Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

April 2000

Subject 109 — Financial Economics

EXAMINERS' REPORT

1 (i) CAPM: $E(R_i) = r_f + \beta_i(E(R_{\text{market}}) - r_f)$

where r_f = risk-free rate, R_{market} is the return on the market and β_i is a parameter.

APT: $E(R_i) = \mu_{io} + \sum_{k=1}^{n} \beta_{ik}\, E(\gamma_k)$

where μ_{io} , β_{ik} are security-specific parameters and $\gamma_1 \ldots \gamma_n$ are the rates of change on the common factors.

(ii)

- APT identifies several key systematic factors in the returns generating process while the CAPM assumes a single factor
- APT recognises that these key factors can change over time whereas the CAPM's single factor is unchanging.
- APT makes fewer assumptions about investor preferences than the CAPM.
- The efficiency of the market portfolio is critical for the CAPM.
- However, APT does not tell us what factors to include or how many.

2 (i) Consider surplus of assets over liabilities, i.e.

$$S = A\sum_{i=1}^{n} x_i(1 + R_i) - L$$

and apply mean-variance theory to this quantity effectively treating liabilities as negative assets.

The estimates of the covariance of L with the asset returns will typically require an ALM to be run.

(ii) In most cases analytically tractable ALMs are impossible to construct. Instead Monte Carlo simulation is used.

Multiple scenarios are calculated with the required distribution to provide the stochastic variation. For example, to calculate the net cash flow into a pension fund during a single period, the values of contributions received, income generated by the assets held and benefit outgo, and interest rate are all random variables which are generated using a specific distribution. The exercise is repeated many times (say 10,000) to obtain a distribution of the net cash flow.

3 (i) $$\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

where

μ = mean return
$f(x)$ = p.d.f. of return

(ii) $$\int_{-\infty}^{\mu} (x-\mu)^2 f(x)\,dx$$

(iii) $$\int_{-\infty}^{L} f(x)\,dx$$

where L = benchmark level

4 According to the Black-Scholes model, the value of a European option is a function of 5 parameters: stock price, strike price, interest rate, time to maturity, and volatility. The first 4 parameters can be observed directly in the market. The only parameter which is not observed is volatility.

Given the market price of the option, and the observed values for the 4 parameters, we can work out the market figure for volatility. This figure is called implied volatility.

5 (i) $U'(W) > 0$ implies that the investor is non-satiated (i.e. s/he prefers more wealth to less).

$U''(W) > 0$ implies the investor is risk-seeking, i.e. will always accept a fair gamble.

(ii) Project B exhibits a first order stochastic dominance over project A. First order stochastic dominance guarantees that the expected utility of wealth offered by project B will be greater than that offered by A for all non-satiated investors.

Since the investor in (i) is non-satiated, they will prefer B.

6 The payoff from the call is $\max(S_T - K, 0)$.

Let $\sigma = 0$, S will grow at rate r to $Se^{r(T-t)}$ at time T. Discounting at r,

$$c = e^{-r(T-t)} \max[Se^{r(T-t)} - K, 0]$$

$$c = \max[S - Ke^{-r(T-t)}, 0] \qquad (1)$$

The Black-Scholes pricing formula is

$$c = S\Phi(d_1) - Ke^{-r(T-t)}\,\Phi(d_2) \qquad (2)$$

If $S > Ke^{-r(T-t)}$, then $\ln(S/K) + r(T-t) > 0$.

As σ tends to zero, d_1 and d_2 tend to $+\infty$.

Accordingly, $\Phi(d_1)$ and $\Phi(d_2)$ tend to 1.

Equation 2 becomes $c = S - Ke^{-r(T-t)}$.

If $S < Ke^{-r(T-t)}$, then $\ln(S/K) + r(T-t) < 0$.

As σ tends to zero, d_1 and d_2 tend to $-\infty$.

Accordingly, $\Phi(d_1)$ and $\Phi(d_2)$ tend to zero.

Equation 2 gives a call price of zero.

7 Let x be the proportion invested in the risky asset.

$$\bar{r}_p = xr + (1-x)\,r_f$$

$$\sigma_p^2 = x^2\sigma^2$$

$$\max_x \; U(\bar{r}_p) = xr + (1-x)\,r_f - \tfrac{1}{2}x^2\sigma^2$$

$$\frac{\partial U(\bar{r}_p)}{\partial x} = r - r_f - x\sigma^2 = 0$$

$$\Rightarrow x = \frac{r - r_f}{\sigma^2}$$

The second order test

$$\frac{\partial^2 U(\bar{r}_p)}{\partial x^2} = -\sigma^2 \text{ (which is } < 0 \text{ since } \sigma^2 \text{ is +ve)}$$

Therefore, the optimal percentage to invest in the risky asset is $\dfrac{r - r_f}{\sigma^2}$

$$x = \frac{8-5}{4} = 0.75$$

The investor should invest 75% in the risky asset and 25% in the risk free asset.

8 (i) Delta: the change in option price with respect to change in the price of the underlying asset.

Gamma: the rate of change of delta as the price of the underlying asset change.

Theta: measures how quickly the time value of the option changes as the option moves towards expiration.

(ii) Delta hedging is the attempt to set up a riskless portfolio consisting of a position in a derivative on a stock and a position in the stock.

Assume that the delta (Δ) of a call option is 0.6. This means for a small change in the stock price, the option changes by about 60% of the change.

Imagine the investor sold 20 option contract, that is, options to buy 2,000 shares. The investor's position could be hedged by buying $0.6 \times 2{,}000 = 1{,}200$ shares.

If the stock price goes up by £1, the investor will make a gain of £1,200. However, he will also make a loss of $0.6 \times -2{,}000 = -£1{,}200$ on the options written.

In general, the gain (loss) on the option position would offset the loss (gain) on the stock position.

9 (i) Weak-form hypothesis: stock prices reflect all information that can be derived from studying past market trading data.

Semi strong: stock prices reflect all publicly available information about the stock.

Strong form: stock prices reflect all information relevant to the firm, even including information available only to company "insiders".

(ii) Even in a perfectly efficient market, portfolio managers would have the important role of constructing and implementing an integrated set of steps to create and maintain appropriate combinations of investment assets.

These steps are:

(1) Specify and quantify investors risk tolerance, required returns, time horizon, tax considerations, form of income needs (capital gains or dividends), liquidity, legal and regulatory constraints) ... etc.

(2) Monitoring and evaluating market conditions. Relevant factors such as the economy, political situation ... etc.

(3) Monitoring the investor's circumstances.

(4) Portfolio adjustment as a result of significant changes in any or all relevant variables.

10 (i) $\bar{R}_i = \alpha_i + \beta_i \bar{R}_m$

$\bar{R}_I = 0.04 + (1.20)(0.16) = 23.2\%$

$\bar{R}_Z = 0.09 + (1.50)(0.16) = 33\%$

$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_e^2$

$\sigma_I^2 = (1.2)^2 (0.04) + 0.0625 = 0.1201$

$\sigma_Z^2 = (1.5)^2 (0.04) + 0.16 = 0.25$

(ii) $\sigma_{IZ} = \beta_I \beta_Z \sigma_m^2 = (1.20)(1.50)(0.04) = 0.072$

(iii) $\beta_p = (0.5)(1.20) + (0.5)(1.5) = 1.35$

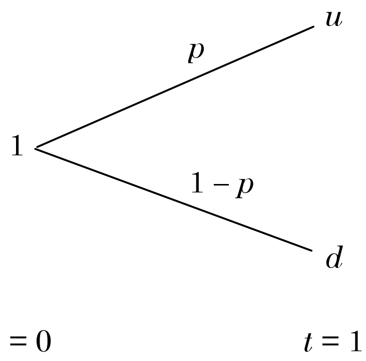
(iv) $E(R_p) = (0.5)(23.2) + (0.5)(33) = 28.1\%$

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2$$

$$= (1.35)^2 (0.04) + (0.5)^2 (0.0625 + 0.16)$$

$$= 0.1285$$

11 (i)



Solve $\phi u + \psi e^r = u - k$ (1)

$\phi d + \psi e^r = 0$ (2)

$$\Rightarrow \qquad \phi = \frac{-\psi e^r}{d}$$

Substituting in (1)

$$-\frac{\psi e^r}{d} u + \psi e^r = u - k$$

$$\Rightarrow \qquad \psi e^r \left(1 - \frac{u}{d}\right) = u - k$$

$$\Rightarrow \qquad \psi = e^{-r}\, d\, \frac{u-k}{d-u}$$

$$\Rightarrow \qquad \phi = \frac{u-k}{u-d}$$

Therefore replicating portfolio is

ϕ units of stock, ψ units of cash invested at the risk-free rate

(ii) Value of option at $t = 0$ = value of replicating portfolio at $t = 0$

$$c = \phi S_0 + \psi = \phi + \psi$$

$$= \frac{u-k}{u-d} + e^{-r} d\, \frac{u-k}{d-u}$$

$$= (u-k)\; e^{-r} \left[\frac{d}{d-u} - \frac{e^r}{d-u}\right] + 0$$

$$= e^{-r}\left[(u-k)\left(\frac{e^r-d}{u-d}\right)+0\left\{1-\left(\frac{e^r-d}{u-d}\right)\right\}\right]$$

$$= e^{-r}[q(u-k) + (1-q)\,0] \qquad *$$

$$= e^{-r}E_Q[c_{t=1} \mid F_0]$$

noting that the option has a value of $u - k$ or 0 at time $t = 1$.

$$* \; q = \frac{e^r - d}{u-d}$$

(iii) Under Q,

$$E_Q[S_1] = S_0(qu + (1-q)\,d)$$

$$= \frac{e^r-d}{u-d}u + \left(1-\frac{e^r-d}{u-d}\right)d$$

$$= \frac{(e^r-d)\,u}{u-d} + \frac{(u-e^r)\,d}{u-d}$$

$$= e^r$$

i.e. the expected return on the stock is the risk-free rate. This implies that under Q (the risk-neutral measure) investors are not requiring a higher expected return for bearing risk.

Under the real world measure, P, investors would typically demand a higher return for bearing risk, i.e.

$$E_P(S_1) > e^r$$

and the expectation would normally depend on u, d and p.

For risk-seeking investors, $q > p$ and $E_P(S_1) < e^r$.

For risk-averse investors $p > q$ and $E_P(S_1) > e^r$.

For risk-neutral investors $p = q$ and $E_P(S_1) = e^r$.

12 (i) Total return = $\left[1+ {}^{1}\!\!\diagup_{R(t+1)}\right] R(t)\,\dfrac{Q(t+1)}{Q(t)}$

or, equivalently,

$$R(t)\,\frac{Q(t+1)}{Q(t)} + \frac{R(t)}{R(t+1)}\,\frac{Q(t+1)}{Q(t)}$$

(ii) EI is a weighted average of this year's "force" of inflation and EI from one year ago. This is equivalent to an exponentially-weighted average of the "forces" of inflation in all previous years. To the extent that past rates of inflation persist and become entrenched in financial decision making, so the historical average might be used as a proxy for expected future inflation.

(iii) (a) Log of real rate of interest is first order autoregressive around a long term mean with a normally distributed error term. Because logarithms are modelled, the distribution of $R(t)$ will be lognormal and hence positive. This might be regarded as too restrictive.

Continuously compounded long bond yield equals expected inflation plus the continuously compounded real yield plus a "risk premium". This model accords reasonably with economic theory. However the structure of the risk premium term allows for negative nominal yields that many would regard as implausible.

(b) Continuously compounded long bond yield equals a constant plus a proportion of the difference between current expected inflation and a mean rate of inflation plus a normally distribution innovation term.

Nominal yields can become negative.

No mean-reverting characteristics so if inflation were nearly constant then the yields would follow a random walk, probably implausible.

Continuously compounded real yield is first order autoregressive plus a trend equal to the difference between current real interest rates (as measured by the difference between current nominal yields and expected inflation) and the mean real interest rate.

Negative real yields might emerge (not problematic).

Nominal yields treated exogenously to real yields, which could be considered unusual.