INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2012 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

D C Bowie
Chairman of the Board of Examiners

December 2012

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the September 2012 paper

The general performance was of a lower standard compared with the previous two exams. Well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q3(ii), Q4(ii) and Q9(iii) were less well answered than those that just involved calculation. This is an area to which attention should be paid. Candidates should note that it is important to explain and show understanding of the concepts and not just mechanically go through calculations. At the other end of the spectrum, there was a difficulty for many candidates when it came to answering questions involving introductory ideas.

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 Let d be the annual simple rate of discount.

The discounted value of 100 in the deposit account would be x such that:

$$x = 100(1.04)^{-91/365} = 99.0269$$

$\therefore$ to provide the same effective rate of return a treasury bill that pays 100 must have a price of 99.0269 and $100\left(1-\frac{91}{365}\times d\right) = 99.0269$

$$\therefore\ 1-\frac{91}{365}\times d = \frac{99.0269}{100} = 0.990269$$

$$d = (1\text{-}0.990269) \times \frac{365}{91} = 0.03903$$

Many candidates scored full marks on this question but many others failed to score any marks at all. Some candidates incorrectly used (1-*nd*) *as an accumulation factor*

2 (i) $e^{-\delta/4} = 1-\frac{0.08}{4} = 0.98 \quad \therefore \delta = 0.080811$

(ii) $(1+i)^{-1} = \left(1-\frac{0.08}{4}\right)^4 = 0.92237 \therefore\ i = 0.084166$

(iii) $\left(1-\frac{d^{(12)}}{12}\right)^{12} = \left(1-\frac{0.08}{4}\right)^4 = 0.92237 \therefore d^{(12)} = 0.080539$

A lot of marginal candidates scored very badly on this question even though it was covering an introductory part of the syllabus.

3 (i) $(1+i)^{2.5} = \frac{140}{120}\times\frac{600}{140+200} = 2.05882$

$\therefore\ 1 + \mathrm{i} = 1.33490$

$\therefore\ i = 33.49\%$ p.a. effective.

(ii) The money weighted rate of return weights performance according to the amount of money in the fund. The fund was performing better after it had been given the large injection of money on 1/1/2011.

Part (i) was answered well. The type of explanation asked for in part (ii) is commonly asked for in CT1 exams. To get full marks, candidates should address the specific situation given in the question rather than just repeat the bookwork.

4 (i) Present value of dividends, I, is:

$$I = \left(v^{1/4} + v^{1/2} + v^{3/4}\right)$$

Calculated at $i'\%$ when $(1+i') = (1.04)^2 = 1.0816$

So $I = 1.0816^{-1/4} + 1.0816^{-1/2} + 1.0816^{-3/4}$
$= 2.88499$

Hence, forward price, F, is:

$$F = (10 - 2.88499)(1+i')^{10/12} \text{ at } 8.16\%$$
$$= \left(10 - 2.88499\right) \times 1.0816^{10/12} = £7.5956$$

(ii) The price of the forward can be determined from the price of the share (for which it is a close substitute). The forward is like the share but with delayed settlement and without dividends.

5 (i) The characteristics of a Eurobond are:

- Medium- or long-term borrowing
- Unsecured
- Regular coupon payments
- Redeemed at par
- Issued and traded internationally/not in the jurisdiction of any one country
- Can be denominated in any currency (e.g. not the currency of issuer)
- Tend to be issued by large companies, governments or supra-national organisations
- Yields depend on issue size and issuer (or marketability and risk)
- Issue characteristics may vary – market free to allow innovation

(ii) (a) The characteristics of a certificate of deposit are:

- Tradable certificate issued by banks stating that money has been deposited
- Terms to maturity between one and six months
- Interest payable on maturity/issued at a discount
- Security and marketability will depend on issuing bank
- Active secondary market

(b) Answer is i such that $(1.01)^{12}\left(1+\frac{i^{(12)}}{12}\right)^{12}=(1.02)^{24}$ giving

$i^{(12)}=36.119\%$

6 (i) Amount of loan is:

$100(Ia)_{\overline{10|}}+100a_{\overline{10|}}$ at 6% p.a.

= 100 ×36.9624 + 100 × 7.3601
= 3696.24 + 736.01 = £4,432.25

(ii) (a) the o/s loan after sixth instalment is:

$100(Ia)_{\overline{4|}}+700\,a_{\overline{4|}}$

=100 × 8.4106 + 700 × 3.4651 = 841.06 + 2425.57 = £3,266.64

The interest component is therefore:

0.06 × 3266.64 = £196.00

(b) The capital component =

800-196.00 = £604.00

(iii) The capital remaining after the seventh instalment is
3266.64 – 604.00 = 2662.64

Let the new instalment = X

$Xa_{\overline{8|}}=2{,}662.64$ at 8%

$a_{\overline{8|}}=5.7466$; X = 2,662.64/5.7466 = £463.34

7 (i) Expected annual interest rate in both ten-year periods =
0.04 ×0.3 + 0.06 × 0.7 = 0.054 or 5.4%

Amount of the investment would be X such that:

$X\,(1.054)^{20}=200{,}000$

$\therefore X$ = £69,858.26

(ii) Expected accumulation factors in both ten-year periods are:

$0.3\,(1.04)^{10} + 0.7(1.06)^{10} = 1.697667$

The accumulation factors in each ten-year period are independent.

Therefore the expected accumulation is:

$69{,}858.26 \times 1.697667 \times 1.697667$

$= £201{,}336.55$

Therefore the value of investment over and above £200,000 = £1,336.55.

(iii) The extreme outcomes for the investment are:

$69{,}858.26 \times 1.04^{20} = 153{,}068.06$
$69{,}858.26 \times 1.06^{20} = 224{,}044.91.$

Therefore the range is: £70,976.85

Many candidates struggled with this question and seemed to have difficulty particularly with part (ii). Part (iii) was also badly answered even though part (ii) was not needed to answer part (iii).

8 (i) $t \le 9$

$$v(t) = e^{-\int_0^t (0.03+0.01s)\,ds}$$
$$= e^{-\left[0.03s+\frac{0.01s^2}{2}\right]_0^t}$$
$$= e^{-\left[0.03t+0.005t^2\right]}$$

$t > 9$

$$V(t) = e^{-\left[\int_0^9 \delta(s)\,ds+\int_9^t 0.06\,ds\right]}$$
$$= V(9).e^{0.06(t-9)}$$
$$= e^{-0.675}.e^{-0.06(t-9)}$$
$$= e^{-(0.135+0.06t)}$$

(ii) (a) $PV = 5,000\, e^{-(0.135+0.06\times 15)}$

$$= 5,000\, e^{-1.035}$$
$$= £1,776.13$$

(b) $1,776.13\, e^{\delta\times 15} = 5,000$

$$e^{\delta\times 15} = 2.81511$$

$$15\delta = \ln 2.81511$$

$$\delta = \frac{\ln 2.81511}{15} = 0.0690$$

(iii)

$$P.V. = \int_{11}^{15} e^{-(0.135+0.06t)} \times 100\, e^{-0.02t}\, dt$$
$$= \int_{11}^{15} 100\, e^{-0.135-0.08t}\, dt$$
$$= 100\, e^{-0.135} \left[\frac{e^{-0.08t}}{-0.08}\right]_{11}^{15}$$
$$= 100\, e^{-0.135}(5.18479 - 3.76493)$$
$$= 124.055$$

Generally answered well but some candidates lost marks in part (i) by not deriving the discount factor for t < 9.

9 (i) Expectations theory: yields on short and long-term bonds are determined by expectations of future interest rates as it is assumed that a long-term bond is a substitute for a series of short-term bonds.
[If interest rates are expected to rise (fall) long-term bonds will have higher (lower) yields that short-term bonds.]

Liquidity preference: it is assumed that investors have an inherent preference for short-term bonds because interest-rate sensitivity is lower. As such, (there is an upward bias on the expectations-based yield curve) and longer-term bonds will offer a higher expected return than implied by expectations theory on its own. *N.B. the part in brackets is not in core reading.*

Market segmentation: bonds of different terms to redemption are attractive to different investors with different liabilities.

The supply of bonds of different terms to redemption will depend on the strategy of the relevant issuer. The term structure is determined by the interaction of supply and demand in each term-to-redemption segment.

(ii) Duration = $\dfrac{\sum tC_t \, v^t}{\sum C_t \, v^t} = \dfrac{4\times(Ia)_{\overline{n|}} + 100n\,v^n}{4\times a_{\overline{n|}} + 100\,v^n}$

For $n = 1$ to 5. Clearly duration on one-year bond is one year.

Term	$(Ia)_{\overline{n\|}}$	$100v^n$	$a_{\overline{n\|}}$	$n100v^n$	$4(Ia)_{\overline{n\|}}$
3	5.3580	86.384	2.7232	259.152	21.432
5	12.5664	78.353	4.3295	391.765	50.2656

Duration of three-year bond:

$$\frac{21.432 + 259.152}{4\times 2.7232 + 86.384} = 2.884 \text{ years}$$

Duration of five-year bond:

$$\frac{50.2656 + 391.765}{4\times 4.3295 + 78.353} = 4.620 \text{ years}$$

(iii) The duration of a bond is the average time of the cashflows weighted by present value. The coupon payments of the 8% coupon bond will be a higher proportion of the total proceeds than for the 4% coupon bond. Thus, a greater proportion of the total proceeds of the 8% coupon bond will be received before the end of the term. The average time of the cashflows will be shorter and hence the duration will be lower.

(iv) **Option 1**

The equation of value would be:

$$95 = 4a_{\overline{4|}} + 79v^5$$

The rate of return is zero (incoming and outgoing cash flows are equal).

Option 2

The equation of value would be:

$$95 = 4a_{\overline{4|}} + v^4\,a_{\overline{8|}} + 100v^{12}$$

$i = 2.5\%$

RHS = 4 × 3.762 + 0.90595 × 7.1701 + 100 × 0.74356
= 15.0479 + 6.4958 + 74.3556 = 95.8993

i = 3%

RHS = 4 × 3.7171 + 0.88849 × 7.0197 + 100 ×0.70138
= 14.8684 + 6.2369 + 70.1380
= 91.2433

By interpolation:

$$i = 0.005 \times \left(\frac{95.8998 - 95}{95.8998 - 91.2433} \right) + 0.025$$

= 0.025966 or 2.6% per annum effective.

Hence Option 2 would provide the higher rate of return

(v) Two of the following:

- Option 2 creates a higher duration bond which might not be suitable for the investor
  ….e.g. alternative investments may be available in the longer term
- The credit risk over the longer duration may be greater
- The inflation risk over the longer duration may be greater
- There may be tax implications because of the differing capital and income combinations.
- the institution could reinvest the proceeds from option 1 at whatever rate of return prevails.

Part (i) was often poorly answered even though this was bookwork and candidates also struggled with part (ii). In part (ii) it is important to include the correct units for the duration (in this case, years). Most candidates made a good attempt at part (iv) even if some made calculation errors (e.g. in the calculation of the outstanding term of the bond under Option 2). Marginal candidates scored badly on parts (iii) and (v).

10 (i) The payback period simply looks at the time when the total incoming cash flows are greater than the total outgoing cash flows. It takes no account of interest at all.

Though the discounted payback period takes account of interest that would have to be paid on loans, it only looks at when loans used to finance outgoing cash flows would be repaid and not at the overall profitability of the projects.

(ii) (a) Outgoing cash flow = £3m

In £m, at time t, total incoming cash flows are £0.64t

We need t such that $3 = 0.64t$

$$t = \frac{3}{0.64} = 4.6875 \text{ years}$$

(b) Present value of incoming cash flows at time t is:

$$0.64\bar{a}_{\overline{t}|} = 0.64\left(\frac{1-v^t}{\delta}\right) \text{where } \delta = 0.039221$$

Require t such that:

$$0.64\left(\frac{1-v^t}{0.039221}\right) = 3$$

$1 - v^t = 0.183848$
$v^t = 0.816152$
$t \ln v = \ln 0.816152$

$$t = \frac{\ln 0.816152}{\ln v}$$

$$= -\frac{-0.203155}{-0.039221} = 5.1798 \text{ years}$$

(iii) Crossover point is the rate of interest at which the n.p.v. of the two projects is equal. As the present value of the cash outflows for both projects is the same at all rates of interest, the crossover point is the rate of interest at which the present value of the cash inflows from both projects is equal.

P.V of cash inflows from Project B = $0.64\bar{a}_{\overline{6}|}$
P.V of cash inflows from Project A =

$$0.5\, v^{1/2} + 1.1 \times 0.5 \times v^{1^{1/2}} + \cdots + 1.1^5 \times 0.5 \times v^{5^{1/2}}$$
$$= 0.5v^{1/2}\left[\frac{1-1.1^6 \times v^6}{1-1.1 \times v}\right]$$

Therefore require i such that:

$$0.64\, \bar{a}_{\overline{6}|} - 0.5\, v^{1/2}\left[\frac{1-1.1^6 v^6}{1-1.1v}\right] = 0$$

Let $i = 4\%$

$a_{\overline{6}|} = 5.2421 \quad i/\delta = 1.019869$

$v^{1/2} = 0.98058 \quad v = 0.96154$

$v^6 = 0.79031$

$1.1^6 = 1.77156$

$$\text{LHS} = 0.64 \times 5.2421 \times 1.019869 - 0.5 \times 0.98058 \left[\frac{1 - 1.77156 \times 0.79031}{1 - 1.1 \times 0.96154} \right]$$

$= 3.4216 - 0.49029 \times 6.93490 = 3.4216 - 3.4001$
$= 0.0215$

Let $i = 0\%$

$$\text{LHS} = 0.64 \times 6 - 0.5 \times \left[\frac{1 - 1.77156}{1 - 1.1} \right] = 3.84 - 3.8578 = -0.0178$$

Given that NPV of Project A is greater than that of project B at 0% per annum effective and the reverse is true at 4% per annum effective, the NPV of the two projects must be equal at some point between 0% and 4%.

(iv) **Project A**

Duration is:

$$\frac{v^{1/2} 0.5 (0.5 + 1.1 \times 1.5 v + 1.1^2 \times 2.5 v^2 + 1.1^3 \times 3.5 v^3 + 1.1^4 \times 4.5 v^4 + 1.1^5 \times 5.5 \times v^5)}{0.49029 \times 6.93490}$$

Term in brackets is

0.5 + 1.58654 + 2.79678 + 4.14139 + 5.63183 + 7.28047 = 21.93702.

$$\therefore \text{ Duration} = \frac{0.98059 \times 0.5 \times 21.93702}{0.49029 \times 6.93490} = 3.163 \text{ years}$$

Project B

$$\text{Duration is}: \frac{0.64 \int_0^6 t v^t \, dt}{0.64 \int_0^6 v^t \, dt} = \frac{(\bar{I}\bar{a})_{\overline{6}|}}{\bar{a}_{\overline{6}|}} = \frac{\left(i/\delta \; a_{\overline{6}|} - 6v^6 \right) / \delta}{i/\delta \; a_{\overline{6}|}}$$

$$= \frac{\left(1.019869 \times 5.2421 - 6 \times 0.79031\right)/0.039221}{1.019869 \times 5.2421}$$

$$\frac{15.41000}{5.3462} = 2.882 \text{ years}$$

(v) Project A has a longer duration and therefore the present value of its incoming cash flows is more sensitive to changes in the rate of interest. As such, when the interest rate rises, the present value of incoming cash flows falls more rapidly than for Project B.

Most candidates could calculate the discounted payback period but struggled with the undiscounted equivalent. As in Q9, the units should be included within the answer. The working of many candidates in part (iii) was often unclear even when the formulae were correctly derived. In part (iv) many candidates incorrectly thought the duration should be $\frac{(I\bar{a})_{\overline{6|}}}{\bar{a}_{\overline{6|}}}$ *for Project B.*

END OF EXAMINERS' REPORT