EXAMINATIONS

September 2004

Subject 109 — Financial Economics

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

23 November 2004

1 (i) The three forms are:

Strong — stock prices reflect all current information relevant to the stock, including information which is not public.

Semi-strong — stock prices reflect all current, publicly available information relevant to the stock.

Weak — stock prices reflect all information available in the past history of the stock price.

(ii) Tests need to make assumptions (which may be invalid) such as normality of returns or stationarity.

Transaction costs may prevent the exploitation of anomalies, so that the EMH might hold net of transaction costs.

Allowance for risk: the EMH does not preclude higher returns as a reward for risk; however the EMH does not tell us how to price such risks.

2 (i) An Arrow-Debreu security is a contract which pays 1 unit of currency if and only if a particular state of the world occurs at a specified time in the future.

(ii) Let the real world probabilities of the three states be denoted p_1, p_2 and p_3, the state prices be denoted by s_1, s_2 and s_3, and the deflators by d_1, d_2 and d_3, then, $s_i = d_i\, p_i$.

The state prices satisfy

$$100 = 200s_1 + 110s_2 + 50s_3,$$
$$90 = 100s_1 + 100s_2 + 60s_3$$
$$75 = 75s_1 + 75s_2 + 75s_3$$

hence

$s_1 = 1/18$, $s_2 = 25/36$ and $s_3 = 1/4$. Now $d_1 = 0.5$ so $p_1 = 1/9$ and $p_2 = 1/2$, hence $p_3 = 7/18$.

3 (i) With a flat yield curve there is an opportunity to buy short and long bonds and short sell medium bonds. This constructs a portfolio which generates a profit for any small change in the interest rate. Thus arbitrageurs will follow this strategy increasing the price of long and short bonds (and thus depressing their yields) whilst the price of medium length bonds will fall (raising their yield).

(ii) Both of these models are mean-reverting and arbitrage-free. The CIR model has the advantage that interest rates cannot go negative (indeed it usually forces a positive lower bound on yields). The Vasicek model is more mathematically tractable. The parameters in both models are not time dependent.

4 (i) The Greeks are the derivatives of the price of a derivative security with respect to the different parameters needed to calculate the price:

$$\Gamma = \frac{\partial^2 f}{\partial s^2};\ \kappa = \frac{\partial f}{\partial \sigma};\ \rho = \frac{\partial f}{\partial r};\ \Theta = \frac{\partial f}{\partial t}$$

where f is the value of the derivative, s is the price of the underlying security, σ is the volatility, r is the interest rate and t is time. In each case the relevant Greek measures sensitivity (rate of change) of the option price to change in that variable.

(ii) At time $2\Delta t$, the three possible values of the stock are 1.00137, 1 and 0.99863 and of the derivative are 1.53613, 1.529831 and 1.523557 correspondingly.

Estimate delta to be

$$\frac{1.53613 - 1.529831}{1.00137 - 1} = 4.59579$$

$$\frac{1.52983 - 1.523557}{1 - 0.99863} = 4.58321$$

Hence, an estimate of Γ is

$$\frac{4.59579 - 4.58321}{1 - 0.99863} = 9.1853$$

or $$\frac{4.59579 - 4.58321}{1.00137 - 1} = 9.1727$$

The theoretical value is $6S\exp(2rt + 3\sigma^2 t) = 9.2005$

These are fairly close, the difference is, of course, caused by the approximation.

5 (i) The pricing measure **Q** must satisfy:

$$\mathbf{E_Q}\left[\frac{1}{1+r}S_{t+1}\middle|\mathbf{F_t}\right] = S_t$$

so, if we set

$$q_t = \mathbf{Q}(S_{t+1} = 1.25S_t\,|\mathbf{F}_t),$$

then

$$1.05 = 1.25q_t + 0.8(1 - q_t) \Leftrightarrow q_t = q = \tfrac{5}{9}.$$

Thus the unique risk-neutral measure makes S a multiplicative random walk with up-jump probability of $\frac{5}{9}$.

(ii) (a) The price of the option is

$$\mathbf{E_Q}\left[\frac{1}{(1+r)^2}(S_2 - M_2)\right] = \frac{1}{1.05^2}(q^2 562.5 + q(1-q)200) = 202.3p.$$

(b) If ϕ and ψ are the respective values of stock and cash in the hedging portfolio at time 1, we need to solve

$$0.8\phi + 1.05\psi = 0$$

$$1.25\phi + 1.05\psi = 200,$$

which implies that $\phi = 444.4$p and $\psi = -388.6$p.

[The corresponding holding is .5555 shares.]

[Check: if $S_1 = 800$, then the option price at time 1 is $\frac{200q}{1.05} = 105.8 = 444.4 - 338.6$.]

6 (i) If the model is an EMM then $\mu = r$

(ii) $S_t = S_0 \exp((\mu - \sigma^2/2)\, t + \sigma Z_t) = f(Z_t, t)$, where $f(x, t) = S_0 \exp((\mu - \sigma^2/2)\, t + \sigma x)$, hence $d\, S_t = (\sigma^2/2\, S_t + (\mu - \sigma^2/2)\, S_t)\, dt + \sigma\, S_t\, d\, Z_t = \mu\, S_t dt + \sigma\, S_t\, d\, Z_t$

(iii) In the model, μ is a mean multiplicative return, and σ is a multiplicative volatility. This makes log S a (continuous time) continuous random walk with IID Normal increments.

Real asset price returns show evidence of trends, mean reversion and jumps, and also seem to exhibit fatter than Normal tails in the return distribution — each of these is inconsistent with a Brownian motion.

(iv) Z_t has IID Normal increments with mean 0 and variance equal to the length of the time increment so monthly returns are IID LogNormal($\mu/12$, $\sigma^2/12$); consequently if we can simulate IID standard Normals X_1, … then the simulated monthly returns will be $\exp(\mu/12 + \sqrt{(\sigma^2/12)}\, X_1)$,…

7 (i) Typical ARIMA equations of the form found in the Wilkie model are:

$$x_t = \alpha(x_{t-1} - \mu_X) + \mu_X + \beta(y_{t-1} - \mu_Y) + e_t\sigma_X$$

$$y_t = \mu_Y + \gamma(y_{t-1} - \mu_Y) + u_t\sigma_Y + u_{t-1}\delta$$

where

x_t is the value of the modelled variable at time t

y_t is the value of another modelled variable at time t

u_t and e_t are i.i.d. random normal

α, β, γ, δ, μ_X, μ_Y, σ_Y and σ_X are the eight parameters that appear to have to be estimated

These equations can be rewritten as a VARMA model

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix}\begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma_X & 0 \\ 0 & \sigma_Y \end{bmatrix}\begin{bmatrix} e_t \\ u_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix}\begin{bmatrix} e_{t-1} \\ u_{t-1} \end{bmatrix}$$

All the 0's in the 2×2 parameter matrices are effectively parameters that have been set to 0, either because they were estimated to be not significantly different from 0 or for parsimony. There are 14 parameters evident in this formulation. Since the Wilkie model involves many variables with autoregressive and moving average terms of up to order 2, there will be many such "0" parameters in the VARMA formulation.

(ii) (a) Saying that the dividend process has unit gain with respect to changes in inflation means that a change in inflation ultimately produces the same change in the dividend process.

(b) We need $DX = 1 - DW$.

8 (i) Assuming no market frictions and no arbitrage, we can apply put-call parity:

$$C - P = S_0 - e^{-rt} K,$$

where C and P are the call and put prices, S_0 is the current stock price, K is the common strike price, r is the risk-free rate and t is the exercise date of the options. So

$$P = 41 - 400 + 420e^{-.035} = 46.6\text{p}.$$

(ii) With $\sigma = 20\%$: $d_1 = -.0268$, $d_2 = -.1682$, $\Phi(d_1) = .4893$, $\Phi(d_2) = .4332$, $e^{-rt} = .9656$

so $f = 20.03$p

with $\sigma = 50\%$: $d_1 = .1378$, $d_2 = -.2158$, $\Phi(d_1) = .5548$, $\Phi(d_2) = .4146$, $e^{-rt} = .9656$

so $f = 53.78$

Linear interpolation gives an estimate of

$20 + 30(41 - 20.03) / (53.78 - 20.03) = 38.6\%$ for the implied volatility. To check accuracy, using 38.6% gives $f = 41.00$, so the estimate is clearly accurate enough.

Other initial estimates may require further iterations.

(iii) Δ is the rate of change of the option price with respect to change in the price of the underlying; i.e. $\Delta = \dfrac{\partial f}{\partial s}$.

$$\begin{aligned}\Delta &= \Phi(d_1) + s\phi(d_1)\frac{\partial d_1}{\partial s} - Ke^{-rt}\phi(d_2)\frac{\partial d_2}{\partial s}\\ &= \Phi(d_1) + s\phi(d_1)/s\sigma\sqrt{t} - Ke^{-rt}\phi(d_2)/s\sigma\sqrt{t}\\ &= \Phi(d_1)\end{aligned}$$

$d_1 = .0879$, $\Phi(d_1) = .5350$ so

$\Delta = .5350$.

(iv) The hedging portfolio holds Δ units of stock per unit of the option and the rest of the option price in cash, so the hedging portfolio holds 5,350 shares (value £21,400) and –£17,300 in cash.

9 (i) The theorem states that a rational agent with a preference ordering $\geq$ on certain outcomes maximises their expected utility, where their utility function u (over certain outcomes) models their preference in the sense that

$$u(A) \geq u(B) \Leftrightarrow A \geq B.$$

The axioms are:

(1) Comparability: for any outcomes A and B, either $A \geq B$ or $B \geq A$ (or both).

(2) Transitivity: $A \geq B \geq C \Rightarrow A \geq C$.

(These two can be summarised as: "preference is a total ordering on outcomes".)

(3) Independence: if an agent is indifferent between outcomes A and B, then he is also indifferent between the following gambles:

(G1) A with probability p and C with probability $1 - p$

and

(G2) B with probability p and C with probability $1 - p$.

(4) Certainty equivalence: if $A \geq B \geq C$ then there is a p such that the agent is indifferent between B and G, where G is the gamble: A with probability p and C with probability $1 - p$.

(ii) An agent is risk-averse if, for any amount of wealth w, they prefer a certain outcome of w to any gamble with expected outcome w.

An agent with wealth w is non-satiated if they strictly prefer wealth v to w for any $v > w$.

(iii) (a) If she opts for deadline T, her probability of succeeding is $T - 1$ and her expected utility is

$$f(T) = \frac{T-1}{T^3},$$

while her expected reward is £$(1 + (T - 1)/T^6)$ *Million*.

(b) It follows that

$$f'(T) = \frac{T-3(T-1)}{T^4} = \frac{3-2T}{T^4}.$$

It follows that f is concave on $[1, 2]$ $(f'' < 0)$, and achieves its maximum at $\frac{3}{2}$, so this is her optimal choice of T.

10 (i) This is a single index model. A general model would require the estimation of $n(n + 3)/2$ parameters ($n(n + 1)/2$ covariances and n means)whereas this model requires the estimation of 1 mean (the mean of the market index), n betas and $n + 1$ standard deviations (the n σ_i s and the s.d. of the market index) = $2(n + 1)$ parameters.

Moreover, the nature of the estimates required from analysts conforms much more closely to the way they traditionally work.

(ii) (a) The CAPM formula for expected returns is

$$\overline{r_i} = r_f + \beta_i(\overline{r_M} - r_f) \text{ for } i = 1 \ldots n$$

where, $\overline{r_i}$ is the expected return on security i, $\overline{r_M}$ is the expected return on the market index, $\beta_i = \mathrm{Cov}(R_i, R_M) / \mathrm{Var}(R_M)$.

The APT formula for expected returns is

$$r_i = r_0 + \beta_{i,1}\lambda_1 + \ldots \beta_{i,k}\lambda_k \text{ for } i = 1 \ldots n \text{, where } \lambda_j$$

is the expected return on index j and $\beta_{i,j}$ is the sensitivity of security i to index j.

(ii) (b) Taking expectations in the returns-generating model leads to equations for the expected return that are consistent with the CAPM and single-index APT pricing formulae.

The CAPM assumes (in addition) that all investors have the same assessment of risk and return (means, variances and covariances), the same one-period investment horizon and that the market is perfect.

The CAPM also requires the expected return on the market index to be identical to the expected return on the actual market.

For the APT no assumptions are made about investor preferences other than that more is preferred to less and that the market is arbitrage free. The APT also requires n to be large.

(iii) (a) The covariance matrix V is:

$$V = \begin{bmatrix} .04 & .01 \\ .01 & .01 \end{bmatrix},$$

so V^{-1} is proportional to $\begin{bmatrix} .01 & -.01 \\ -.01 & .04 \end{bmatrix}$.

The global variance minimiser is proportional to $V^{-1}1$ and hence to $(0, .03)^T$, so the global minimising portfolio is 100% invested in asset B.

Alternatively, if proportion p is invested in asset A then the variance of return on the portfolio is

$$.04p^2 + .02p(1 - p) + .01\,(1 - p)^2 = .01 + .03p^2,$$

which is minimised at $p = 0$, hence result.

(b) The general portfolio is $(0, 1)^T + c(-1, 1)^T$ with corresponding expected return of

$$r(c) = 0.1 + 0.05c$$

and s.d. of

$$\sigma(c) = \sqrt{0.01 + 0.03c^2}\,.$$

The efficient frontier is the curve traced out by this pair for $c \geq 0$.

(iv) We need to determine the tangent (in mean-s.d. space) to the previous frontier which passes through (0, 0.07). Now $\sigma' = \dfrac{0.03c}{\sqrt{0.01+0.03c^2}}$ and $r' = .05$, while the desired tangent occurs at the value of c satisfying

$$\frac{dr}{d\sigma}(c) = \frac{r(c)-0.07}{\sigma(c)},$$

i.e. $$\frac{5\sqrt{0.01+0.03c^2}}{3c} = \frac{0.03+0.05c}{\sqrt{0.01+0.03c^2}}.$$

The solution is $\bar{c} = \frac{5}{9}$. Thus the new efficient frontier is of the form:

$$\left(r, \frac{r-0.07}{r(\bar{c})-0.07}\sigma(\bar{c})\right) = (r,\ 2.4019(r-0.07)),$$

for $r \geq 0.07$.

END OF EXAMINERS' REPORT