EXAMINATIONS

September 2002

Subject 109 — Financial Economics

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners
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1 (a) A: $1 \begin{smallmatrix} \nearrow 4 \\ \searrow 0 \end{smallmatrix}$ Expected wealth = 2

B: Expected wealth = 1.5

Risk neutral: allocated all to A because it has higher expected terminal wealth.

(b) Risk averse: invest X_A in A and $(1 - X_A)$ in B.

$$\Rightarrow E[U] = 0.5 \ln(4X_A + (1 - X_A)1.5) + 0.5(\ln(0 + (1 - X_A)\, 1.5))$$

$$= 0.5 \ln[(2.5X_A + 1.5)(-1.5X_A + 1.5)]$$

$$= 0.5 \ln[1.5^2 + 1.5X_A - 1.5 \times 2.5X_A^2]$$

Maximum when $\dfrac{\partial E[U]}{\partial X_A} = 0$, noting that $\ln(\cdot)$ is monotonic and the quadratic form is concave (or explicitly calculate the second derivative).

$$\Rightarrow \frac{1}{1.5 + X_A - 2.5X_A^2} \times (1 - 5X_A) = 0$$

$$\Rightarrow X_A = 0.2$$

i.e. 20% in A, 80% in B

2 (a) $E[R_t] = ½\left(E[r_t] + E[r_{t-1}]\right)$

$$= E[r_t] \text{ (i.i.d.)}$$

$$\text{Var}(R_t) = \left(\frac{1}{2}\right)^2 \text{Var}(r_t) + \left(\frac{1}{2}\right)^2 \text{Var}(r_{t-1}) + 2 \times \frac{1}{2} \times \frac{1}{2} \times \text{Cov}(r_t, r_{t-1})$$

$$= 2 \times \left(\frac{1}{2}\right)^2 \times \sigma^2 + 0 \text{ (i.i.d.)}$$

$$\Rightarrow \text{std}(R_t) = \frac{\sigma}{\sqrt{2}}$$

(b) For $x < \mu = E[R_t]$

$$F_R(x) = \Pr[R_t < x] = \Pr\left[Z < \frac{x-\mu}{\frac{\sigma}{\sqrt{2}}}\right] < \Pr\left[Z < \frac{x-\mu}{\sigma}\right] = F_r(x)$$

Hence, for $x < \mu$, $\int_{-\infty}^{x} F_R(x)dx < \int_{-\infty}^{x} F_r(x)dx$

For $x = \mu + \delta$, $\delta > 0$

$F_r(\mu + \delta) = 1 - F_r(\mu - \delta)$ (and similarly for $F_R(x)$)

So,

$$\int_{\mu}^{\mu+\delta} (F_R(y) - F_r(y))dy = -\int_{\mu-\delta}^{\mu} (F_R(y) - F_r(y))dy$$

Therefore:

$$\int_{-\infty}^{\mu+\delta} (F_R(y) - F_r(y))dy = \int_{-\infty}^{\mu-\delta} (F_R(y) - F_r(y))dy < 0$$

Therefore:

$$\int_{-\infty}^{x} F_R(x)dx < \int_{-\infty}^{x} F_r(x)dx \text{ for } x < \infty$$

and hence R_t dominates r_t to second order.

3 (i) Vasicek and CIR encompass mean reverting short rates.
Both generate aribtrage-free yield curves.
In both models, the parameters are time invariant.
Vasicek permits negative rates, CIR does not.
Vasicek is more mathematically tractable.
CIR enforces a non-negative lower bound on yields.

(ii) (a) Spot yield: $$-\frac{\ln P}{t} = \frac{1}{t}\left[\frac{1-e^{-\alpha t}}{\alpha}R + \frac{(\alpha t - 1 + e^{-\alpha t})}{\alpha}L + \frac{B}{2}\left(\frac{1-e^{-\alpha t}}{\alpha}\right)^2\right]$$

(b) Forward yield: $-\dfrac{\partial}{\partial t}\ln P(t) = e^{-\alpha t}R + \left(L + \dfrac{\beta}{\alpha}e^{-\alpha t}\right)(1-e^{-\alpha t})$

(c) As $t \to 0$, forward and spot rates $\to R$ as can be seen from expanding the exponentials.

4 (i) (a) "Greeks" are differentials of the price of an option with respect to the underlying variables needed to price the option.

(b) $\Delta = \dfrac{\partial f}{\partial s}$ where f = value of option and s is the share price.

(ii) If $f = s\Phi(d_1) - ke^{-ru}\,\Phi(d_2)$

Then $\dfrac{\partial f}{\partial s} = \Phi(d_1) + s\dfrac{\partial \Phi(d_1)}{\partial s} - ke^{-ru}\dfrac{\partial \Phi(d_2)}{\partial s}$

$$= \Phi(d_1) + s\frac{\partial \Phi}{\partial d_1}\frac{dd_1}{ds} - ke^{-ru}\frac{\partial \Phi}{\partial d_2}\frac{dd_2}{ds}$$

$$= \Phi(d_1) + s\psi(d_1)\frac{\partial d_1}{ds} - ke^{-ru}\psi(d_2)\frac{dd_2}{ds}$$, where ψ is the standard normal density function

Since $d_2 = d_1 - \sigma\sqrt{u}$, $\dfrac{dd_1}{ds} = \dfrac{dd_2}{ds}$

So $\Delta = \dfrac{\partial f}{\partial s} = \Phi(d_1) + \dfrac{dd_1}{ds}(s\psi(d_1) - ke^{-ru}\psi(d_2))$

Express d_2 as function of d_1 to force terms to cancel.

$$\Delta = \Phi(d_1) + \frac{dd_1}{ds}\psi(d_1)\left(S - Ke^{-ru}\,e^{d_1\sigma\sqrt{u}}\,e^{-\frac{\sigma^2 u}{2}}\right)$$

Substitute for $d_1 = \dfrac{\ln\dfrac{S}{K} + \left(r + \dfrac{\sigma^2}{2}\right)u}{\sigma\sqrt{u}}$

$$\Delta = \Phi\big(d_1\big) + 0.$$

and all the terms cancel in the brackets.

When $r = 0, S = K, u = 1$

$d_1 = \tfrac{1}{2}\sigma = .1,\ \Delta = \Phi(.1) = .5398$

(iii) Initially investor B is worth £1m + £1m($\Phi(.1) - \Phi(-.1)$)

= £1.08m

After the rise, B is worth approximately £1.08m + .54 × .2m = £1.19m

A will be worth £1.2m so A is worth more.

Candidates recalculating the value of the option at the higher equity price to determine the value of B's holdings were given full credit if correct.

5 (i) A replicating portfolio is a portfolio of assets that produces the same payoffs as the option in all states of the world. By the principle of no arbitrage, this portfolio must have the same value as the option. In the case that the assets all have an observable market value, the value of the option can be determined by calculating the value of the portfolio. The replicating portfolio can also be used to hedge the risks of the option using the same principle.

(ii)

underlying	derivative	cash
$S_0 \nearrow S_0u$, $S_0 \searrow S_0d$	$C \swarrow C_u$, $C \nwarrow C_d$	e^r

(a) Using the principle of risk neutrality (or by constructing the replicating portfolio explicitly)

$$C = e^{-r}\left[qC_u + (1-q)C_d\right];\ q = \frac{e^r - d}{u-d}$$

If pay off is $S_0 - S$, $C_u = S_0\,(1-u)$
$C_d = S_0\,(1-d)$

$$C = e^{-r}\left[\frac{e^r - d}{u-d}\,S_0(1-u) + \frac{u-d-e^r+d}{u-d}\,S_0(1-d)\right]$$

$$= S_0\,e^{-r}(-e^r + 1) = S_0\left(e^{-r} - 1\right)$$

(b) If the stock pays a dividend yield, q, then the stock price S_0 includes a non-random component that is the present value of q.

The owner of the stock gets the benefit of dividends, the owner of the derivative does not.

Since this derivative has negative value (it is in effect a short future) the price of the derivative will <u>rise</u> (i.e. become less negative).

6 (i) Weak form efficiency: historical price information and patterns already priced into shares.

Semi-strong: all publicly available information already priced into shares.

Strong: all information, including insider information already priced

(ii) Weak: historical data, test whether patterns repeat themselves, try to fit time series models, test whether trading rules based on fitted models would yield higher than average returns.

Semi-strong: test whether all off-balance sheet items priced, test whether changes to accounting standards led to changes in prices, run regressions of price changes on lagged accounting and economic data items

Strong: assess whether holders of privileged information made higher than average return.

Measure trading of directors before key events.

Assess whether any abnormal return generated before key event e.g. announcement of merger should provide a one-off charge in abnormal returns, rather than any gradual build up of abnormal returns.

(iii) Theoretical:

test of efficiency = joint test of efficiency and pricing model so that if you reject efficiency, it might just be because you have the wrong pricing model (or vice versa)

Practical and statistical problems:

obtaining data
time period specificity about any conclusions reached
limited sample sizes
low statistical power
survivorship bias

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(i) Return on portfolio = $P = wA + (1-w)B$

$$V(P) = w^2 V(A) + (1-w)^2 V(B) + 2w(1-w)\, p_{AB}\sqrt{V(A)V(B)}$$

$$= w^2 .2^2 + (1-w)^2 .04^2 + 2w(1-w)\, 0.5 \times .2 \times .04$$

$$= w^2(.04) + (1-w)^2(.0016) + w(1-w) \times (.008)$$

$\dfrac{dV(P)}{dw} = (.08)w - (.0032)(1-w) + (.008)(1-2w) = 0$ at min. noting the convexity of $V(P)$

$$\Rightarrow .0672w = -.0048$$

$$\Rightarrow w = -.07143$$

−7.1% in A, 107.1% in B.

(Minimum variance position includes a short position in A.)

(ii)

(a) $V(S) = V(1000(1 + w\text{A} + (1-w)\text{B}) - 1000\,(1 + X))$

$$= 1000^2(V(w\text{A} + (1-w)\text{B} - X))$$

$$\psi = V(w\text{A} + (1-w)\text{B} - X) = w^2 V(\text{A}) + (1-w)^2\, V(\text{B}) + V(X)$$

$$+ 2w(1-w)\, p_{AB}\sqrt{V(A)V(B)}$$

$$- 2w p_{AX}\sqrt{V(A)V(X)}$$

$$- 2(1-w)\, p_{BX}\sqrt{V(B)V(X)}$$

$$\frac{d\psi}{dw} = (.08)w - (.0032)(1-w) + .008(1-2w) - 0.032 + 0.04^2 = 0$$

$$\Rightarrow (.08 + .0032 - .016)w = .0032 - .008 + .032 - (.0016)$$

$$\Rightarrow w = .38095$$

Invest 38.1% in A, 61.9% in B.

(b) $S = \text{Surplus} = 1050(1 + wA + (1 - w)B) - 1000(1 + X)$

$= 1000(wA + (1 - w)B - X) + 50(1 + wA + (1 - w)B)$

$= 1000(1.05wA + 1.05(1 - w)B - X) + 50$

$f(w) = E[S \mid w]$

$= 1000\,[.1w + (1 - w)\,.05 - .08]$

$+ 50\,[.1w + (1 - w).05] + 50$

$= 105w + 52.5(1 - w) - 30$

$= 52.5w + 22.5$

$g(w) = V[S \mid w]$

$= 1000^2[w^2(.04)1.05^2 + (1 - w)^2(.0016)1.05^2 + w(1 - w)(.008)1.05^2$

$+ .1^2 - 1.6 \times 1.05 \times .2 \times .1 \times w$

$- .4(1 - w)\,.04 \times .1 \times 1.05]$

$= 1000^2[(.21)^2w^2 + (.042)^2\{1 - 2w + w^2\}$

$+ w\,.00882 - w^2\,.00882$

$+(.01) - .0336w$

$- .00168 + .00168w]$

$= 1000^2[w^2\,.037044 - .026628w + .010084]$

(c) $P[S < 0] = \Phi\left(-\dfrac{E[S]}{\text{std}[S]}\right)$

Minimizing $\Phi\left(-\dfrac{E[S]}{\text{std}[S]}\right)$ is achieved by maximizing $\dfrac{E[S]}{\text{Std}[S]}$

i.e. maximize

$$\varphi = \frac{f(w)}{[g(w)]^{\frac{1}{2}}}$$

$$\frac{\partial \varphi}{\partial w} = 0$$

$$\Rightarrow \frac{d}{dw} f(w)[g(w)]^{-\frac{1}{2}} = 0$$

$$\Rightarrow f'(w)[g(w)]^{-\frac{1}{2}} - \frac{1}{2}\left[g(w)\right]^{-\frac{3}{2}} f(w)\, g'(w) = 0$$

$$\Rightarrow f'(w) - \frac{1}{2}\frac{f(w)}{g(w)} g'(w) = 0 \,.$$

8 (i) $R_i = \alpha_i + \beta_i R_m + \varepsilon_i$

R_i = return on stock i

α_i, β_i are single index model parameters for stock i

R_m = return on index

ε_i = unique, or stock-specific component of return

$E(\varepsilon_i) = 0$, $\text{Cov}(\varepsilon_i, \varepsilon_j) = 0 \qquad i \neq j$

$\text{Var}(R_j) = \beta_i^2 \text{Var}(R_m) + \text{Var}(\varepsilon_i)$

$\beta_i^2 \text{Var}(R_m)$ = systematic risk

$\text{Var}(\varepsilon_i)$ = stock-specific risk

(ii) $R_p = \sum x_i R_i$, where x_i is the proportion of initial wealth invested in stock i

$= \sum x_i (\alpha_i + \beta_i R_m + \varepsilon_i)$

$$\text{Var}(R_p) = \text{Var}(R_m)\left(\sum x_i\beta_i\right)^2 + \sum_i\sum_j x_i x_j \,\text{Cov}(\varepsilon_i\varepsilon_j)$$

$$= \text{Var}(R_m)\beta_p^2 + \sum x_i^2 \,\text{Var}(\varepsilon_i) \qquad (*)$$

where $\beta_p = \sum x_i\beta_i$

In a well diversified portfolio all x_i are of the order of $\frac{1}{N}$, where N is the number of stocks. Provided N is large, the second term of the RHS of (*) is nearly zero. Hence the risk of the portfolio depends only on the β_i (i.e. the systematic component).

(iii) (a) $$\text{Var}(R_I) = .04 + .04\sum\left(\frac{1}{20}\right)^2$$

$$= .04\left(1+\frac{1}{20}\right)$$

$= .042$, systematic component = .04
unique component = .002

[Systematic risk for all stocks is the same, so all betas must be the same, so index systematic risk must be the same, 0.04.]

(b) $$R_I^* = \frac{20}{23}R_I + \frac{3}{23}R_{21}$$

$$\text{Var}(R_I^*) = .04 + .04 \times \left[20\times\left(\frac{1}{23}\right)^2 + \left(\frac{3}{23}\right)^2\right]$$

$$= .04[1 + .05482]$$

$= .0422$, systematic component = .04
unique component = .0022

Risk has increased with the addition of the stock.

$$\left[\begin{aligned}\text{Or Var}(R_I^*) &= \left(\frac{20}{23}\right)^2 \text{Var}(R_I) + \left(\frac{3}{23}\right)^2 \text{Var}(R_{21}) \\ &\qquad + 2\times\frac{20}{23}\times\frac{3}{23}\times\text{Cov}(R_I, R_{21}) \\ &= 0.0422\end{aligned}\right]$$

(iv) SIM is a returns-generating model, CAPM is an asset pricing model.

SIM makes no assumptions about equilibrium; CAPM requires an equilibrium assumption.

SIM: $R_i = \alpha_i + \beta_i R_I + \varepsilon_i \Rightarrow E[R_i] = \alpha_i + \beta_i\, E[R_I]$

CAPM: $E[R_i] = r_f + \beta_i^* \,[E[R_M] - r_f]$, r_f = risk-free rate
R_M = return on market portfolio

In the SIM, the index used will not usually be the same as the market portfolio (which includes ALL assets).

In the SIM, if $\alpha_i \approx r_f(1 - \beta_i)$ and if $E[R_I] \approx E[R_M]$, then $\beta_i \approx \beta_i^*$, i.e. they both use the notion of systematic risk as being the only important risk. Both models are single time period models in their basic form.