EXAMINATIONS

April 2004

Subject 105 — Actuarial Mathematics 1

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

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In general terms this was a relatively straightforward paper of standard questions with the possible exceptions of Questions 4 and 8. It was well done by the well prepared students. The Examiners' noted, however, that many students appeared unprepared for this examination and often their marks were well short of the required pass mark resulting in overall a disappointing pass ratio.

1 (a) $$\text{ACF} = \frac{\sum_x {}^sE^c_{x,t}\,{}^sm_{x,t}}{\sum_x {}^sE^c_{x,t}} \Bigg/ \frac{\sum_x E^c_{x,t}\,{}^sm_{x,t}}{\sum_x E^c_{x,t}}$$

where

$E^c_{x,t}$: Central exposed to risk in population being studied between ages x and $x + t$

${}^sE^c_{x,t}$: Central exposed to risk in standard population between ages x and $x + t$

${}^sm_{x,t}$: central rate of mortality either observed or from a life table in standard population for ages x to $x + t$

(b) When multiplied by the crude death rate for the population or area under consideration, the ACF provides a standardised mortality rate ("the indirectly standardised rate"). This approach is often favoured when data required by other methods, usually local age-specific mortality rates, are unavailable.

Question 1 was generally well done although clearly many students could not remember the standard formula.

2 (a) Expected present value of a benefit of 12,000 p.a. payable continuously to a life now aged x and healthy whenever x is sick, with the benefit ceasing at age 65

(b) EPV of a benefit of 10,000 p.a. payable continuously to a life now aged 35 and healthy throughout their first period of sickness, ceasing at age 65 in any event

This question was done reasonably well. In part (b) of the question there was an erroneous symbol x in the formula which should have been 35. The examiners gave full credit for using either x or 35 in the answer above.

3 Conventional assumes all eligible lives exercise the option, experience ultimate mortality according to some table and pay premiums on the option policy based on select mortality from the same table as if underwriting took place at the time of commencement of the option policy.

The North American approach assumes that only a certain proportion of eligible lives exercise the option. Opters and non-opters are subject to different mortality levels. This is normally achieved by having a double decrement table of mortality / exercise of option for original policyholders and also a mortality table for post-option mortality for those who exercise the option.

Question done well. Credit was given for other appropriate comments.

4 Total Fertility Rate = $\sum_x f_x$ where f_x is age specific fertility rate at age x.

For calendar years, we use the period rate approach where we sum the f_x's observed in that year.

For women born in a calendar year, we sum across the f_x's observed over their lifetime, each x coming from the rate observed in the calendar year in which they were aged x. (None of this is required from the student, it is just explanation for the following results).

Up to the end of 2003, $f_x = 1$ for $x = 21, 26, 31, 36$ and $f_x = 0$ otherwise.

Therefore, the answer to (a) and (c) = 4, seeing as all relevant births occur before the change at the end of 2003.

From 1 January 2004, $f_x = 1$ for $x = 23, 28, 33, 38$ and $f_x = 0$ otherwise.

The answer to (d) is also 4. They will have babies when they are aged 21 (in 2003), 28, (in 2010), 33 (in 2015) and 38 (in 2020).

The answer to (b) is zero. There are no women who will have babies in 2004. Those aged 23, 28, 33, and 38 all had babies during 2002 when aged 21, 26, 31 and 36, and therefore will not have their next baby until 2009 when they are aged 28, 33 and 38 respectively.

This question was not done well and many students failed to understand the concept of a Total Fertility Rate attempting often to construct probabilities.

The solution above is a full one. One mark was awarded for each part if the student just wrote down the correct numerical answer.

5 (a) $_{n}q^{2}_{[x][y]}$ represents the probability that a select life, now aged y, will die within n years, having been predeceased by a select life now aged x

(b) $$_{25}q^{2}_{[40][40]} = \tfrac{1}{2} * {}_{25}q_{\overline{[40][40]}} = \tfrac{1}{2} * \{({}_{25}q_{[40]})^2\} = \tfrac{1}{2} * \{(1 - \frac{l_{65}}{l_{[40]}})^2\}$$

$$= \tfrac{1}{2} * \{(1 - \tfrac{8,821.2612}{9,854.3036})^2\} = 0.005495$$

Question was done reasonably well.

6 x nearest birthday at death $\Rightarrow$ x – ½ at start of rate interval (life year from x – ½ to x + ½) during which life dies or x at mid-point when the force of mortality is estimated. No assumption necessary.

r at policy anniversary preceding death means exact duration r at the anniversary before death (the start of the policy year rate interval for duration) and hence r + ½ mid-year. No assumption necessary

The average age at entry [y] is therefore [(x – ½) – r], but we must assume an even spread of birthdays over the policy year because the two rate intervals are not the same type and therefore not coincident. (Based on the information we have the age at entry could range from (x – ½) – (r + 1) to (x + ½) – (r) i.e. x – r – 1½ to x – r + ½, on average x – r – ½.)

Therefore we get an estimate of $\mu_{[x-r-½]+r+½}$

Well prepared students scored well on this question. For full marks all comments regarding assumptions needed to be stated.

7 (i)

$$_{3|5}q_{xy} = {}_{3}p_{xy} - {}_{8}p_{xy} = {}_{3}p_{x} * {}_{3}p_{y} - {}_{8}p_{x} * {}_{8}p_{y}$$

$$= \exp[\int_0^3 0.02dt] * \exp[\int_0^3 0.03dt] - \exp[\int_0^8 0.02dt] * \exp[\int_0^8 0.03dt]$$

$$= \exp[\int_0^3 0.05dt] - \exp[\int_0^8 0.05dt] = e^{-0.15} - e^{-0.4} = .8607 - .6703 = 0.1904$$

Alternatively, the joint life status has constant hazard rate 0.02+0.03 = 0.05 giving a probability of the first death occurring between time 3 and 8:

$$_{3|5}q_{xy} = \int_3^8 {}_{t}p_{xy}\mu_{x+t:y+t}dt = \int_3^8 0.05e^{-0.05t}dt = e^{-0.15} - e^{-0.4} = 0.1904$$

(ii)

$${}_{3|5}q_{\overline{xy}} = {}_3p_{\overline{xy}} - {}_8p_{\overline{xy}} = ({}_3p_x + {}_3p_y - {}_3p_x * {}_3p_y) - ({}_8p_x + {}_8p_y - {}_8p_x * {}_8p_y)$$
$$= (e^{-.06} + e^{-.09} - e^{-0.15}) - (e^{-.16} + e^{-.24} - e^{-0.40})$$
$$= (.9418 + .9139 - .8607) - (.8521 + .7866 - .6703) = .9950 - .9685 = 0.0265$$

Alternatively,

$${}_{3|5}q_{\overline{xy}} = {}_{3|5}q_x + {}_{3|5}q_y - {}_{3|5}q_{xy} = \int_3^8 (0.02e^{-0.02t} + 0.03e^{-0.03t} - 0.05e^{-0.05t})dt = 0.0265$$

Although this question was a simple application of probabilities it was surprisingly not done well overall.

8 (a) Insurer makes a loss if either the policyholder dies or the asset value's 5-year return is less than 40% for survivors

Probability of loss = ${}_5q_{60} + {}_5p_{60} * \Phi[\frac{40-50}{25}] = {}_5q_{60} + {}_5p_{60} * \Phi[-0.4]$

$${}_5p_{60} = \frac{l_{65}}{l_{60}} = \frac{87,093}{91,732} = 0.94942877$$
$$\Phi[-0.4] = 1 - \Phi[0.4] = 1 - 0.65542 = 0.34458$$

$\Rightarrow$ Probability of loss = $0.05057123 + (0.94942877) * (0.34458) = 0.3777$

(b) Terminal bonus is received if both the policyholder is alive and the asset value exceeds 155.556% of single premium

Probability of terminal bonus = ${}_5p_{60} * (1 - \Phi[\frac{55.556-50}{25}]) = {}_5p_{60} * (1 - \Phi[.2222])$

$\Phi[0.2222] = 0.58792$ interpolating linearly between values for 0.22 and 0.23

$\Rightarrow$ Probability of terminal bonus = 0.94942877*(1-.58792)=0.3912

This question was done very poorly overall. Even though the question defined a Normal Distribution very few students appreciated how to apply this in this case.

9 Dependent decrement rates from independent using:

$(aq)_x^r = q_x^r(1 - \frac{1}{2}q_x^d)$ etc. assuming a uniform distribution of decrements in the single decrement tables.

Age	q^r	q^d	$(aq)^r$	$(aq)^d$	(ap)
63	0.1	0.009189	0.099541	0.00873	0.89173
64	0.06	0.010604	0.059682	0.010286	0.930032

Probability of retiring at 64 last birthday = (.89173)*(.059682) = 0.05322

Probability of retiring at 65 = (.89173)*(.930032) = 0.829338

Assuming that those retiring age 63 or 64 last birthday can be represented as retiring on average half-way through the year then we get

Age	*Benefit*	*Discount*	*Probability*	*EPV*
63	45,000	0.971286	0.099541	4,350.73
64	46,500	0.916307	0.05322	2,267.61
65	48,000	0.889996	0.829338	35,429.16
			Total	42,047.50

Despite the definitions given in the question, many students failed to appreciate that they needed to use dependent decrements and produced an answer based merely on independent decrements. Limited credit was given for a solution based on independent decrements and to score well the dependent approach was necessary.

10

Category	*Example*	*Pricing*
Initial	Commission	Allow for directly, usually premium related
	Marketing, promotional	Per policy on estimated volumes
	Underwriting / Processing proposal / Issue of policy documentation	Usually per policy, although some elements might be tied to other driver (e.g. medical expenses might be sum assured related)
Renewal	commission	Allow for directly, usually premium related
	administration	Per policy per annum, allow for inflation
Claim	Calculation and payment of benefit	Per policy, allow for inflation

Overhead	Central services e.g. IT, legal	Per policy per annum

The solution above are the main items the Examiners were seeking. The question was open to wide interpretation as the word Costs was used as opposed to Expenses. Thus the Examiners gave full credit within the total marks for other valid references. Allowing for this the question was well done.

11 (i) (a) the force of sickness $\bar{z}_x$ is the probability that a person aged exactly x is sick at that moment

(b) the annual rate of sickness s_x is the expected number of weeks sickness that a life aged exactly x will experience in the year of age x to $x + 1$

(ii)

$$P\bar{a}_{35:\overline{30|}} = 10,000\bar{A}_{35:\overline{30|}} + 200\bar{a}_{35}^{HS(0/26)} + 150\bar{a}_{35}^{HS(26/26)} + 100\bar{a}_{35}^{HS(52/all)} + \frac{P}{52.18}\bar{a}_{35}^{HS(0/all)}$$

using S(ID) notation and where all sickness benefit functions are understood to terminate at age 65.

Using values from S(MU) tables, and noting that $\bar{A}_{35:\overline{30|}} = 1 - \delta\bar{a}_{35:\overline{30|}}$ we get

$$P*16.979 = 10,000\{(1-(0.039221)(16.979)\}$$
$$+200(10.813+1.931)+150(2.203)+100(2.972+11.859)+\frac{P}{52.18}(29.778)$$

$\Rightarrow P$(16.979-0.571) = 3,340.72+(2,548.80+330.45+1,483.10) = 7,703.07

$\Rightarrow P$ = 469.47 per annum

(Theoretically, some adjustment to the age should be made to reflect the fact that a healthy 35 year-old cannot receive the 2nd / 3rd levels of benefits immediately, but the usual adjustments are approximate and have only minor influence on the result, so are ignored here.)

Question done well by well prepared students.

12 Premium per 100,000 given by:

$$100,000A_{[35]:\overline{25|}} = 0.95P\ddot{a}_{[35]:\overline{25|}} - .55P$$
$$\Rightarrow 100,000*(.24198) = P*[(0.95)(13.392) - 0.55] \Rightarrow P = 1,987.94$$

Reserve per 100,000 for fully in force policy at 31 12 2002 given by:

$${}_{10}V = 100,000A_{45:\overline{15|}} - 0.95P\ddot{a}_{45:\overline{15|}}$$
$$\Rightarrow {}_{10}V = (100,000)(.42556) - (0.95)(1,987.94)(10.149) = 23,389.18$$

Reserve per 100,000 for fully in force policy at 31 12 2003 given by:

$${}_{11}V = 100,000A_{46:\overline{14|}} - 0.95P\ddot{a}_{46:\overline{14|}}$$
$$\Rightarrow {}_{11}V = (100,000)(.45028) - (0.95)(1,987.94)(9.712) = 26,686.47$$

SV at 31 12 2003 per 100,000 SA given by

$${}_{11}SV = \frac{D_{[35]}}{D_{46}}(0.95P\ddot{a}_{[35]:\overline{11|}} - .55P - 100,000A^{1}_{[35]:\overline{11|}}) \ @4\%$$

$$\ddot{a}_{[35]:\overline{11|}} = \frac{N_{[35]} - N_{46}}{D_{[35]}} = \frac{52,662.65 - 29,905.96}{2,507.02} = 9.0772$$

$$A^{1}_{[35]:\overline{11|}} = \frac{M_{[35]} - M_{46}}{D_{[35]}} = \frac{481.53 - 460.84}{2,507.02} = 0.0082528$$

$$\Rightarrow {}_{11}SV = \frac{2,507.02}{1,611.07}[\{1,987.94\}*\{(.95*9.0772) - 0.55\} - \{100,000*0.0082528\}] = 23,690.44$$

Cost of PUPs at 31 12 2003 per 100,000 SA given by:

$${}_{11}PUPV = (\tfrac{11}{25})100,000A_{46:\overline{14|}} = 44,000*0.45028 = 19,812.32$$

Total funds available at 31 12 2003 before paying any claims or setting up reserves:

$[\{500 * (P+{}_{10}V)\} - 100,000] * (1.07) = 13,469,759.20 = \text{A}$

Total claims paid on 31 12 2003 = (200,000 + 25 * ${}_{11}$SV) = 792,261 = B

Total closing reserves required:

$(500 - 2 - 25 - 10){}_{11}V + 10 * {}_{11}\text{PUPV} = 12,553,958.81 = \text{C}$

Profit for 2003 = A – B – C = 123,539.39

Question not done well and very few complete answers were presented

13 Equivalence principle $\Rightarrow$ EPV premiums = EPV benefits + EPV expenses

Let P = quarterly premium

EPV premiums: $4P\ddot{a}^{(4)}_{[45]:\overline{20}|} = 4P[\ddot{a}_{[45]:\overline{20}|} - \frac{3}{8}(1 - \frac{D_{65}}{D_{[45]}})]$

$$= 4P[13.785 - \tfrac{3}{8}(1 - \tfrac{689.23}{1{,}677.42})] = 54.2563P$$

EPV death benefit:

$$210{,}000\bar{A}^{\;1}_{[45]:\overline{20}|} - 10{,}000(I\bar{A})^{\;1}_{[45]:\overline{20}|}$$

$$= 210{,}000*(1.04)^{0.5}[\frac{M_{[45]} - M_{65}}{D_{[45]}}] - 10{,}000*(1.04)^{0.5}[\frac{R_{[45]} - R_{65} - 20M_{65}}{D_{[45]}}]$$

$$= 210{,}000*(1.04)^{0.5}[\frac{462.68 - 363.82}{1{,}677.42}] - 10{,}000*(1.04)^{0.5}[\frac{13{,}987.39 - 5{,}441.07 - 20*363.82}{1{,}677.42}]$$

$$= 12{,}621{,}61 - 7{,}720.60 = 4{,}901.01$$

EPV annuity: $\frac{D_{65}}{D_{[45]}}(23{,}000\ddot{a}_{65} + 2{,}000(I\ddot{a})_{65})$

$$= (0.410887)(\{23{,}000*12.276\} + \{2{,}000*113.911\})$$

$$= 209{,}622.20$$

EPV expenses: Death claim: 250 * ${}_{20}q_{[45]}$ = 250 * (1 – 0.90030) = 24.92
Annuity .02 * EPV annuity = 4,192.44

Premium related:

$$(0.05)(4P\ddot{a}^{(4)}_{[45]:\overline{20}|}) + (0.30)(4P\ddot{a}^{(4)}_{[45]:\overline{1}|})$$

$$= (0.05)(54.2563P) + (0.3)(4P)(1 - \tfrac{3}{8}[1 - \tfrac{D_{[45]+1}}{D_{[45]}}])$$

$$= 2.712815P + 1.182171P = 3.8950P$$

Other:

$$160 + 40\ddot{a}_{[45]:\overline{20}|} @ 0\% = 160 + 40(\{1 + e_{[45]}\} - \frac{l_{65}}{l_{[45]}}\{1 + e_{65}\})$$

$$= 160 + 40[35.282 - (0.90030)(17.645)] = 935.85$$

$54.2563P = 4{,}901.01 + 209{,}622.20 + 24.92 + 4{,}192.44 + 3.8950P + 935.85$

$50.3613P = 219{,}676.42$

Hence $P = 4{,}362.01$

Quarterly Premium is 4,362 to nearer whole unit.

Very few complete answers were presented but many well prepared students did successfully complete a number of parts.

For the item of Renewal Expenses, credit was also given if the student took the alternative approaches of:

1. *Assuming that this particular expense applied throughout life.*
2. *If the inflation escalator of 4% applied from year 3 rather than year 2*

14 (i) (a)

Year t	$q_{[60]+t-1}$	$p_{[60]+t-1}$	${}_{t-1}p_{[60]}$	$NUCF_t$	*Profit signature*	*NPV @ 5%*	*NPV @ 6%*
1	0.005774	0.994226	1	–400	–400.00	–380.95	–377.36
2	0.008680	0.991320	0.99423	210	208.79	189.38	185.82
3	0.010112	0.989888	0.98560	–190	–187.26	–161.77	–157.23
4	0.011344	0.988656	0.97563	450	439.03	361.19	347.76
Total						7.85	–1.01

Because there is a change in sign in NPV between 5% and 6%, there must be a solution to NPV = 0 for an interest rate between 5% and 6%.

(b) ${}_3V = 0$ since policy has positive cash flow in year 4.
${}_2V = 190 / 1.075 = 176.74$

Clearly ${}_1V = 0$ since 210 ($NUCF_2$) > $p_{[60]+1} * {}_2V$

(i) (c) It will increase it. The rate on non-unit reserves exceeds the IRR so in this case the deferral of profits, by introducing reserves, will increase NPV and IRR.

(Usually the discount rate exceeds the non-unit rate of return and allowing for reserves would then reduce NPV and IRR)

(ii) These preliminary calculations, while also an alternative way to get the answer required in (ii)(d), are presented here as background calculations for (ii)(a), (b) and (c).

Unit fund

Year	*Cost of allocation*	*Fund brought forward*	*Interest*	*Mgmt. charge*	*Fund at end*
t	a_t	b_t	c_t	d_t	e_t
	5000*.975*.96	e_{t-1}	$.09*(a_t+b_t)$	$.01*(a_t+b_t+c_t)$	$a_t+b_t\text{-}c_t\text{-}d_t$
1	4,680.00	0.00	421.20	51.01	5,050.19
2	4,680.00	5050.19	875.72	106.06	10,499.85

Non-unit fund

Year	*Unallocated premium*	*Expense*	*Interest*	*Death cost*	*Mgmt. charge*	*Cash flow*	*Prof Sig.*	*NPV*
t	f_t	g_t	h_t	i_t	j_t	k_t	l_t	m_t
	$5{,}000–a_t$		$.06*(f_t–g_t)$	$q_{[60]+t–1}*(40{,}000–e_t)$	d_t	$f_t–g_t+h_t–i_t+ j_t$	${}_{t–1}p_{[60]}*k_t$	$1.12^{–t}* l_t$
leading to								
1	320.00	250.00	4.20	201.80	51.01	–76.59	–76.59	–68.38
2	320.00	50.00	16.20	256.06	106.06	136.20	135.41	107.95
Total								39.57

(a) Yr 1: 320 – 250 +4.20 + 51.01 = 125.21
Yr 2: 320 – 50 +16.20 + 106.06 – (40,000 – 10,499.85) = –29,107.89

(b) NPV = $125.21v – 29{,}107.89v^2 = –23{,}092.84$ (@12%)

(c) Die Yr 1 Cash flow: 320–250+4.20+51.01–(40,000–5,050.19) = –34,824.60
NPV = $–34{,}824.6v = –31{,}093.39$ (@12%)

Survive: Yr 1 = 125.21
Yr 2 = 320 – 50 + 16.20 + 106.06 = 392.26

NPV = $125.21v + 392.26v^2 = 424.50$ (@12%)

(d) NPV for contract = $–31{,}093.39q_{[60]} – 23{,}092.84p_{[60]}q_{[60]+1} + 424.50\,{}_2p_{[60]}$
= 39.57

(same NPV as in preliminary calculations above in non-unit cash flows)

A very straightforward question done very well by well prepared students many of whom scored virtually full marks.

END OF EXAMINERS' REPORT