Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

April 2004

Subject 109 — Financial Economics

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

J Curtis
Chairman of the Board of Examiners

22 June 2004

1 Let A and B be two investment portfolios.

Portfolio A will dominate B if:

- the mean return of A is greater than the mean return of B, and

  $\int_a^x \int_a^t (F_A(y) - F_B(y))\, dy\, dt \leq 0$ for all x. The strict inequality must hold for some value of x between a and b where b is the largest return that could be provided by either portfolio and a is the smallest return that could be provided by either portfolio.

$F_A(y)$ and $F_B(y)$ are the cumulative probability distribution functions of A and B respectively.

Conditions:

investors exhibit decreasing absolute risk aversion
investors are risk averse
investors prefer more to less

2 (a) Strong form — all information is in the price.

(b) Semi-strong — all publicly available information is in the price but not insider information.

(c) Weak form — all information contained in the price history is contained in the price.

Technical analysis — where price history is used make trading decisions.

If this produces excess returns then the market is not weak form efficient.

Fundamental analysis — Analysis of all publicly available information (balance sheets, company strategy…) to make trading decisions.

If this produces excess returns then the market is not semi-strong form efficient.

Insider trading — illegal in UK and involves trading on the basis of information that has not been published or known to the public.

If this produces excess returns then the market is not strong form efficient.

3 (i) Variance of return

$$\int_{-\infty}^{\infty} (\mu - x)^2 f(x)dx$$

Semi variance of return

$$\int_{-\infty}^{\mu} (\mu - x)^2 f(x)dx$$

Expected shortfall

$$\int_{-\infty}^{L} (L - x) f(x)dx$$

where L is the chosen benchmark level, μ = mean and $f(x)$ is the probability density function.

(ii) The investor's utility function will be quadratic below the level of expected return and linear above it.

4 (i) Equity return

$$\frac{D(t+1).\left(1+\frac{1}{Y(t+1)}\right)}{D(t)/Y(t)} = Y(t)\left(\frac{1}{Y/(t+1)}+1\right)\exp(K(t+1))$$

Index linked gilts return

$$\frac{Q(t+1)}{Q(t)/R(t)}\left(1+\frac{1}{R(t+1)}\right) = R(t)\left(\frac{1}{R(t+1)}+1\right)\exp(I(t+1))$$

(ii) The equity risk premium is defined as the conditional expectation of the log relative return on equities and index linked gilts, conditional on the state vector $U(t)$.

Thus

$$\mathbf{E}\left(\log\left(\frac{Y(t)\left(\frac{1}{Y(t+1)}+1\right)\exp(K(t+1))}{R(t)\left(\frac{1}{R(t+1)}+1\right)\exp(I(t+1))}\right)\middle| U(t)\right)$$

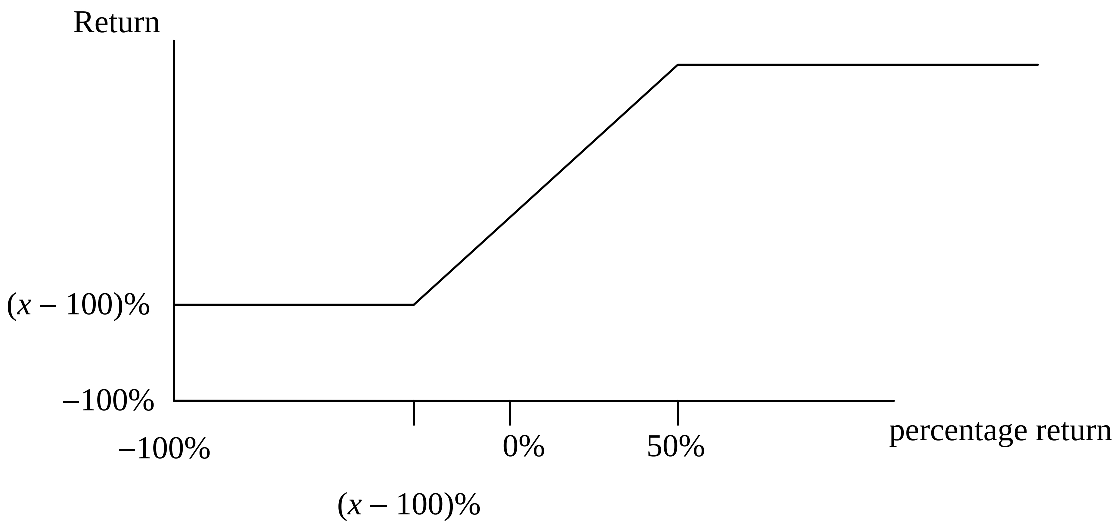
where $U(t)$ is the state vector.

$$= \log\left(\frac{Y(t)}{R(t)}\right) + \mathbf{E}\left(\log\left(\frac{1}{Y(t+1)}+1\right) - \log\left(\frac{1}{R(t+1)}+1\right) + K(t+1) - I(t+1) \,\middle|\, U(t)\right)$$

(iii) If markets are not efficient, excess profits could be earned by holding equities when risk premium is high (in the absence of changes in the level of risk) and holding index linked gilts otherwise because a high risk premium means equities are expected to out perform.

For the expression above to represent an efficient market, its value must be (approximately) constant. For example, the difference in yields $\log\left(\frac{Y(t)}{R(t)}\right)$ must be roughly equal to the difference in expected growth rates $K(t) - I(t)$ (adjusted for a constant premium).

5 (i) The payoff function from the bond is as follows:



Consider £1 invested in this bond, this is used to purchase £1 of the FTSE 100 index, a put option with exercise price $\frac{£x}{100}$ and sell a call option with exercise price £1.50. (Other strategies are also possible that achieve the same payoffs.)

The value of the call option is

$$d_1 = \frac{\left(\ln\left(\frac{1}{1.5}\right) + (0.04 + ½0.3^2)3\right)}{0.3\sqrt{3}} = –0.28957$$

$$d_2 = –0.28957 – 0.3\sqrt{3} = –0.80919$$

$$c = 100 \times \Phi(–0.28957) – 150e^{–0.04\times 3}\,\Phi(–0.80919)$$

$$= 100 \times 0.38607 – 150e^{–0.04\times 3} \times 0.20920$$

$$c = 10.775$$

Let p_3 denote the value of the put option

$\Rightarrow$ £1 = £1 – 10.775 + p_3

$\Rightarrow p_3 = 10.775$

Using x = 80 gives, value of put option

$$d_1 = \frac{\left(\ln\left(\frac{1}{0.80}\right) + \left(0.04 + \frac{0.3^2}{2}\right)3\right)}{0.3\sqrt{3}} = 0.92019$$

$$d_2 = 0.92019 – 0.3\sqrt{3} = 0.40057$$

Value of put option

$$p_t = 0.80e^{–0.04\times 3}\,\Phi(–0.40057) – 100\Phi(–0.92019)$$

$$= 80 \times 0.88692 \times 0.34437 – 100 \times 0.17874$$

$$= 6.56$$

Using x = 100 gives, the value of put option

$$d_1 = \frac{\left(\ln\left(\frac{1}{1}\right) + \left(0.04 + \frac{0.3^2}{2}\right)3\right)}{0.3\sqrt{3}} = 0.49075$$

$$d_2 = 0.49075 – 0.3\sqrt{3} = –0.02887$$

Value of put option

$$p_t = 100e^{-0.04\times 3}\Phi(+0.02887) - 100\Phi(-0.49075)$$

$$= 100 \times 0.88692 \times 0.51151 - 100 \times 0.311802$$

$$= 14.186$$

Interpolating $\frac{14.186 - 10.775}{14.186 - 6.56} \times 20 = 8.94$

$\Rightarrow x = 100 - 8.94 = 91$

Therefore $x = 91\%$.

(ii) The Building Society can construct a hedging portfolio that replicates the payoff from the three year bond.

The existence of the hedging portfolio means that the Building Society's position is protected no matter what the return on the FTSE 100 Share Index is.

Therefore, an assumption about the future growth of the FTSE 100 Share Index is not needed.

6 (i) Market portfolio has expected return of

$$\frac{5\% \times 250,000 + 7\% \times 250,000 + 9\% \times 750,000 + 10\% \times 50,000}{(250,000 + 250,000 + 750,000 + 50,000)}$$

$$= 7.88\%$$

$\Rightarrow$ Require $x\% \times 7.88\% + (1 - x\%) \times 3\% = 6\%$

$\Rightarrow$ $x = 61.5\%$

So the investor holds 38.5% of risk free asset and 61.5% of market portfolio, where constituents are held in proportion to the market capitalisation.

(ii) The mathematical form of the CAPM is

$$E[R_p] = \beta_p[E[R_m] - r_f] + r_f \text{ for each asset } p$$

Returns will vary because of

(a) randomness around expectation ($E[R_p]$)

(b) changes in expected returns (betas) of assets (β_p)

(c) errors in estimating expected returns (betas)

(d) changes in expected market returns $E[R_m]$

7 (i) Let α and β be the parameters of model

τ = term to maturity

$P(\tau)$ = price of zero coupon bond with τ years to maturity

R = short rate

L = long rate

$$P(\tau) = \exp(-D(\tau)R - (\tau - D(\tau))\, L - \frac{\beta}{2} D(\tau)^2)$$

where $D(\tau) = \dfrac{1 - e^{-\alpha\tau}}{\alpha}$ is the modified duration

(ii) The model:

- is arbitrage free provided L is constant
- does not prescribe the short rate process.
- is tractable, i.e. allows closed form analytical solutions to a wide range of derivatives
- encompasses mean reversion
- allows for a wide range of yield curves

(iii) The model:

- does not prevent negative interest rates.
- is difficult to use to obtain humped yield curves.
- has a lack of time dependence of parameters which is not compatible with empirical evidence.

- gives, in long run, spot rates normally distributed not compatible with empirical evidence.

- implies perfect instantaneous correlation of bond prices, which is not compatible with empirical evidence.

- as the yield curve evolves the model will need to be reparameterised

(iv) $$s(\tau) = \frac{-\ln(P(t))}{\tau}$$

$$= \frac{R}{\tau}D(\tau) + \frac{L}{\tau}(\tau - D(\tau)) + \frac{\beta}{2\tau}D(\tau)^2$$

$$D(\tau) = \frac{1-e^{-\alpha t}}{\alpha}$$

$$= \frac{1}{\alpha}\left(1-\left(1-\alpha\tau+\frac{\alpha^2\tau^2}{2!}-...\right)\right)$$

$$\Rightarrow \quad \frac{D(\tau)}{\tau} = \frac{1}{\alpha}\left(\alpha-\frac{\alpha^2\tau}{2!}+...\right)$$

$$\Rightarrow \quad \lim_{\tau\to 0} D(\tau) = 0$$

$$\lim_{\tau\to 0}\frac{D(\tau)}{\tau} = 1$$

$$\lim_{\tau\to 0}\frac{D(\tau)^2}{\tau} = 0$$

so $\lim_{\tau\to 0} s(\tau) = R$

8 (i) Lagrangian function, W

$$W = V - \lambda(E - 0.07) - \mu(\Sigma_i x_i - 1)$$

where λ and μ are Lagrangian parameters

$E = \Sigma x_i E_i$
$V = \Sigma_i \Sigma_j x_i x_j C_{ij}$
E_i = expected return on asset i
C_{ij} = covariance between i and j
x_i = proportion invested in i

(ii) $$\frac{\partial W}{\partial x_i} = 2\Sigma_j x_j C_{ij} - \lambda E_i - \mu$$

$$\frac{\partial W}{\partial \lambda} = -(\Sigma_i E_i x_i - 0.07)$$

$$\frac{\partial W}{\partial \mu} = -(\Sigma_i x_i - 1)$$

Set equal to zero:

(1) $x_1 6\% + x_2 7\% + x_3 8\% = 7\%$

(2) $x_1 + x_2 + x_3 = 1$

(3) $2(x_1(5\%)^2 + x_2 \times 0.5 \times 5\% \times 15\% + x_3 \times 0.5 \times 5\% \times 20\%)$
$= \lambda 6\% + \mu$

(4) $2(x_1 \times 0.5 \times 5\% \times 15\% + x_2 \times (15\%)^2 + x^3 \times 0.5 \times 15\% \times 20\%)$
$= \lambda 7\% + \mu$

(5) $2(x_1 \times 0.5 \times 5\% \times 20\% + x_2 \times 0.5 \times 15\% \times 20\% + x_3(20\%)^2)$
$= \lambda 8\% + \mu$

(iii) Corner portfolio where $x_1 = 0$. Equations become

$1 = x_2 + x_3$

(1) $2\sigma_1(\rho\sigma_2 x_2 + \rho\sigma_3 x_3) - \lambda E_1 - \mu = 0$

(2) $2\sigma_2(\sigma_2 x_2 + \rho\sigma_3 x_3) - \lambda E_2 - \mu = 0$

(3) $2\sigma_3(\rho\sigma_2 x_2 + \sigma_3 x_3) - \lambda E_3 - \mu = 0$

becomes

(1) $2 \times 0.05(0.5 \times 0.15x_2 + 0.5 \times 0.2 \times (1 - x_2)) - \lambda 6\% - \mu = 0$
$= -0.0025x_2 + 0.01 - \lambda 6\% - \mu = 0$

(2) $2 \times 0.15(0.15x_2 + 0.5 \times 0.2(1 - x_2)) - \lambda 7\% - \mu = 0$
$= 0.015x_2 + 0.03 - \lambda 7\% - \mu = 0$

(3) $2 \times 0.2(0.5 \times 0.15x_2 + 0.2(1 - x_2)) - \lambda 8\% - \mu = 0$
$-0.05x_2 + 0.08 - \lambda 8\% - \mu = 0$

(1) – (2) gives

† $-0.0025x_2 + 0.01 - \lambda 6\% - 0.015x_2 - 0.03 + \lambda 7\% = 0$

(1) – (3) gives

‡ $-0.0025x_2 + 0.01 - \lambda 6\% + 0.05x_2 - 0.08 + \lambda 8\% = 0$

† $\Rightarrow \lambda 1\% - 0.02 - 0.0175x_2 = 0$

‡ $\Rightarrow 2\%\lambda + 0.0475x_2 - 0.07 = 0$

$\Rightarrow$ $0.0475x_2 - 0.07 + 0.04 + 2 \times 0.0175x_2 = 0$

$\Rightarrow$ $0.0825x_2 = 0.03$

$x_2 = 36.36\%$

$\Rightarrow$ $x_3 = 63.64\%$

9 (i)

- Comparability — investors can state a preference between all available certain outcomes.

- Transitivity — *A* is preferred to *B*, *B* is preferred to *C* then *A* is preferred to *C*.

- Independence — If an investor is indifferent to certain outcomes *A* or *B* then they are indifferent to gambles

  A with probability *p* and *C* with probability $(1 - p)$ and
  B with probability *p* and *C* with probability $(1 - p)$

- Certainty equivalence — If *A* is preferred to *B* and *B* is preferred to *C* there is a unique probability *p* such that the investor is indifferent between *B* and a gamble giving *A* with probability *p* and *C* with probability $(1 - p)$.

(ii) Non satiation: prefer more to less

$$\frac{du}{dw} > 0.$$

Risk seeking: incremental increase in wealth more highly valued than incremental decrease

$$\frac{d^2u}{dw^2} > 0.$$

(iii) (a) For risk aversion, require $U''(w) < 0$

$$U'(w) = b + 2cw$$

$$U''(w) = 2c$$

$$\Rightarrow \quad c < 0$$

(b) For non satiation, $U'(w) > 0$

$$\Rightarrow \quad b + 2cw > 0$$

$$\Rightarrow \quad 2cw > -b$$

$$\Rightarrow \quad w < \frac{-b}{2c} \qquad (\text{as } c < 0)$$

(iv) Absolute risk aversion

$$= \frac{-(-\tfrac{1}{2})w^{-3/2}}{w^{-1/2}} = \frac{1}{2w}$$

$\Rightarrow$ Decreasing absolute risk aversion.

This is desirable as an investor will invest a larger absolute amount in a risky asset as their wealth increases.

10 (i) The payoff function can be written as the sum of two payoff functions, as follows:

$$g(S_t) = \begin{cases} 0 & S_t < 100 \\ S_t - 100 & \text{otherwise} \end{cases}$$

$$f(S_t) = \begin{cases} 0 & S_t < 200 \\ S_t - 200 & \text{otherwise} \end{cases}$$

The payoff function for the special call option is:

$$g(S_t) + \tfrac{1}{2}f(S_t) = \begin{cases} 0 & S_t \le 100 \\ S_t - 100 & 100 < S_t < 200 \\ 1.5S_t - 200 & 200 \le S_t \end{cases}$$

Therefore, the special call option is equivalent to holding a portfolio of a call option with exercise price 100 and half a call option with exercise price 200.

(ii) The above relationship shows that the value of the special call option is the sum value of the two call options with payoff functions *g*(.) and *f*(.) (using no arbitrage arguments).

The value of a call option with exercise price 100 is

v_1 =

$$120\Phi\left(\frac{½(5\% + ½0.25) + \ln\left(\frac{120}{100}\right)}{\sqrt{0.25 \times ½}}\right) - 100 \times \Phi\left(\frac{\ln\left(\frac{120}{100}\right) + ½(5\% - ½0.25)}{\sqrt{0.25 \times ½}}\right) e^{-0.05½}$$

= 120Φ(0.76317) – 100Φ(0.40962) × 0.97531

= 120 × 0.77732 – 100 × 0.65896 × 0.97531

= 29.01

[Note σ = 0.5, giving σ^2 = 0.25.]

The value of a call option with exercise price 200 is

v_2 =

$$120\Phi\left(\frac{\ln\left(\frac{120}{200}\right) + ½(½ \times 0.25 + 5\%)}{\sqrt{0.25 \times ½}}\right) - 200\Phi\left(\frac{\ln\left(\frac{120}{200}\right) + ½(5\% - ½ \times 0.25)}{\sqrt{½ \times 0.25}}\right) e^{-0.05½}$$

= 120Φ(–1.19735) – 200Φ(–1.55100) × 0.97531

= 120(1 – Φ(1.19735)) – 200(1 – Φ(1.55100)) × 0.97531

= 120(1 – 0.88441) – 200(1 – 0.93957) × 0.97531

= 2.08

⇒ Value of special call option is 29.01 + ½2.08 = 30.05.

(iii) The delta of a vanilla call option is $\frac{\partial v_1}{\partial s} = \Phi(d_1)$.

The delta of the special option is the weighted sum of the deltas of the two call options in the portfolio

At S_0 = 150:

$$\Phi\left(\frac{\ln\left(\frac{150}{100}\right)+(5\%+½0.25)½}{\sqrt{½\times 0.25}}\right)+½\Phi\left(\frac{\ln\left(\frac{150}{200}\right)+(5\%+½0.25)½}{\sqrt{½\times 0.25}}\right)$$

$= \Phi(1.39431) + ½\Phi(–0.56620)$

$= 0.918389 + ½0.28563$

$= 1.0612$

At S_0 = 250:

$$\Phi\left(\frac{\ln\left(\frac{250}{100}\right)+(5\%+½0.25)½}{\sqrt{½0.25}}\right)+½\Phi\left(\frac{\ln\left(\frac{250}{200}\right)+(5\%+½0.25)½}{\sqrt{½0.25}}\right)$$

$= \Phi(2.83915) + ½\Phi(0.87863)$

$= 0.99774 + ½(0.81020)$

$= 1.4028$

(iv) Consider the increase in delta from S_0 = 150 to S_0 = 250

$$\frac{1.4028}{1.0612} = 1.32$$

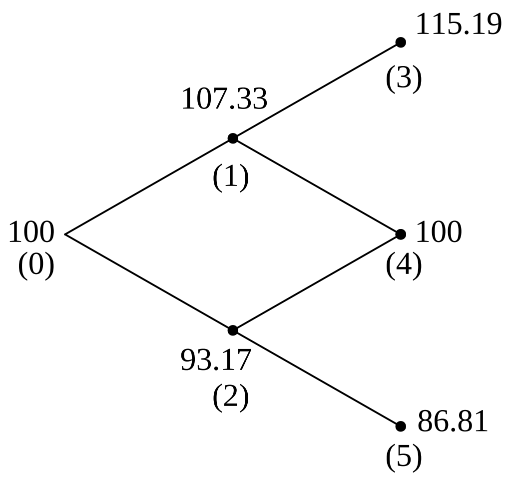
Compare this to a vanilla call option with exercise price 100

$$\frac{0.99774}{0.91839} = 1.086$$

The gamma for the special call option is higher than for a vanilla call option. Therefore, the hedging portfolio for the special call option is likely to require more rebalancing as the underlying share price rises than the hedging portfolio of a simple call option with exercise price 100.

11 (i) $u = \exp\left(\frac{0.1}{\sqrt{2}}\right) = 1.073271.$

The binomial tree of total shareholder return is



The risk neutral probabilities are

$$d = \frac{1}{u} = 0.93173$$

$$q = \frac{e^{0.03\times½} - 0.93173}{1.073271 - 0.93173} = 0.58911$$

and 1 – 0.58911 = 0.41089

(ii) Value at node (1) $= e^{-0.03\times½}(0.58911\times(115.19-90)+(1-0.58911)\times(100-90))$

$= 18.6665$

Value at node (2) $= e^{-0.03\times½}(0.58911 \times (100 - 90))$

$= 5.8034$

Value at node (0) $= e^{-0.03\times½}(0.58911 \times 18.6665 + (1 - 0.58911) \times 5.8034)$

$= 13.1820$

(iii) (a) and (b)

Node	*Total shareholder return*	*Share price*	*Payoff*
(3)	115.19	95.19	5.19
(4)	100	80	0
(5)	86.81	66.81	0

(iv) Value of option at node (1) is

$$e^{-0.03\times½}(0.58911 \times (95.19 - 90))$$

$$= 3.012$$

(v) (a) Payoff at node (1) is 107.37 – 90 = 17.37

(b) The holder of an American call option should exercise it at node (1) since the payoff from doing so exceeds the value of holding the option.

(c) The value of the American option at time 0 is strictly greater than the value of the European option. This is because exercising at node (1) increases the value delivered to the option holder. Therefore, this discounted value of the payoff must be increased.

(vi) Black's approximation involves calculating the prices of European options that are exercisable at the final expiry date *T* and the final ex dividend date and setting the American price equal to the greater of the two.

END OF EXAMINERS' REPORT