REPORT OF THE BOARD OF EXAMINERS

April 2003

Subject 109 — Financial Economics

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

J Curtis
Chairman of the Board of Examiners

3 June 2003

1 (i) $E(R) - r = \beta(E(R_{\text{market}}) - r)$

$7\% - 5\% = \beta(6\% - 5\%)$

$\Rightarrow \beta = 2.$

(ii) Systematic risk relates to the whole market. It cannot be diversified away and investors, in return for accepting it, should receive on average higher returns.

Specific risk relates to a particular security's factors. It can be diversified away and therefore investors do not require any compensation for accepting it.

(iii) $E(R) - r = \beta(E(R_{\text{market}}) - r)$

$5.5\% - 5\% = \beta(6\% - 5\%)$

$\Rightarrow \beta = ½.$

$\text{Var} = ½^2 * 0.2^2 = 0.01$ (systematic component)

$\text{Var(portfolio)} = 0.3^2 - 0.01 = 0.08$

$\Rightarrow$ specific component = 0.08

2 (i)

1. Specify the time horizon and objectives
2. choose the parameters to be optimised
3. for a particular set of input parameters, run a number of simulations
4. record the distribution of the outputs
5. vary the inputs and repeat steps 3 and 4
6. repeat steps 3, 4 and 5 until efficient opportunity set has been mapped
7. present results and decide on choice of input parameters and outcome

(ii)

- Investment model parameters

  – expected return, variances and covariances
  – inputs need to be realistic
  – consider historical data and relevance

- Liability data
  - starting values
  - demographic assumptions and all other factors ignored by normal valuation
- Starting value of assets
  - adjusted if not consistent with long term assumed return on model
- Objective function
  - function and future liability valuation methodology
  - fixed or dynamic depending on future projection
- Dynamic modelling rules
  - rules for future funding rates or investment policy

(iii) Practical considerations

- number of simulations
- complexity of model
- time involve
- significance of modelling
  - parameter risk
  - model risk

3 (i) Excessive volatility means that the change in market value of stocks (i.e. volatility) could not simply be justified by the flow of new information.

(ii)

- considered discounted cashflow model of equities over 100 year period
- determined "perfect foresight price(PFP)" (i.e. the correct market price if markets could correctly predict future dividends)
- difference between PFP and actual price was due to forecasting errors

- If markets are rational expect no systematic errors. Also if markets are efficient broad movements in PFP should be correlated with moves in actual price.

Conclusion: strong evidence that the level of observed volatility contradicted Efficient Market Hypothesis.

Criticisms: Later studies with different formulations found that violation of Efficient Market Hypothesis only borderline, reasons cited are:

- difficult to choose terminal value for stock price
- Schiller used constant discount rate
- bias in estimates of the variances because of autocorrelation
- possible non-stationarity of series measured

4 (i) To prove that the flat yield curve produces arbitrage opportunities, consider a situation where at time 0 where bonds with varying maturities yield an interest rate i_0. Consider an investor who:

1. issues 2-year bonds with a nominal value of $1{,}000(1 + i_0)$

2. buys 1-year and 3-year bonds with nominal values of 500 and $500(1 + i_0)^2$ respectively

The net initial cost of this portfolio is:

$$PV(0) = \frac{1{,}000(1+i_0)}{(1+i_0)^2} - \frac{500}{(1+i_0)} - \frac{500(1+i_0)^2}{(1+i_0)^3}$$

$$= \frac{1{,}000}{(1+i_0)} - \frac{500}{(1+i_0)} - \frac{500}{(1+i_0)}$$

$$= 0$$

Suppose at time 1 the yields have changed to i_1.

The value of this portfolio at that point in time will be:

$$PV[1] = -\frac{1,000(1+i_0)}{(1+i_1)} + 500 + \frac{500(1+i_0)^2}{(1+i_1)^2}$$

$$= 500\left[1 - 2\frac{(1+i_0)}{(1+i_1)} + \left(\frac{1+i_0}{1+i_1}\right)^2\right]$$

$$= 500\left[1 - \frac{(1-i_0)}{(1+i_1)}\right]^2 > 0$$

Clearly the above relationship holds for all values of i_1 which implies an arbitrage opportunity being there regardless of the value of i_1.

This shows that a flat-yield curve always allows for arbitrage opportunities.

(ii) (a) $P(\tau) = \exp\left(-\int f(t)dt\right)$

$f(t) = e^{-\alpha t}R + (1 - e^{-\alpha t})L$

$\int f(t)dt = L\tau + (R - L) \int e^{-\alpha t}dt$

$= L\tau - (R - L)\dfrac{(e^{-\alpha t} - 1)}{\alpha}$

$P(\tau) = \exp[-RD(\tau) - \{\tau - D(\tau)\}L]$

where $D(\tau) = \dfrac{1 - e^{-\alpha t}}{\alpha}$

(b) Spot yield $(\tau) = S(\tau) = -\dfrac{\ln P(\tau)}{\tau}$

$= \dfrac{RD(\tau)}{\tau} + \dfrac{\{\tau - D(\tau)\}L}{\tau}$

$= L + \dfrac{D(\tau)}{\tau}(R - L)$

$$\lim_{\tau\to\infty} \frac{1 - e^{-\alpha\tau}}{\alpha\tau} = 0$$

$$\therefore \lim_{\tau \to \infty} S(\tau) = L$$

5 (i) (a) Absolute risk aversion: $A(w) = \dfrac{-U''(w)}{U'(w)}$

(b) Relative risk aversion: $R(w) = -w \, . \dfrac{U''(w)}{U'(w)}$

(ii) (a)

$$U'(w) = ae^{-aw}$$

$$U''(w) = -a^2 e^{-aw}$$

$$\Rightarrow \quad A(w) = -\frac{(-a^2 e^{-aw})}{ae^{-aw}} = a$$

$\Rightarrow$ Constant absolute risk aversion

Relative risk aversion $R(w) = aw$

$$\Rightarrow \quad R'(w) = a > 0$$

Increasing relative risk aversion.

(b)

$$U'(w) = w^{\gamma-1}$$

$$U''(w) = (\gamma - 1)\, w^{\gamma-2}$$

$$\Rightarrow \quad A(w) = \frac{(1-\gamma)}{w} \qquad A'(w) = -\frac{(1-\gamma)}{w^2} < 0$$

Decreasing absolute risk aversion.

$$R(w) = \frac{w(1-\gamma)}{w} = (1-\gamma), \qquad R'(w) = 0$$

$\Rightarrow$ Constant relative risk aversion.

(iii) Decreasing absolute risk aversion is a more desirable property than increasing absolute risk aversion.

An investor who has the choice between a risky asset and a risk free asset will typically risk more wealth in absolute terms in the risky asset as the amount of their wealth increases.

6 (i)

- investors select their portfolios on the basis of expected return and variances of return (or the returns on assets are distributed with a multi-variate normal distribution)
- investors are risk-averse
- investors are never satiated
- investors make their investment decisions over a single time horizon
- no taxes or transaction costs
- assets are infinitely divisible

(ii) $1 = X_A + X_B$

$$\sigma^2 = X_A^2\sigma_A^2 + (1-X_A)^2\sigma_B^2 + 2X_A(1-X_A)\rho\sigma_A\sigma_B$$

$$\frac{\partial\sigma^2}{\partial X_A} = 0 \text{ for stationary point}$$

$$\Rightarrow 2X_A\sigma_A^2 - 2(1-X_A)\sigma_B^2 + 2\rho\sigma_A\sigma_B - 4X_A\rho\sigma_A\sigma_B = 0$$

$$\Rightarrow X_A(2\sigma_A^2 + 2\sigma_B^2 - 4\rho\sigma_A\sigma_B) - 2\sigma_B^2 + 2\rho\sigma_A\sigma_B = 0$$

$$\Rightarrow X_A = \frac{\sigma_B^2 - \rho\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B} = \frac{(\sigma_B^2 - \rho\sigma_A\sigma_B)}{(\sigma_B^2 - \rho\sigma_A\rho\sigma_B) + (\sigma_A^2 - \rho\sigma_A\sigma_B)}$$

(iii) $X_A > 0 \Rightarrow \sigma_B^2 - \rho\sigma_A\sigma_B \gtreqless 0$ and $(\sigma_B^2 - \rho\sigma_A\sigma_B) + (\sigma_A^2 - \rho\sigma_A\sigma_B) \gtreqless 0$

<u>and</u>

$1 - X_A > 0 \Rightarrow \sigma_A^2 - \rho\sigma_A\sigma_B \gtreqless 0$ and $(\sigma_B^2 - \rho\sigma_A\sigma_B) + (\sigma_A^2 - \rho\sigma_A\sigma_B) \gtreqless 0$

Taking the case $\sigma_B^2 - \rho\sigma_A\sigma_B + \sigma_A^2 - \rho\sigma_A\sigma_B > 0$

$$\Rightarrow \sigma_B^2 - \rho\sigma_A\sigma_B > 0 \text{ and } \sigma_A^2 - \rho\sigma_A\sigma_B > 0$$

$$\Rightarrow \frac{\sigma_B}{\sigma_A} > \rho \text{ and } \frac{\sigma_A}{\sigma_B} > \rho \quad (*)$$

Taking the case $\sigma_B^2 - \rho\sigma_A\sigma_B + \sigma_A^2 - \rho\sigma_A\sigma_B < 0$

$$\Rightarrow \sigma_B^2 - \rho\sigma_A\sigma_B < 0 \text{ and } \sigma_A^2 - \rho\sigma_A\sigma_B < 0$$

$$\Rightarrow \frac{\sigma_B}{\sigma_A} < \rho \text{ and } \frac{\sigma_A}{\sigma_B} < \rho$$

$$\Rightarrow \rho > 1, \text{ which is not possible}$$

Hence (*) holds.

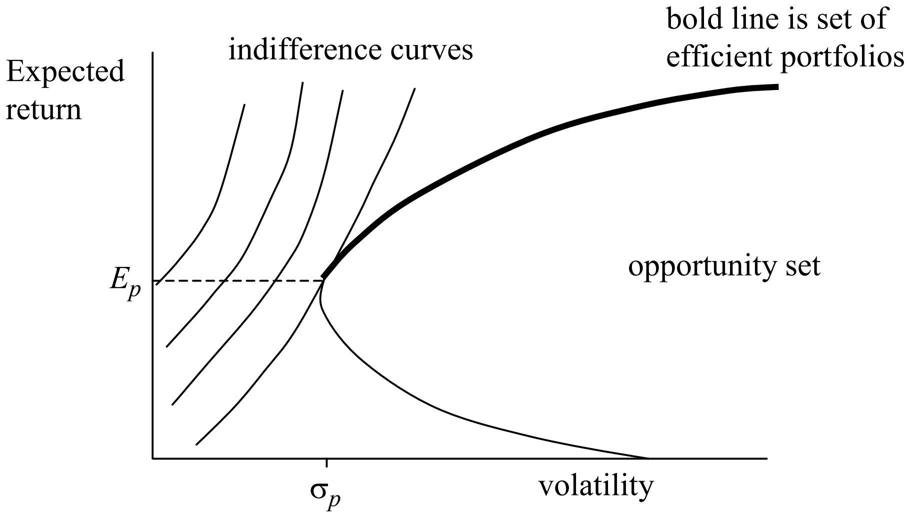
(iv) Indifference curve: Curves where each point represents a portfolio that gives the same expected utility.

Optimal portfolio: Point where the indifference curve is tangential to the efficient frontier. Requires quadratic utility or distribution of returns to be normal. Or, the portfolio with the highest attainable expected utility.

Opportunity set: Characteristics of each and every portfolio available to investors.

Efficient frontier: Collection of efficient portfolios. A portfolio is efficient if we cannot find one with lower variance for same expected return or higher expected return for same variance.

(v)



7 (i)

Let x = proportion of portfolio in asset A, and therefore $(1 - x)$ is the proportion in B

$$\bar{r}_p = x\bar{r}_A + (1 - x)\,\bar{r}_B$$

$$\sigma_p^2 = x^2\sigma_A^2 + (1-x)^2\sigma_B^2 + 2x(1-x)\sigma_A\sigma_B\rho$$

$$\Rightarrow E_\alpha(U) = x\bar{r}_A + (1-x)\bar{r}_B - \alpha(x^2\sigma_A^2 + (1-x)^2\sigma_B^2 + 2x(1-x)\sigma_A\sigma_B\rho)$$

maximum will occur when $\dfrac{\partial E_\alpha(U)}{\partial x} = 0$ and $\dfrac{\partial^2 E_\alpha(U)}{\partial x^2} < 0$

$$\frac{\partial E_\alpha(U)}{\partial x} = \bar{r}_A - \bar{r}_B - \alpha(2x\sigma_A^2 - 2(1-x)\sigma_B^2 + 2(1-2x)\sigma_A\sigma_B\rho)$$

setting $= 0 \Rightarrow \bar{r}_B - \bar{r}_A - 2\alpha\sigma_B^2 + 2\alpha\sigma_A\sigma_B\rho$

$$= x(-2\alpha\sigma_A^2 - 2\alpha\sigma_B^2 + 4\alpha\rho\sigma_A\sigma_B)$$

$$\Rightarrow x = \frac{\bar{r}_B - \bar{r}_A - 2\alpha\sigma_B(\sigma_B - \rho\sigma_A)}{2\alpha(2\rho\sigma_A\sigma_B - \sigma_A^2 - \sigma_B^2)}$$

Second order test $\dfrac{\partial^2 E_\alpha(U)}{\partial x^2} = -\alpha(2\sigma_A^2 + 2\sigma_B^2 - 4\rho\sigma_A\sigma_B)$

< 0

$\Rightarrow$ maximum.

Substituting for parameters

$$\sigma_A^2 = 4\%\% \Rightarrow \sigma_A = 2\%$$

$$\sigma_B^2 = 25\%\% \Rightarrow \sigma_B = 5\%$$

$$\Rightarrow x = \frac{8\% - 6\% - 2\alpha \times 0.5(5\% - 0.5 \times 2\%)}{2\alpha(2 \times 0.5 \times 2\% \times 5\% - 4\%\% - 25\%\%)}$$

$$= \frac{0.02 - 0.04\alpha}{\alpha(-0.038)} = \frac{1}{19}\left(20 - \frac{100}{\alpha}\right)$$

(ii) $\dfrac{dx}{d\alpha} = +\dfrac{100}{19}\cdot\dfrac{1}{\alpha^2} > 0 \Rightarrow x$ = proportion invested in asset A increases as α increases

8 (i) The risk neutral probability is given by

$$q = \frac{e^{r/365} - d}{u - d} = \frac{e^{0.05/365} - 0.9844}{1.01583 - 0.9844} = 0.5007$$

where $u = \exp(0.3/\sqrt{365}) = 1.01583$ and $d = 1/u = 0.9844$, $r = 0.05$

State prices are

A $\quad 1 \times 0.5007^2 \times e^{\frac{-0.05\times 2}{365}} = 0.25063$

B $\quad 1 \times 2 \times 0.5007 \times 0.4993 \times e^{\frac{-0.05\times 2}{365}} = 0.49986$

C $\quad 0.4993^2 \times e^{\frac{-0.05\times 2}{365}} = 0.24923$

(ii) Deflator

A $\quad 0.25063/0.6^2 = 0.69619$

B $\quad 0.49986/(2 \times 0.6 \times 0.4) = 1.041375$

C $\quad 0.24923/0.4^2 = 1.55768$

(iii) The three share prices are

$100 \times u^2 = 103.19$ A

$100 \times u \cdot d = 100$ B

$100 \times d^2 = 96.91$ C

Value at time zero of call option is

$$(103.19 - 99) \times 0.6^2 \times 0.69619 + (100 - 99) \times 0.6 \times 0.4 \times 2 \times 1.041375$$

$$= 1.05013 + 0.49986 = 1.55$$

(iv) Value of option at time 1 if up jump

$v(1) = (4.19 \times 0.5007 + 1 \times 0.4993) \times e^{-0.05/365} = 2.597;$

value of option at time 1 if down jump $v(0) = 0.5007 / e^{0.05/365}$

$= 0.5006$

$$\Delta = \frac{v(1) - v(0)}{S_0 e^{-qt}(u-d)} = \frac{2.597 - 0.5006}{100e^{\frac{-0.5007}{365}}(1.01583 - 0.9844)}$$

$= 0.6679$

(v) Also $1.55 = 0.6679 \times 100 - \beta$

$\Rightarrow \beta = 65.24$

$\Rightarrow$ Optimal portfolio is 66.79% of a share and short 65.24 cash.

(vi) This is the process of determining the present value of a future, uncertain, cashflow as the expected discounted value. However, the probability measure employed is not the "real world" but rather a specially constructed "risk neutral measure". This risk neutral measure is chosen to obtain arbitrage free prices.

9 (i) For notational convenience, s_1 and s_2 refer to both the stock and the price of a share in the stock at time 0.

Consider the stock s_1, priced in terms of units of s_2. Let $S' = \frac{s_1}{s_2}$. This has lognormal distribution with volatility given by $v^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$.

The value of a call option over stock S', using the Black-Scholes formula with exercise price of 1 is

$S'\Phi(d_1) - 1 \cdot e^{-rT}\Phi(d_2)$ where

$$d_1 = \frac{r + \frac{1}{2}v^2 T + \log(S')}{v\sqrt{T}}$$

$$d_2 = d_1 - v\sqrt{T}$$

We also note that $r = 0$. To see this consider a 1 year forward contract (expressed in units of s_2) that delivers one unit of s_2 in a year's time.
This has value s_2, i.e. the price of one unit of s_2, now. Therefore, we have the equation of value

$$1 = e^{-rT} \cdot 1 \Rightarrow r = 0.$$

Since the payoff function from the call option is $\max(S' - 1, 0)$ and the above formula gives its value in units of s_2 we require s_2 times this to give the value of our exchange option.

$s_2 \times S'\Phi(d_1) - s_2 \times \Phi(d_2)$ where

$$d_1 = \frac{\log(S') + \tfrac{1}{2}v^2 T}{v\sqrt{T}}$$

$$d_2 = d_1 - v\sqrt{T}$$

value of exch. option $= f = s_1 \Phi\left(\dfrac{\log\left(\dfrac{s_1}{s_2}\right) + \tfrac{1}{2}v^2 T}{v\sqrt{T}}\right) - s_2 \Phi\left(\dfrac{\log\left(\dfrac{s_1}{s_2}\right) - \tfrac{1}{2}v^2 T}{v\sqrt{T}}\right)$

(ii) Delta of f with respect to s_1 is

$$\frac{\partial f}{\partial s_1} = \Phi(d_1) + s_1\Phi'(d_1)\frac{\partial d_1}{\partial s_1} - s_2\Phi'(d_2)\frac{\partial d_2}{\partial s_1}$$

Note: $\dfrac{\partial d_1}{\partial s_1} = \dfrac{\partial d_2}{\partial s_1}$ by symmetry and $\Phi'(d_1) = \phi(d_1) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2}$.

We can note that:

$$\frac{s_1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log\left(\frac{s_1}{s_2}\right) + \frac{1}{2}v^2 T}{v\sqrt{T}}\right)^2} = \frac{s_2}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log\left(\frac{s_1}{s_2}\right) - \frac{1}{2}v^2 T}{v\sqrt{T}}\right)^2}$$

because

$$\log\left(\frac{s_1}{s_2}\right) - \tfrac{1}{2}\left(\frac{\log\left(\frac{s_1}{s_2}\right) + \tfrac{1}{2}v^2T}{v\sqrt{T}}\right)^2 = -\tfrac{1}{2}\left(\frac{\log\left(\frac{s_1}{s_2}\right) - \tfrac{1}{2}v^2T}{v\sqrt{T}}\right)^2$$

Therefore:

$$\frac{\partial f}{\partial s_1} = \Phi\left(\frac{\log\left(\frac{s_1}{s_2}\right) + \tfrac{1}{2}v^2T}{v\sqrt{T}}\right)$$

$$\frac{\partial f}{\partial s_2} = s_1\Phi'(d_1)\frac{\partial d_1}{\partial s_2} - s_2\Phi'(d_2)\frac{\partial d_2}{\partial s_2} - \Phi(d_2)$$

Therefore using the results used in deriving $\frac{\partial f}{\partial s_1}$:

$$\frac{\partial f}{\partial s_2} = -\Phi\left(\frac{\log\left(\frac{s_1}{s_2}\right) - \tfrac{1}{2}v^2T}{v\sqrt{T}}\right)$$

(iii) The replicating portfolio consists of

$$\Phi\left(\frac{\log\left(\frac{s_1}{s_2}\right) + \tfrac{1}{2}v^2T}{v\sqrt{T}}\right) \text{ of } s_1 \text{ and } -\Phi\left(\frac{\log\left(\frac{s_1}{s_2}\right) - \tfrac{1}{2}v^2T}{v\sqrt{T}}\right) \text{ of } s_2.$$

(iv) Because of rounding in the given information, the implied volatility for s_1 is not unique and any value less than 22% was accepted provided an appropriate methodology or line of reasoning was used. (There is also technically an arbitrage represented by the quoted prices because of rounding.) The appropriate methodology is to interpolate between two Black-Scholes prices for the European call option specified in the question calculated at two "trial and error" volatilities. The remainder of this solution assumes a volatility for s_1 of 15% p.a.

$v^2 = 0.15^2 + 0.45^2 - 2 \times ½ \times 0.15 \times 0.45 = 0.1575$

$$\frac{\log\left(\frac{s_1}{s_2}\right) + ½v^2T}{v\sqrt{T}} = \frac{\log\left(\frac{2}{1.75}\right) + ½ \times 0.1575 \cdot ½}{\sqrt{0.1575 \times ½}}$$

$$= 0.61615$$

$\Rightarrow \Phi(0.61615)$

Similarly $\dfrac{\log\left(\frac{s_1}{s_2}\right) - ½v^2T}{v\sqrt{T}} = 0.3355$

Value

$$= 2\Phi(0.61615) - 1.75 \times \Phi(0.3355)$$

$$= 2 \times 0.7311 - 1.75 \times 0.63137$$

$= 0.357 \quad = 36\text{p}$