Faculty of Actuaries Institute of Actuaries

EXAMINATIONS

April 2001

Subject 109 — Financial Economics

EXAMINERS' REPORT

1 (i) For a utility function $U(w)$, where w is the wealth.

(a) The absolute risk aversion is measured by

$$A(w) = -\frac{U''(w)}{U'(w)}$$

(b) Relative risk aversion is measured by

$$R(w) = -w\frac{U''(w)}{U'(w)}$$

(ii) $U(w) = a + bw + cw^2$

This can be re-written as

$$U(w) = w + dw^2 \quad \text{where } d = \frac{c}{b} < 0$$

as multiplying by a constant or adding a constant does not alter decision

$$U'(w) = 1 + 2dw$$

$$U''(w) = 2d$$

$$\therefore\ A(w) = \frac{-2d}{1+2dw}$$

$$A'(w) = \frac{(2d)^2}{(1+2dw)^2} > 0 \Rightarrow \text{increasing absolute risk aversion}$$

$$R(w) = -\frac{2dw}{1+2dw} \rightarrow R'(w) = \frac{-2d}{(1+2dw)^2}$$

$R'(w) \geq 0 \Rightarrow$ increasing relative risk aversion

If $U(w) = \ln w$

$$U'(w) = \frac{1}{w}$$

$$U''(w) = -\frac{1}{w^2}$$

$$A(w) = \frac{1}{w} \Rightarrow A'(w) = -\frac{1}{w^2} < 0$$

$\Rightarrow$ declining absolute risk aversion

$R(w) \quad = 1 \quad \Rightarrow$ constant relative risk aversion

(iii) (a) Let x be the loss.

Hence equivalence of

$$E[u(100 - x)] = u[94.5]$$

for $u(w) = w + dw^2$ we have

$$100 - E(x) + d(100^2 - 2E(x)100 + E(x^2)) = 94.5 + d94.5^2$$

$$d = \frac{94.5 - 100 + E(x)}{100^2 - 2E(x).100 + E(x^2) - 94.5^2} = \frac{-0.5}{94.75}$$

$d = -0.002567$

(b) For non-satiation

$$-\infty < w < -\frac{1}{2d} = 194.78$$

Therefore utility function cannot help for wealth in excess of 194.78. Can not use the utility function.

(c) Quadratic utility function is considered inappropriate as

- only demonstrates non-satiation over a limited range
- has increasing absolute and relative risk aversion
- not supported by empirical evidence

2 Greeks

(i) Greeks are differentials of the price of an option with respect to different underlying variables needed to calculate the price.

$$\Delta = \frac{\partial f}{\partial s} \qquad \gamma = \frac{\partial^2 f}{\partial s^2} \qquad \kappa = \frac{\partial f}{\partial \sigma} \qquad \rho = \frac{\partial f}{\partial r} \qquad \theta = \frac{\partial f}{\partial t}$$

where s is the price
f is the value of the derivative
σ is the volatility
r is the interest rate
t is time

(ii) (a) $$p + \Delta p = 25.04 + \frac{\partial f}{\partial r}\Delta r + \frac{\partial f}{\partial \sigma}\Delta\sigma + \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial s}\Delta s + \tfrac{1}{2}\frac{\partial^2 f}{\partial s^2}\Delta s^2 = 32.26p$$

(b) Cross-terms have been ignored.

3 CAPM

(i) $\sigma_p^2 = \beta^2\sigma_m^2 + \sigma_{ss}^2$

Systematic risk, that part of portfolio variance attributable to correlation with the market.

Specific risk, that part of portfolio variance attributable to individual stocks. Specific risk can be diversified, i.e. goes to zero in a diversified portfolio.

(ii) $$\sigma^2 = \sum_i \sum_j w_i^2 \sigma_{ij} = \sum_i w_i^2 \sigma_i^2 = \frac{1}{n^2}\sum_i \sigma^2 = \frac{n\bar{\sigma}^2}{n^2} = \frac{\bar{\sigma}^2}{n} \Rightarrow = \sigma \ \frac{\bar{\sigma}}{\sqrt{n}}$$

[*there was a typographical error in part (ii) of this question: the word "increases" should have read "decreases". This was taken into account by the examiners in the marking of the papers*.]

(iii) If correlated

$$\sigma^2 = \sum_i \sum_j w_i^2 \sigma_{ij} = \sum_i w_i^2 \sigma_i^2 + \sum_i \sum_{i \neq j} w_i w_j \sigma_{ij}$$

$$= \frac{1}{n}\bar{\sigma}^2 + \overline{\sigma_{ij}} \sum_i \sum_{i \neq j} \frac{1}{n^2}$$

$$= \frac{1}{n}\bar{\sigma}^2 + f\bar{\sigma}^2 \frac{1}{n^2}\sum_i \sum_{i \neq j} 1$$

$$= \frac{1}{n}\sigma^2 + f\bar{\sigma}^2 \frac{1}{n^2}(n^2 - n)$$

$$= \frac{1-f}{n}\bar{\sigma}^2 + f\bar{\sigma}^2$$

where $\lim_{n \to \infty} \sum_{i \neq j} \sigma_{ij} = f\bar{\sigma}^2$

(iv) A broad spectrum of UK stocks is uncorrelated.

A narrow theme is highly correlated.

The risk in the portfolio has risen.

[Marks were awarded for other reasonable and reasoned suggestions]

4 (i) The Wilkie model has been described as a cascade or hierarchical model, with inflation being the key model component. Variables such as dividend yield and growth and interest rates are affected not only by current shocks in the inflation model, but also moving averages of past inflation.

There is no feedback mechanism for shocks in the interest rate process, or any other process, to affect future inflation.

(ii) (a) $I_\infty = a + bI_\infty$

$$\Rightarrow I_\infty = \frac{a}{1-b}$$

(b) AR(1) $\Rightarrow$ inflation tends to fall after a period of high inflation, and tends to rise after a period of below average inflation.

Inflation is an important economic indicator, and in the UK and other developed economies, treasuries and central bankers often attempt to stabilise inflation around a target rate or with a specified range. As inflation increases, for example, interest rates may be raised to try to encourage price rises to slow down or even fall.

If inflation were not "pulled back" to some "reasonable" level, then market economics would collapse, e.g. Germany during the inter war period.

(c) Comments on:

- predictable (on average) prices
  $\Rightarrow$ high returns with little risk
  (not observed in the real world)

- no mechanism in a market of mildly rational investors for driving down prices

- lots of evidence of non-normality and "jumps" in share price movements

- share prices have trended upwards over time, the AR(1) process is a stationary one

5 (i) By put-call parity we know that

$$c_t + ke^{-r(T-t)} = p_t + s_t$$

$$\therefore \; p_t = 187.06 + 5{,}250e^{-0.05\times.5} - 5{,}000$$

$$= 307.44$$

(ii) Implied volatility

$$c_t = s_t\phi(d_1) - ke^{-r(T-t)}\,\phi(d_2)$$

$$d_1 = \frac{\ln\dfrac{s_t}{k} + (r + \tfrac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\text{if} \quad \sigma = 0.15 \Rightarrow c_t = 158.96\begin{cases} d_1 = -.1713 \Rightarrow \phi(d_1) = .4320 \\ d_2 = -.2773 \Rightarrow \phi(d_2) = .3908 \end{cases}$$

$$\sigma = 0.18 \Rightarrow c_t = 200.72\begin{cases} d_1 = -.1233 \Rightarrow \phi(d_1) = .4509 \\ d_2 = -.2506 \Rightarrow \phi(d_2) = .4011 \end{cases}$$

$k = 5{,}250 \quad s_t = 5{,}000 \quad r = .05 \quad ke^{-r(T-t)} = 5120.3769$

linear interpolation $\Rightarrow \sigma = 0.17019$

$$\Rightarrow c_t - 187.03$$

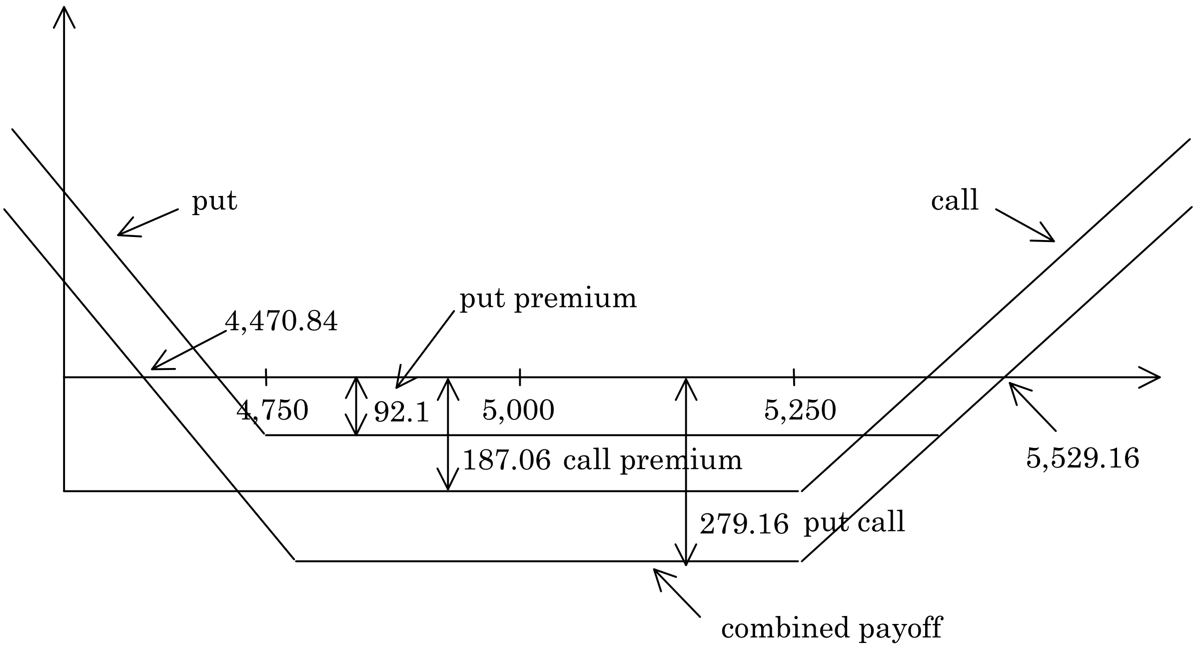
say $\sigma = 0.17$ $\Rightarrow c_t = 187.06 \begin{cases} d_1 = .1378 \Rightarrow \phi(d_1) = .4452 \\ d_2 = .2580 \Rightarrow \phi(d_2) = .3982 \end{cases}$

Can therefore price a call with strike of 4,750 and volatility = 0.17 using formula in (ii)

$$c_t\ (k = 4{,}750) = 459.38 \begin{cases} d_1 = .6948\ \phi(d_1) = .7564 \\ d_2 = .5746\ \phi(d_2) = .7172 \end{cases}$$

$$p_t\ (k = 4{,}750) = 92.10$$

$k = 4{,}750$ $s_t = 5{,}000$ $r = .05$ $ke^{-r(T-t)} = 4632.722$



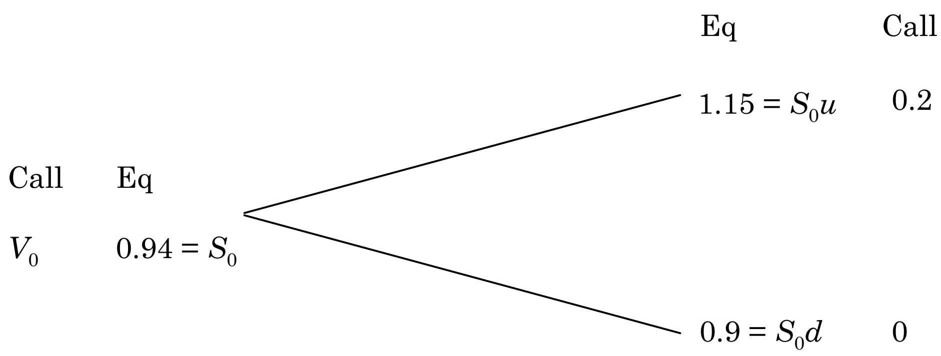
(iii) Main reason is the volatility smile, or how volatility depends on the strike price.

Other specifications in the B-S model may cause market price to differ from theoretical price.

Arguably the volatility curve is the market's correction for the B-S implications.

6 (i)

	Good	*Bad*
Equity	1.15	0.9
Bond	1.00	1.00
Probability	0.6	0.4



Assume hold ϕ units of stock and
ψ units of the bond

$V_1 = \phi\, 1.15 + \psi$ if good state

$V_1 = \phi\, .9 + \psi$ if bad state

where V_1 is the price of the call

$\therefore \quad .2 = \phi \times 1.15 + \psi$

$\quad\; 0 = \phi \times .9 + \psi$

$\Rightarrow \quad .2 = \phi \times 1.15 + (-\phi \times .9)$

$\quad\quad = .25\phi$

$\Rightarrow \quad \phi = 0.8$

$\quad\; \psi = -0.72$

$\Rightarrow \quad V_0 = .94[\phi + \psi] = .0752$

(ii) (a) We know $P_t K_t = E_t[P_s K_s]$

In this case $s = t + 1$

For the bond

$$.94\,K_0 = E[1\,.\,K_{t+1}]$$

$$= 0.6K_g + 0.4K_b$$

where K_g is the kernel at $t = 1$ in the good state of the economy and K_b the kernel at $t = 1$ in the bad state.

For the equity

$$.94\,K_0 = E[\text{Equity Price}\,.\,K_{t+1}]$$

$$.94\,K_0 = 1.15 \times K_g \times .6 + 0.9K_b \times .4$$

without loss of generality

set $K_0 = 1$, then

$$.94 = 1.15\,K_g \times .6 + 0.9 + (.94 - 0.6K_g)$$

$$0.094 = .15K_g \Rightarrow K_g = 0.6267$$

$$\therefore \quad K_b = 1.41$$

(b) Strike price is .95.

Option pays out .20 in good state

0 in bad state

$$P_{\text{option}} = E[kP_{\text{option}}(1)]$$

$$= .6 \times k_1 \times .20 + .4 \times k_2\,0$$

$$= 0.0752$$

7 (i) Cov (A, B) = $\sigma_{A,B} = E[(A - E(A))\,(B - E(B))]$

$$E(A) = 0.2 \times .18 + 0.3 \times .11 = 6.9\%$$

$$E(B) = 0.2 \times .13 + 0.3 \times .06 = 4.4\%$$

$$\sigma_{AB} = 0.2(.031 \times (-.064) + (0.011 \times 0.106)$$

$$+ (0.3(.181 \times (-0.044) + (-.209) \times 0.016)$$

$$= -0.001636 - .003392$$

$$= -0.003556$$

$$\sigma_A^2 = E[(A - E(A))^2] = E[A^2] - E[A]^2$$

$$= 0.2(.1^2 + .08^2) + .3(.25^2 + .14^2) - 0.069^2 = .00328 + .02463 - .004761$$

$$= .023149$$

$$\sigma_B^2 = 0.2(.02^2 + .15^2) + 0.3\,(0^2 + .06^2) - .044^2$$

$$= .00458 + .00108 - .001936$$

$$= .003724$$

$$\text{Corr(A, B)} = \rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B} = \frac{-.003556}{.152148 \times .061025}$$

$$= \underline{\underline{-.383}}$$

(ii) Assume proportion "a" of assets are in asset A.

Let Portfolio be $P = a\text{A} + (1 - a)\,\text{B}$

Return on Portfolio is R_p

$$V(R_p) = a^2\sigma_A^2 + (1 - a)^2\,\sigma_B^2 + 2a(1 - a)\,\sigma_{AB}$$

$$\frac{dV(R_p)}{da} = 2a\sigma_A^2 + (1 - a)\,(-1)\,\sigma_B^2 + \sigma_{AB}\,(1 - 2a)$$

set = 0

$\Rightarrow \quad 0 = a(\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}) - \sigma_B^2 + \sigma_{AB}$

$$a = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}} \doteqdot 0.2142$$

$\therefore$ invest 21.42% of portfolio in asset A to get minimum risk portfolio.

(iii) No diversification benefits remain when the variance of the portfolio equals the variance from holding only asset B.

$\therefore$ when $V(R_p) = V(\text{B})$

$\therefore$ i.e. $a^2\sigma_A^2 + (1-a)^2\sigma_B^2 + 2a(1-a)\,\sigma_{AB} = .003724$

$\Rightarrow \quad \sigma_{AB} = 0.000486$

$$\Rightarrow \quad \rho_{AB} = \frac{.000486}{.152148 \times .061025}$$

$$= 0.052$$

8 Liabilities

(i) (a) $S(t) = A(t) - L(t) \Rightarrow \dfrac{\partial S(t)}{\partial t} = \dfrac{\partial A(t)}{\partial t} - \dfrac{\partial L(t)}{\partial t}$

$$\Rightarrow \frac{\partial S(t)}{\partial t} = A(t)r_p - L(t)r_L$$

$$= S(t)r_p + L(t)\,(r_p - r_L)$$

(b) negative eventually if $r_L > r_p$, but might be positive initially depending on $S(t)$ and $L(t)$.

(ii) (a) $\dfrac{\partial F}{\partial t} = Fr_p - L$

$$\frac{\partial Fr_p}{Fr_p - L} = \partial t r_p$$

$\Rightarrow \quad \ln(Fr_p - L) = r_p t + C$

$\Rightarrow \quad (Fr_p - L) = (F(0)r_p - L)\; e^{r_p t}$

(b) $\frac{\partial F}{\partial t} = 0$ if $Fr_p = L$

(c) If L is fixed, we must either have a large enough $F(0)$ or a return r_p that is sufficiently high so that $F(0)r_p > L$.

(iii) In cases (i) and (ii) the liabilities and liability cash flows, respectively, are deterministic so there is no need to model randomness in them.

The asset return will, however, require an investment model to be built, typically incorporating statistical parameters such as the mean, variance and correlations of the assets in which the portfolio is invested.

Project assets or fund, respectively, many times and measure the distribution of surplus or fund size, respectively, at a chosen time horizon.