + {}_{n+1}P_{y+2,\,r+1})$$

assuming ${}_nP_{y,r}$ varies linearly over the calendar year

(rest of part (i) solution is as above)

(ii) Pooling the date will give rise to more credible estimates of true underlying mortality rates, since greater exposure means lower variance.

However, one must be wary of heterogeneity in the data from the two offices:

e.g. differing geographical coverage
differing underwriting standards
different distribution or target market etc.

13 (i) Simple bonus version:

$$L = 250 + (S[1 + (.06)K_x] + 150)\ v^{T_x} - \left\{P(.98)\ddot{a}_{\overline{\min[1+K_x, 65-x]|}} + .02P\right\}$$

Compound bonus version:

$$L = 250 + (S[(1.04)^{K_x} + 150)\ v^{T_x} - \left\{P(.98)\ \ddot{a}_{\overline{\min[1+K_x, 65-x]|}} + .02P\right\}$$

(ii) Equivalence principle $\Rightarrow E[L] = 0$

Also we shall assume $E[T] \doteqdot E[K] + ½$

Simple bonus:

$$\Rightarrow \underbrace{250 + (S+150)\ \bar{A}_{[x]} + (.06S)\ (I\bar{A})_{[x]+1}\frac{D_{[x]+1}}{D_{[x]}}}_{\substack{\text{more easily valued as} \\ (.94S+150)\ \bar{A}_{[x]} + .06S(I\bar{A})_{[x]}}} = P\left[(.98)\ \ddot{a}_{[x]:\overline{65-x}|} + .02\right]$$

In this case:

$$250 + (1.04)^{½} \left[\{(.94)\,(200{,}000) + 150\}\, A_{[40]} + \frac{R_{[40]}}{D_{[40]}} (.06)\,(200{,}000) \right]$$

$$= P[(.98)\, \ddot{a}_{[40]:\overline{25}|} + .02]$$

$$\Rightarrow\ 250 + (1.04)^{½} \left[188{,}150\,(.27284) + (12{,}000)\, \frac{57{,}705.359}{6{,}981.5977} \right]$$

$$= P[(.98)\,(15.609) + .02]$$

$$\Rightarrow 250 + (1.04)^{½}\ [51{,}334.85 + 99{,}184.22] = P[15.31682]$$

$$\Rightarrow P = 153{,}749.94 \div 15.31682 = £10{,}038 \text{ p.a.}$$

Compound bonus:

$$250 + (1.04)^{½} \left[\frac{A^{*}_{[40]}}{1.04} 200{,}000 + 150 A_{[x]} \right] = 15.31682P$$

$$* \text{ at } \frac{i-b}{1+b} \text{ i.e. } 0\%$$

$$\Rightarrow\ (250) + (1.04)^{½} \left[\frac{200{,}000}{1.04} + (150)\,(.27284) \right] = 15.31682P$$

$$\Rightarrow 250 + (1.04)^{½}\ [192{,}307.69 + 40.926] = 15.31682P$$

$$\Rightarrow P = 196{,}407.87/15.31682 = £12{,}823\text{p.a.}$$

(iii) Net Premium Reserve for WP policies

(i) allows for accrued bonuses only
(ii) net premium ignoring any bonuses

$$\Rightarrow {}_{10}V = 290{,}000\bar{A}_{50} - (\text{NP})\, \ddot{a}_{50:\overline{15}|}$$

$$\text{where NP} = 200{,}000\ \frac{\bar{A}_{40}}{\ddot{a}_{40:\overline{25}|}} = \frac{(200{,}00)\,(1.04)^{½}\,(.27331)}{15.599}$$

$$= 3{,}573.60 \text{ p.a.}$$

$\Rightarrow {}_{10}V$ = (290,000) $(1.04)^{1/2}$ (.38450) – (3,573.60) (10.995)

= 113,713.23 – 39,291.73 = 74,421.50

14 (i) Reserves $\quad {}_1V_{55:\overline{5}|} = 1 - \dfrac{\ddot{a}_{56:\overline{4}|}}{\ddot{a}_{55:\overline{5}|}} = 1 - \dfrac{3.720}{4.547} = 0.18188$

Similarly
${}_2V$ = .37189
${}_3V$ = .57115
${}_4V$ = .78007
${}_5V$ = 0 assuming all claims paid in cash flow outgo

Also

$q_{[55]}$ = .00447362	$p_{[55]}$ = .99552638 ⇒	${}_0p_{[55]}$ = 1
$q_{[55]+1}$ = .00625190	$p_{[55]+1}$ = .99374810	${}_1p_{[55]}$ = .995526
q_{57} = .01049742	p_{57} = .98950358	${}_2p_{[55]}$ = .989302
q_{58} = .01168566	p_{58} = .98831434	${}_3p_{[55]}$ = .978917
q_{59} = .01299373	p_{59} = .98700627	${}_4p_{[55]}$ = .967478

Year	*Premium*	*Expense*	*Opening Reserve*	*Interest*	*Claim*	*Closing Reserve*	*Profit Vector*
1	P	300	0	.075P–22.5	447.36	18106.63	1.075P–18876.49
2	P	.025P+30	18188	.073125P+1361.8	625.19	36956.50	1.048125P–18061.89
3	P	.025P+30	37189	.073125P+2786.9	1049.74	56515.44	1.048125P–17619.28
4	P	.025P+30	57115	.073125P+4281.4	1168.57	77095.44	1.048125P–16897.61
5	P	.025P+30	78007	.073125p+5848.3	100000	0	1.048125P–16174.70

⇒ *Profit Signature*		*NPV of Profit Signature*		
1	1.075P – 18,876.49	$\times v$	=	.95982P – 16,854.01
2	1.043446P – 17,981.08	$\times v^2$	=	.83183P – 14,334.41
3	1.036912P – 17,430.79	$\times v^3$	=	.73805P – 12,406.89
4	1.026027P – 16,541.36	$\times v^4$	=	.65206P – 10,512.33
5	1.014038P – 15,648.67	$\times v^5$	=	.57539P – 8,879.48
				3.75715P – 62,987.12

NPV = .15P = 3.75715P – 62,987.12

⇒ P = 62,987.12 ÷ (3.75715 – .15) = £17,462

(ii) (a) Lower interest rate ⇒ larger reserves
Larger reserves ⇒ profit deferred
Profit deferred ⇒ NPV reduced
(since risk rate (12%) > earned (7.5%))
⇒ profit falls below 15% premium
⇒ need to increase premium to re-establish 15% margin

(b) Higher risk rate ⇒ lower NPV etc.
⇒ need higher premium to meet profit requirement

(c) Ultimate mortality ⇒ more death claims
⇒ earlier claims (NPV claims increases) and reduced premium income
⇒ NPV falls
⇒ need to increase premium