REPORT OF THE BOARD OF EXAMINERS

April 2003

Subject 105 — Actuarial Mathematics 1

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

J Curtis
Chairman of the Board of Examiners

3 June 2003

Overall Comments

The standard this year was generally good, slightly improved from last year. Candidates seemed to cope well with the new areas that were examined this time (mainly questions 6 and 12). However, the following areas continue to prove the most difficult for candidates:- estimation of select forces of mortality (question 11), mortality options (question 8), and contingent assurances / reversionary annuities (question 7 and 13(ii)), despite the questions asked being very standard for these topics.

Comments for individual questions follow after each question which we hope will assist students.

1 $s_{26}^{26|26} = z_{26}^{26|26} \int_0^1 {}_t p_{26} dt \quad (\approx z_{26}^{26|26} \; {}_{0.5} p_{26})$

Well answered. The main error, if one was present, was to confuse the exact and approximate relationships.

2 (i) The super compound bonus method is a method of allocating annual bonuses under which two bonus rates are declared each year. The first rate, usually the lower, is applied to the basic sum assured and the second rate is applied to the bonuses added in the past.

(ii) The sum assured and bonuses increase more slowly than under other methods for the same ultimate benefit, enabling the office to retain surplus for longer and thereby providing greater investment freedom.

This method also rewards longer standing policyholders and discourages surrenders, relative to other methods.

Very well answered overall. In part (ii), other reasons, where valid, were accepted.

3 The death benefit in year 10 is £15,000

Profit emerging per policy in force at the start of the year is:

$([{}_9V + P]*1.05) - (15{,}000*0.04) - ([1 - 0.04]*{}_{10}V) =$
$([11{,}300 + 1{,}500]*1.05) - (15{,}000*0.04) - (0.96*13{,}200) = £168$

Well answered. Two common errors recurred, using a wrong death benefit (usually nine times the premium) and omitting the survival probability of 0.96 for closing reserves.

4 The component method builds up recursively year on year, allowing explicitly for each of the 3 key elements: births, deaths and net emigration. Each of these can be modelled separately to incorporate changing trends, although to do so relies on detailed data and / or assumptions, usually split by age and sex.

The logistic model is easy to apply, but is restricted in the variation it can allow for a population, relying on 2 parameters which give a limiting population and an initial growth rate, which reduces as population increases. The model does not lend itself to understanding mechanisms of population changes. In reality, growth varies over time in a different manner and most recent projections using the logistic and similar models have tended to overestimate the population.

Also well answered. Occasionally candidates gave extremely lengthy and detailed descriptions of the two methods, too much for the marks available, while at the same time overlooked the comparison of the two approaches, which was the main thrust of the question asked.

5 (i) Crude rates are easily calculated, relying only on total population at risk and total deaths for each cause of death in this case.

However, the relative results for different countries can vary widely if the death rate for a certain cause of death (a) varies by age — as most do — and (b) population structures vary by age between countries. Differences in the crude rates for a cause of death would then be confounded with differences in population structures.

(ii) The rates could be standardised. Direct standardisation is best, whereby each countries actual age-specific death rates are applied to a common population.

Any reasonable standard population could be chosen, but where possible it should have some relevance to the study e.g. a European study could standardise according to the population in Europe sorted by age as follows:

Directly standardised death rate for cause A for a given country = $\dfrac{\sum_x {}^sE_x^c m_x^A}{\sum_x {}^sE_x^c}$

Where ${}^sE_x^c$ is the central exposed to risk at age x in the standard population and m_x^A is the central mortality rate from cause A at age x in the country in question.

Generally very well answered, especially part (i). While many candidates did not relate their answers to the specific question which concerned a cause of death study and wrote about mortality rates generally, this was accepted by the examiners. In part (ii), alternative suggestions were also accepted, where justified.

6 (a) (0.242488)(100)(52.18) using $\overset{-}{a}{}_{45}^{HS(1/all)}$ = £1,265.30

(b) (5.4952)(100)(52.18) using $\overset{-}{a}{}_{\overline{55,2}}^{SS}$ = £28,673.95

Very well answered. The only common error was the omission of the 52.18 factor. Candidates seemed clearly familiar with the new examination tables.

7 Value $= 10,000 * 1.02 I + 300$ where

$$I = \bar{a}^{m}_{60:\overline{20}|} + \bar{a}^{f}_{60:\overline{20}|} - 2 * \bar{a}^{m\,f}_{60:60:\overline{20}|}$$

$$\bar{a}^{m}_{60:\overline{20}|} = \ddot{a}_{60} - \tfrac{1}{2} - v^{20}\frac{l_{80}}{l_{60}}\left(\ddot{a}_{80} - \tfrac{1}{2}\right) = 15.132 - (0.456387)\frac{6,953.536}{9,826.131}(7.006) = 12.869$$

$$\bar{a}^{f}_{60:\overline{20}|} = \ddot{a}_{60} - \tfrac{1}{2} - v^{20}\frac{l_{80}}{l_{60}}\left(\ddot{a}_{80} - \tfrac{1}{2}\right) = 16.152 - (0.456387)\frac{7,724.737}{9,848.431}(8.489) = 13.113$$

$$\bar{a}^{m\,f}_{60:60:\overline{20}|} = \ddot{a}_{60:60} - \tfrac{1}{2} - v^{20}\frac{l^{f}_{80}}{l_{60}}\frac{l^{m}_{80}}{l_{60}}\left(\ddot{a}^{m\,f}_{80:80} - \tfrac{1}{2}\right) = 13.590 - (0.456387)\frac{6,953.536}{9,826.131}\frac{7,724.737}{9,848.431}(5.357) = 12.233$$

I = 12.869+13.113-2*12.233=1.516

Value =10000*1.02*1.516+300

$\Rightarrow$ Premium = £15,763

This question caused considerable problems to candidates . Common errors were to only allow for one reversion (usually on death of male), omit the factor of 2 for joint life annuity, use a factor of 0.98 instead of 1.02 for expenses, or assuming that the annuity ran for 20 years from the first death. A surprisingly high proportion of candidates used erroneous formulae to convert annuities from annually in advance to continuous, often dividing by $1.04^{0.5}$. This is a basic actuarial function which is given in the examination tables.

8 Let the full single premium at commencement = P

The premium (based on normal mortality) payable at the time of exercising the option on the 2nd anniversary =

$$(1.05)^{0.5}250{,}000 q_{57} v = 250{,}000 v^{0.5}\frac{d_{57}}{l_{57}} = 250{,}000 v^{0.5}\frac{554}{93{,}583} = 1{,}444.30$$

Therefore the premium required at duration 2, if the option is exercised, is £1,444.30

Thus equating the expected present value of all premium income with the expected present value of all claims, we get:

$$P+(0.4)\,{}_2p_{55}v^2(1{,}444.30)=(1.05)^{0.5}250{,}000(q_{55}v+{}_{1|}q_{55}v^2+{}_2p_{55}[(.6)q_{57}v^3+(.4)(2)(3q_{57}v^3)])$$

$$\Rightarrow P+(0.4)\frac{l_{57}}{l_{55}}v^2(1{,}444.30)=(1.05)^{0.5}\frac{250{,}000}{l_{55}}(d_{55}v+d_{56}v^2+3d_{57}v^3)$$

$$\Rightarrow P+(0.4)\frac{93{,}583}{94{,}532}v^2(1{,}444.30)=(1.05)^{0.5}\frac{250{,}000}{94{,}532}(450v+499v^2+3(554)v^3)$$

$$\Rightarrow P+518.75=6{,}278.54 \text{ leading to } P=5{,}759.79$$

<u>Alternative approach based on non-option policy</u>

If the policy were a simple 3-year term assurance without any options, the single premium at commencement would be:

$$250{,}000\bar{A}^{\,1}_{55:\overline{3|}}=(1.05)^{0.5}250{,}000[q_{55}v+p_{55}q_{56}v^2+{}_2p_{55}q_{57}v^3]$$

$$=(1.05)^{0.5}\frac{250{,}000}{l_{55}}(d_{55}v+d_{56}v^2+d_{57}v^3)$$

$$=(1.05)^{0.5}\frac{250{,}000}{94{,}532}(450v+499v^2+554v^3)=3{,}684.80$$

To allow for the option, the initial single premium needs to be increased by:

$$0.4\,{}_2p_{55}v^2\{1.05^{0.5}[250{,}000(q^*_{57}-q_{57})v+250{,}000(q^*_{57}-q_{57})v]\}$$

q^*_{57} represents the mortality of optioners post-option = $3q_{57}$

(The 1st term in square brackets represents the extra mortality of optioners on the original SA, and the 2nd term represents the extra mortality on the additional SA over and above that paid for by the normal rates premium paid at the time of exercising option, t=2)

$$=(0.4)({}_2p_{55})v^3(1.05)^{0.5}250{,}000(4q_{57})=(0.4)(1.05)^{0.5}250{,}000\frac{4d_{57}}{l_{55}}v^3$$

$$=(0.4)(1.05)^{0.5}250{,}000\frac{4*554}{94{,}532}v^3=2{,}075.00$$

The total single premium at outset = 3,684.80+2,075.00 = 5,759.80 (same as above, allowing for rounding)

The premium payable by policyholders at t=2 when exercising their option is (unchanged from original solution):

$$(1.05)^{0.5}250{,}000q_{57}v=250{,}000v^{0.5}\frac{d_{57}}{l_{57}}=250{,}000v^{0.5}\frac{554}{93{,}583}=1{,}444.30$$

This proved the most difficult question for students, with few fully correct answers. A number of candidates seemed to misread the question and tried to calculate the cost of the option (instead of the premiums) while others treated the policy as annual premium. Many students calculated the basic premium for a policy with no option and tried to calculate the additional premium required for the option so the examiners have provided an alternative solution along these lines.

9 (i) To zeroise future negative cash flows.

The office must meet all future outgo (additional to unit liabilities) e.g. death claims in excess of units, expenses, maturity guarantees. It can take credit for future income to the non-unit fund but cannot assume recourse to future capital.

If there are negative cash flows, we cannot assume that they will be met from subsequent positive cash flows or future capital (lapse risk, regulations). They are future losses which we need to reserve for now. With adequate non-unit reserves established, the minimum expected cash flow in future years, allowing for release of reserves, is zero, hence the "zeroisation" of cash flows.

(ii) (−2, 0, 0, 0, +1, 0, 0, 0, 0, +1)

(iii) Cash flow approach is more flexible in general and allows for clarity of thought and ease of presentation of results
Allows for complex policies (varying benefits, options)
Permits variable or stochastic premium basis e.g. interest basis
Best (often only) approach for multiple state model situations
Allows amount and timing of cash flows to be observed
Provides net cash flows useful for investment strategy
Allows for explicit amount of profit to be calculated.
Makes explicit allowance for cost of capital
Only way to calculate non-unit reserves
Facilitates repeating with altered basis for sensitivity testing (once spreadsheet or program set up)

Generally well answered, especially part (ii). Some candidates only gave examples of outgo in part (i), without considering offsetting income while in part (iii) some candidates tended to concentrate on only one reason.

10 (i)

$$\frac{2}{3}(30{,}000)\frac{s_{40}}{s_{39}}\frac{{}^{z}M_{40}^{ia}}{{}^{s}D_{40}} = \frac{2}{3}(30{,}000)\frac{7.814}{7.623}\frac{58{,}094}{25{,}059} = £47{,}527.51$$

(ii)

$$\frac{30{,}000}{80}\frac{s_{40}}{s_{39}}\left(\frac{18\,{}^{z}M_{40}^{ra} + {}^{z}\overline{R}_{40}^{ra} - {}^{z}\overline{R}_{62}^{ra}}{{}^{s}D_{40}}\right)$$

$$= \frac{30{,}000}{80}\frac{7.814}{7.623}\frac{(18)(128{,}026) + 2{,}884{,}260 - 159{,}030}{25{,}059}$$

$$= £77{,}153.73$$

(iii) $$50{,}000\left(\frac{M_{40}^{i}+M_{40}^{r}}{D_{40}}\right)=50{,}000\left(\frac{369+782}{3{,}207}\right)=£17{,}945.12$$

Well answered throughout. The commonest mistakes related to omitting the salary adjustment, treating (i) as service-related, omitting the factor for age 62 in the future service part of (ii). In (iii), some candidates used annuity functions and / or omitted one of the types of retirement.

11 (i) We are estimating $\mu_{[x]+t}$

From ${}^{1}\theta_{y,r}$ y is policy year rate interval and lives are aged between y and $y + 1$ at the start of the interval in which death occurs, giving an average age at the policy anniversary before death of $y + .5$, assuming an even spread of birthdays over the policy year.

r is also a policy year rate interval, and is the same as a duration of $r - 1$ years exact at the policy anniversary before death, without assumption.

The age at entry is $y + .5 - (r - 1) = y - r + 1.5$ and the duration midway through the rate interval (needed for the duration when estimating forces of mortality) is $r - 1 + .5 = r - .5$ so we are estimating $\mu_{[y-r+1.5]+r-.5}$. No further assumptions are required.

From ${}^{2}\theta_{y,r}$ y is age last birthday at death giving a life year rate interval, with lives y exact at the start of the interval without assumptions needed.

r is again a policy year rate interval, giving duration r years exact at the policy anniversary before death, without assumption.

The average age at entry is $y - r$, but we must assume an even spread of birthdays over the policy year because the 2 rate intervals are not the same (the age at entry could range from $y - r - 1$ to $y - r + 1$ based on the information we have) and the duration midway through the rate interval is $r + .5$ so we are estimating $\mu_{[y-r]+r+.5}$.

(ii) Census A gives a life year for age, with y last birthday, and a policy year for duration with r curtate.

Census B gives y next birthday at next policy anniversary, which is also $y - 2$ last birthday at previous policy anniversary. It also gives duration r at policy anniversary following census or $r - 1$ curtate at census.

For the ${}^{1}\theta_{y,r}$ deaths, census B fits perfectly but we just need to be careful with age labels. To get y last birthday at previous policy anniversary, and $r - 1$ curtate, we need $P_{y+2,r}$.

The approximate exposed to risk is $\left[\frac{1}{2}\,{}^{B}P_{y+2,r}^{2000} + {}^{B}P_{y+2,r}^{2001} + \frac{1}{2}\,{}^{B}P_{y+2,r}^{2002}\right]$ for estimating $\mu_{[y-r+1.5]+r-.5}$

For the ${}^{2}\theta_{y,r}$ deaths, census A fits perfectly. To get y last birthday, and r curtate, we need $P_{y,r}$.

The approximate exposed to risk is $\left[\frac{1}{2}\,{}^{A}P_{y,r}^{2000} + {}^{A}P_{y,r}^{2001} + \frac{1}{2}\,{}^{A}P_{y,r}^{2002}\right]$ for estimating $\mu_{[y-r]+r+.5}$

We assume that $P_{x,t}$.varies linearly between census dates.

Generally not well answered , especially part (ii). In (i), some candidates based their answer on the census data rather than on the death data, listed standard assumptions regardless of if they applied here. Others, having defined the age and duration labels correctly did not define the force of mortality at all or incorrectly.

In part (ii), while many students correctly matched the censuses to the death tabulations, almost none got the correct age / duration labels for census B matched with deaths method 1.

There was a slight discrepancy in the question between the number of years of death data (3) and the time period spanned by the censuses (2). This was not central to any of the answers required, but the examiners accepted all valid interpretations / assumptions made by students in this regard.

12 (i) Death claims in 2002 get SA, no reversionary bonus, and terminal bonus = 125,000

Discontinuances in 2002 get 0.25*4,300 = 1,075

2002 money flows:

Premium income:	5000*4,300 =	21,500,000
Expenses:		15,000,000
Balance:		6,500,000
Interest during 2002 @ 6.5%:		422,500
Balance @ 31/12/2002 before claims:		6,922,500
Death claims 2002:	4*125,000	500,000
Surrender claims:	200*1,075	215,000

Total funds 31/12/2002: 6,207,500

No. of policies in force 31/12/2002: 5000 – 4 – 200 4,796

Asset share per policy in force at 31/12/2002 = $\dfrac{6,207,500}{4,796}$ = £1,294

(ii) The basis for net premium reserves and the 2002 reversionary bonus declaration were the unnecessary items.

Neither affected the cash flows during 2002 nor therefore the year end asset share.

Well answered, especially as this was the first time an asset share calculation had appeared. The main error was to allow for reserves in some way. Some students tried to do the calculation per policy sold but this usually led to errors.

13 (i) A (contingent) whole life assurance with benefit of £100,000 paid immediately on the death of (y) providing it occurs after (x)'s death

(ii) Reserve before alteration

$$V = 100,000\int_0^\infty e^{-\delta t}(1-{}_tp_x)\,{}_tp_y\mu_{y+t}dt$$

$$= 100,000\int_0^\infty e^{-.04t}(1-e^{-.02t}).03e^{-.03t}dt = 3,000\left\{\int_0^\infty e^{-.07t}dt - \int_0^\infty e^{-.09t}dt\right\}$$

$$= 3,000\left\{\left(\frac{1}{.07}\right)-\left(\frac{1}{.09}\right)\right\} = 9,523.81$$

Reserve post alteration:

$$100,000\bar{A}_{xy} - (0.975P)\ddot{a}_{xy}$$

$$\bar{A}_{xy} = \int_0^\infty e^{-\delta t}\,{}_tp_x\,{}_tp_y(\mu_{x+t}+\mu_{y+t})dt$$

$$= \int_0^\infty e^{-.04t}e^{.02t}e^{-.03t}(0.05)dt = 0.05\int_0^\infty e^{-.09t}dt$$

$$= .05\left(\frac{1}{.09}\right) = 0.555556$$

$$\ddot{a}_{xy} = \sum_0^\infty e^{-\delta t}\,{}_tp_x\,{}_tp_y = \sum_0^\infty e^{-.04t}e^{-.02t}e^{-.03t} = \sum_0^\infty e^{-.09t} = \frac{e^{0.09}}{e^{0.09}-1} = 11.6186$$

or alternatively

$$\sum_0^\infty e^{-.09t} = \ddot{a}_{\overline{\infty}|} \text{ (at } i = e^{.09} - 1 = 0.094174) = \frac{1}{d} = \frac{1+i}{i} = 11.6186$$

Reserve before = Reserve after + alteration expense

9,523.81 = (100,000)(.555556) – P(.975)(11.6186) + 100

so P = £4,072.32 p.a.

(iii) Both lives should be underwritten at this time.
The proposed change increases the probability of claim payout by the insurer substantially with regard to life y. Previously if y was worse than assumed mortality, it was a margin for the office, but now the office is at immediate risk in relation to y. The risk with regard to x is similar to that before the alteration as regards the likelihood of a claim arising, but because the claim would now be paid immediately on x's death, the present value could increase significantly.

Part (i) was well answered. In part (ii), many candidates made a good effort but many omitted or could not calculate the pre-alteration reserve. In part (iii), many candidates made general comments about underwriting without explaining why in the context of this particular alteration.

14 (i) Gross future loss random variable (GFL r.v.) =

$$(v^{K_{[60]}+1})\{(200,000-(50,000)(K_{[60]})\} + 300 + 30a_{\overline{K_{[60]}}|} - P(.975\ddot{a}_{\overline{K_{[60]}+1}|} - .225)$$

for $K_{[60]} < 4$

or $300 + 30a_{\overline{3}|} - P(.975\ddot{a}_{\overline{4}|} - .225)$ for $K_{[60]} \geq 4$

(ii) E(GFL r.v.) = 0

$$\Rightarrow 250,000A^{\,1}_{[60]:\overline{4}|} - 50,000(IA)^{\,1}_{[60]:\overline{4}|} + 300 + 30(\ddot{a}_{[60]:\overline{4}|} - 1) = P(.975\ddot{a}_{[60]:\overline{4}|} - .225)$$

$$A^{\,1}_{[60]:\overline{4}|} = \frac{M_{[60]} - M_{64}}{D_{[60]}} = \frac{400.74 - 372.69}{880.56} = 0.0318547$$

$$(IA)^{\,1}_{[60]:\overline{4}|} = \frac{R_{[60]} - R_{64} - 4M_{64}}{D_{[60]}} = \frac{7380.21 - 5813.76 - 4(372.69)}{880.56} = 0.08595666$$

$$\ddot{a}_{[60]:\overline{4|}} = \frac{N_{[60]} - N_{64}}{D_{[60]}} = \frac{12475.24 - 9186.74}{880.56} = 3.734555$$

leading to $7{,}963.68 - 4{,}297.83 + 300 + 82.04 = 3.41619P \quad P = £1{,}184.91$

(iii)

$q_{[60]}$	0.005774	$p_{[60]}$	0.994226	${}_0p_{[60]}$	1
$q_{[60]+1}$	0.00868	$p_{[60]+1}$	0.99132	${}_1p_{[60]}$	0.994226
q_{62}	0.010112	p_{62}	0.989888	${}_2p_{[60]}$	0.985596
q_{63}	0.011344	p_{63}	0.988656	${}_3p_{[60]}$	0.97563

Year	*Prem*	*Expense*	*Interest*	*Claim*	*Cash flow*	*Profit Signature*	*NPV*
1	P	$0.25P+300$	$0.03P-12$	1154.8	$0.78P-1466.8$	$0.78P-1466.8$	$0.75P-1410.38$
2	P	$0.025P+30$	$0.039P-1.2$	1302	$1.014P-1333.2$	$1.008145P-1325.5$	$0.932087P-1225.5$
3	P	$0.025P+30$	$0.039P-1.2$	1011.2	$1.014P-1042.4$	$0.999394P-1027.39$	$0.888458P-913.342$
4	P	$0.025P+30$	$0.039P-1.2$	567.2	$1.014P-598.4$	$0.989289P-583.817$	$0.845648P-499.049$

Total NPV = $3.416193P - 4{,}048.28$

So $P = £1{,}185.03$
(same as above except for rounding due to use of commutation functions)

(iv) (a) Profit is deferred but as earned interest and risk discount rate are equal, there is no impact on NPV or premium.

(b) Profit is deferred but because the discount rate exceeds earned rate, NPV falls and premium would have to increase to satisfy the same profit criterion.

Parts (ii) and (iii) were handled well throughout, with only the death benefit element of part (ii) causing any difficulty. In part (i), a number of students gave the expectation of the random variable, and among those who did give a random variable many omitted the select notation and / or struggled with the benefit element. In part (iv), many gave correct answers for (b), but in (a) very few students recognised that there would be no impact on the premium because the earned interest rate equalled the discount rate.