Faculty of Actuaries                                        Institute of Actuaries

# REPORT OF THE BOARD OF EXAMINERS ON THE EXAMINATIONS HELD IN

April 2002

## Subject 105 — Actuarial Mathematics 1

**Introduction**

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

K Forman
Chairman of the Board of Examiners

11 June 2002

## EXAMNINER'S COMMENTS

*The overall standard of scripts was better than in recent sittings. However answers were very disappointing for questions 5, 10 and 13(iii) in particular, where the question posed a problem not seen in recent examinations. It is also clear that many candidates' statistical knowledge or understanding is not up to the standard required. Finally candidates are urged to read the questions carefully. In many cases the answers for questions 5, 8, 9, 10 and 12 omitted elements asked for or added details not required for the question.*

*Comments on individual questions follow after the solution to each question.*

**1** $_{3|4}q_{[40]+1}$ is the probability that a life now aged 41, who entered the population of interest a year ago subject to select mortality at that time, will survive for 3 more years, and die during the following 4, when aged between 44 and 48.

$= ({}_3p_{[40]+1})({}_4q_{44})$ for a 2 year select table

$$= \frac{l_{44}}{l_{[40]+1}} \cdot \frac{l_{44} - l_{48}}{l_{44}} = \frac{l_{44} - l_{48}}{l_{[40]+1}} = (33{,}309.271 - 32{,}934.221) / 33{,}484.739$$

$= 0.0112006$

***Comment on Question 1***
*Well answered, with only a small minority of candidates mixing up the survival and death periods.*

**2** $_{10|20}q^{1}_{35:35} = (.5)({}_{10|20}q_{35:35}) = .5[{}_{10}p_{35:35}\{1 - {}_{20}p_{45:45}\}]$

$= .5[{}_{10}p_{35}{}^2 - {}_{30}p_{35}{}^2] = .5[(e^{-.2})^2 - (e^{-.6})^2] = .5(.67032 - .30119) = .1846$

***Comment on Question 2***
*Answers were generally of a reasonable standard. The commonest errors related to the factor of .5, and a range of errors in evaluating the required integrals.*

**3** Total fertility rates summarise the age specific fertility rates $f_x$ (i.e. the ratio of births to population of women aged $x$ generating them). The summation is over all ages for which $f_x > 0$, often taken as 15-49.

Cohort: fertility rates are summed (over a period of time) for women born in a specified period e.g. all those born in the same calendar year

Period: fertility rates are summed at a point of time (e.g. the rates experienced in one calendar year) for women of different ages

Cohort rates are generally preferred for their greater stability and their smooth rate of change over time
or
Period rates are quicker and easier to obtain, and therefore suitable for immediate use

Other sensible reasons also gained credit.

***Comment on Question 3***

*Good standard, although some candidates mixed up cohort and period rates while others provided formulae that dealt with numbers of births rather than fertility rates.*

**4** Logistic model

$\{1/P(t)\}\{dP(t)/dt\} = \rho - kP(t)$ or also $P(t) = [Ce^{-\rho t} + (k/\rho)]^{-1}$ or $= \rho/[C\rho e^{-\rho t} + k]$

Current rate: $\rho - k100{,}000 = .25$
Limiting population: $\rho - k300{,}000 = 0$
leading to $k = (.25/200{,}000) = 1/800{,}000$ and $\rho = .375$ (=3/8)

We want $t$ such that $P(t) = 200{,}000$

From $t = 0$ (now) $100{,}000 = [C + (1/300{,}000)]^{-1}$ so $C = 2/300{,}000$ or 0.00000666667

$200{,}000 = [(2/300{,}000)\, e^{-.375t} + (1/300{,}000)]^{-1}$

$(1/200{,}000) - (1/300{,}000) = (2/300{,}000)\, e^{-.375t}$

$e^{-.375t} = (1/4)$ so using logs $t = 3.70$ years

***Comment on Question 4***
*Overall standard was quite good, although a surprising number of candidates did not seem to know the logistic model at all. The commonest error was to use $\rho = 0.25$.*

**5** Insurance company received $P$ so guaranteed maturity benefit
$= [(1.05)^4] * P = 1.21550625P$

The company invests $P$ @5.25% so is due to receive $1.2915479P$ in 5 years.

On death, the office breaks even because it pays out exactly the value of asset available. This occurs with probability ${}_4q_{56} = (1 - [l_{60}/l_{56}]) = 0.0690$

At maturity ($t = 4$) office loses money only if yields at the time are $j$ such that
$\{1.2915479P / (1 + j)\} < 1.21550625P$ i.e $(1 + j) > 1.06256$

Prob $(1 + j > 1.06256)$ for lognormal $(1 + j)$
= Prob ($z > [\text{Ln } 1.06256 - 0.05] / 0.01$) from standard normal
= Prob ($z > 1.07$) = 1 − .85769 = 0.14231

Maturity occurs with probability 1 − .0690 = .9310 so the overall probability of a loss is 0.9310 * 0.14231 = 0.1325 = 0.13

***Comment on Question 5***
*Very poor standard of answers. Many made no reasonable attempt. Of those who did, some tried to calculate a surrender profit or loss, even though this was clearly zero. Many tried to calculate the value of the zero coupon bond at the end of 4 years (one year short of redemption) by considering the distribution of $(1+i)^4$ and accumulating rather than using the distribution of $1+i$ directly and discounting.*

**6** Pension A

EPV = (20,000/1.009709) $(a_{\overline{4|}} + v^4{}_4p_{[60]}\, a_{64})$ @ (1.04/1.009709) = 3%

= (20,000/1.009709)[3.7171 + (.88849)(.947214)(11.962 −1)] = 256,363

Pension B

EPV = 12,000 $a_{[60]}$ + 1,000$(Ia)_{[60]}$ = 12,000 $a_{[60]}$ + (1,000$S_{[60]+1}/D_{[60]}$)

= 12,000[12.710 − 1] +1,000 [307,254.58/2,815.3028] = 249,657

***Comment on Question 6***

*Well answered. Common mistakes in A were not getting initial level correct (missing divisor of 1.009709) and using 4% interest for the deferred period until life annuity commences. In B, many evaluated $(Ia)_{[60]}$ using $S_{[60]}$.*

**7** Nutrition influences morbidity and (in longer term) mortality.
Lack of nutrition leads to general weakening of body
Poor quality increases risk of disease / hinders recoveries.
Excessive / inappropriate can lead to obesity and associated diseases (e.g. hypertension, heart disease). This can arise from social factors e.g. ready processed food / fast food etc.
Poor / lack of nutrition can arise from adverse economic circumstances.

Education (covering formal and also general awareness from public health campaigns).
It influences awareness of elements of healthy lifestyle. This can affect behaviour in many areas e.g. nutrition / diet; personal health and hygiene; awareness of effects of tobacco, alcohol, drugs;
Education level will also have a bearing on income level, occupation , standard of housing and general lifestyle, all of which are themselves regarded as influencers of mortality..

Other reasonable points also received credit.

***Comment on Question 7***

*Well answered overall. Some candidates were inclined to repeat the same point rather than identifying distinct influences on mortality.*

**8** (i) Age last birthday = $x$ at start of rate interval in which dies
Curtate duration = $r$ at start of rate interval in which dies
No assumptions needed

$x$ + .5 at mid-point of interval, with duration $r$ +.5 so we are estimating

$$\mu_{[x+.5-(r+.5)]+r+.5} = \mu_{[x-r]+r+.5}$$

This does require an assumption of an even spread of retirements over the year of age, because we have no other information about ages at entry (we can only deduce that they can range from $x - r + 1$ to $x - r - 1$).

(ii) Retired aged 62 years and 3 months. Dies aged 64 years and 11 months so total exposure is 2 years 8 months.

| | |
|---|---|
| (62,0) | 9 months |
| (63,0) | 3 months |
| (63,1) | 9 months |
| (64,1) | 3 months |
| (64,2) | 8 months |

***Comment on Question 8***
*Very poorly answered, especially part (ii). Some otherwise correct answers omitted assumptions completely while others gave "standard" assumptions e.g. policy anniversaries spread evenly over the year of age when there are no policies (only retirements). Overall, the understanding of the different rate intervals and the associated assumptions seems confused. In part(ii), many candidates calculated the total exposure incorrectly, including in some cases not even calculating the age at death correctly.*

**9** (a)

**Capital units no actuarial funding**

| *Year* | *Cost of investment* | *Fund at end before m.c.* | *Management Charge* | *Fund at end* |
|---|---|---|---|---|
| 1 | 969 | 1,041.67 | 52.08 | 989.59 |

**Non-unit fund**

| *Year* | *Premium less cost of allocation* | *Interest* | *Death cost* | *Management charge* | *Cashflow* |
|---|---|---|---|---|---|
| 1 | 31 | 1.55 | 0 | 52.08 | 84.63 |

(b)

***A* funding factors**

$A_{[61]:\overline{4}|}$ 0.85697

$A_{[61]+1:\overline{3}|}$ 0.89045 from $(M_{[61]+1} - M_{65} + D_{65}) / D_{[61]+1}$

$= (1{,}337.8829 - 1{,}258.7316 + 2{,}144.1713) / 2{,}541.7641$

***A* funded capital unit fund**

| *Year* | *Cost of investment* | *Fund available at end* | *Fund needed at end* | *Management Charge* |
|---|---|---|---|---|
| 1 | 830.40 | 892.68 | 881.18 | 11.50 |

**Non-unit fund**

| *Year* | *Premium less cost of allocation* | *Interest* | *Death cost* | *Management charge* | *Cashflow* |
|---|---|---|---|---|---|
| 1 | 169.60 | 8.48 | 0.78 | 11.50 | 188.80 |

The death cost is $q_{[61]}$*(full capital unit fund – *A* funded capital unit fund @ $t = 1$)
i.e. 0.00723057*(989.59-881.21)

***Comment on Question 9***
*Handled very well. Errors, where they occurred, were to include a death cost in (a), use of the wrong funding factor at t=1 and incorrect calculation of the death cost in (b). Some candidates completed a full profit test for each year of the contract, wasting valuable time.*

**10** (a) $$\text{EPV} = 1{,}000\int_0^{30} e^{-\delta t}\, p_{35,t}^{HS}\,dt$$

(b) $$\text{EPV} = 1{,}000\int_0^{\infty}\int_1^{6} e^{-\delta t}\, {}_t p_{35,z}^{HS}\,dz\,dt$$

or

$$1{,}000\int_0^{\infty} e^{-\delta t}\, \mathrm{p}_{35,t}^{HH}\,\sigma_{35+t}\int_1^{6} e^{-\delta r}\, p_{35+t,r}^{\overline{SS}}\,dr\,dt$$

(c) $$\text{EPV} = 1{,}000\int_0^{\infty} e^{-\delta t}\, p_{35,t}^{\overline{HH}}\,\sigma_{35+t}\int_0^{\infty} e^{-\delta r}\, p_{35+t,r}^{\overline{SS}}\,dr\,dt$$

or

$$1{,}000\int_0^{\infty} e^{-\delta t}\, \mathrm{p}_{35,t}^{\overline{HH}}\,\sigma_{35+t}\int_0^{\infty} \bar{a}_{\overline{r}|}\, p_{35+t,r}^{\overline{SS}}(\rho_{35+t+r}+\nu_{35+t+r})\,dr\,dt$$

***Comment on Question 10***
*This was a testing question that was not answered well at all. Most attempted (a) but often got it wrong, while very few candidates made any real attempt at (b) and (c), even though (b) in particular just required direct use of a formula given in the appropriate Core Reading. Where an attempt was made at (b) or (c), candidates often used the benefit ceasing age for (a), although none applied in (b) or (c).*

**11** (i) The retrospective and prospective reserves equal each other on the premium basis. We want the SV calculation to result in a lower reserve.

Retrospective reserve needs to be done at a smaller interest rate, as it is accumulating past excess premiums over claims/ expenses.

Prospectively, the interest rate needs to be higher than the premium basis, so that the discounting of the excess of future outgo (claims / expenses) over premium income results in a lower answer.

(ii) SV = 41,000

PUPSA = $54{,}000\,\bar{A}_{60:\overline{5}|} = 54{,}000((1.06)^{.5}\,\{A_{60:\overline{5}|} - A_{60:\overline{5}|}^{\;\;\;\;1}\} + A_{60:\overline{5}|}^{\;\;\;\;1})$

where $A_{60:\overline{5}|}^{\;\;\;\;1} = v^5\,{}_5p_{60} = (.747258)(27{,}442.681/30{,}039.787) = .68265$

EPV of PUPSA = $54{,}000(1.02956\{.75477 - .68265\} + .68265) = 40{,}873$

Whole Life option 100,000 $\bar{A}_{60} = (1.06)^{.5}\,(.39136) = 40{,}293$

So SV is best

***Comment on Question 11***

*Well answered. Those who got (i) wrong often wrestled with reserving formulae rather than considering the underlying concept needed. In part (ii), a surprisingly large number of candidates overvalued the paid up option by multiplying the entire endowment factor by* $1.06^{0.5}$ *rather than just the death element.*

**12** EPV of past pensions: $(n/60)(\text{Sal})({}^zM_x^{ia} + {}^zM_x^{ra})/{}^sD_x$

EPV future pensions: $(1/60)(\text{Sal})({}^z\bar{R}_x^{ia} + {}^z\bar{R}_x^{ra})/{}^sD_x$

EPV of contributions @ 1% of salary: $(.01)(\text{Sal})({}^s\bar{N}_x)/{}^sD_x$

| *age* | *salary* | *past service* | ${}^sD_x$ | ${}^zM_x^{ia}$ | ${}^zM_x^{ra}$ | ${}^z\bar{R}_x^{ia}$ | ${}^z\bar{R}_x^{ra}$ | ${}^s\bar{N}_x$ |
|---|---|---|---|---|---|---|---|---|
| 30 | 25,000 | 5 | 28,043 | 8,636 | 88,345 | 231,941 | 2,915,486 | 540,020 |
| 35 | 20,000 | 6 | 22,276 | 8,513 | 88,345 | 188,977 | 2,473,760 | 417,224 |

| | *EPV past pension* | *EPV future pension* | *EPV cont. 1%sal* |
|---|---|---|---|
| | 7,204.78 | 46,764.89 | 4,814.21 |
| | 8,696.18 | 39,844.63 | 3,745.95 |
| Total | 15,900.96 | 86,609.52 | 8,560.16 |

Total Liability = 15,900.96+86,609.52 = 102,510.48

Contribution rate needed = 102,510.48/8,560.16 = 11.98% of salary

New employee

| *Age* | *salary* | *past service* | ${}^sD_x$ | ${}^z\bar{R}_x^{ia}$ | ${}^z\bar{R}_x^{ra}$ | ${}^s\bar{N}_x$ |
|---|---|---|---|---|---|---|
| 40 | 30,000 | 0 | 18,629 | 147,045 | 2,032,033 | 317,121 |

| *EPV past pension* | *EPV future pension* | *EPV cont. 1%sal* |
|---|---|---|
| 0.00 | 58,486.18 | 5,106.89 |

Contribution needed = 58,486.18 / 5,106.89 = 11.45%.

Therefore the contribution rate of 11.98% established for the original 2 members is more than that required to meet the costs of the new entrant, and the scheme is in surplus.

***Comment on Question 12***
*Answered very well, but a disappointing number of candidates overlooked the ill-health retirement benefits.*

**13** All values at $t = 0$

(i) Future loss random variable =
$$100,000v^{\min(K_{50}+1,15)} + 300 - .975P\ddot{a}_{\overline{\min(K_{50}+1,15)}|}$$

$$= 100,000v^{\min(K_{50}+1,15)} + 300 - .975P\frac{1-v^{\min(K_{50}+1,15)}}{d}$$

(ii) (a) Mean for single policy just take expected value of random variable

$X$ = EV one policy = 100,000 $A_{50:\overline{15}|}$ + 300 – .975 $P$ $\ddot{a}_{50:\overline{15}|}$

= (100,000)(.44395) + 300 –(.975)$P$(9.823)

= 44,695 – 9.577425$P$

$Y$ = Variance one policy

Variance = (using 2nd form of loss r.v.)

= [100,000 + (.975P/d)]$^2$Var ($v^{\min(K_{50}+1,15)}$)

= [100,000 + (.975P/d)]$^2$ $({}^2A_{50:\overline{15}|} - [A_{50:\overline{15}|}]^2)$

where the 2 superscript denotes at $i^2 + 2i$

= [100,000 + (17.225$P$)]$^2$(.007168397)

or Standard Deviation = (100,000 + 17.225$P$)(.084666)

(b) 100 policies

Mean = 100$X$

Variance = 100$Y$ assuming the lives are independent or

Standard Deviation = 10 Std Dev above

(iii) Using Central Limit Theorem ($n = 100$) we can assume normality of portfolio loss.

We want Prob (loss > 0) < .025

Prob ([loss – mean]/ Std Dev > [0 – mean]/std Dev) < .025

Prob ($z$ > – mean/Std Dev) < .025

This means that (– mean / Std Dev) > 1.96 or (mean/ std Dev) < –1.96

$(100)(44{,}695 - 9.577425\,P) < (-1.96)\,(84{,}666 + 14.5837P)$

$P \geq 463{,}5445/929.158448 = 4{,}988.86$ say 4,989

***Comment on Question 13***

*A very mixed standard. It is clear some candidates do not have a good understanding of the difference between a random variable and its expectation, at least in this context. Common errors in (i) were to use assurance or life annuity functions, to miss the +1 in the $K_x$+1 terms or to give a profit (rather than loss) random variable.*

*In (ii), very few got the variance correct for a single policy, usually not making the conversion of the annuity into $(1-v^n)/d$ format used in the model solution, and then missing the cross-product or covariance term between the benefit and premium random variables. A surprising number of candidates missed the independence of lives within the portfolio and therefore concluded that the variance of the portfolio was $100^2$ times the variance of one policy.*

*In (iii), of the few candidates who attempted this part, many started with considering a loss < 0, when the loss has to be > 0 to be a loss.*

**14** (i)

| *age* | $q_x$ | $p_x$ | ${}_{t-1}p_x$ |
|---|---|---|---|
| 60 | 0.01443246 | 0.98556754 | 1 |
| 61 | 0.01601356 | 0.98398644 | 0.98556754 |
| 62 | 0.01774972 | 0.98225028 | 0.96978510 |

| *Year* | *Prem.* | *Expense* | *Opening reserve* | *Interest* | *Death claim* | *Closing reserve* | *Profit vector* | *Profit signature* | *NPV* |
|---|---|---|---|---|---|---|---|---|---|
| 1 | $P$ | $0.35P$ +200 | 0 | $0.0455P$ –14 | 2,886.49 | $0.985568P$ | $-0.290068P$ –3,100.49 | $-0.290068P$ –3,100.49 | $-0.252233P$ –2,696.08 |
| 2 | $P$ | $0.03P$ +25 | $P$ | $0.1379P$ –1.75 | 3,202.71 | $0.983986P$ | $1.123914P$ –3,229.46 | $1.107693P$ –3,182.85 | $0.837575P$ –2,406.69 |
| 3 | $P$ | $0.03P$ +25 | $P$ | $0.1379P$ –1.75 | 3,549.94 | 0 | $2.107900P$ –3,576.69 | $2.044210P$ –3,468.62 | $1.344101P$ –2,280.67 |
| | | | | | | | | | $1.929443P$ –7,383.44 |

Therefore $1.929443P - 7383.44 = .25P \Rightarrow$ Premium = 4,396.36 = 4,396.

(ii) If we use this premium, and ignore reserves, the cash-flows per policy in force at the start of each year are (–43, 1,334, 986).

(iii) As the cash flows in years 2 and 3 are all positive, there is no need to establish reserves at the end of any year.

In such a scenario, the profits emerge earlier and because the discount rate exceeds the earned rate of interest, the NPV increases.

***Comment on Question 14***

*Answered well overall. The most common error was mishandling of reserves. A disappointing number of students started from a commutation function approach when a cashflow model was needed. In (iii), a number of candidates made the general statement that it was not necessary to hold reserves for term assurance contracts because the probability of death was low, without any reference to the specifics of the cashflows in this case.*