INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2014 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

D C Bowie
Chairman of the Board of Examiners

June 2014

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the April 2014 paper

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates

1 We can ignore the fund values given at 30 June.

Working in £000s:

$$870(1+i)^3 + 26(1+i)^{2\frac{1}{2}} + 27(1+i)^{1\frac{1}{2}} + 33(1+i)^{\frac{1}{2}} = 990$$

Approximate i comes from:

$$(870+26+27+33)(1+i)^3 = 990$$

$$\Rightarrow i = 1.2\%$$

Try 1%, LHS = 983.587

Try 2%, LHS = 1011.713

So

$$i = 0.01 + (0.02 - 0.01) \times \frac{990 - 983.587}{1011.713 - 983.587}$$

$$= 0.0123$$

Answer = 1.2% p.a.

Well answered although many candidates ignored the instruction to give the answer to the nearest 0.1%, and were penalised accordingly.

2 (a) **Debentures**

Debentures are part of the loan capital of companies.
The term "loan capital" usually refers to long-term borrowings rather than short-term.
Payments consist of regular coupons…
…and a final redemption payment
The issuing company provides some form of security to holders of the debenture…
…e.g. via a fixed or floating charge on the company's assets
Debenture stocks are considered more risky than government bonds…
…and are considered less marketable than government bonds.
Accordingly the yield required by investors will be higher than for a comparable government bond.

(b) **Unsecured loan stocks**

Issued by various companies.
They are unsecured – holders rank alongside other unsecured creditors.
Yields will be higher than on comparable debentures issued by the same company…
…to reflect the higher default risk.

This question was poorly answered despite being completely based on bookwork.

The above shows the variety of points that could be made (and not all were required for full marks). Many marginal candidates either made no significant attempt at the question or did not make enough distinct points.

3 (i)

$$900 \times \left(1 + \frac{i^{(2)}}{2}\right)^{2 * \frac{4}{12}} = 925$$

$$\Rightarrow \left(1 + \frac{i^{(2)}}{2}\right)^{\frac{8}{12}} = \frac{925}{900} \Rightarrow 1 + \frac{i^{(2)}}{2} = \left(\frac{925}{900}\right)^{\frac{12}{8}} = 1.041954693$$

$$\Rightarrow i^{(2)} = 8.39\% \quad (8.3909385)$$

(ii)

$$900 = 925 \times \left(1 - \frac{d^{(4)}}{4}\right)^{4 * \frac{4}{12}}$$

$$\Rightarrow \left(1 - \frac{d^{(4)}}{4}\right)^{\frac{16}{12}} = \frac{900}{925} \Rightarrow 1 - \frac{d^{(4)}}{4} = \left(\frac{900}{925}\right)^{\frac{12}{16}} = 0.979660466$$

$$\Rightarrow d^{(4)} = 8.14\% \quad (8.1358136)$$

(iii)

$$900 \times \left(1 + \frac{4}{12} i'\right) = 925 \Rightarrow i' = 8.33\% \left(8.\dot{3}\right)$$

Where i' is the simple rate of interest per annum.

This question was answered very well although some candidates calculated $i^{(4)}$ rather than $d^{(4)}$ for part (ii).

4 Firstly we must consider $i^{(2)}$ and $(1-t)\dfrac{D}{R}$

where $i^{(2)}$ is evaluated at the net yield rate (6% p.a.) = 5.9126%

$t = 0.30$, the income tax rate

$$\frac{D}{R} = \frac{8}{1.03} = 7.7670 \text{ p.a.} \quad \Rightarrow (1-t)\frac{D}{R} = 5.4369\%$$

We have $i^{(2)} > (1-t)\dfrac{D}{R}$

$\Rightarrow$ there is a capital gain and the stock will be redeemed at the last possible date if the minimum yield is received. i.e. at the end of 25 years.
Hence, let P be price per £100 nominal, then

$$P = (1-0.3)8\, a_{\overline{25}|}^{(2)} + \left(103 - (103-P)\times 0.4\right)v^{25} \text{at 6\% p.a.}$$

$$= 5.6a_{\overline{25}|}^{(2)} + (61.8 + 0.4P)v^{25}$$

$$\Rightarrow P = \frac{5.6\dfrac{i}{i^{(2)}}\, a_{\overline{25}|} + 61.8v^{25}}{1-0.4v^{25}}$$

$$= \frac{5.6\times 1.014782\times 12.7834 \; + 61.8\times 0.23300}{1-0.4\times 0.23300}$$

$$= \frac{72.6452 + 14.3994}{1-0.0932}$$

= £95.99

Generally well-answered although some candidates' arguments for choosing the latest possible date were unclear.

5 (i) The amounts of cash flows:

Coupon on 25/4/2013

$$= 10,000 \times \frac{0.03}{2} \times \frac{RPI_{4/2013}}{RPI_{10/2008}}$$

$$= 10000 \times \frac{0.03}{2} \times \frac{171.4}{149.2} = £172.319$$

Coupon on 25/10/2013

$$= 10000 \times \frac{0.03}{2} \times \frac{RPI_{10/2013}}{RPI_{10/2008}}$$

$$= 10000 \times \frac{0.03}{2} \times \frac{173.8}{149.2} = £174.732$$

Redemption on 25/10/2013 $= 10000 \times \frac{173.8}{149.2}$

$$= £11,648.794$$

(ii) Purchase Price at 25/10/2012 = PV at real rate of $3\frac{1}{2}\%$ p.a. effective of future cash flows.

= PV at $3\frac{1}{2}\%$ p.a. effective of "25/10/2012 money values" of future cash flows.

Future cash flows expressed in 25/10/2012 money values

Coupon at 25/4/2013 $= 172.319 \times \frac{RPI_{10/2012}}{RPI_{4/2013}}$

$$= 172.319 \times \frac{169.4}{171.4} = £170.308$$

Coupon at 25/10/2013 $= 174.732 \times \frac{RPI_{10/2012}}{RPI_{10/2013}}$

$$= 174.732 \times \frac{169.4}{173.8} = £170.308$$

(same as 25/4/2013, as expected)

Redemption at 25/10/2013 = $11648.794 \times \dfrac{169.4}{173.8} \quad = \quad £11,353.888$

$$\left[\text{or } 10000 \times \frac{RPI_{10/2012}}{RPI_{10/2008}} = 10000 \times \frac{169.4}{149.2} = 11353.888 \right]$$

Hence Price at 25/10/2012

$$= 170.308 \times \frac{1}{(1.035)^{\frac{1}{2}}} + \frac{170.308 + 11353.888}{(1.035)}$$

$$= £11,301.89$$

Many candidates had difficulty in recognising that the real yield would be based on using the inflation-adjusted cashflows as at the time of purchase. Some candidates made no adjustment at all whereas others incorrectly assumed that the inflation rate would be constant throughout the holding period.

6 (i) Redington's first condition states that the *pv* of the assets should equal the *pv* of the liabilities.

Working in £ million:

pv of assets
$$\begin{aligned} &= 7.404v^2 + 31.834v^{25} \text{ at } 7\% \\ &= 7.404 * 0.87344 + 31.834 * 0.18425 \\ &= 6.467 + 5.865 \\ &= 12.3323 \end{aligned}$$

pv of liabilities
$$\begin{aligned} &= 10v^{10} + 20v^{15} \text{ at } 7\% \\ &= 10 * 0.50835 + 20 * 0.36245 \\ &= 5.0835 + 7.249 \\ &= 12.3324 \end{aligned}$$

Allowing for rounding, Redington's first condition is satisfied.

Redington's second condition states that the DMT of the assets should equal the DMT of the liabilities. Given denominator of DMTs of assets and liabilities have been shown to be equal, we only need to consider the numerators.

Numerator of DMT of assets
$$\begin{aligned} &= 7.404 * 2 * v^2 + 31.834 * 25 * v^{25} \text{ at } 7\% \\ &= 6.467 * 2 + 5.865 * 25 \\ &= 159.569 \end{aligned}$$

Numerator of DMT of liabilities $= 10*10*v^{10} + 20*15*v^{15}$ at 7%
$= 5.0835*10 + 7.249*15$
$= 159.569$

Allowing for rounding, Redington's 2nd condition is satisfied.

(ii) Profit $= 7.404v^{2} + 31.834v^{25} - 10v^{10} - 20v^{15}$ at 7.5%
$= 6.40692 + 5.22011 - 4.85194 - 6.75932$
$= 0.015772$ i.e. a profit of £15,772

(iii) It can be seen that the spread of the assets is greater than the spread of the liabilities. This will mean that Redington's third condition for immunization is also satisfied, and that therefore a profit will occur if there is a small change in the rate of interest. Hence we would have anticipated a profit in (ii).

Parts (i) was answered well. Equating volatilities instead of DMTs was perfectly acceptable in this part. Part (ii) was also generally answered well although some candidates estimated the answer by using an estimation based on volatility rather than calculating the answer directly as asked. Part (iii) was less well answered with some candidates ignoring this part completely and others stating that Redington's 3rd condition was satisfied without further explanation.

7 Let K_t and S_t denote the forward price of the contract at time t, and the stock price at time t respectively.

Let r be the risk-free rate per annum at time $t = \frac{1}{2}$

Then, $K_0 = S_0 e^{0.04}$

and $K_{\frac{1}{2}} = 0.98\ S_0\ e^{\frac{1}{2}r}$

The value of the contract $V_{\frac{1}{2}}$ is $\left(K_{\frac{1}{2}} - K_0\right)e^{-\frac{1}{2}r}$

Hence $V_{\frac{1}{2}} = \left(K_{\frac{1}{2}} - K_0\right)e^{-\frac{1}{2}r}$

$= S_0 \times \left(0.98e^{\frac{1}{2}r} - e^{0.04}\right)e^{-\frac{1}{2}r}$

And

$V_{\frac{1}{2}} > 0$ when $0.98e^{\frac{1}{2}r} > e^{0.04}$

which is when $r > 2\ln\left(\dfrac{e^{0.04}}{0.98}\right) = 12.041\%$ p.a.

One of the worst answered questions on the paper. Some candidates, who did not complete the question, lost some of the marks that would have been available to them by not showing clear working e.g. writing down one half of a formula without explaining what the formula was supposed to represent.

8 (i) Let DPP be t. We want (all figures in £000s)

$$50,000 = 6,000a_{\overline{t|}}^{(2)} \quad \text{at 9\% p.a.}$$

$$= 6,000 \times \frac{i}{i^{(2)}} \times a_{\overline{t|}}$$

$$\Rightarrow a_{\overline{t|}} = \frac{50}{6 \times 1.022015}$$

$$= 8.1538268$$

$$\Rightarrow v^t = 1 - 8.1538268 \times 0.09$$

$$\Rightarrow t = \frac{\ln(1 - 8.1538268 \times 0.09)}{\ln 1.09^{-1}} = 15.360$$

$\therefore$ Take DPP as 15.5 years

(ii) Profit at the end of 20 years is

$$-50,000 \times (1.09)^{15.5} \times (1.07)^{4.5} + 6,000 \times s_{\overline{15.5|}}^{(2)} \times (1.07)^{4.5} + X$$

where

$$s_{\overline{15.5|}}^{(2)} = \frac{(1+i)^{15.5} - 1}{i^{(2)}} \quad \text{at 9\%}$$

$$= \frac{1.09^{15.5} - 1}{0.088061}$$

$$= 31.8285476$$

and to find X we work in half-years:

$$X = 3,000 s_{\overline{9}|} \text{ at } j\% \text{ where } (1+j)^2 = 1.07$$

$$= 3,000 \times \frac{(1+j)^9 - 1}{j}$$

$$= 3,000 \times \frac{(1.07)^{9/2} - 1}{(1.07)^{1/2} - 1}$$

$$= 31,030.35528$$

$\therefore$ Profit = -257,814.7272+258,937.5717+31,030.35528

= 32,153.20

$(= £32,153,200)$

Part (i) was answered well although candidates lost marks for not recognising that the DPP could only be at the time of income receipt i.e. at the end of a half-year. Part (ii) was answered badly with some candidates ignoring the initial profit obtained at the end of the DPP. A common error in the calculation of the profit arising after the DPP was to calculate the present value rather than the accumulated value.

9 (i) We can find the one-year forward rates $f_{1,1}$ and $f_{2,1}$ from the spot rates y_1, y_2 and y_3:

$$(1+y_2)^2 = (1+y_1)(1+f_{1,1})$$

$$\Rightarrow (1+0.037)^2 = (1+0.036)(1+f_{1,1})$$

$$\Rightarrow f_{1,1} = 3.800\% \text{ p.a.}$$

and

$$(1+y_3)^3 = (1+y_2)^2(1+f_{2,1})$$

$$\Rightarrow (1.038)^3 = (1.037)^2(1+f_{2,1})$$

$$\Rightarrow f_{2,1} = 4.000\,\% \text{ p.a.}$$

(ii) (a) Price per £100 nominal

$$= 4\left(\underset{3.6\%}{v} + \underset{3.7\%}{v^2} + \underset{3.8\%}{v^3}\right) + 105 \underset{3.8\%}{v^3}$$

$$= 4 \times 2.78931 + 105 \times 0.89414$$

$$= £105.0425$$

(b) Let yc_2 = two-year par yield

$$1 = yc_2\left(\underset{3.6\%}{v} + \underset{3.7\%}{v^2}\right) + \underset{3.7\%}{v^2}$$

$$\Rightarrow yc_2 = 3.6982\ \%\ \text{p.a.}$$

Questions on the term structure of interest rates have caused significant problems for candidates in past years but this question was generally answered very well.

10 (i) Let X = initial payment

$$20000 = (X - 50)a_{\overline{25|}} + 50(Ia)_{\overline{25|}}$$

$$= (X - 50) \times 10.6748 + 50 \times 98.4789$$

$$= 10.6748X - 533.74 + 4923.95$$

$$\Rightarrow X = \frac{15609.80}{10.6748} = £1,462.31.$$

(ii) After 3 years, capital o/s is:

$$1562.31a_{\overline{22|}} + 50(Ia)_{\overline{22|}}$$

$$= 1562.31 \times 10.2007 + 50 \times 87.1264$$

$$= £\,20,293.01$$

(iii) The loan has actually increased from £20,000 to £20,293.01. The reason for this is that the loan is being repaid by an increasing annuity and, in the early years, the interest is not covered by the repayments (e.g. 1st year: Interest is 0.08 × 20000 = £1,600 but 1st instalment is £1462.31 and so interest is not covered).

(iv) Total of instalments paid

$$= 25\times 1462.31+\frac{24\times 25}{2}\times 50 = 51557.66$$

$\Rightarrow$ Total interest = 51557.66 − 20000 = £31557.66

Parts (i) and (ii) were answered well, although in part (ii) some candidates incorrectly calculated the instalment that would be paid in the fourth year. Part (iii) was also answered relatively better than similar explanation questions in previous years. Many candidates failed to include the effect of the increasing payments in the calculation of the total instalments in part (iv) despite having correctly allowed for this in earlier parts.

11 (i) $PV = \int_4^{10} 3{,}000\nu(t)\,dt$

where $\nu(t)$ is as follows:

$0 \le t < 4$

$$\nu(t) = e^{-\int_{o}^{t}(0.03+0.01t)dt} \quad = e^{-\left[0.03t+\frac{1}{2}x0.01t^2\right]}$$

$4 \le t < 6$

$$\nu(t) \ = e^{-0.20}.e^{-\int_4^t 0.07dt} = e^{-0.20}.e^{(-0.07t+0.28)}$$

$$= e^{0.08-0.07t}$$

$t \ge 6$

$$\nu(t) \ = e^{-0.34}.e^{-\int_6^t 0.09dt} = e^{-0.34}.e^{(-0.09t+0.54)}$$

$$= e^{(0.20-0.09t)}$$

$$\Rightarrow PV \ = 3{,}000\int_4^6 \left(e^{0.08-0.07t}\right)dt \ + 3{,}000\int_6^{10} e^{(0.20-0.09t)}dt$$

$$= \frac{3{,}000e^{0.08}}{-0.07}\left[e^{-0.42} - e^{-0.28}\right] + \frac{3{,}000e^{0.20}}{-0.09}\left[e^{-0.90} - e^{-0.54}\right]$$

$$= 4584.02 + 7172.83 = \$11{,}756.85$$

(ii) $11.75685 = 3\left(\bar{a}_{\overline{10}|} - \bar{a}_{\overline{4}|}\right)$

at i = 6%, RHS = 3(1.029709)[7.3601 − 3.4651] = 12.03215

at i = 7%, RHS = 3(1.034605)[7.0236 − 3.3872] = 11.28671

by interpolation

$$\therefore i = 0.06 + \left(\frac{12.03215 - 11.75685}{12.03215 - 11.28671} \times 0.01\right) = 0.06369 \text{ i.e. } 6.4\%$$

(actual answer is 6.36%)

One of the worst answered questions on the paper with the different formulation of a question based on varying forces of interest causing problems for many candidates. It is also possible to answer part (i) as a combination of continuous deferred annuities. Part (ii) was poorly answered even by candidates who had made a good attempt to part (i).

12 (i) $1 + i_t \sim \text{LogNormal}\left(\mu, \sigma^2\right)$

$$S_{12} = \prod_{1}^{12} (1 + i_t)$$

$$\Rightarrow \ln S_{12} = \sum_{1}^{12} \ln(1 + i_t) \sim N\left(12\mu, 12\sigma^2\right)$$

$$E(1 + i_t) = 1.08 \quad = \exp\left\{\mu + \sigma^2 / 2\right\}$$

$$\text{Var}(1 + i_t) = 0.05^2 \quad = \exp\left\{2\mu + \sigma^2\right\}\left(\exp\left(\sigma^2\right) - 1\right)$$

$$= 1.08^2\left(\exp\left(\sigma^2\right) - 1\right)$$

$$\Rightarrow e^{\sigma^2} \quad = 1 + \frac{0.05^2}{1.08^2}$$

$$\Rightarrow \sigma^2 \quad = 0.002141053$$

$$\mu \quad = \ln 1.08 - \sigma^2 / 2$$

$$= 0.075890514$$

Hence S_{12} has LogNormal distribution with parameters 0.910686 and 0.025692636

(ii) PV of annuity at time 12:

$$PV = 4000\underbrace{\ddot{a}^{(12)}_{\overline{4}|}}_{7\%} + 5000\underbrace{\ddot{a}^{(4)}_{\overline{2}|}v^4}_{7\%} + 5000\underbrace{\ddot{a}^{(4)}_{\overline{2}|}}_{9\%}\underbrace{v^6}_{7\%} + 6000\underbrace{\bar{a}_{\overline{4}|}v^2}_{9\%}\underbrace{v^6}_{7\%}$$

$$\begin{aligned} &= 1000\,(4\times1.037525\times3.3872+5\times1.043380\times1.8080\times0.76290 \\ &\qquad +5\times1.055644\times1.7591\times0.66634 \\ &\qquad +6\times1.044354\times3.2397\times0.84168\times0.66634) \end{aligned}$$

$$\begin{aligned} &= 1000\times(14.057219+7.195791+6.186911+11.385358) \\ &= 38{,}825.28 \end{aligned}$$

Hence

$$\begin{aligned} \text{Prob}\,(18{,}000S_{12} \geq 38{,}825.28) &= \text{Prob}\,(S_{12} \geq 2.15696) \\ &= \text{Prob}\left(Z \geq \frac{\ln\left(2.15696\right)-0.910686}{\sqrt{0.025692636}}\right) \\ &= \text{Prob}\,(Z \geq -0.8858) \\ &= \Phi\left(0.89\right) \\ &= 0.81 \\ &\text{i.e. } 81\% \end{aligned}$$

This question provided the greatest range of quality of answers. Many candidates scored well on part (i) although common errors included assuming that $E\left(1+i_t\right)=0.08$ *and/or that* $\text{Var}\left(1+i_t\right)=0.05$. *Few candidates calculated the correct value of the required present value in part (ii) and candidates who made errors in this part lost further marks by not showing clear working or sufficient intermediate steps (although the examiners recognise that some candidates might have been under time pressure by the time they attempted this question). The probability calculation was often answered well by candidates who attempted this part.*

END OF EXAMINERS' REPORT