Faculty of Actuaries **Institute of Actuaries**

REPORT OF THE BOARD OF EXAMINERS ON THE EXAMINATIONS HELD IN

April 2002

Subject 109 — Financial Economics

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

K Forman
Chairman of the Board of Examiners

11 June 2002

1 (i)

(a) Variance $= \int_{-\infty}^{\infty} (\mu - x)^2 \; f(x)dx$

(b) Downside semi-Var $= \int_{-\infty}^{\mu} (\mu - x)^2 \; f(x)dx$

(c) Expected shortfall $= \int_{-\infty}^{L} (L - x) \; f(x)dx \Big/ \int_{-\infty}^{L} f(x)dx$

L = benchmark level

(ii)

(a) Var $= \dfrac{1}{\lambda^2} = 4$

(b) Downside semi-Var (D S-V) $= \int_{-\infty}^{2} (2 - x)^2 \; f(x)dx$

$$= \int_{0}^{2} (4 - 4x + x^2) \, \lambda e^{-\lambda x} \, dx$$

$$\int_b^a xe^{-\lambda x} dx = \left. \frac{xe^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right|_b^a$$

$$\therefore \text{ D S-V } = \left[-4e^{-\lambda x} - 4\lambda\left(\frac{xe^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right) + \lambda\left(\frac{x^2 e^{-\lambda x}}{-\lambda} - \int \frac{2xe^{-\lambda x}}{-\lambda} \, dx \right) \right]_0^2$$

$$= \left[-4e^{-\lambda x} + 4xe^{-\lambda x} + \frac{4e^{-\lambda x}}{\lambda} - x^2 e^{-\lambda x} + 2\left(\frac{xe^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right) \right]_0^2$$

$$= 4 - 8e^{-1} = 1.05696$$

(c) Amount returned when a shortfall occurs is:

$$\int_0^k x\lambda e^{-\lambda x}dx = \left[-xe^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda}\right]_0^k$$

$$= -ke^{-\lambda k} - \frac{e^{-\lambda k}}{\lambda} + \lambda^{-1}$$

Probability of a shortfall:

$$\int_0^k \lambda e^{-\lambda x}dx = \left[-e^{-\lambda x}\right]_0^k$$

Conditional expected shortfall $= k - \dfrac{\dfrac{1}{\lambda} - e^{-\lambda k}\left(k + \dfrac{1}{\lambda}\right)}{1 - e^{-\lambda k}}$

$$= 2 - \frac{2 - 4e^{-1}}{1 - e^{-1}}$$

$= 1.16$

2 (a) Excessively volatile markets are those where the volatility of prices is greater than can be justified by the arrival of new information and any other returns to investors.

(b)

- You would require
  - a long history of prices and cashflows from a market — i.e. 100 years of equity returns and details of all dividend payments and any other returns to investors
- This enables the calculations of the perfect foresight price (PFP) which is the present value of all future cashflows and the terminal value of the asset.
- The difference between the PFP and the actual price arises from forecast errors of future dividends. If markets are rational there should be no systematic pattern to the errors.

- The PFP and the actual price should be correlated because in an efficient market actual prices should respond to anticipated future increased cashflows which are reflected in PFP.

(c) Criticism of the approach

- How to choose the terminal value of the asset.
- What discount rate to use/changes in expected return or growth.
- Biases due to autocorrelations
- Non-stationarity of the time-series data.
- Distributional assumptions used in the statistical tests.
- Dependence on a dividend model that does not change over time.

3 (i) $\frac{s_u}{s_t} \sim LN(\mu(u-t), \sigma^2(u-t)]$ $u > t$, s_u = price of the market at time u

$E[s_1] \quad = s_t\, e^{\mu(u-t)+\frac{1}{2}\sigma^2(u-t)} = 1.08$

$V[s_1] \quad = E[s_u]^2\ (e^{\sigma^2(u-t)} - 1) = .16^2$

Solve the simultaneous equations to obtain: $\mu = 0.066106$ and $\sigma^2 = 0.02171$

(ii)

The approach used in setting question 3(ii)(a) was not consistent with Core Reading and in particular involved solving the simultaneous equations in (i) with $E[s_1] = 0.08$. This results in alternative values for the parameters of $\mu = -3.33$ and $\sigma^2 = 1.61$. The examiners apologise for the confusion that this might have caused. This inconsistency and its ramifications were taken into account in grading the paper.

$$\text{(a)}P[s_1 > 1.5s_0] = P\left[\ln\frac{s_1}{s_0} > \ln 1.5\right] = P\left[N(0,1) > \frac{\ln 1.5 - \mu}{\sigma}\right]$$

From standard normal tables and using the correct values for μ and σ, this is 1.06%. The solution based on the alternative values for μ and σ give a value of 0.161%.

(b) $$\ln\frac{s_{t+1}}{s_t} \sim N\left(\mu\frac{1}{200}, \frac{\sigma^2}{200}\right)$$

$$P[s_{t+1} > 1.0025\, s_t] \quad = P\left[\ln\frac{s_{t+1}}{s_t} > \ln 1.0025\right]$$

$$= P\left[N(0,1) > \frac{\ln 1.0025 - \frac{\mu}{200}}{\left(\frac{\sigma^2}{200}\right)^{½}}\right]$$

Using normal tables and the correct values for μ and σ give a value of 41.8%. The solution based on the alternative parameters is 41.5%.

(c) Find X s.t.

$$P[s_{t+1} > X s_t] \quad = 0.161\%$$

$$\Rightarrow P\left[\ln \frac{s_{t+1}}{s_t} > \ln X\right] = 0.161\%$$

$$\Rightarrow P\left[N(0,1) > \frac{\ln X - \mu/200}{\left(\sigma^2/200\right)^{½}}\right] = .00161 \qquad ,$$

$$\Rightarrow \frac{\ln X - \mu/200}{\left(\sigma^2/200\right)^{½}} = 1 - \Phi^{-1}(0.00161)$$

where Φ is the distribution function for the standard normal distribution, and hence solve for X. Using the correct values for μ and σ implies a value for X of 1.0315, i.e. a rate of return of 3.15%. The solution based on the alternative parameters is 1.2808, i.e. a rate of return of 28.1%.

(iii) Mean scales with time, t, but standard deviation of the return scales with $\sqrt{t}$.

The standard deviation and the mean return determines how likely an event is and hence this controls the probabilities.

Candidates were then expected to make a comment using the figures that they had calculated. Typically this would involve contrasting the figures calculated using the log-normal distribution in (ii) with how well they thought the answers matched up with actual returns and consequently questioning the appropriateness of the log-normal model for stock market returns.

4 From a flat yield curve there exists the opportunity to buy short and long bonds and to sell medium bonds. A portfolio can then be constructed so for any small change in the interest rates, d_i, the portfolio increases in value using Redington's theory of immunisation.

If arbitrageurs operated by buying short and long bonds the yields on these bonds would be expected to fall as the price responded to increased demand. Conversely yields on medium bonds would rise as arbitrageurs shorted (sold) these bonds.

This would result in convexity in the yield curve with a characteristic humped shape.

5 (i) (a) In a risk-neutral world an investor is neutral to assuming greater risk in their portfolio. Hence they do not require additional returns for holding risky assets as opposed to risk free assets.

A model is guaranteed to be arbitrage free if (subject to certain conditions) there exists a risk neutral formulation of a model, i.e. the probabilities of possible events can be "adjusted" (called a change of measure) so that the prices of assets can be determined by discounting expected values at the risk free rate.

Hence if a model, under a change of measure, is risk neutral it is arbitrage free. This provides a method of generating models that do not permit arbitrage opportunities.

(b) An equilibrium model recognises that, in aggregate, all assets are held by investors. All investors are assumed to hold portfolios that maximise their utilities (at the prices in the market and subject to any constraints affecting their holdings). Hence current market conditions represent an equilibrium where prices of assets provide sufficient expected rewards to compensate investors for their risks. An equilibrium model can be constructed by assuming a utility function for the market as a whole.

(ii) Equilibrium models were considered more general as they provided models of many asset classes and provided additional constraints on the rewards required for bearing risk. These economic conditions can be relevant for macro-economic models.

However, arbitrage free (risk-neutral) models are often used in pricing (e.g. bond options). It has been shown that various models can be derived from either risk-neutral or equilibrium arguments.

Overriding concern for pricing models is that they are arbitrage free.

In some cases (such as incomplete markets) equilibrium approaches can help decide among the multiple equivalent martingale measures, or can be used to find solutions where arbitrage models do not provide unique prices.

6 (i) (a) Using Equivalent martingale measure the expected return must be the return on a currency unit.

$$\therefore\ qu + (1-q)\,d = e^{r\Delta t}$$

(b) Want the variance of the discrete model to be the same as that for the continuous model.

Using discrete model

$$V\left[S_{n\Delta t}\,\middle|\,S_{(n-1)\Delta t}\right] = E\left[S^2_{n\Delta t}\,\middle|\,S_{\Delta t(n-1)}\right] - E\left[S_{n\Delta t}\,\middle|\,S_{\Delta t(n-1)}\right]^2$$

$$= u^2S^2q + d^2S^2(1-q) - (uSq + dS(1-q))^2$$

Using the continuous model

$$V\left[S_{n\Delta t}\,\middle|\,S_{(n-1)\Delta t}\right] = E\left[S^2_{n\Delta t}\,\middle|\,S_{n-1\Delta t}\right] - E\left[S_{n\Delta t}\,\middle|\,S_{(n-1)\Delta t}\right]^2$$

$$= S^2e^{2r\Delta t}\ (e^{\sigma^2\Delta t} - 1)$$

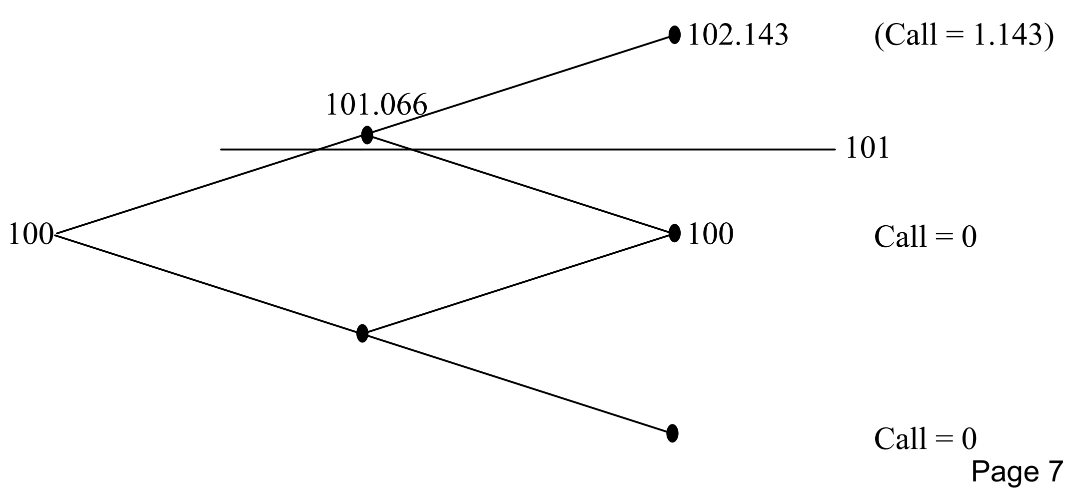
$$= S^2e^{2r\Delta t+\sigma^2\Delta t} - \left(E[S\,|\,S]\right)^2$$

Using the expression (a), and equating the expressions for ∀ the variance.

$$\therefore\ e^{2r\Delta t+\sigma^2\Delta t} = u^2q + d^2(1-q)$$

[Marks were awarded even if $E[S\,|\,S] = Suq + Sd(1-q)$ was not used.]

(ii) (a)



$u \doteqdot e^{\sigma\sqrt{\Delta t}} = 1.01066 \qquad d = 0.98945$

From (i)(a) $q = \dfrac{e^{r\Delta t} - d}{u - d} = 0.51155$

Value of call option

$= \exp(-r.\Delta t)^* E_Q[\max(0, C_T)]$

$= e^{(-.06*2/200)} {}^* q^2 \times 1.143$

$= 0.299$

Other discounting conventions also acceptable, i.e. 2/365.

(ii) (b) Just Black-Scholes Formula

$f(t, S_t) = S_t\, \Phi(d_1) - k\Phi(d_2)\, e^{-r(T-t)}$

$d_1 = \dfrac{\log\dfrac{S_t}{k} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$

$T - t = \dfrac{2}{200}$

$k = 101$

$S_t = 100 \quad \Rightarrow \quad d_1 = -0.61586$

$d_2 = d_1 - \sigma\sqrt{T-t} = -0.63086$

$r = 0.06$

$\sigma = .15$

$\Phi(d_1) = 1 - .0.73100 \qquad \Phi(d_2) = 1 - 0.73593$

$f(t, s_t) = 26.90 - 26.67e^{-r(T-t)}$

$= 0.245$

So the percentage error is

$\dfrac{.299}{.245} - 1 = 22\%$

7 (i) Application of Itôs lemma

$$dS_t = S_t(\mu dt + \sigma dZ_t)$$

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S_t}dS_t + ½\frac{\partial^2 f}{\partial S_t^2}(dS_t)^2$$

$$= \left(\frac{\partial f}{\partial t} + \mu S_t\frac{\partial f}{\partial S_t} + ½\sigma^2 S_t^2\frac{\partial^2 f}{\partial S_t^2}\right)dt + \sigma S_t\frac{\partial f}{\partial S_t}dZ_t$$

(ii) Consider holding -1 derivatives
Y shares

$\therefore$ value of portfolio $V(t, s_t) = -f(t, s_t) + Ys_t$

Change in the value of the portfolio over dt is

$$-\partial f(t, S_t) + Y\partial S_t = \left(-\frac{\partial f}{dt} - \mu S_t\frac{\partial f}{\partial S_t} - ½\sigma^2 S_t^2\frac{\partial^2 f}{\partial S_t^2} + Y\mu S_t\right)dt$$

$$+\left(-\sigma S_t\frac{\partial f}{\partial S_t} + Y\sigma S_t\right)dZ_1$$

Hence if $Y_t = \dfrac{\partial f}{\partial S_t}$ then no dependency on dZ, i.e. risk free.

(iii) $\Delta = \dfrac{\partial f}{\partial S}$ = rate of change of the derivative's value with the change in share price

$\Gamma = \dfrac{\partial^2 f}{\partial S^2}$ = rate of change of Δ with the share price

$\theta = \dfrac{\partial f}{\partial t}$ = rate of change of the derivative's value with time

$K = \dfrac{\partial f}{\partial \sigma}$ = rate of change of the derivative's value with volatility

These assume all other parameters are kept constant.

(iv) $Y_t = \dfrac{\partial f}{\partial S_t}$

Change in portfolio value is risk free hence

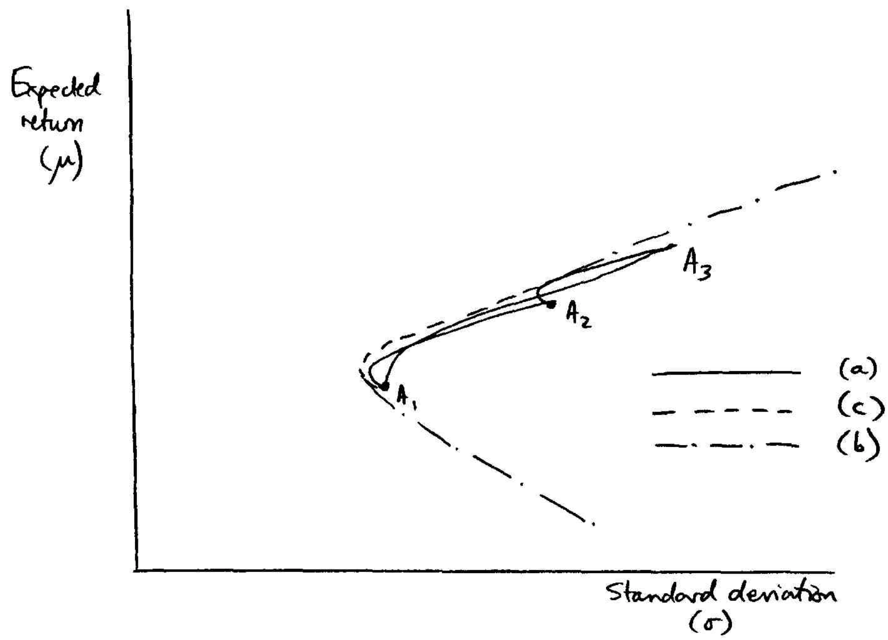
$$rV(t, S_t)dt = \left(-\frac{\partial f}{\partial t} - \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2}\right)dt$$

$$r\left(-f + \frac{\partial f_t}{\partial S_t} S_t\right)dt = \left(-\frac{\partial f}{\partial t} - \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2}\right)dt$$

$$-rf + r\Delta S_t = (-\theta - \tfrac{1}{2}\sigma^2 S_t^2 \Gamma)$$

$$\theta + rS\Delta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = rf$$

8 (i)



(ii) (a) All efficient portfolios are linear combinations of any two efficient portfolios in the unconstrained case.

To find corner portfolios we need combinations of the 2 portfolios on the efficient frontier such that the holding on an asset is zero (and other holdings positive).

Hence, need to find $\gamma_1, \gamma_2, \gamma_3$ such that

$$\gamma_1 0.986 + (1 - \gamma_1) \times (-.220) = 0$$
$$\gamma_2 0.043 + (1 - \gamma_2) \times (0.661) = 0$$
$$\gamma_3(-0.029) + (1 - \gamma_3) \times (0.559) = 0$$

to get corner portfolios

Hence $\gamma_1 = 0.183$
$\gamma_2 = 1.069 \rightarrow$ hence this leads to negative holdings in asset 3
$\gamma_3 = 0.951$

Two corner portfolios (with non-negative holdings) are:

$$.183\begin{bmatrix} .986 \\ .043 \\ -.029 \end{bmatrix} + (1-.183)\begin{bmatrix} -.220 \\ .661 \\ .559 \end{bmatrix} = \begin{bmatrix} 0 \\ .548 \\ .452 \end{bmatrix}$$

$$.951\begin{bmatrix} .986 \\ .043 \\ -.029 \end{bmatrix} + (1-.951)\begin{bmatrix} -.220 \\ .661 \\ .559 \end{bmatrix} = \begin{bmatrix} .927 \\ .073 \\ 0 \end{bmatrix}$$

Expected returns are 8.452% and 6.146%

(b) All portfolios of non-negative holdings are linear combinations of corner portfolios.

From (ii) (a) we know that the corner portfolio $P_C = \begin{bmatrix} 0.927 \\ 0.073 \\ 0 \end{bmatrix}$ has an expected return of 6.146% and we are looking for the minimum variance portfolio that has an expected return less than 6.146%. The adjacent corner portfolio with a lower expected return is $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, 100% investment in asset 1.

Consider linear combination of P_c and x_1 and let p be the amount invested in x_1 and denote the minimum variance portfolio by P.

$$P = px_1 + (1 - p)\, P_c.$$

This can be re-expressed as a linear combination of x_1 and x_2.

$$P = px_1 + (1-p)\left[0.927x_1 + .073x_2\right],$$

which can be re-expressed as

$$P = \hat{p}x_1 + (1-\hat{p})x_2,$$

where $\hat{p} = \left[p + (1-p)0.927\right]$ and $0.927 \le \hat{p} \le 1$.

[also note that it is a combination of x_1 and x_2 and should contain at least 92.7% of x_1]

$$V[P] = .07^2\,\hat{p}^2 + .15^2(1-\hat{p})^2 + 0.0042\,\hat{p}(1-\hat{p})$$

Therefore:

$$\frac{\partial V[P]}{\partial \hat{p}} = 2\times 0.07^2\,\hat{p} + 2(1-\hat{p})(-1)0.15^2 + 2(1-2\hat{p})\times 0.2\times 0.07\times 0.15$$

$\dfrac{\partial V[P]}{\partial \hat{p}} = 0.00464\hat{p} - 0.0408$, which is positive for $\hat{p}$ greater than 0.927. Hence volatility increases as $\hat{p}$ increases. Hence the minimum variance portfolio with expected return less than 6.146% is immediately adjacent to the corner portfolio, P_C.

$$P_{\min} = \begin{bmatrix} 0.927+\varepsilon \\ 0.073-\varepsilon \\ 0 \end{bmatrix} \text{ for small } \varepsilon.$$

Full marks awarded if minimum variance portfolio stated to be the corner portfolio itself ($P_{min} = P_C$), i.e. treating "less than" as "less than or equal to" is acceptable.