Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

September 2001

Subject 109 — Financial Economics

EXAMINERS' REPORT

1 (i) APT requires the returns on a stock to be linearly related to a set of factors

$$R_i = a_i + \sum_{j=1}^{n} b_{ij} \, I_j + c_i$$

R_i = return on security i

a_i = constant

c_i = random component of return unique to stock i

I_j = return on factor j

b_{ij} = sensitivity of security i to factor j

$E[c_i] = 0$

$E[c_i \, c_j] = 0 \qquad \forall \; i, j$ with $i \neq j$

$E[c_i(I_j - E[I_j])] = 0 \;\; \forall \; i, j$

The market is assumed to be perfectly competitive and frictionless.

When portfolios of stocks are formed the unique random components will be diversified away. This means that the portfolio returns will be dependent only on the systematic factors. In fact all diversified portfolios will lie on the hyperplane $E(R_p) = a_p + \sum_{j=1}^{n} b_{pj} \, E[I_j]$.

Consequently many portfolios will be close substitutes and as such must have the same value so as to prevent arbitrage. If the market is assumed to be in equilibrium, this insight allows individual securities to be priced.

(ii) Under APT

$$E[R_p] = a_p + \sum_{j=1}^{n} b_{pj} \, E[I_j]$$

By CAPM

$$E[R_p] = (E[R_m] - r) \, \frac{\sigma_{pm}}{\sigma_m} + r$$

R_m is the market portfolio

σ_{pm} is the covariance of the return on portfolio p with the returns on the market

σ_m is the s.d. on the return on the market

r is the risk free return

$R_m - r$ can be thought of as a factor, hence the CAPM is a one factor model with

$$\text{“}I_1\text{”} = R_m - r$$

$$b_{p1} = \frac{\sigma_{pm}}{\sigma_m^2}$$

and $a_p = r$, which is the same for all portfolios.

Hence CAPM is a one factor implication of the APT.

Important assumption: the market must itself be well-diversified.

A valid alternative is $I_1 = R_m$ with $a_p = r(1 - b_{p1})$, which is different for each portfolio, p.

2 (i) $P(\tau) = \exp[-D(\tau)r - (\tau - D(\tau))\,L - \frac{\beta}{2}\,D(\tau)^2]$

$$D(\tau) = \frac{1 - e^{-\alpha\tau}}{\alpha}$$

Spot rate $s(\tau)$

$$s(\tau) = -\,\frac{\ln P(\tau)}{\tau}$$

$$= \frac{r}{\tau}\,D(\tau) + \frac{L}{\tau}\,(\tau - D(\tau)) + \frac{\beta}{2\tau}\,D(\tau)^2$$ (*in the question paper, the final "+" sign appeared as a "–" symbol; this was taken into account by the examiners, and nocandidates lost marks as a consequence*)

(ii) $D(\tau) \doteqdot \frac{1}{\alpha}(1 - (1 - \alpha\tau + \frac{\alpha^2\tau^2}{2!} - \ldots)$

$$\therefore \quad \frac{D(\tau)}{\tau} \doteqdot \frac{1}{\alpha}\left(\alpha - \frac{\alpha^2\tau}{2!} + \frac{\alpha^3\tau^2}{3!}\ldots\right)$$

$\tau \to 0 \qquad \dfrac{D(\tau)}{\tau} \to 1 \qquad \dfrac{D(\tau)^2}{\tau} \to 0$

$\tau \to \infty \qquad \dfrac{D(\tau)}{\tau} \to 0$

$\therefore \qquad \lim\limits_{t \to 0} \quad s(\tau) = r$

$\lim\limits_{t \to \infty} \quad s(\tau) = L$

(iii) Forward $F(\tau)$

$$F(\tau) = -\frac{d}{d\tau} \ln P(\tau)$$

$$= +\frac{d}{d\tau}\left(rD(\tau) + L(\tau - D(\tau)) + \frac{\beta}{2} D(\tau)^2 \right)$$

$$= re^{-\alpha\tau} + L(1 - e^{-\alpha\tau}) + \frac{\beta}{2}\, 2D(\tau)e^{-\alpha\tau}$$

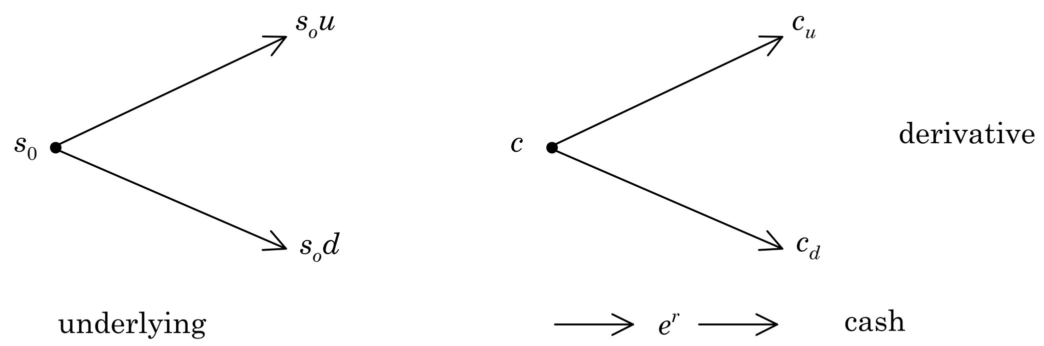
$\lim\limits_{\tau \to 0} \quad F(\tau) = r$

$\lim\limits_{t \to \infty} \quad F(\tau) = L \qquad$ i.e. the same as for the spot yield

3 (i) Replicating Portfolio — a portfolio that takes on the same value as the option whatever the future state of the world is.

(ii) Risk neutral probability measure — a measure in which the expected returns of all securities are the same. It is the way to calculate derivative prices such that there is no arbitrage between the derivative and the underlying.

(iii) (a)



ϕ stock and ψ cash

$$\left.\begin{aligned} s_o u\phi + \psi e^r &= c_u \\ s_o d\phi + \psi e^r &= c_d \end{aligned}\right\} \text{ simultaneous equations}$$

$$c = \phi s_0 + \psi = e^{-r}[qc_u + (1-q)\,c_d]\ ,\ q = \frac{e^r - d}{u-d}\,.$$

(b) Payoff is $|s_1 - s_0| \Rightarrow c_u = |s_0 u - s_0| = s_0(u-1)$

$$c_d = |s_0 d - s_0| = s_0(1-d)$$

$$\Rightarrow \qquad c = e^{-r}\left[\frac{e^r - d}{u-d}\,s_0(u-1) + \frac{u - e^r}{u-d}\,s_0(1-d)\right]$$

(c) This is in fact a long call and long put, struck at s_0, by inspection of the above price. Alternatively candidates may sketch the payoff function or describe it in words.

4 (i) (a) $\dfrac{dS}{S} = \alpha dt + \sigma dz$, where dz is a Brownian motion

(b) This is a random walk, with IID, normally distributed log returns.
α and σ are constants in this model, but there is some evidence that they vary in the real world.
Markets exhibit trending and mean reversion, jumps and fat tails that are inconsistent with a diffusion process and particularly a Brownian motion.

(ii) (a) $c = s_t\,\phi(d_1) - ke^{-r(T-t)}\,\phi(d_2)$

$$d_1 = \frac{\ln(s/k) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

$r = .05 \quad \sigma = 0.2 \quad T - t = 1 \quad s = k = 100$

$$d_1 = \frac{\ln(1) + \left(.05 + \dfrac{.2^2}{2}\right)1}{0.2} = \frac{.07}{0.2} = .35 \qquad \phi(d_1) = .63683$$

$$d_2 = 0.35 - 0.2 = 0.15 \qquad \phi(d_2) = .55962$$

$$c = 100\phi(0.35) - 100e^{-0.05}\phi(0.15)$$

$= 100(0.6368) - 100e^{-0.05}(0.5596)$

$= 100(0.6368 - 0.9512 \times 0.5596) = 100(0.1045)$

$= 10.45$p

(b) Differences from the theoretical price:

Buy/Sell spread — market maker makes a margin.

Odd results can arise from different views on implied volatility.

Market participants may make other adjustments to allow for the extent to which the BS assumptions and reality diverge.

(c) Price of 8p $\Rightarrow$ volatility must be lower than 20% p.a.

If volatility is 16%, then price of option is:

$$d_1 = \left(0.05 + \frac{0.0256}{2}\right) \Big/ 0.16 = 0.3925 \qquad \phi(d_1) = .6517$$

$$d_2 = 0.3925 - 0.16 = 0.2325 \qquad \phi(d_2) = .5910$$

$$\Rightarrow \text{Price} = 0.6517 - 0.9512 \times 0.5910 = 8.95\text{p}$$

If volatility is 13% then price of option is

$$d_1 = \left(0.05 + \frac{.0169}{2}\right) \Big/ .13 = .4496 \quad \phi(d_1) = .6736$$

$$d_2 = .4496 - .13 = .3196 \qquad \phi(d_2) = .6255$$

$$\Rightarrow \text{price} = .6736 - .9512 \times .6255$$

$$= 7.86\text{p}$$

Interpolating gives an implied volatility of 13.4% p.a.

5 (i) Strong form — all information is in the price, or the conditional distribution of future returns is unaffected by any research process.

Semi-strong — all publicly available information is in the price, but not insider information.

Weak — all information contained in the price history of the asset is reflected in the current price.

It is also acceptable to define the forms by stating that it is impossible to outperform the market in risk-adjusted terms given the levels of information.

(ii) Fundamental analysis is "semi-strong" play. There is publicly available information to be researched out, and some are better at finding it or using it than others. Fundamental analysis is only worthwhile if you believe the market to be semi-strong inefficient since all the information you can legally find and use is already priced otherwise. Note that it might be argued that making better use of the information costs more.

The fact that insider trading is illegal suggests that price sensitive information exists, but which is not yet embedded in the price of the asset. There is therefore a high likelihood that the market is not strong-form efficient. Note that it could be argued that because insider trading is illegal, the information cannot be acted on and so the market is made efficient by legislation.

If the price history can be used to provide better than average predictions of asset returns then the market is not even weak-form efficient.

6 **Utility Theory**

(i) *Axioms*

(a) 1. Comparability — investor can state a preference between all available certain outcomes

2. Transitivity $A > B, B > C \Rightarrow A > C$

3. Independence

If an investor is indifferent to certain outcomes A or B

Then indifferent to gambles pA or $(1-p)C$
and pB or $(1-p)C$

4. Certainty equivalence if $A > B$ and $B > C$

There is a unique probability p such that

pA or $(1 - p)$ C is equivalent to B

(b) Non-satiation: investors prefer more to less

$$\frac{du}{dw} > 0$$

Risk seeking: incremental increase in wealth more highly valued than incremental decrease, $d^2u/dw^2 > 0$ will seek a fair gamble.

(c) Insurance $\Rightarrow$ investors are prepared to overpay to avoid a large loss $\Rightarrow$ extremely risk averse for large losses.

Lottery $\Rightarrow$ investors are prepared to overpay to risk a large amount $\Rightarrow$ extremely risk seeking for large gains.

Note: this is inconsistent with a conventional utility curve.

(ii) (a) On an expected return basis,

$3 \times 0.2 + 1 \times 0.6 = 1.2 > 1$, the risk free rate,
so the project is worth doing in principle.

(b) Utility basis: invest proportion X_p in the project and remainder in risk-free asset.

$$E[U] = 0.2\,(\ln(3X_p + (1 - X_p))) + 0.6\,(\ln(X_p + (1 - X_p))) + 0.2\ln(1 - X_p)$$

$$= 0.2\ln[-2X^2 + X_p + 1]$$

$$\frac{\partial E[U]}{\partial X_p} = 0.2\frac{1}{(2X_p + 1)(-X_p + 1)}\,[-4X_p + 1] = 0 \text{ if } X_p = 0.25$$

7 (i) $r_p = X_A\, r_A + X_B\, r_B$

(ii) (a) $E[r_p] = X_A\, E[r_A] + X_B\, E[r_B]$

(b) We have assumed r_A and r_B are stationary,
and rebalancing at the start of each period.
In fact there is serial correlation in markets, so IID fails

(iii) $\sigma_p = (X_A^2\sigma_A^2 + X_B^2\,\sigma_B^2 + 2X_AX_B\sigma_A\sigma_B\rho_{AB})^{½}$

(iv) (a) $\sigma_p^2 = X_A^2\sigma_A^2 + X_B^2\,\sigma_B^2 + 2X_A X_B\sigma_A\sigma_B\rho_{AB}$

Substitute out X_B (= $1 - X_A$)

$$= X_A^2\sigma_A^2 + (1 - 2X_A + X_A^2)\,\sigma_B^2 + 2X_A\sigma_A\sigma_B\rho_{AB} - 2X_A^2\sigma_A\sigma_B\rho_{AB}$$

Minimise risk

where $\dfrac{\partial\sigma_p^2}{\partial < r_p >} = \dfrac{\partial\sigma_p^2}{\partial X_A}\dfrac{\partial X_A}{\partial < r_p >} = 0$

$$= \frac{2}{< r_A > - < r_B >}\;[X_A(\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho_{AB}) - (\sigma_B^2 - \sigma_A\sigma_B\rho_{AB})]$$

$$= 0 \text{ when } X_A = \frac{\sigma_B^2 - \sigma_A\sigma_B\rho_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B\rho_{AB}} = X_{A0}$$

$$X_{B0} = 1 - X_{A0}$$

Note to markers: many will probably proceed by finding $\dfrac{\partial\sigma_p^2}{\partial X_A}$.

This is acceptable.

(b) Zero risk

Now, consider with risk portfolio X_{A0} and X_{B0}.

Consider $\sigma_p^2 = X_{A0}^2\sigma_A^2 + X_{B0}^2\sigma_B^2 + 2X_{A0}X_{B0}\sigma_A\sigma_B\rho_{AB}$.

Complete the square = $(X_{A0}\sigma_A + X_{B0}\sigma_B)^2 + \delta$

$$\Rightarrow \sigma_p^2 = (X_{A0}\sigma_A + X_{B0}.\sigma_B)^2 + 2X_{A0}X_{B0}\sigma_A\sigma_B(\rho_{AB} - 1).$$

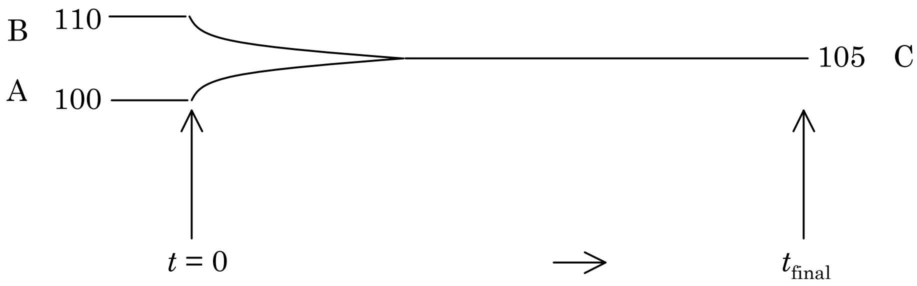
$$= 0 \text{ if } \frac{1}{1 - \rho_{AB}} = \frac{2X_{A0}X_{B0}\sigma_A\sigma_B}{(X_{A0}\sigma_A + X_{B0}\sigma_B)^2}$$

Either if ρ has a special value, or the sign of X_A is opposite to that of X_B.

For example $2X_{A(0)} = -X_{B(0)}$, $\tfrac{1}{2}\sigma_A = \sigma_B$, $\rho = 1$ (*)

$X_{A(0)} = X_{B(0)}$, $\sigma_A = \sigma_B$, $\rho = -1$

(v)



Go long 1 share of A, short 1 share of B. Wait until A turns to C, return C for short B. Profit is 10p. We are using the case (*) above. The dangers are in the assumptions of $\rho = 1$ and $\sigma_A = \sigma_B$. If new information emerges, this will alter the prices to the extent that profit might be eroded. The risk that the merger is called off means that the prices of A and B do not converge immediately.

8 (i) Consider asset a. Assume a normal distribution for $r_a - r_f$, with mean $\langle r_a \rangle - r_f$ and standard deviation of σ.

Chance of making a loss over one period is

$$= \frac{\int_{-\infty}^{r_f} \exp\{-((r_a - \langle r_a \rangle)/2\sigma)^2\} dr_a}{\int_{-\infty}^{\infty} \exp\{-((r_a - \langle r_a \rangle)/2\sigma)^2\} dr_a} = \Phi\left(\frac{r_f - \langle r_a \rangle}{\sigma}\right)$$

(ii) (a) CAPM states that stock specific risk is not rewarded, and therefore should be diversified away.

That is the stance of the index manager. Active managers believe that some parts of stock specific risk are researchable. If this is the case, then those who are better at research will outperform those who are poor.

Some investors will be less risk averse than others and will want to take on more systematic risk. The benchmark may simply be a reflection of how much systematic risk they want to take on.

(b) Tracking Error^2 = Var($R_{\text{fund}} - R_{\text{cash}}$)

$$= \text{Var}(R_{\text{fund}}) + \text{Var}(R_{\text{cash}}) + 2\ \text{Cov}(R_{\text{fund}}, R_{\text{cash}})$$

$$= \text{Var}(R_{\text{fund}}) + 0 + 0$$

Hence TE = SD(R_{fund})

(iii) (a) Outperformance $= r_p - r_{bm}$

$$= Xr_{\text{corp}} + (1 - X)\, r_{\text{gilt}} - r_{\text{gilt}}$$

$$= Xr_{\text{corp}} + r_{\text{gilt}} - Xr_{\text{gilt}} - r_{\text{gilt}}$$

$$= X(r_{\text{corp}} - r_{\text{gilt}})$$

TE $= \text{X.SD}\{r_{\text{corp}} - r_{\text{gilt}}\}$

$= X\{\sigma^2_{\text{corp}} + \sigma^2_{\text{gilt}} - 2\sigma_{\text{corp}}\, \sigma_{\text{gilt}}\, \rho_{\text{corp gilt}}\}^{½}$ i.e. linear in X.

(b) Expected relative return $= X \times (.06 - .05) = .01 \times X$

Tracking Error $= X\,(.12^2 + .08^2 - 2 \times .9 \times .08 \times .12)^{½}$

$= 0.0593 \times X$

$\Pr[R < -.01] \leq .05$

$$\Rightarrow \quad \Pr\left[Z < \frac{-.01 - .01 \times X}{.0593 \times X}\right] \leq .05$$

$$\Rightarrow \quad \frac{.01 + .01 \times X}{.0593 \times X} \geq 1.645$$

$$\Rightarrow \quad X \leq .1142$$