INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2012 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and who are using past papers as a revision aid, and also those who have previously failed the subject. The Examiners are charged by Council with examining the published syllabus. Although Examiners have access to the Core Reading, which is designed to interpret the syllabus, the Examiners are not required to examine the content of Core Reading. Notwithstanding that, the questions set, and the following comments, will generally be based on Core Reading.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report. Other valid approaches are always given appropriate credit; where there is a commonly used alternative approach, this is also noted in the report. For essay-style questions, and particularly the open-ended questions in the later subjects, this report contains all the points for which the Examiners awarded marks. This is much more than a model solution – it would be impossible to write down all the points in the report in the time allowed for the question.

T J Birse
Chairman of the Board of Examiners

July 2012

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the April 2012 paper

The general performance was broadly similar to the previous two exams. Well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q2(iii), Q5(iii) and Q6(iv) were less well answered than those that just involved calculation. Marginal candidates should note that it is important to explain and show understanding of the concepts and not just mechanically go through calculations. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 (i) Price, P, of £100 nominal stock is:

$$P = 3v_{y_1} + 3v_{y_2}^2 + 103v_{y_3}^3$$

where

$y_1 =$ 0.041903

$y_2 =$ 0.043625

$y_3 =$ 0.045184

$\Rightarrow P = 95.845$

And gross redemption yield, i%, solves:

$$95.845 = 3a_{\overline{3|}} + 100v^3 \text{ at } i\%$$

Try 4% RHS = 97.225
5% RHS = 94.554

$$\Rightarrow i = 0.04 + 0.01 \times \frac{97.225 - 95.845}{97.225 - 94.554}$$

= 0.0452

i.e. 4.5% p.a.

(ii) y_1, y_2 and y_3 as above. $y_4 = 0.046594$

$$1 = (yc_4)\left(v_{y_1} + v_{y_2}^2 + v_{y_3}^3 + v_{Y_4}^4\right) + v_{y_4}^4$$

$$\Rightarrow 1 = yc_4 \times 3.587225 + 0.8334644$$
$$\Rightarrow yc_4 = 0.04642 \text{ i.e. } 4.642\% \text{ p.a.}$$

2 (i) TWRR, i, is given by:

$$\frac{2.9}{2.3} \times \frac{4.2}{2.9 + 1.5} = 1 + i \Rightarrow i = 0.204 \text{ or } 20.4\% \text{ p.a.}$$

(ii) MWRR, i, is given by:

$$2.3 \times (1+i) + 1.5(1+i)^{8/12} = 4.2$$

Then, we have:

$$\left.\begin{array}{lll} i=12\% & \Rightarrow & LHS=4.1937 \\ i=13\% & \Rightarrow & LHS=4.2263 \end{array}\right\} \Rightarrow i=0.12+\left(0.13-0.12\right)\times\left(\frac{4.2-4.1937}{4.2263-4.1937}\right)$$

$= 0.122$

or 12.2% p.a.

(iii) The MWRR is lower as fund performs better before the cash inflow than after. Then, as the fund is larger after the cash inflow on 1 May 2011, the effect of the poor investment performance after this date is more significant in the calculation of the MWRR.

The calculations were performed well but the quality of the explanations in part (iii) was often poor. This type of explanation is commonly asked for in CT1 exams. To get full marks, candidates should address the specific situation given in the question rather than just repeat the bookwork.

3 (i) Let R = annual repayment

$$500,000 = R\, a_{\overline{10|}}^{9\%} = R \times 6.4177$$

$$\Rightarrow R = 77,910.04$$

and total interest $= 10 \times 77,910.04 - 500,000$

$$= 279,100$$

(ii) (a) Capital outstanding at beginning of 8th year is:

$$77910.04\, a_{\overline{3|}}^{9\%} = 77909.53 \times 2.5313$$

$$= 197,213.28$$

Let R' be new payment per annum then

$$R'\, a_{\overline{4|}}^{(4)} = R' \times 1.043938 \times 3.0373 = 197,213.28$$

$$\Rightarrow \quad R' = 62,196.62$$

and quarterly payment is £15,549.16

(b) Interest content of 2nd quarterly payment is:

$$15549.16\times\left(1-v_{12\%}^{3\frac{3}{4}}\right)=5383.41$$

[Or Capital in 1st quarterly payment is

$$15549.16-197213.28\times\left(1.12^{\frac{1}{4}}-1\right)=9,881.77$$

So capital outstanding after 1st quarterly payment

$$=197213.28-9881.77=187331.51$$

$\Rightarrow$ Interest in next payment is

$$187331.51\times\left(1.12^{\frac{1}{4}}-1\right)=5383.41\,]$$

Generally answered well but a number of candidates made errors in calculating the remaining term in part (ii)

4 (i) The "no arbitrage" assumption means that neither of the following applies:

(a) an investor can make a deal that would give her or him an immediate profit, with no risk of future loss;

nor

(b) an investor can make a deal that has zero initial cost, no risk of future loss, and a non-zero probability of a future profit.

(ii) The current value of the forward price of the old contract is:

$$7.20\times(1.025)^4-1.20\,a_{\overline{5}|}^{2\frac{1}{2}\%}$$

whereas the current value of the forward price of a new contract is:

$$10.45-1.20\,a_{\overline{5}|}^{2\frac{1}{2}\%}$$

Hence, current value of old forward contract is:

$$10.45-7.20\times(1.025)^4=£2.5025$$

(iii) The current value of the forward price of the old contract is:

$$7.20(1.025)^{4}(1.03)^{-9} = 6.0911$$

whereas the current value of the forward price of a new contract is:

$$10.45(1.03)^{-5} = 9.0143$$

$\Rightarrow$ current value of old forward contract is:

$$9.0143 - 6.0911 = £2.9232$$

This was the most poorly answered question on the paper but well-prepared candidates still scored full marks. Some candidates in part (ii) assumed that the dividend income was received during the lifetime of the forward contract. Whilst the examiners did not believe that such an approach was justified, candidates who assumed this alternative treatment of the income were not penalised. It was very clear that the poor performance on the question was not as result of this alternative interpretation.

5 (i) The equation of value is:

$$1309.5 = 100\left(a_{\overline{5}|}^{(4)} + (1.05)^{5} v^{5} a_{\overline{5}|}^{(4)} + \ldots + (1.05)^{20} v^{20} a_{\overline{5}|}^{(4)}\right) - 12a_{\overline{25}|}^{(4)}$$

Rearranging:

$$1309.5 = 100a_{\overline{5}|}^{(4)}\left(\frac{1-(1.05v)^{25}}{1-(1.05v)^{5}}\right) - 12a_{\overline{25}|}^{(4)}$$

At 9%, RHS is:

$$100 \times 1.033144 \times 3.8897 \times \left(\frac{1-\left(\frac{1.05}{1.09}\right)^{25}}{1-\left(\frac{1.05}{1.09}\right)^{5}}\right) - 12 \times 1.033144 \times 9.8226$$

$$= 401.8570 \times \frac{0.607292}{0.170505} - 121.7779$$

$= 1309.53 \Rightarrow$ IRR is 9% p.a.

(ii) For Project B the equation of value is

$$1000 = 85\ddot{a}_{\overline{20}|}^{(4)} + v^{20}\ 90\ddot{a}_{\overline{5}|}^{(4)}$$

Roughly $1000 \simeq 85a_{\overline{25}|} \Rightarrow i \simeq 7\%$

At 7% RHS is 1039.05
= 85 × 1.043380 × 10.5940 + 0.25842 × 1.043380 × 4.1002 × 90

8% RHS is 956.78
= 85 × 1.049519 × 9.8181 + 0.21455 × 1.049519 × 3.9927 × 90

$\Rightarrow i \simeq 7.5\%$ p.a.

(iii) Project A is more attractive since it has the higher IRR. However, the investor will also need to take into account other factors such as:

- the outlay is much higher for Project A than Project B
- the interest rate at which the investor might need to borrow at to finance a project since it will affect the net present values and discounted payback periods of the projects
- the risks for each project that the rents and expenses will not be those assumed in the calculations.

In part (i) candidates were asked to demonstrate that the internal rate of return was a given value. In such questions, candidates should set up the equation of value and clearly show each stage of their algebra and their calculations (including the evaluation of all factors that make up the equation). Many candidates claimed that they had shown the correct answer despite obvious errors and/or insufficient working. Candidates who tried to create a "proof" where the arguments didn't follow logically gained few marks. In this type of question, if you can't complete a proof, it is better to show how far you have got and be open about being unable to proceed further. This will generally gain more intermediate markst.

Part (ii) was answered well but in part (iii) few candidates came up with any of the other factors that should be considered.

6 (i) Price per £100 nominal is given by:

$$P = 5 \times a_{\overline{18}|}^{3.158\%} + 100v_{3.158\%}^{18} = 5 \times \left(\frac{1 - v_{3.158\%}^{18}}{0.03158} \right) + 100v_{3.158\%}^{18} = 125.00$$

(ii) As coupons are payable annually and the gross redemption yield is equal to the annual coupon rate, the new price per £100 nominal is £100.

i.e. $P = 5a_{\overline{13}|}^{5\%} + 100v_{5\%}^{13} = 5\left(\frac{1 - v_{5\%}^{13}}{0.05} \right) + 100v_{5\%}^{13} = 100.00$

(iii) Equation of value is:

$$125.00 = 5a_{\overline{5}|} + 100v^5 \Rightarrow i = 0\%$$

Thus, the investor makes a return of 0% per annum over the period.

(iv) Longer-dated bonds are more volatile.

Thus, as a result of the rise in gross redemption yields from 3.158% per annum on 1 March 2007 to 5% on 1 March 2012, the fall in the price of the bond would be greater.

Thus, as the income received over the period would be unchanged, the overall return achieved would be reduced (as a result of the greater fall in the capital value).
[In fact, the price on 1 March 2007 would have been £133.91 per £100 nominal falling to £100 per £100 nominal on 1 March 2012.

i.e. in this case, we need to find *i* such that $133.91 = 5a_{\overline{5}|} + 100V^5 \Rightarrow i < 0\%$.]

The first three parts were generally well-answered although relatively few candidates noticed that parts (ii) and (iii) could be answered quickly and consequently many candidates made avoidable calculation errors.

7 (i) $E(1+i) = e^{\mu + \frac{1}{2}\sigma^2}$

$= e^{0.05 + \frac{1}{2} \times 0.004}$

$= 1.0533757$

$\therefore E[i] = 0.0533757$ since $E(1+i) = 1 + E(i)$

Let A be the accumulation of £5000 at the end of 20 years

then $E[A] = 5000\,\ddot{s}_{\overline{20|}}$ at rate $j = 0.0533757$

$$= 5000\frac{\left((1+j)^{20}-1\right)}{j}\times(1+j)$$

$$= 5000\frac{\left(1.0533757^{20}-1\right)}{0.0533757}\times 1.0533757$$

$$= £180{,}499$$

(ii) Let the accumulation be S_{20}

S_{20} has a log-normal distribution with parameters 20μ and $20\sigma^2$

$$\therefore E[S_{20}] = e^{20\mu+\frac{1}{2}\left(20\sigma^2\right)} \qquad \left\{\text{or}\,(1+j)^{20}\right\}$$

$$= \exp(20\times 0.05 + 10\times 0.004)$$

$$= e^{1.04} \quad = 2.829217$$

$$\ln S_{20} \sim N\left(20\mu, 20\sigma^2\right)$$

i.e. $\ln S_{20} \sim N(1,\ 0.08)$

$$P\left(S_{20} > e^{1.04}\right) = P(\ln S_{20} > 1.04)$$

$$= P\left(Z > \frac{1.04-1}{\sqrt{0.08}}\right) \quad \text{where} \quad Z \sim N(0,1)$$

$$= P(Z > 0.14) = 1-\Phi(0.14)$$

$$= 1-0.56 = 0.44$$

Questions regarding annual investments are comparatively rarely asked on this topic and students seemed to struggle with part (i). Part (ii) was answered better in general than equivalent questions in previous exams.

8 (i) for $t > 8$

$$v(t) = \exp -\left\{\int_0^5 0.04 + 0.003t^2 dt + \int_5^8 0.01 + 0.03t dt + \int_8^t 0.02 dt\right\}$$

$$= \exp -\left\{\left[0.04t + 0.001t^3\right]_0^5 + \left[0.01t + 0.015t^2\right]_5^8 + \left[0.02t\right]_8^t\right\}$$

$$= \exp -\left\{0.2 + 0.125 + 0.01\times 3 + 0.015\left(8^2 - 5^2\right) + 0.02t - 0.02\times 8\right\}$$

$$= \exp -\left\{0.325 + 0.615 + 0.02t - 0.16\right\}$$

$$= e^{-(0.78+0.02t)}$$

Hence PV of £1,000 due at t = 10 is:

$$1000\times \exp -\left(0.78 + 0.02\times 10\right) = £375.31$$

(ii)

$$1000\left(1 - \frac{d^{(4)}}{4}\right)^{4\times 10} = 375.31$$

$$\left(1 - \frac{d^{(4)}}{4}\right)^{40} = \frac{375.31}{1000}$$

$$d^{(4)} = 4\left(1 - \left(\frac{375.31}{1000}\right)^{1/40}\right)$$

$$\underline{= 0.09681}$$

(iii)

$$PV = \int_{10}^{18} \rho(t) v(t)\, dt$$

$$= \int_{10}^{18} 100e^{0.01t} \times e^{-(0.78+0.02t)}$$

$$= 100e^{-0.78}\int_{10}^{18} e^{-0.01t} dt$$

$$= 100e^{-0.78}\left\{\left[\frac{e^{-0.01t}}{-0.01}\right]_{10}^{18}\right\}$$

$$= \frac{100}{0.01} e^{-0.78} \left(e^{-0.1} - e^{-0.18} \right)$$

$$= £318.90$$

Parts (i) and (ii) were answered well. Some candidates made errors in part (iii) by not discounting the payment stream back to time 0.

9

(i)

$$PV = 100 \times 0.35 \left(1.03v + 1.03 \times 1.05v^2 + 1.03 \times 1.05 \times 1.06v^3 + 1.03 \times 1.05 \times 1.06^2 v^4 + \cdots \right)$$

$$= 35 \left(1.03v + 1.03 \times 1.05v^2 + \frac{1.03 \times 1.05 \times 1.06v^3}{1 - 1.06v} \right) \quad @ \ 8\%$$

$$= 35 \left(\frac{1.03}{1.08} + \frac{1.03 \times 1.05}{1.08^2} + \frac{1.03 \times 1.05 \times 1.06}{1.08^3} \times \frac{1.08}{0.02} \right)$$

$$= 35 \left(0.95370 + 0.92721 + 49.14223 \right)$$

$$= £1785.81$$

(ii) Real rate of return is i such that:

$$1720 = 35 \left(1.03 \times \frac{110}{112.3} v + 1.03 \times 1.05 \times \frac{110}{113.2} v^2 + 1.03 \times 1.05 \times 1.06 \times \frac{110}{113.8} v^3 \right) + 1800 \times \frac{110}{113.8} v^3$$

$$= 35 \left(1.0089047v + 1.050928v^2 + 1.108110v^3 \right) + 1739.894552v^3$$

$$= 35.3116645v + 36.78248v^2 + 1778.678402v^3$$

For initial estimate, assume all income received at end of 3 years:

1720 ≈ 1850.77v3

$\Rightarrow v \approx 0.9758696 \quad \Rightarrow i \approx 2.4727$

Try i = 2.5%, RHS = 1721.14 ≈ 1720

so i = 2.5%

Most candidates made a good attempt at part (i) although slight errors in setting up the equation and/or in the calculation were common. Many candidates struggled with setting up the required equation in part (ii).

10 (i) Working in 000's

$$PV_L = 200a_{\overline{20}|} + 300v^{15} \quad @ \ 8\%$$
$$= 200 \times 9.818147 + 300 \times 0.315242$$
$$= 2058.20199$$

i.e. £2,058,201.99

(ii)

$$DMT_L = \frac{200v + 200 \times 2v^2 + 200 \times 3v^3 + \cdots + 200 \times 20v^{20} + 300 \times 15v^{15}}{200a_{\overline{20}|} + 300v^{15}}$$
$$= \frac{200\left(Ia\right)_{\overline{20}|} + 300 \times 15v^{15}}{2058.20199} \quad @ \ 8\%$$
$$= \frac{200 \times 78.9079 + 300 \times 15 \times 0.31524}{2058.20199}$$
$$= \frac{17200.175}{2058.20199} = 8.3569 \text{ years}$$

(iii) Redington's first two conditions are:

$$\Rightarrow PV_L = PV_A$$
$$\Rightarrow DMT_L = DMT_A$$

Let the nominal amount in securities A and B be *X* and *Y* respectively.

$$PV_A = PV_L \Rightarrow X\left(0.09a_{\overline{12}|} + v^{12}\right) + Y\left(0.04a_{\overline{30}|} + v^{30}\right) = 2058201.99 \ @ \ 8\%$$
$$\Rightarrow X\left(0.09 \times 7.5361 + 0.39711\right) + Y\left(0.04 \times 11.2578 + 0.09938\right)$$
$$\Rightarrow 1.075361X + 0.549689Y = 2058201.99$$
$$\Rightarrow X = \frac{2058201.99 - 0.549689Y}{1.075361}$$

$$DMT_A = DMT_L \Rightarrow \frac{X\left(0.09\left(Ia\right)_{\overline{12}|} + 12v^{12}\right) + Y\left(0.04\left(Ia\right)_{\overline{30}|} + 30v^{30}\right)}{2058201.99} = 8.3569$$
$$\Rightarrow X\left(0.09\left(Ia\right)_{\overline{12}|} + 12v^{12}\right) + Y\left(0.04\left(Ia\right)_{\overline{30}|} + 30v^{30}\right) = 17200175 \quad @ \ 8\%$$
$$\Rightarrow X\left(0.09 \times 42.17 + 12 \times 0.39711\right) + Y\left(0.04 \times 114.7136 + 30 \times 0.09938\right) = 17200175$$
$$\Rightarrow 8.56066X + 7.56986Y = 17200175$$
$$\Rightarrow 8.56066 \times \frac{\left(2058201.99 - 0.549689Y\right)}{1.075361} + 7.56986Y = 17200175$$
$$\Rightarrow 3.19394Y = 815370.9$$
$$\Rightarrow Y = 255287, \ X = 1783470$$

Hence company should purchase £1,783,470 nominal of security A and £255,287 nominal of security B for Redington’s first two conditions to be satisfied.

(iv) Redington’s third condition is that the convexity of the asset cash flow series is greater than the convexity of the liability cash flow series. Therefore the convexities of the asset cash flows and the liability cash flows will need to be calculated and compared.

Generally well answered but candidates’ workings in part (iii) were often unclear which made it difficult for examiners to award marks when calculation errors had been made.

END OF EXAMINERS’ REPORT