Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

September 2004

Subject 105 — Actuarial Mathematics 1

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M Flaherty
Chairman of the Board of Examiners

23 November 2004

In general, well prepared candidates did well on this examination which contained reasonably standard questions. Indeed some students scored high marks testifying to the fairly straightforward nature of the paper. The Examiners noted however that there were many candidates who were just not well prepared for the examination and this resulted in a large number being quite a few marks below the required pass level.

Questions without further comment below were those that were in general done well by candidates.

1 The premium is given by:

$$P = 10000\ddot{a}^{(12)}_{\overline{65:62}}$$

$$\ddot{a}^{(12)}_{\overline{65:62}} = \ddot{a}_{65} + \ddot{a}_{62} - \ddot{a}_{65:62} - \tfrac{11}{24}$$

$$= 13.666 + 15.963 - 12.427 - 0.458$$

$$= 16.744$$

$$\therefore \quad P = £167{,}440$$

2 The expected present value is given by:

$$= \frac{50000}{0.5*60*\left(s_{49}+s_{50}\right)*D_{50}}\left(20\,{}^{z}M^{ia}_{50} + {}^{z}\overline{R}^{ia}_{50}\right)$$

$$= \frac{50000}{0.5*60*\left(9.031+9.165\right)*1796}\left(20*45392+363963\right)$$

$$= £64{,}861$$

3 Let P be the weekly premium. P is given by

$$52.18*\left[\left(100+P\right)\overline{a}^{HS(0/all)}_{50} - 50\overline{a}^{HS(2/all)}_{50}\right] = 0.95*52.18P*\overline{a}_{50:\overline{15}|}$$

$$\Rightarrow \left[\left(100+P\right)*0.456447 - 50*0.184025\right] = 0.95*P*9.516$$

$$\Rightarrow P = £4.25$$

4 (i) Risk classification is used as an underwriting tool by life insurance companies. The company divides policyholders into different risk groups according to factors that affect mortality. The company's expectation is that policyholders in the same risk group are homogeneous with respect to mortality risk. The groups are defined by the use of rating factors, e.g., age, sex, smoking habit.

(ii) In theory the company should add rating factors to its underwriting system until the all mortality differences are fully accounted for, apart from random variation. In reality, the ability of prospective policyholders to provide accurate responses to questions and the cost of collecting information limit the extent to which rating factors can be used. In addition, from a marketing point of view, proposers are anxious that the process of underwriting should be straightforward and speedy.

In setting underwriting terms, companies compromise between the conflicting requirements of risk classification and marketing and use a limited number of rating factors. It is important for a company not to omit a significant rating factor that is used by other companies in the market: otherwise, there would be a risk of selection against the company.

Credit was given for other suitable points and description.

5 Let P =monthly premium

G = annual equivalent premium (=12P)

e = annual regular expenses

f = claim expenses

$${}_tV' = (100000 + f)\,A^1_{x+t:\overline{20-t}|} - (G-e)\,\ddot{a}^{(12)}_{x+t:\overline{20-t}|}$$

$$A^1_{x+t:\overline{20-t}|} = vq_{x+t} + vp_{x+t}A^1_{x+t+1:\overline{20-t-1}|}$$

and $$\ddot{a}^{(12)}_{x+t:\overline{20-t}|} = \ddot{a}^{(12)}_{x+t:\overline{1}|} + vp_{x+t}\ddot{a}^{(12)}_{x+t+1:\overline{20-t-1}|}$$

$$\Rightarrow {}_tV' = (100000 + f)\left(vq_{x+t} + vp_{x+t}A^1_{x+t+1:\overline{20-t-1}|}\right) - (G-e)\left(\ddot{a}^{(12)}_{x+t:\overline{1}|} + vp_{x+t}\ddot{a}^{(12)}_{x+t+1:\overline{20-t-1}|}\right)$$

$$= (100000 + f)\,vq_{x+t} - (G-e)\,\ddot{a}^{(12)}_{x+t:\overline{1}|} + vp_{x+t}\Big[(100000 + f)\,A^1_{x+t+1:\overline{20-t-1}|} - (G-e)\,\ddot{a}^{(12)}_{x+t+1:\overline{20-t-1}|}\Big]$$

$$= (100000 + f)\,vq_{x+t} - (G-e)\,\ddot{a}^{(12)}_{x+t:\overline{1}|} + vp_{x+t}\left({}_{t+1}V'\right)$$

$$\therefore \left[{}_tV' + (G-e)\,\ddot{a}^{(12)}_{x+t:\overline{1}|}\right](1+i) - q_{x+t}(100000 + f)$$

$$= p_{x+t} * {}_{t+1}V'$$

Many students attempted to just write down the relationship which was not satisfactory. To score well the relationship had to be derived from 1st principles and the nature of the monthly premium effect clearly brought out.

6 The gross premium prospective policy value is given by:

$$500,200\bar{A}_{\overset{1}{\overbrace{53:53}}:\overline{7}|} - 1,000 * 0.97 * \ddot{a}_{53:53:\overline{7}|}$$

$$\ddot{a}_{53:53:\overline{7}|} = \ddot{a}_{53:53} - \frac{l_{60}^m}{l_{53}^m} * \frac{l_{60}^f}{l_{53}^f} * v^7 * \ddot{a}_{60:60}$$

$$= 16.716 - \frac{9826.131}{9922.995} * \frac{9848.431}{9934.574} * \left(\frac{1}{1.04}\right)^7 * 14.090$$

$$= 16.716 - 0.745975 * 14.09$$

$$= 6.205$$

$$\bar{A}_{\overset{1}{\overbrace{53:53}}:\overline{7}|} = (1.04)^{1/2} * \left(1 - d\ddot{a}_{53:53:\overline{7}|} - \frac{l_{60}^m}{l_{53}^m} * \frac{l_{60}^f}{l_{53}^f} * v^7\right)$$

$$= (1.04)^{1/2} * (1 - 0.038462 * 6.205 - 0.745975)$$

$$= 0.015676$$

$\therefore$ the gross premium policy value is:

$$500,200 * 0.015676 - 1,000 * 0.97 * 6.205 = £1822 \text{ to nearer } £$$

7 (i) Let $l^{\alpha}_{x+t} = l^{\alpha}_x - t^2 d^{\alpha}_x$ and $l^{\beta}_{x+t} = l^{\beta}_x - t^3 d^{\beta}_x$

$$\Rightarrow {}_t p^{\alpha}_x = 1 - t^2 q^{\alpha}_x \quad \text{and} \quad \Rightarrow {}_t q^{\alpha}_x = t^2 q^{\alpha}_x$$

$$\Rightarrow {}_t p^{\alpha}_x \mu^{\alpha}_{x+t} = -\frac{\partial\, {}_t p^{\alpha}_x}{\partial t} = 2tq^{\alpha}_x$$

(ii) Therefore $(aq)^{\alpha}_x = \int_0^1 {}_r p^{\alpha}_x \cdot {}_r p^{\beta}_x \cdot \mu^{\alpha}_{x+r} dr$

$$= \int_0^1 \left(1 - r^3 q^{\beta}_x\right) 2rq^{\alpha}_x dr$$

$$= q^{\alpha}_x \int_0^1 \left(2r - 2r^4 q^{\beta}_x\right) dr$$

$$= q^{\alpha}_x \left(r^2 - \frac{2r^5}{5} q^{\beta}_x \right) \Bigg|_0^1$$

$$= q^{\alpha}_x \left(1 - \frac{2}{5} q^{\beta}_x \right)$$

This question was done very poorly and few candidates derived satisfactory answers.

8 The expected present value of the benefits is given by

$$\sum_{t=0}^{9} S \frac{(ad)^d_{x+t}}{(al)_x} v^{t+1} + \sum_{t=1}^{2} \frac{(ad)^w_{x+5t-1}}{(al)_x} v^{5t} A'_{x+5t}, \text{ (I) where}$$

$$A'_x = \sum_{t=0}^{\infty} S\, {}_t p'_x q'_{x+t} v^{t+1}$$

The expected present value of the premium income is given by

$$P_x \ddot{a}_{x:\overline{10|}} + \sum_{t=1}^{2} P'_{x+5t} \frac{(ad)^w_{x+5t-1}}{(al)_x} v^{5t} \ddot{a}'_{x+5t}, \text{ (II) where}$$

$$\ddot{a}_{x:\overline{10|}} = \sum_{t=0}^{9} {}_t (ap)_x v^t \quad \text{and} \quad \ddot{a}'_x = \sum_{t=0}^{\infty} {}_t p'_x v^t$$

P_x is the premium for the term assurance and P'_{x+5} or P'_{x+10} is the premium for the whole life assurance at the date on which the option is effected.

The additional single premium is given by (I) – (II).

A double decrement table is constructed for all lives that effect the term assurance policy, with decrements of death and exercising the option, with the following definitions:

$(ad)_x^d$, the number of decrements due to death aged x last birthday;

$(ad)_{x+4}^w$ and $(ad)_{x+9}^w$, the number of decrements due to exercise of the option at the fifth policy anniversary and at the expiry of the 10-year term respectively; and

$(al)_x$, the number of lives aged exactly x in the double decrement table.

$${}_t(ap)_x = \frac{(al)_{x+t}}{(al)_x}$$

The dashed functions represent the mortality of those who have exercised the option.

The above solution is just one of a number of possible approaches and credit was given to candidates whose chosen method showed clear definitions. It was not totally necessary to adopt a multiple decrement approach as movements took place at discrete points and again credit was given for other methods.

9 (i) $$L = 250 + \left(S\left[1+(0.06)K_{[x]}\right]+150\right)v^{T_{[x]}} - \left\{(0.98)P\ddot{a}_{\overline{\min[1+K_{[x]},65-x]}|} + 0.02P\right\}$$

(ii) Equivalence principle $\Rightarrow E[L]=0$

Assume $E[T] \approx E[K] + \frac{1}{2}$

$$\Rightarrow 250 + (0.94S+150)\bar{A}_{[40]} + 0.06S\left(I\bar{A}\right)_{[40]} = \quad 0.98P\ddot{a}_{[40]:\overline{25}|} + 0.02P$$

$$\therefore\ 250 + (1.04)^{\frac{1}{2}}\left[\left\{(0.94)(200{,}000)+150\right\}A_{[40]} + (IA)_{[40]}(0.06)(200{,}000)\right]$$

$$= \ 0.98P\ddot{a}_{[40]:\overline{25}|} + 0.02P$$

$$\Rightarrow 250 + (1.04)^{\frac{1}{2}}\left[188{,}150*(0.23041)+(12{,}000)\ \ *7.95835\right]$$

$$= P[0.98*15.887+0.02]$$

$$\Rightarrow 250+1.04^{1/2}[43351.64+95500.2] = P[15.58926]$$

$$\Rightarrow P = £9,099.32$$

10 $\theta_{x,d}$ is classified as x nearest birthday at entry and duration d at policy anniversary following death. Define a census taken at time t after the start of the period of investigation (1.1.2000), $P'_{x,d}(t)$, of those lives having an in force policy at time t, who were aged x nearest birthday at entry and will be duration d on the policy anniversary following time t.

The Central Exposed to Risk is then given by

$$E^c_{x,d} = \int_{t=0}^{t=3.75} P'_{x,d}(t).dt$$

Then assuming that $P'_{x,d}(t)$ varies linearly between the census dates (1.1.2000, 30.9.2000, 30.9.2002, 30.9.2003) the integral can be approximated by

$$\tfrac{1}{2}*\tfrac{3}{4}\{P'_{x,d}(0)+P'_{x,d}(0.75)\}$$

$$+\tfrac{1}{2}*2\{P'_{x,d}(0.75)+P'_{x,d}(2.75)\}$$

$$+\tfrac{1}{2}*1\{P'_{x,d}(2.75)+P'_{x,d}(3.75)\}$$

However the censuses $P'_{x,d}(t)$ have not been recorded. The recorded censuses $P_{x,d}(t)$ have lives classified by x nearest birthday at entry and curtate duration d at time t. We can write

$$P'_{x,d}(t) = P_{x,d-1}(t)$$

Substituting into the previous formula gives an expression for the required Central Exposed to Risk.

Then: $\hat{\mu}_{x,d} = \dfrac{\theta_{x,d}}{E^c_{x,d}}$ estimates $\mu_{[x]+d-0.5}$

because the average age at entry is x assuming birthdays are uniformly distributed over the policy year, and the exact duration at the mid-point of the rate year (policy year) of deaths is d – 0.5 for all lives (no assumptions are necessary).

This question was generally done well by well prepared students but many did not appreciate the relatively straightforward triangulation method.

11 If S_t is surplus in year t per policy in force at begin year t then:

$$({}_{t-1}V+P-E_t)*1.07 = q_t(10000 + {}_tV) + (1-q_t)*{}_tV + S_t$$

Where ${}_tV$ etc is relevant reserve, P the required premium, E_t is expenses for year t and q_t the relevant mortality for year t

So $S_t = (P - E_t)*1.07 + {}_{t-1}V*1.07 - {}_tV - 10000*q_t$

We need to sum ${}_{t-1}p_{[x]}*S_t*v^t$ at 10% for t = 1, 2, 3 and set to zero.

${}_1V = 10000*(1 - \ddot{a}_{[57]+1:\overline{2}|} / \ddot{a}_{[57]:\overline{3}|}) = 10000*(1-(1+v*l_{59}/l_{[57]+1})/2.873)$

= 10000*(1 – 1.956/2.873) = 3191.79

${}_2V$=10000*(1 – 1/2.873) = 6519.32 and ${}_3V$=10000 using 4% interest.

The following table can now be completed:

Year end t	*1*	*2*	*3*
Prem-Expense	$0.8*P$	$0.95*P$	$0.95*P$
${}_{t-1}V_{[57]}$	0	3191.79	6519.32
$10000*q_{[57]+t-1}$	41.71	61.80	71.40
Interest	$0.056*P$	$0.0665*P+223.43$	$0.0665*P+456.35$
${}_tV_{[57]}$	3191.79	6519.32	10000.00
S_t	$0.856P-3233.50$	$1.0165*P-3165.90$	$1.0165*P-3095.73$
${}_{t-1}p_{[57]}$	1.00000	0.99583	0.98967
${}_{t-1}p_{[57]}*S_t$	$0.856*p-3233.50$	$1.0123*P-3152.70$	$1.006*P-3063.75$

Therefore:

$(0.856*P-3233.5)*v+(1.0123*P-3152.70)*v^2+(1.006*P-3063.75)*v^3=0$ at 10%

i.e. $2.3706*P=7846.92$

P = £3,310.10

Very few students produced a full answer here. Although most solutions attempted were as above, it was also acceptable to take the 3rd year reserve as zero i.e. assuming the £10000 maturity value had been paid. This approach would have given a numerical answer of £3287.7

12 The inflation rate of 2.8846% p.a. combined with the valuation rate of 7% p.a. means that all benefits can be valued at 4% per annum effective.

The expected present value of the member's pension is given by

Benefit a

$$10000 * \frac{l_{65}}{l_{62}} * \frac{1}{1.04^3} * \ddot{a}^{(12)}_{65:\overline{5}|}$$

$$\ddot{a}^{(12)}_{65:\overline{5}|} = \ddot{a}^{(12)}_{\overline{5}|} + \frac{l_{70}}{l_{65}} * v^5 * \ddot{a}^{(12)}_{70} \text{ at } i = 4\%$$

$$\ddot{a}^{(12)}_{\overline{5}|} = \frac{i}{d^{(12)}} * a_{\overline{5}|} = 1.021537 * 4.4518 = 4.5477$$

$$\frac{l_{70}}{l_{65}} = \frac{9238.134}{9647.797} = 0.957538$$

$$v^5 = 0.82193$$

$$\ddot{a}^{(12)}_{70} = 11.562 - 0.458 = 11.104$$

$$\therefore \ddot{a}^{(12)}_{65:\overline{5}|} = 13.287$$

$\therefore$ the expected present value is given by

$$10000 * \frac{9647.797}{9773.083} * 0.889 * 13.287 = 116{,}607$$

Benefit b

The expected present value of the spouse's pension on death before retirement is given by

$$5000 \sum_{t=0}^{2} \frac{1}{1.04^{t+0.5}} * h_{62+t+0.5} * \frac{d_{62+t}}{l_{62}} * \ddot{a}^{(12)}_{59+t+0.5}$$

$$\ddot{a}^{(12)}_{59.5} = 0.5 * (16.982 + 16.652) - 0.458 = 16.359$$

Similarly,

$$\ddot{a}^{(12)}_{60..5} = 16.024$$

$$\ddot{a}^{(12)}_{61.5} = 15.679$$

$\therefore$ the value is given by

$$5000 * \left(0.980581 * 0.9 * \frac{34.694}{9773.083} * 16.359 + 0.942866 * 0.9 * \frac{41.398}{9773.083} * 16.024 \right.$$
$$\left. + 0.906602 * 0.9 * \frac{49.193}{9773.083} * 15.679 \right)$$

= 866

$\therefore$ the total expected present value is

116,607+866 = £117,473.

13 (i) The Standardised Mortality Ratio is the ratio of the actual deaths in a population compared with the expected deaths, based on standard mortality rates.

The formula is

$$\frac{\sum_x E^c_{x,t} m_{x,t}}{\sum_x E^c_{x,t}\, {}^s m_{x,t}}, \text{ where}$$

$E^c_{x,t}$ is the central exposed to risk in the population between ages x and $x+t$

$m_{x,t}$ is the central rate of mortality for the population between ages x and $x+t$

${}^s m_{x,t}$ is the central rate of mortality for a standard population between ages x and $x+t$

(ii) The Ratio may be written in the form

$$\frac{\sum_x E^c_{x,t}\, {}^s m_{x,t} \dfrac{m_{x,t}}{{}^s m_{x,t}}}{\sum_x E^c_{x,t}\, {}^s m_{x,t}}$$

which is the weighted average of the age-specific mortality differentials between the population being studied and the standard population.

i.e. $\dfrac{m_{x,t}}{{}^s m_{x,t}}$,

weighted by the expected deaths in the population being studied based on standard mortality.

i.e. $E^c_{x,t}\, {}^s m_{x,t}$

(iii) The SMR for 2000–2001 is $\dfrac{1.8*10+0.9*20}{30}=1.2$

The SMR for 2002–2003 is $\dfrac{2*10+0.8*20}{30}=1.2$

(iv) (a) A formula for the CMF is

$$\frac{\sum_x {}^s E^c_{x,t} m_{x,t}}{\sum_x {}^s E^c_{x,t}\, {}^s m_{x,t}}$$

which may be written in the form

$$\frac{\sum_x {}^s E^c_{x,t}\, {}^s m_{x,t} \dfrac{m_{x,t}}{{}^s m_{x,t}}}{\sum_x {}^s E^c_{x,t}\, {}^s m_{x,t}}.$$

This is simply a weighted average of

$$\frac{m_{x,t}}{{}^s m_{x,t}},$$

weighted by

$$ {}^{s}E^{c}_{x,t}\ {}^{s}m_{x,t} \,. $$

The differences between the SMR and CMF figures indicates that the Standard Population A and the observed population have different proportions in the two age ranges.

As the CMF<SMR, this indicates that Standard Population A is more heavily weighted to the older age group.

(b) In my opinion, use of the SMR gives better results for comparing the population in each of the two periods. The mortality experience in the two periods is compared using Standard Population A exposed to risk in the CMF calculations and the observed population exposed to risk in the SMR calculations. Standard Population A appears to have a significantly different composition from the observed population. Therefore, using the Standard Population A exposed to risk in the weight calculations could introduce differences in the results which have nothing to do with underlying mortality differences. Use of the observed population exposed to risk removes this difficulty and results should be more reliable.

(c) I disagree with the committee's conclusion. The SMR figures indicate that the mortality experience has not changed between 2000–2001 and 2002–2003.

In part (iv) other acceptable comments were given credit.

END OF EXAMINERS' REPORT