EXAMINATIONS

September 2000

Subject 109 — Financial Economics

EXAMINERS' REPORT

1 (i) Put-call parity expresses a relationship between the price of a put option and the price of a call option on a stock where the options have the same exercise dates and strike prices.

(ii) Consider a portfolio A which contains one European call and an amount of cash $D + Xe^{-r(T-t)}$

where X = strike price
r = risk-free rate
$T - t$ = time to exercise of the option
D = present value of dividends payable

At the exercise date if the share price $S_T \geq X$ then call will be exercised and portfolio A will have a value of

$$De^{r(T-t)} + S_T$$

If at T we have $S_T < X$ then the call will not be exercised and portfolio A will be worth

$$De^{r(T-t)} + X$$

Now consider portfolio B consisting of one European put and a share.

At the exercise date if $S_T \geq X$ then the put will not be exercised and portfolio B will have value of

$$S_T + De^{r(T-t)}$$

If at the exercise date T, we have $S_T < X$ then the put will be exercised and portfolio B will have a value of

$$X + De^{r(T-t)}$$

Clearly portfolios A and B have the same value in all circumstances at the exercise date T. Hence they must be equivalent at all earlier times
$\Rightarrow$ the portfolios are of equal value

$$\therefore\ c + D + Xe^{-r(T-t)} = p + S_t$$

c = value of European call with strike X and exercise date T

p = value of European put with strike X and exercise date T

S_t = value of stock at time t

(iii) Let D be the present value of dividends payable and consider

$$c + D + Xe^{-r(T-t)} < p + S_t$$

then for some amount A

$$A + c + D + Xe^{-r(T-t)} = p + S_t$$

Hence we can short one share and sell a put and receive $p + S_t$. At the exercise date we know the value of this portfolio is $\max[S_T + De^{r(T-t)}, X + De^{r(T-t)}]$.

However we know that the value of a portfolio invested in a European call and $D + Xe^{-r(T-t)}$ at time t will be worth

$$\max[S_T + De^{r(T-t)}, X + De^{r(T-t)}] \text{ at } T.$$

This is the same as the amount we must repay at time T.

Hence we are left with a profit of $Ae^{r(T-t)}$

$\therefore$ strategy is

Short 1 share and sell a put.

Buy 1 call and put on deposit $A + D + Xe^{-r(T-t)}$

If the inequality is reversed also reverse investment (i.e. swap long positions for short positions and vice versa).

2 Some examples of valid points are:

- Some managers do appear to generate returns in excess of the market returns on a regular basis.

- This outperformance is not consistent, in particular a manager can not guarantee to produce excess performance in any given year.

- The outperformance is usually prior to charges being taken into consideration. Once charges are included there is extremely limited evidence of consistent outperformance.

- The risk of positions must be taken into account. A higher risk portfolio should provide, on average, higher returns to compensate for the risk. For sensible comparisons risk adjusted returns are required.

- Fund managers are also employed to build and maintain diversified portfolios or specialist portfolios with specific mandates (ethical or high risk). An efficient market does not mean that tailored portfolios will not be required by some investors.

- Given the diversity of investment services we would expect by pure chance that some managers would have above average track records over short/medium time periods.

- Capital markets are closer to the idealised "perfect markets". More likely that inefficiencies arise in the market for buying and selling investment services rather than the markets for buying/selling securities.

3 (i) Let $Q(t)$ be the inflation index at t

$$\frac{Q(t-1)}{Q(t-2)} = 1.025 = e^{I(t-1)}$$

$$\therefore\ I(t-1) = \ln 1.025$$

$\therefore$ The 97.5% percentile for $I(t)$, $I(t)_{0.975}$ is

$$I(t)_{0.975} = 0.047 + 0.58[\ln 1.025 - 0.047] + 0.0425 \times 1.96$$

$$= 0.03406 + 0.0833$$

$$= 0.11736$$

and

$$I(t)_{0.025} = 0.03406 - 0.0833$$

$$= -0.04924$$

$\therefore$ The 95% CI for the rate of inflation is:

$$(e^{-0.04924} - 1,\ e^{.11736} - 1)$$

$$(-4.8\%,\ 12.5\%)$$

(ii) The range appears very wide given the previous year's inflation rate of 2.5%.

The calculation in (i) represents the conditional property of the inflation model looking one year ahead. The parameters QSD etc. were set by analysis of historical data, the longitudinal property. As the longitudinal property represents unconditional values it forms an upper bound for the conditional values.

The Wilkie model was designed as a long-term model for actuarial use, hence a one year projection could be considered a mis-use of the model.

The inflation rate in the previous year was below the long term mean of 0.047 hence the expected inflation rate next year is higher than that just experienced. The modelled symmetry of inflation is not realistic.

4 (i) Opportunity set describes the characteristics of all assets, and hence all portfolios that are available to investors.

A portfolio is efficient if the investor cannot find an alternative portfolio that has higher expected return for the same variance. The efficient frontier is the set of all efficient portfolios.

Here we are implicitly assuming that investors are never satisfied and they are risk averse.

An indifference curve plots points which represent portfolios between which an investor is indifferent, i.e. they provide the same expected utility. This set of points is a function of the investor's utility function and shows how they trade off risk and reward.

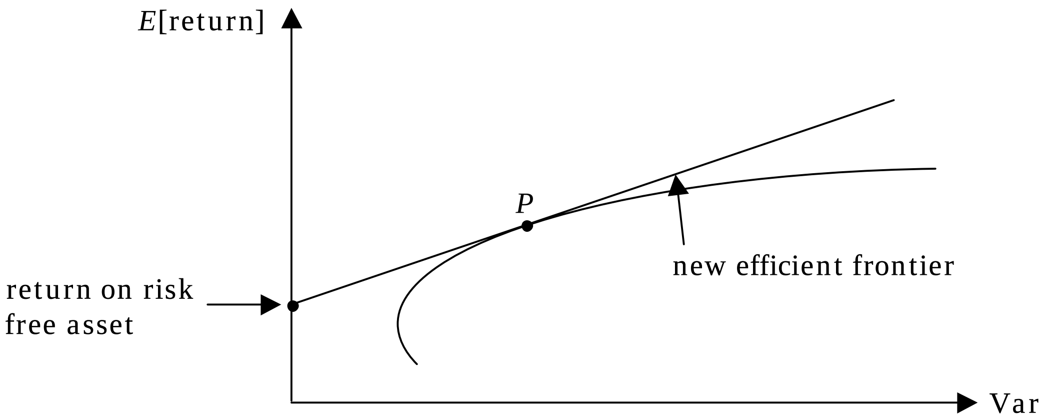
An optimal portfolio is the point where the indifference curve is tangential to the efficient frontier. Optimal portfolios can be found in this way if

- the investor has a quadratic utility function

or

- distribution of returns can be specified in terms of the mean and variance (e.g. normally distributed).

(ii) The risk free asset creates a new efficient frontier which is a straight line connecting the rate of return on the risk-free asset to the point where the line marks a tangent with the efficient frontier, say P.



(iii) The efficient portfolios now contain varying proportions of the risk-free asset and P, the optimal portfolio of risky assets.

This means that all investors, no matter what their risk preferences, will all hold the risky assets in the same proportion, i.e. there is a separation of risk preference and the decision of the portfolio of risky assets to hold.

The proportion of risky assets within the investor's total portfolio will vary with the risk tolerance of the investor. An investor with high risk tolerance will borrow funds at the risk free rate and invest them in higher holdings of the risky assets. This will give them a portfolio with a higher expected return and variance that portfolio P. A low risk tolerance investor will invest some funds in the risk free asset and reduce holdings in the risky assets.

5

Comparability — state a preference between certain outcomes

Transitivity — if $A > B$ and $B > C$ then $A > C$ (where “>” indicates preferred to)

Independence — if investor is indifferent between 2 outcomes A and B, then she is also indifferent between two gambles

A with probability p and C with probability $(1 - p)$
B with probability p and C with probability $(1 - p)$

Certainty Equivalent — if $A > B$ and $B >$ C

then there exists a unique probability p such that the investor is indifferent between receiving B with certainty or a gamble with A with probability p and C with probability $(1 - p)$. B is known as the certainty equivalent of the above gamble.

6 (i) Non-satiation $\Rightarrow U'(w) > 0$

Investor is risk averse if $U''(w) < 0$

$$\begin{aligned} & U(w) &=& 1 - e^{-aw} \\ \Rightarrow\; & U'(w) &=& ae^{-aw} \\ \Rightarrow\; & U''(w) &=& -a^2 e^{-aw} \end{aligned}$$

Non-satiation $\Rightarrow a > 0$

Risk aversity puts no further constraints on a

(ii) Absolute risk aversion $A(w) = -\dfrac{U''(w)}{U'(w)}$

$$A(w) = -\frac{-a^2 e^{-aw}}{ae^{-aw}} = a$$

Relative risk aversion

$$R(w) = \frac{-wU''(w)}{U'(w)}$$

$$R(w) = -w\ \frac{-a^2 e^{-aw}}{ae^{-aw}} = aw$$

(iii) $A(w) = a$

$\therefore$ Investor has constant absolute risk aversion. She will hold the same amount of her wealth in risky assets as her level of wealth changes.

$$R'(w) = a > 0$$

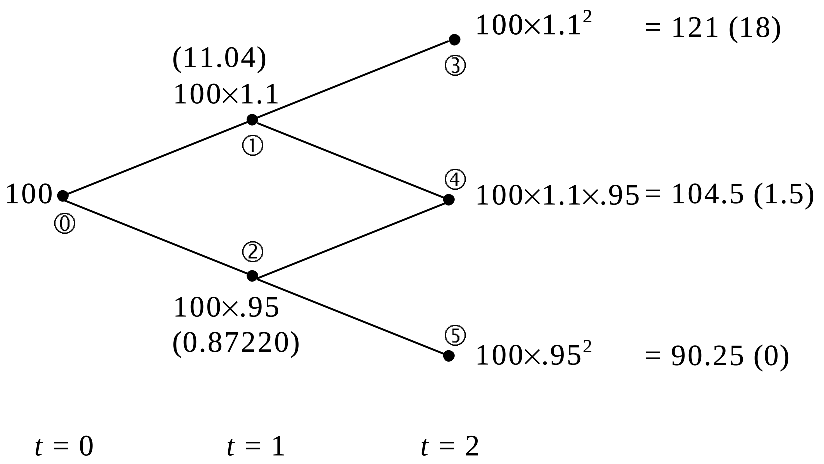
Investor has positive relative risk aversion. She will hold a smaller proportion of her wealth in risky assets as her total wealth increases.

7 (i) Systematic risk relates to the market as a whole. This risk is non-diversifiable. In return for accepting this risk investors are compensated by receiving, on average, higher returns (risk premium).

Specific risk relates to factors that affect a particular security, independently of all other securities.

(ii) By holding a well diversified portfolio the specific risk of the portfolio is reduced leaving only the non-diversifiable systematic risk. As specific risk is diversifiable investors do not require additional reward for accepting it, hence specific risk does not provide a compensating risk premium to investors.

8 The binomial tree is:



S_i = stock price at node i

c_i = option value at i

$u = 1.1$

$d = .95$

$\therefore\ S_1 = uS_0$

Write the value of the option in brackets on the tree.

At $t = 2$ the options value is known with certainty.

Consider node ① on the tree.

Need to calculate holdings of cash and stock at ① which will replicate the value of the option at $t = 2$.

Let cash holding be ψ_k
stock holding be ϕ_k } in general for mode k

$$C_{\text{up}} = \phi_{\text{now}}\, S_{\text{now}}\, u + \psi_{\text{now}}\, e^r$$

$$C_{\text{down}} = \phi_{\text{now}}\, S_{\text{now}}\, d + \psi_{\text{now}}\, e^r$$

Hence need

$$18 = \phi_1 \times 121 + \psi_1\, e^{0.04} \quad \text{if stock goes up}$$

$$1.5 = \phi_1 \times 104.5 + \psi_1\, e^{0.04} \text{if stock goes down}$$

$$\Rightarrow \qquad \phi_{\text{now}} = \frac{C_{\text{up}} - C_{\text{down}}}{S_{\text{now}}(u-d)}$$

$$\psi_{\text{now}} = e^{-0.04}\left(\frac{C_{\text{down}}\, u - C_{\text{up}}\, d}{u-d}\right)$$

$$\Rightarrow \qquad \phi_1 = \frac{18 - 1.5}{121 - 104.5} = 1$$

$$\psi_1 = e^{-0.04}\left(\frac{1.5 \times 1.1 - 18 \times .95}{1.1 - .95}\right)$$

$$= -98.96$$

Hence the value of the call at node ① is

$$= 1 \times 110 - 98.96 = \underline{\underline{11.04}}$$

Similarly at node ②

$$\phi_2 = \frac{1.5 - 0}{104.5 - 90.25} = 0.10526$$

$$\psi_2 = e^{-0.04}\left(\frac{0 \times 1.1 - 1.5 \times .95}{1.1 - .95}\right) = -9.12750$$

Value of call at node ② is

$$= 95 \times .10526 - 9.12750$$

$$= 0.87220$$

Hence replicating portfolio at $t = 0$ is given by

$$\phi_0 = \frac{11.04 - 0.8722}{110 - 95} = 0.67785$$

$$\psi_0 = e^{-0.04}\left(\frac{0.8722 \times 1.1 - 11.04 \times .95}{1.1 - .95}\right) = -61.03306$$

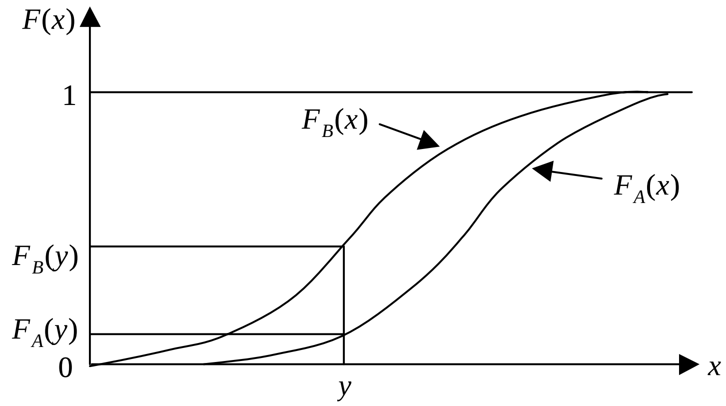
Hence value of the option at $t = 0$ is

$$.67785 \times 100 - 61.03306$$

$$= 6.75194$$

9 (i) Distribution A will have first order stochastic dominance over distribution B if

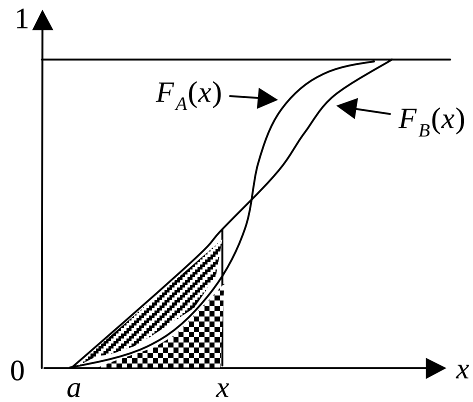
$$F_A(x) \le F_B(x) \qquad \forall x \text{ and}$$
$$F_A(x) < F_B(x) \qquad \text{for some } x.$$



A will have 2nd order stochastic dominance over B if

$$\int_a^x F_A(y)dy \le \int_a^x F_B(y)dy \qquad \forall x$$

where a is the lowest return and the strict inequality holds for some x.



Area under F_B ▨ + ▩

> area under F_A ▩

(ii) (a) If an investor prefers more to less (is non-satiated) then they will always prefer an investment that first order stochastically dominates another.

(b) If an investor is non-satiated and risk averse then they will prefer an investment that second-order stochastically dominates another.

10 (i) The model is equivalent to geometric Brownian motion. This model is commonly used for share <u>prices</u>. Under this model the short term interest rates would, in the long term, increase (if $\mu > 0$) geometrically. Changes in the rate of interest are log-normally distributed and are independent of the current rate. The volatility of the changes is constant.

These modelled characteristics are very unlike the real world. Interest rates do not increase geometrically. The change in rate is also dependent on the current rate — typically rates increase after periods when they have been low and vice versa, i.e. they are mean reverting.

The log-normal distribution is perhaps not too unreasonable, although volatility tends to be lower when rates are low and higher when rates are high.

(ii) (a) Vasicek model incorporates mean reversion — the drift is negative/positive as the current rate is higher/lower than the long term average. Arbitrage free.

(b) CIR model incorporates mean reversion as well as an adjustment to the volatility. Volatility is high/low when rates are high/low. Arbitrage free.

11 (i) (a) Under SIM

$\text{Var}(R_i) = \beta_i^2 \text{ Var}(R_m) + \text{Var}(\varepsilon_i)$

$E(R_i) = \alpha_i + \beta_i E(R_m)$

$\text{Cov}(R_i , R_j) = \beta_j \beta_j \text{ Var}(R_m)$

Require: β_j, α_i, $\sigma_{\varepsilon_i}^2$ per security

plus $\text{Var}(R_m)$ and $E[R_m]$, i.e. $3N + 2$ parameters

For unstructured covariance matrix, require $\frac{N(N+1)}{2}$ unique covariance matrix elements + N means

i.e. $\frac{N(N+3)}{2}$ items.

(b) $N = 50$ SIM requires: 152 parameters
Full model: 1325 parameters

i.e. difference of 1173

(ii) SIM is an empirically-based return generating model. It implies no price for a security, nor can it be used to identify over or under-valued securities without considerable additional assumptions.

CAPM is an asset-pricing model that models simultaneously all the expected returns of all securities in the market.

(iii) (a) $E(R) - r_f = \beta(E(R_m) - r_f)$

$\Rightarrow E(R) = 5 + 1.5(10 - 5)$

$= 12½\%$

(b)

1. Randomness. The CAPM gives the <u>expected</u> return, not the predicted actual return.

2. Error of estimation of β. Statistical estimation error may mean that the true β was nothing like 1.5.

3. Market index return is NOT the same thing as the <u>expected</u> market (the whole market, not just the securities in the index) return.

4. β coefficient may have changed over time as the debt structure or operational nature of the underlying firm may have changed since the β was measured.