EXAMINATIONS

April 2000

Subject 105 — Actuarial Mathematics 1

EXAMINERS' REPORT

1 "Prospective service benefit" means a benefit not dependent on either past or future service explicitly, although it may depend on total expected service.

Examples include — lump sum death benefit of 4 × salary or spouse's pension death in service of $\frac{n}{120}$ × final salary where n is based on deceased member's total potential service to NPA, including any past service.

Many candidates confused "prospective" with "future".

2 x next birthday at entry $\Rightarrow x - ½$ on average at entry assuming birthdays uniformly distributed over policy year.

r at policy anniversary after death means exact duration $r - 1$ at the anniversary before death (the start of the policy year rate interval for duration) and hence $r - ½$ mid-year when the force of mortality is estimated. No assumptions are necessary.

The force estimated is $\mu_{[x-½]+r-½}$, so $y = x - ½$, $t = r - ½$.

3 If its age/sex profile is such that if it experienced the same age/sex specific mortality rates as the country, then its crude death rate would be twice that of the country, i.e. the region has a much older age structure (and/or higher male proportion) than the country.

4
$$ {}_tV^{\text{Zillmer}} = 1 - \frac{\ddot{a}_{[x]+t:\overline{n-t}|}}{\ddot{a}_{[x]:\overline{n}|}} - I\,\frac{\ddot{a}_{[x]+t:\overline{n-t}|}}{\ddot{a}_{[x]:\overline{n}|}} $$

$$ \text{Here } {}_{10}V = 1 - \frac{\ddot{a}_{50:\overline{15}|}}{\ddot{a}_{[40]:\overline{25}|}} - (.025)\left(\frac{\ddot{a}_{50:\overline{15}|}}{\ddot{a}_{[40]:\overline{25}|}}\right) $$

$$ = 1 - \frac{11.671}{17.180} - (.025)\left(\frac{11.671}{17.180}\right) $$

$$ = 0.30368 $$

5 (a) $({}_tV' + GP - e_t)(1+i) = q_{x+t}(S) + p_{x+t}({}_{t+1}V')$

where ${}_tV'$ = gross premium reserve @ time t

GP = office premium

e_t = expenses incurred at time t

i = interest rate in premium/valuation basis

S = Sum Assured

p_{x+t} (q_{x+t}) probability life aged $x+t$ survives (dies within) one year on premium/valuation mortality basis.

(b) Income (opening reserve plus interest on excess of premium over expense, and reserve) equals outgo (death claims and closing reserve for survivors) if assumptions are borne out.

6 $$A^{1}_{30:30:\overline{30}|} = \tfrac{1}{2}A_{\overset{1}{\overbrace{30:30}}:\overline{30}|} = \tfrac{1}{2}\left(A_{30:30} - \frac{D_{60:60}}{D_{30:30}}A_{60:60}\right)$$

$$= \tfrac{1}{2}\left[1-\left(\frac{.04}{1.04}\right)\ddot{a}_{30:30} - \frac{D_{60:60}}{D_{30:30}}\left\{1-\left(\frac{.04}{1.04}\right)\ddot{a}_{60:60}\right\}\right]$$

$$= \tfrac{1}{2}\left[1-\left(\frac{.04}{1.04}\right)(19.701) - \frac{2487.2117}{10236.789}\left\{1-\left(\frac{.04}{1.04}\right)(9.943)\right\}\right]$$

$$= \tfrac{1}{2}[1 - .75773 - (.24297)(.61758)]$$

$$= .0461$$

7 Future service = 18 + 14 past $\Rightarrow$ total = 32 > max of 30.

$$\Rightarrow \text{Value of benefit } = \left(\frac{2}{3}\right)(40{,}000)\,\frac{s_{47}}{s_{46}}\,\frac{{}^{z}C^{ra}_{65}}{{}^{s}D_{47}}$$

$$= \left(\frac{2}{3}\right)(40{,}000)\left(\frac{4.28}{4.18}\right)\left(\frac{35846}{15778}\right) = £62{,}033$$

Most candidates allowed for retirement at any age, not just 65, and many failed to notice that service exceeded 30 years so the maximum of 2/3rds applied.

8 (i) A continuous annuity of £1 p.a. payable for a minimum of n years and continuing thereafter until the death of the survivor of x and y.

(ii) $E[g(T)] = \bar{a}_{\overline{\overline{xy}}:\overline{n}|}$.

Rather than defining asset share, some candidates discussed bonuses and policy payouts.

9 The asset share for a with-profit policy is the accumulated value of premiums less deductions plus an allocation of profits from non-profit business. The accumulation is at actual earned rates of return.

The deductions include expenses, cost of benefits, tax, transfers to shareholders, cost of capital and contribution to free assets.

Rather than defining asset share, some candidates discussed bonuses and policy payouts.

10 In logistic model $P(t) = \left[Ce^{-\rho t} + \frac{K}{\rho}\right]^{-1}$ or $\left[\frac{\rho}{C\rho e^{-\rho t} + K}\right]$

As $t \to \infty$ $\quad P(t) \to \frac{\rho}{K} \Rightarrow K = \frac{.05}{250{,}000}$

$$P(0) = \left[C + \frac{1}{250{,}000}\right]^{-1} = 100{,}000 \Rightarrow C = 0.000006$$

$$\Rightarrow P(10) = \left[(.000006)\, e^{-(.05)(10)} + \frac{1}{250{,}000}\right]^{-1}$$

$$= 130{,}904$$

Only a minority of candidates seemed familiar with the logistic model.

11 Under UDD in **single** decrement table

$$(aq)_x^{\alpha} = q_x^{\alpha}\,(1 - ½q_x^{\beta}) = q_x^{\alpha} - ½q_x^{\alpha}q_x^{\beta} = 0.2$$

$$(aq)_x^{\beta} = q_x^{\beta}\,(1 - ½q_x^{\alpha}) = q_x^{\beta} - ½q_x^{\alpha}q_x^{\beta} = 0.05$$

$$\Rightarrow q_x^{\alpha} - q_x^{\beta} = 0.15 \Rightarrow q_x^{\alpha} = q_x^{\beta} + 0.15$$

$$(q_x^{\beta} + 0.15) - ½(q_x^{\beta} + 0.15)\, q_x^{\beta} = 0.2$$

$$\Rightarrow -½(q_x^{\beta})^2 + .925q_x^{\beta} + 0.15 = 0.2$$

OR $(q_x^\beta)^2 - 1.85q_x^\beta + 0.1 = 0$

Roots are $\dfrac{1.85 \mp \sqrt{1.85^2 - 0.4}}{2} \Rightarrow 0.05573$ (and $q > 1$ is invalid)

$q_x^\beta = 0.05573$ and $q_x^\alpha = 0.20573$

Alternatively, $q_x^\alpha = (aq)_x^\alpha \div \left(1 - ½q_x^\beta\right)$ and $q_x^\beta = (aq)_x^\beta \div \left(1 - ½q_x^\alpha\right)$

Using iteration, and taking starting values in denominators of $q_x^\alpha \approx (aq)_x^\alpha$ etc.

1st iteration $q_x^\alpha = 0.2 \div [1 - (.5)(.05)] = .205128$

$q_x^\beta = 0.05 \div [1 - (.5)(.2)] = .055556$

Similarly, 2nd iteration $q_x^\alpha = .20571$, $q_x^\beta = .05571$

3rd iteration $q_x^\alpha = .20573$, $q_x^\beta = .05573$

4th iteration $q_x^\alpha = .20573$, $q_x^\beta = .05573$

Hence $q_x^\alpha = .20573$, $q_x^\beta = .05573$

A large number of candidates used formulae appropriate when decrements are uniform in the multiple decrement table, but the question specified that independent decrements were uniform in the single decrement tables.

12 EPV = $500\int_0^{30} e^{-\delta t} p_{35,t}^{hh}\, dt$ (premiums)

$-20{,}000\int_0^{30} e^{-\delta t} p_{35,t}^{hh}\, \mu_{35+t}\, dt$ (death from healthy)

$-30{,}000\int_0^{30} e^{-\delta t} p_{35,t}^{hs}\, \nu_{35+t}\, dt$ (death from sick)

$-3{,}000\int_0^{30} e^{-\delta t} p_{35,t}^{hs}\, dt$ (sickness income)

13 EPV

$$= 10{,}000\left\{\sum_{t=1}^{30} (1 - {}_tp_{60}^f)\, {}_tp_{64}^m\, v^t + \sum_{t=31}^{\infty} ({}_{t-30}p_{60}^f - {}_tp_{60}^f)\, {}_tp_{64}^m\, v^t\right\}$$

$$= 10{,}000\left\{\sum_{t=1}^{30} {}_tp_{64}^m\, v^t + \sum_{t=31}^{\infty} {}_{t-30}p_{60}^f\, {}_tp_{64}^m\, v^t - \sum_{t=1}^{\infty} {}_tp_{60}^f\, {}_tp_{64}^m\, v^t\right\}$$

$$= 10{,}000\left\{a_{64:\overline{30}|}^m + {}_{30}p_{64}^m\, v^{30}\, a_{60:94}^{f\ \ m} - a_{60:64}^{f\ \ m}\right\}$$

$$a^{m}_{64:\overline{30}|} \quad = \quad a_{64} - \frac{D_{94}}{D_{64}} a_{94} = 7.616 - \left(\frac{16.4}{5844.0}\right)(1.707) \ = 7.611$$

$$_{30}p^{m}_{64}\, v_{30} \ = \ \frac{D_{94}}{D_{64}} = \frac{16.4}{5844.0} = 0.002806297$$

$$a^{f:m}_{60:94} \quad = 1.666 \qquad\qquad a^{f\ \ m}_{60:64} = 6.854$$

$$\Rightarrow \text{EPV} \quad = 10{,}000\{7.611 + (.002806297)(1.666) - 6.854\} = £7{,}617$$

*Very few candidates provided a satisfactory answer. Many did not attempt to deal with the term aspect of the question, and most of those who did assumed the annuity ended **30** years after retirement rather than **30** years after the pensioner's death.*

14 (i) **Crude death rate** is heavily influenced by mortality at older ages

(a) OK if population structures by age and sex are reasonably stable.
Therefore beware large scale emigration/immigration. Easy and practical.

(b) Not suitable — age and sex distributions in occupational groups likely to vary significantly.

Standardised Mortality Rate

Again influenced by mortality at older ages.

(a) OK to use but need age specific mortality rates at each time point.

Changing population structure has no effect.

(b) Copes well with age/sex variations provided age specific rates are available for occupational groups.

But use of a fixed age structure may be unrepresentative of given occupation.

Standardised Mortality Ratio

Heavily influenced by relative mortality at older ages.

(a) Fine but ensure standard rates used are same each time.

(b) Good except for possible problems gathering the data on age distributions.

Use of occupational age structure maintains relevance.

(ii) **Occupational A**

Crude Rate

$$= 235 / 37{,}000 = 0.00635$$

Standardised Mortality Rate

$$= (960{,}000 \times \frac{52}{15000}$$

$$+ 1{,}400{,}000 \times \frac{74}{12000}$$

$$+ 740{,}000 \times \frac{109}{10{,}000}) \div 3{,}100{,}000$$

$$= (3{,}328 + 8{,}633.33 + 8{,}066) \div 3{,}100{,}000 = 0.00646$$

Standardised Mortality Ratio

$$= 235 \div \left\{ \begin{array}{l} 15{,}000 \times \dfrac{3{,}100}{960{,}000} \\ + 12{,}000 \times \dfrac{7{,}500}{1{,}400{,}000} \\ + 10{,}000 \times \dfrac{7{,}100}{740{,}000} \end{array} \right\} = 235 \div \left\{ \begin{array}{l} 48.44 + 64.29 \\ + 95.95 \end{array} \right\}$$

$$= 235 \div 208.68$$

$$= 1.126$$

Answered quite well in general, although some students tended to describe the various measures in general rather than relate them to the specific situations described.

15 (i) (a) **North American Method**

Relies on double decrement table with explicit proportions who choose to exercise option and a special mortality table for those people post option.
While theoretically accurate, it is often difficult to obtain sufficient data to estimate experience.

(b) **Conventional Method**

Assumes all eligible lives actually take up option, and that they are subject to Ultimate mortality as opposed to Select if normal underwriting carried out. If there are many option dates etc., then the most costly from the insurers point of view is assumed.

(ii) Insurer charges $\left(\frac{1}{.95}\right)$ $(P_{[65]})$ (100,000) per annum for whole life policy

i.e. (.05254)(100,000) ÷ .95 = £5,530.53 p.a.

At option date (age 65), the value of benefits provided is

$100{,}000\ A_{65} = (100{,}000)(.58705) = £58{,}705$

The insurers net liability at option date present value of benefits – (present value of premiums less expenses)

$= 100{,}000\ A_{65} - (.95)(5{,}530.53)\ \ddot{a}_{65}$

$= 58{,}705 - (.95)(5{,}530.53)(10.737)$

$= 58{,}705 - 56{,}412.20 = £2{,}292.80$

Extra premium, P', spread over term assurance policy term, is from:-

$$.95P'\ \ddot{a}_{[50]:\overline{15}|} = 2{,}292.80\ \frac{D_{65}}{D_{[50]}}$$

$$\Rightarrow P' = (2{,}292.80)\left(\frac{2{,}144.1713}{4{,}581.3224}\right) \div (.95)\ (11.028)$$

$\Rightarrow P' = £102.43$ per annum

(iii) The office needs to decide which option is costlier, not just in the value of the option benefit, but its impact on the overall premium required over the period to the option exercise date.
In this case, it needs to compare the above option cost in premium terms plus the $\overline{15}|$ term assurance premium to the similarly calculated extra premium for the 10 year option combined with a 10 year term insurance premium.
It should then charge the higher combined premium, thereby having option cost at any date more than covered.

Part (i) was well answered, but (ii) and (iii) were very poorly answered. Many candidates treated the contract as a whole life from the start making the option cost the difference between a term assurance and a whole life policy for the life aged 50.

16 (i)

$q_{61} = .016\ 013\ 56$ $\quad p_{61} = .983\ 986\ 44$ $\quad {}_0p_{61} = 1.0$ $\quad A^{5\%}_{61:\overline{4}|} = .82703$

$q_{62} = .017\ 749\ 72$ $\quad p_{62} = .982\ 250\ 28$ $\quad {}_1p_{61} = 0.983\ 986$ $\quad A_{62:\overline{3}|} = .86624$

$q_{63} = .019\ 654\ 64$ $\quad p_{63} = .980\ 345\ 36$ $\quad {}_2p_{61} = 0.966\ 521$ $\quad A_{63:\overline{2}|} = .90792$

$q_{64} = .021\ 743\ 10$ $\quad p_{64} = .978\ 256\ 90$ $\quad {}_3p_{61} = 0.947\ 524$ $\quad A_{64:\overline{1}|} = .95238$

Capital unit fund — fully funded

Year	*Cost of alloc.*	*Fund b/f*	*Y/e fund after 8% growth*	*Management Charge 6%*	*Fund c/f*
1	902.50	–	974.70	58.48	916.22
2	902.50	916.22	1,964.21	117.85	1,846.36
3	–	1,846.36	1,994.07	119.64	1,874.43
4	–	1,874.43	2,024.38	121.46	1,902.92

Capital unit fund — a-funded

Year	*Cost of alloc.*	*Fund b/f*	*Available @y/e after 8%*	*Needed at year end*	*Extra death cost*	*Management charge*
1	746.39	–	806.10	793.67	1.96	10.47
2	781.78	793.67	1,701.49	1,676.35	3.02	22.12
3	–	1,676.35	1,810.46	1,785.17	1.75	23.54
4	–	1,785.17	1,927.98	1,902.92	–	25.06

Premium unit fund

Year	*Cost of alloc.*	*Fund b/f*	*Fund @year end*	1% *Management charge*	*Fund c/f*
3	902.50	–	974.70	9.75	964.95
4	902.50	964.95	2,016.85	20.17	1,996.68

Death cost (using **full** Cap. Units) Yr 1 $\Rightarrow q_{61}$ (4000 – 916.22) = 49.38

Yr 2 $\Rightarrow q_{62}$ (4000 – 1846.36) = 38.23

Yr 3 $\Rightarrow q_{63}$ (4000 – 1874.43 – 964.95) = 22.81

Yr 4 $\Rightarrow q_{64}$ (4000 – 1902.92 – 1996.68) = 2.18

Sterling fund

Year	*Premium less cost of alloc.*	*Expense*	*(4%) Sterling interest*	*Death cost*	*Management charge*	*Profit vector*	*Profit signature*
1	253.61	300.00	(1.86)	49.38	10.47	–87.16	–87.16
2	218.22	20.00	7.93	38.23	22.12	190.04	187.00
3	97.50	21.00	3.06	22.81	33.29	90.04	87.03
4	97.50	22.05	3.02	2.18	45.23	121.52	115.14

$\text{NPV} = -87.16v + 187v^2 + 87.03v^3 + 115.14v^4 = 206.37$

Alternative approach whereby entire death cost is charged to sterling fund is also valid, providing a-funded capital unit management charge is correspondingly increased.

(ii) (a) Given the shape of the cash flows, with the positives after the negatives, a discount rate of 10% would mean larger NPV.

(b) Death cost would reduce, probability of being in force and hence premium income would increase, causing NPV to increase. A-funding factors would also decrease, accelerating the cash flows. Given risk discount rate (12%) > sterling fund rate this will increase NPV.

(c) At 4%, factors will be bigger, unit reserves increase and profit is deferred. Because risk discount rate exceeds sterling fund rate, NPV decreases.

Generally well answered, although candidates often failed to give reasons for their correct conclusions in (ii).

17 (i) $$P\ddot{a}^{6\%}_{30:\overline{35}|} = 50{,}000\left(\frac{1}{1.01923}\overset{4\%}{A}{}^{1}_{30:\overline{35}|} + \overset{4\%}{A}{}_{30:\overset{1}{\overline{35}|}}\right) + 250$$

$$+ .025P\ddot{a}^{6\%}_{30:\overline{35}|} + .575P$$

Because bonuses vest at year end, maturities get an extra bonus compared to deaths in last year, and so the death benefit function is divided by (1 + bonus loading).

$$P\left(.975\ddot{a}^{6\%}_{30:\overline{35}|} - .575\right) = 250 + 50{,}000\left\{\frac{1}{1.01923}\left(A_{30:\overline{35}|} - \frac{D_{65}}{D_{30}}\right) + \frac{D_{65}}{D_{30}}\right\}$$

$$\ddot{a}^{6\%}_{30:\overline{35}|} = 15.019$$

$$A^{4\%}_{30:\overline{35}|} = .27483$$

$$\frac{D_{65}^{4\%}}{D_{30}^{4\%}} = \frac{2144.1713}{10433.31} = .20551$$

$\Rightarrow P(14.0685) = 250 + 50{,}000\{.06801 + .20551\} \Rightarrow P = 989.87 = £990$ p.a.

(ii) Gross future loss = PV future outgo − PV future income

= PV future benefit payment + PV future expenses − PV future premiums

$$= G(K_{30+t}) + (.025)(990)\ \ddot{a}_{\overline{\min[K_{30+t}+1,\,35-t]}|}$$

$$- (990)\ \ddot{a}_{\overline{\min[K_{30+t}+1,\,35-t]}|}$$

where $G(K_{30+t}) = \begin{cases} 50{,}000\,(1.01923)^{t+K_{30+t}}\ v_{.06}^{K_{30+t}+1} & K_{30+t} < 35-t \\ 50{,}000\,(1.01923)^{35}\ v_{.06}^{35-t} & K_{30+t} \geq 35-t \end{cases}$

(iii) Reserve before alteration = reserve after alteration + cost of alteration

Before

$${}_{10}V = 60{,}000\left\{\frac{1}{1.01923}\left(A_{40:\overline{25}|}^{4\%} - \frac{D_{65}}{D_{40}}\right)^{4\%} + \frac{D_{65}}{D_{40}}\right\} - (.975)(990)(\ddot{a}_{40:\overline{25}|}^{6\%})$$

$$= 60{,}000\left\{\frac{1}{1.01923}(.40005 - .30690) + .30690\right\} - (.975)(990)(13.081)$$

$= 23{,}897.55 - 12{,}626.44 = 11{,}271.11$ say £11,271

After

$${}_{10}V = x\ A_{40}^{6\%} - (.975)(990)\ \ddot{a}_{40}^{6\%}$$

$= x\,(.15807) - (.975)(990)(14.874) = (.15807)(x) - 14{,}357$

$\Rightarrow$ $11{,}271 = (.15807)(x) - 14{,}357 + 100 \Rightarrow x = 161{,}498$ say £161,500

(iv) The amount at risk is immediately significantly increased (by £100,000) and the term for which there is a death strain has been extended. There is a grave risk of adverse selection against the office unless it underwrites the alteration as effectively a new business case. A simple declaration of health will not suffice in this case given the size of the change of the immediate risk.

Parts (i), (iii) and if attempted (iv) were well answered although most students missed the different bonus treatment needed for death benefits compared with the maturity benefit. Few candidates seemed familiar with the concept of the gross future loss as a random variable and answers to part (ii) were weak.