Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

September 2000

Subject 105 — Actuarial Mathematics 1

EXAMINERS' REPORT

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1 $A_x \quad = 0.5$

$A_{xx} \quad = 0.8$

$2A^1_{xx} \quad = A_{xx}$

$A^{\ 2}_{xx} \quad = A_x - A^1_{xx} = A_x - ½A_{xx} \quad = 0.5 - ½ \times 0.8 = 0.1.$

2 The standardised mortality ratio (SMR) = $\dfrac{\sum_x E^c_{x,t} m_{x,t}}{\sum_x E^c_{x,t}\ {}^s m_{x,t}}$

$$= \frac{350 + 2{,}295 + 27{,}500}{\left(645 * \dfrac{8{,}347}{13{,}580} + 450 * \dfrac{45{,}360}{8{,}100} + 385 * \dfrac{489{,}860}{6{,}290}\right)}$$

$$= \frac{30{,}145}{32{,}899.93}$$

= 0.9163.

3 (i) $£10{,}000 \int_0^\infty v^t (1 - {}_t p_y)\ {}_t p_x\ \mu_{x+t}\ dt$

where x = age of (x)

y = age of (y)

(ii) The premium should be payable as long as (*x)* is alive, while the benefit is still payable. It does not matter whether (*y)* is alive. The most appropriate annuity factor is, therefore:

$\ddot{a}_x^{(m)}$, where m denotes frequency of payment.

4 Value $= \dfrac{50(K_{35} - K_{60})}{D_{35}}$

$K_{35} \quad = K^{13}_{35} + K^{13/13}_{35} + K^{26/26}_{35} + K^{52/52}_{35} + K^{104/\text{all}}_{35}$

= 462592 + 143625 + 154161 + 179711 + 716291 = 1656380

$K_{60} \quad = 970852.7$

$D_{35} \quad = 23986$

Value = £1429.02

Alternative

$50\left(69.056 - \frac{D_{60}}{D_{35}} 129.405\right)$, based on value of 1 p.w. all periods, whole of life

= £1428.99

5 Required: $Var(a_{\overline{K_{\overline{xy}}}|})$

$$Var(a_{\overline{K_{\overline{xy}}}|}) = Var(\ddot{a}_{\overline{K_{\overline{xy}}+1}|} - 1)$$

$$= Var\left(\frac{1 - v^{K_{\overline{xy}}+1}}{d}\right)$$

$$= \frac{1}{d^2} Var(v^{K_{\overline{xy}}+1})$$

$$= \frac{1}{d^2} ({}^2A_{\overline{xy}} - A_{\overline{xy}})^2)$$

$$= \frac{1}{d^2} ({}^2A_x + {}^2A_y - {}^2A_{xy} - (A_x + A_y - A_{xy})^2)$$

where "2" denotes evaluation at rate of interest $i^2 + 2i$. Other functions are evaluated at rate of interest i.

6 The following are three types of guaranteed reversionary bonuses. The bonuses are usually allocated annually in arrears, following a valuation.

Simple — the rate of bonus each year is a percentage of the initial basic sum assured under a policy. The effect is that the sum assured increases linearly over the term of the policy.

Compound — the rate of bonus each year is a percentage of the basic sum assured and the bonuses previously added. The effect is that the sum assured increases exponentially over the term of the policy.

Super compound — two compound bonus rates are declared each year. The first rate (usually the lowest) is applied to the basic sum assured. The second rate is applied to the bonuses previously added. The sum assured increases exponentially over the term of the policy. The sum assured usually increases more slowly than under a compound allocation in the earlier years and faster in the later years.

7 An asset share is evaluated for an individual policy or for a block of policies, usually for non-unit linked policies.

The asset share is the accumulation of premiums less deductions associated with the contract plus an allocation of profits on non-profit business, all accumulated at the actual rate of return earned on investments. Deductions include all expenditure associated with the contract or contracts.

The asset share may be built up recursively on a year-to-year basis. Initially, the asset share is zero. Each year, the cash flows including premiums received, deductions made to cover actual costs and provisions made to cover other liabilities and provision for profits allocated to the policy or group of policies are recorded. A suitable rate of return is used to accumulate the asset shares plus premiums less deductions plus profit allocations to the year-end to determine the asset share. The process is repeated for subsequent years.

8 (i) Reserve $= 100{,}000\left(1-\dfrac{\ddot{a}_{50:\overline{10}|}}{\ddot{a}_{[40]:\overline{20}|}}\right)-2{,}000\dfrac{\ddot{a}_{50:\overline{10}|}}{\ddot{a}_{[40]:\overline{20}|}}$

$\ddot{a}_{50:\overline{10}|} = 8.207$

$\ddot{a}_{[40]:\overline{20}|} = 13.772$

Reserve = £39,216.24

(ii) Using a Zillmer adjustment has the effect of reducing the policy value. Changing the Zillmer adjustment from 2% of the sum assured to 1% of the sum assured has the effect of reducing the amount of the Zillmer adjustment and hence increasing the policy value, as at 31 December 1999.

9 Premium $= 10{,}000\left(\ddot{a}^{(12)}_{\underset{m}{65}|\underset{f}{60}} - {}_{10|}\ddot{a}^{(12)}_{\underset{m}{65}|\underset{f}{60}}\right)$

$\ddot{a}^{(12)}_{\underset{m}{65}|\underset{f}{60}} = a_{\underset{f}{60}} - a_{\underset{f}{60}:\underset{m}{65}} = 10.996 - ½(7.753 + 7.335)$

$= 3.452$

$$ {}_{10|}\ddot{a}^{(12)}_{\underset{m\ \ f}{65|60}} = \frac{D^m_{75}}{D_{65}}\frac{l^f_{70}}{l_{60}}\left(a_{\underset{f}{70}} - a_{\underset{f\ \ m}{70:75}}\right) $$

$$ = \frac{6809}{17857}\times\frac{780683}{897001}\times(8.328 - ½(4.9 + 4.525)) $$

$$ = 0.381307 \times 0.870326 \times 3.616 $$

$$ = 1.200 $$

$\therefore$ Premium = 10,000(3.452 – 1.2) = £22,520.

10 Define a service table:

l_{x+t} = no. of members aged $x + t$ last birthday

i_{x+t} = no. of members who retire due to ill-health age $x + t$ last birthday

s_{x+t} / s_x = ratio of earnings in the year of age $x + t$ to $x + t + 1$ to the earnings in the year of age x to $x + 1$

Define $z_{x+t} = \frac{1}{3}(s_{x-3} + s_{x-2} + s_{x-1})$; $\bar{a}^i_x$ = value of annuity of 1 p.a. to an ill-health retiree aged exactly $x + t$.

Let (AS) be the member's expected salary earnings in the year of age 25 to 26.

Assume that ill-health retirements take place uniformly over the year of age.

Consider ill-health retirement between ages $25 + t$ and $25 + t + 1$, $t < 35$.

The present value of the retirement benefits related to future service:

$$ \frac{(t+½)(AS)}{60}\frac{z_{25+t+½}}{s_{25}}\frac{v^{25+t+½}}{v^{25}}\frac{i_{25+t}}{l_{25}}\ \bar{a}^i_{25+t+½} = \frac{(t+½)(AS)}{60}\frac{{}^zC^{ia}_{25+t}}{{}^sD_{25}} $$

where ${}^zC^{ia}_{25+t} = z_{25+t+½}\, v^{25+t+½}\, i_{25+t}\ \bar{a}^i_{25+t+½}$

and ${}^sD_{25} = s_{25}\, v^{25}\, l_{25}$

Similarly it may be shown that the present value of the benefits is, in total:

$$ \frac{(AS)}{60^sD_{25}}\left[½^zC^{ia}_{25} + 1½^zC^{ia}_{26} + \ldots + 35^zC^{ia}_{60} + 35^zC^{ia}_{61} + \ldots + 35^zC^{ia}_{64}\right] $$

$$= \frac{(AS)}{60^s D_{25}}\Big[½^z C_{25}^{ia} + 1½^z C_{26}^{ia} + \ldots + 35½^z C_{60}^{ia} + 36½^z C_{61}^{ia} + \ldots + 39½^z C_{64}^{ia} - \left(½^z C_{60}^{ia} + \ldots 4½^z C_{64}^{ia}\right)\Big]$$

$$= \frac{(AS)}{60^s D_{25}}\Big[\left({}^z\bar{M}_{25}^{ia} + {}^z\bar{M}_{26}^{ia} + \ldots {}^z\bar{M}_{64}^{ia}\right) - \left({}^z\bar{M}_{60}^{ia} + {}^z\bar{M}_{61}^{ia} + \ldots + {}^z\bar{M}_{64}^{ia}\right)\Big]$$

[where ${}^z\bar{M}_x^{ia} = \sum_{t=0}^{64-x} {}^zC_{x+t}^{ia} - ½^z C_x^{ia}$]

$$= \frac{(AS)}{60^s D_{25}}\left[{}^z\bar{R}_{25}^{ia} - {}^z\bar{R}_{60}^{ia}\right]$$

where ${}^z\bar{R}_x^{ia} = \sum_{t=0}^{64-x} {}^z\bar{M}_{x+t}^{ia}$

Similarly it may be shown that the present value of benefits related to past service is:

$$\frac{5(AS)}{60^s D_{25}}\,{}^zM_{25}^{ia}$$

where ${}^zM_{25}^{ia} = \sum_{t=0}^{30} {}^zC_{25+t}^{ia}$.

11 $P_x(n)$ = Survivors to n of $P_{x-1}(n-1)$ + migrants during $(n-1, n)$ who survived to be age x at n (net migrants are considered, i.e. migrants less emigrants).

$P_0(n)$ = Births during $(n-1, n)$ + migrants during $(n-1, n)$ who survived to be age 0 at n

where $P_x(n)$ is the population age x last birthday at n, where n refers to mid-year n.

Let

$B(n)$ = births during $(n-1, n)$

$M_x(n)$ = net migrants during $(n-1, n)$ who survive to be age x last birthday at n

$q_{x-½}(n-1)$ = probability that a life aged last birthday $x-1$ at $n-1$ dies in $(n-1, n)$, assuming those aged $x-1$ last birthday at $n-1$ have birthdays uniformly distributed over the calendar year.

${}_{½}q_0(n-1)$ = probability that a life born in $(n-1, n)$ dies in $(n-1, n)$.

Then we have:

$$P_x(n) = P_{x-1}(n-1) \times (1 - q_{x-½}(n-1)) + M_x(n)$$

$$P_0(n) = B(n) \times (1 - {}_{½}q_0(n-1)) + M_0(n)$$

Projections are carried out separately for each sex to give values $P_0(n)$, $P_1(n)$, ..., $P_x(n)$, ...

$B(n)$ and $M_n(x)$ are determined using separate models. Total births in $(n-1, n)$, $B(n)$, are projected using

$$B(n) = \Sigma ½\left\{P_x^f(n-1) + P_x^f(n)\right\} f_x(n)$$

where $P_x^f(n)$ is the number of females aged x last birthday at n.
$½\left\{P_x^f(x-1) + P_x^f(n)\right\}$ gives the average female population aged x last birthday over the year $(n-1, n)$.

$f_x(n)$ is the fertility rate over $(n-1, n)$ for women aged x last birthday at the date of birth.

The summation is taken over all ages where $f_x(n) > 0$.

The sex ratio at birth has been estimated empirically to be 1.06 : 1 (males : females). This ratio is used to obtain male and female births, as follows:

$$B^m(n) = \frac{1.06B(n)}{2.06} \qquad B^f(n) = \frac{B(n)}{2.06}$$

Migration numbers are estimated directly from the International Passenger Survey.

12 Construct multiple decrement tables

For those not exercising the option:

Age	No. alive	No. of deaths
60	100,000	669.90
61	79,464.08	770.94
62	62,954.51	1,117.42

For those exercising the option:

Age	No. alive	No. of deaths
61	19,866.02	477.19
62	35,147.46	935.79

Premiums payable:

$$P_0 = 100{,}000[0.0066990v + (1-.006699)*0.00970168v^2 + (1-.006699)*(1-.00970168)*0.01774972v^3] = 2{,}987.67$$

$$P_1 = 100{,}000[0.00970168v + (1 - .00970168) * 0.01774972v^2] = 2{,}498.85$$

$$P_2 = 100{,}000[0.01774972v] = 1682.44$$

where P_0 is the premium payable at the outset, P_1 is the premium payable at the first anniversary for additional cover and P_2 is the premium payable at the second anniversary for additional cover.

Cost of benefits =

$$\frac{100{,}000}{100{,}000}\left[669.90v + (770.94 + 2*477.19)v^2 + (1{,}117.42 + 2*935.79)v^3\right]$$

=4,730.567

Value of premiums=

$$\frac{1}{100{,}000}[2987.67*100{,}000+2{,}498.85*19{,}866.02v+1{,}682.44*15{,}738.63v^2]$$

=3,696.12.

Option premium=

4,730.57-3,696.12 = £1,034.45

13 (i)

	Outcome	*Cashflow*
(1)	HHH	6,000, 0, 0
(2)	HHS	6,000, 0, –8,000
(3)	HHD	6,000, 0, –16,000
(4)	HSH	6,000, 8,000, 0
(5)	HSS	6,000, –8,000, –8,000
(6)	HSD	6,000, –8,000, –16,000
(7)	HD	6,000, –16,000

"–" indicates cashflow to policyholder

(ii) Complete the set of transition probabilities:

$$p_{55+t}^{HH} = 0.8,\ p_{55+t}^{SS} = 0.1$$

The probability that each outcome occurs is:

Outcome	*Probability*
(1)	0.64
(2)	0.096
(3)	0.064
(4)	0.09
(5)	0.012
(6)	0.018
(7)	0.08
	1.000

(iii) The net present value of each outcome is:

Outcome	*NPV of Profit*
(1)	6,000
(2)	–858.711
(3)	–7,717.421
(4)	–1,407.407
(5)	–8,266.118
(6)	–15,124.829
(7)	–8,814.815

(iv) Mean $= \Sigma \text{ NPV} \times \text{liability}$
$= 6000 \times 0.64 - 858.711 \times 0.096 + \ldots$
$= £2{,}060.36$

Variance $= \Sigma \text{NPV}^2 \times \text{Probability} - (\text{Mean})^2$
$= 34{,}009{,}436.35$

Standard deviation = £5,831.76

14 (i) 1.04 / 1.0097087 = 1.03

$\Rightarrow$ death benefits evaluated at 3% p.a.

Value of death benefits $= 100{,}000\ \bar{A}_{[45]}$

$$= 100{,}000 \times 1.04^{½} \times A_{[45]}^{3\%}$$

$$= 100{,}000 \times 1.04^{½} \times 0.42060$$

$$= 42{,}892.952$$

Value of premiums $= P\ddot{a}_{[45]:\overline{40}|}^{(12)\ 4\%}$

$$\ddot{a}_{[45]:\overline{40}|}^{(12)} = \frac{N_{[45]} - N_{85}}{D_{[45]}} - \frac{11}{24}\left(1 - \frac{D_{85}}{D_{[45]}}\right)$$

$$= \frac{99744.168 - 1095.1562}{5680.3705} - \frac{11}{24}\left(1 - \frac{241.28824}{5680.3705}\right)$$

$$= 17.366651 - 0.438864$$

$$= 16.92779$$

Value of premiums $= 16.92779P$

Value of expenses $= 0.45P + 0.05P\,\ddot{a}_{[45]:\overline{20}|}$

$$= 0.45P + 0.05P \times 17.366651$$

$$= 1.31833P$$

$$\therefore\ 16.92779P = 42892.952 + 1.31833P$$

$$P = 2747.882$$

Monthly premium = £228.99

(ii) Sum assured $= 1.04^8 \times 100{,}000 = 136{,}856.91$

Value of reserves before alteration =

$$136{,}856.91\,\bar{A}_{53} - 228.99 \times 12\ddot{a}^{(12)}_{53:\overline{32|}}$$

$$\bar{A}_{53} = 1.04^{½} A^{4\%}_{53} = 1.04^{½} \times 0.4226 = 0.430969$$

$$\ddot{a}^{(12)}_{53:\overline{32|}} = \frac{N_{53} - N_{85}}{D_{53}} - \frac{11}{24}\left(1 - \frac{D_{85}}{D_{53}}\right)$$

$$= \frac{60363.851 - 1095.1562}{4020.9326} - \frac{11}{24}\left(1 - \frac{241.28824}{4020.9326}\right)$$

$$= 14.74004 - 0.43083$$

$$= 14.30921$$

Value of reserves = 19,661.094.

$= S\bar{A}_{53} = S \times 0.430969$

$\therefore\ S \times 0.430969 + 100 = 19{,}661.094$

$\therefore\ S = £45{,}388.63$

15 (i) **Multiple decrement table**

age (x)	q^d_x	q^s_x	$(al)_x$	$(ad)^d_x$	$(ad)^s_x$
62	0.01774972	0.15	100,000	1774.972	14733.754
63	0.01965464	0.15	83491.274	1640.991	12277.542
64	0.02174310	0	69572.741	1512.727	0
65			68060.014		

Unit Funds (ignoring actuarial funding)

Year, t	*1*	*2*	*3*
Value of Capital Units at start	0	3963.790	4093.703
Premium to CUs	3838	0	0
Interest on CUs	345.42	356.741	368.433
Management charge on CUs	219.630	226.828	234.262
Value of CUs at end	3963.790	4093.703	4227.874
Value of Accumulation Units at start	0	0	4131.127
Premium to AUs	0	3838	3838
Interest on AUs	0	345.42	717.221
Management charge on AUs	0	52.293	108.579
Value of AUs at end	0	4131.127	8577.769
Surrender value of units	3369.222	7815.460	12805.643

Capital Unit Fund (allowing for actuarial funding)

Year, t	*1*	*2*	*3*
Actuarial funding factor	0.89097	0.92528	0.96154
Value of CUs at start	0	3667.616	3936.259
Premium to CUs	3419.543	0	0
Interest on CUs	307.759	330.085	354.263
Management charge on CUs	195.683	209.879	225.252
Value of CUs at end	3531.618	3787.822	4065.270

Sterling Fund

Year, t	*1*	*2*	*3*
Unallocated premium	580.457	162	162
Expenses	300	60	60
Interest	12.621	4.59	4.59
MC on Capital Units	195.683	209.879	225.252
MC on Accumulation Units	0	52.293	108.579
Surrender profit	23.927	15.218	0
Extra death benefit	114.813	40.902	0
Cost of extra allocation	113.546	123.692	162.604
End of year cashflow	284.329	219.297	277.817
Probability in force	1	0.834913	0.695727
Discount factor	0.869565	0.756144	0.657516
Expected present value	247.243	138.445	127.088

Expected p.v. of profit = 512.776

Expected p.v. of premiums = $4000 \times 2.25208 = 9008.323$

Profit margin = 5.69%

(ii) **Revised Sterling Fund (ignoring reserves)**

Year, t	*1*	*2*	*3*
Unallocated premium	162	162	162
Expenses	300	60	60
Interest	–6.21	4.59	4.59
MC on Capital Units	219.630	226.828	234.262
MC on Accumulation Units	0	52.283	108.579
Surrender profit	87.602	60.199	0
Extra death benefit	107.141	34.89	0
End of year cash flow	55.881	411.01	449.449
Reserves at start of year	400	400	400
Interest on reserves	18	18	18
Change in reserves at year end	–66.035	–66.683	–400
Revised cashflow	–260.084	95.693	467.449
Expected present value	–226.160	60.412	213.835

Expected present value of profit = 48.087

Profit margin = 0.53%