INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2011 examinations

Subject CT1 — Financial Mathematics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

July 2011

General comments

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

The general performance was slightly worse than in April 2010 but well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q3(ii) and Q6(iii) were less well answered than those that just involved calculation. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 (i) We want $1000e^{\int_3^7 \delta(s)\,ds}$

$$=1000\,e^{\left[\int_3^5\left(0.04+0.003s^2\right)ds+\int_5^7\left(0.01+0.03s\right)ds\right]}$$

where $\int_3^5\left(0.04+0.003s^2\right)ds=\left[0.04s+0.001\,s^3\right]_3^5$

$= 0.325 - 0.147 = 0.178$

and $\int_5^7\left(0.01+0.03s\right)ds=\left[0.01s+\dfrac{0.03}{2}s^2\right]_5^7$

$=0.805-0.425=0.380$

$\Rightarrow$ accumulation at $t=7$ is

$1000e^{(0.178+0.380)}=1000e^{0.558}=1{,}747.17$

(ii) $1747.17\left(1-\dfrac{d^{(12)}}{12}\right)^{4\times 12}=1000$

$\Rightarrow d^{(12)}=0.138692$

2 Forward price of the contract is $K_0=\left(S_0-I\right)e^{\delta T}=\left(68-I\right)e^{0.14\times 1}$

where I is the present value of income during the term of the contract $=2.5e^{-0.14\times 8/12}$

$\Rightarrow K_0=\left(68-2.5e^{-0.14\times 8/12}\right)e^{0.14}=75.59919$

Forward price a new contract issued at time r (3 months) is
$K_r=\left(S_r-I^*\right)e^{\delta(T-r)}=\left(71-I^*\right)e^{0.12\times 9/12}$

(where I^* is the present value of income during the term of the contract)
$=2.5e^{-0.12\times 5/12}\Rightarrow K_{0.25}=\left(71-2.5e^{-0.05}\right)e^{0.09}=75.08435$

Value of original contract = $\left(K_r - K_0\right)e^{-\delta(T-r)}$

$$=\left(75.08435-75.59919\right)e^{-0.12\times 9/12}$$

$$=-0.47053=-47.053p$$

Many candidates failed to incorporate the change in the value of δ. Another common error was in counting the number of months.

3 (i) $$135,000=7,900\times\frac{121.4}{125.6}.\,v+8,400\times\frac{121.4}{131.8}v^2+8,800\times\frac{121.4}{138.7}v^3$$

$$+9,400\times\frac{121.4}{145.3}v^4+\left(10,100+151,000\right)\times\frac{121.4}{155.2}v^5$$

at i' % where i' = real yield

Approx yield:

$$135,000 = (7635.828+7737.178+7702.379+7853.820+126015.077)\,v^5$$

$$\Rightarrow i'\simeq 3.1\% \text{ p.a.}$$

Try $i'=3\%$, RHS $=137434.955$

Try $i'=3.5\%$, RHS $=134492.919$

$$i'=0.035-0.005\times\frac{135000-134492.919}{137434.955-134492.919}$$

$$=0.03414 \text{ (i.e. 3.4\% p.a.)}$$

(ii) The term:

$$8,800\times\frac{121.4}{RPI\ \text{(June 2008)}}$$

would have a lower value (i.e. the dividend paid on 30 June 2008 would have a lower value when expressed in June 2005 money units). The real yield would therefore be lower than 3.4% p.a.

The most common error on this question was incorrect use of the indices, e.g. many candidates inverted them. Several candidates also had difficulty in setting up the equation of *value. The examiners noted that a large number of final answers were given to excessive levels of accuracy given the approximate methods used.*

4 (i) We can find forward rates $f_{2,1}$ and $f_{2,2}$ from:

$(1+y_3)^3 = (1+y_2)^2(1+f_{2,1})$ and

$(1+y_4)^4 = (1+y_2)^2(1+f_{2,2})^2$

$\Rightarrow (1.033)^3 = (1.032)^2(1+f_{2,1})$

$\Rightarrow f_{2,1} = 3.50029\ \%$ p.a.

and $(1.034)^4 = (1.032)^2(1+f_{2,2})^2$

$\Rightarrow f_{2,2} = 3.60039\ \%$ p.a.

(ii) (a) Price per £100 nominal

$$4\left(v_{3.1\%} + v^2_{3.2\%} + v^3_{3.3\%}\right) + 115\, v^3_{3.3\%}$$

$$= 4(0.969932 + 0.938946 + 0.907192) + 115 \times 0.907192$$

$= 115.59$

(b) Let $yc_3 = 3-$year par yield

$$1 = yc_3\left(v_{3.1\%} + v^2_{3.2\%} + v^3_{3.3\%}\right) + v^3_{3.3\%}$$

$$1 = yc_3(0.969932 + 0.938946 + 0.907192) + 0.907192$$

$\Rightarrow yc_3 = 0.032957$

i.e. 3.2957% p.a.

5 (i) $\left(1+\frac{i^{(2)}}{2}\right)^2 = 1.05 \Rightarrow i^{(2)} = 4.939\%$ (or use tables)

$$g(1-t_1) = \frac{0.06}{1.05} \times 0.80 = 0.0457$$

So $i^{(2)} > g(1-t_1) \Rightarrow$ there is a capital gain on the contract

(ii) Since there is a capital gain, the loan is least valuable to the investor if the repayment is made by the borrower at the latest possible date. Hence, we assume redemption occurs 25 years after issue in order to calculate the minimum yield achieved.

(iii) If A is the price per £100 of loan:

$$A = 100 \times 0.06 \times 0.80\, a_{\overline{25}|}^{(2)} (1.05)^{\frac{2}{12}} + \left(105 - 0.35(105 - A)\right) v^{24\frac{10}{12}} \text{ at } 5\%$$

$$= 4.8 \times 1.012348 \times 14.0939 \times (1.05)^{\frac{2}{12}} + \left(105 - 0.35(105 - A)\right) \times 0.29771$$

Hence $A = \dfrac{69.0452 + 20.3187}{1 - 0.35 \times 0.29771} = 99.759$

$\Rightarrow$ Price of loan = £99,759

The majority of this question was well-answered but most candidates struggled with the two month adjustment. This adjustment needs to be directly incorporated into the equation of value. Calculating the price first without adjustment and then multiplying by $(1+i)^{1/6}$ will lead to the wrong answer.

6 (i) MWRR is given by:

$$10.0 \times (1+i) + 5.5 \times (1+i)^{8/12} = 17.1$$

Try 11%, LHS = 16.996

Try 12%, LHS = 17.132

$$\text{MMRR} = 0.11 + 0.01 \times \frac{17.1 - 16.996}{17.132 - 16.996} = 11.8\% \text{p.a.}$$

(ii) TWRR is given by:

$$\frac{8.5}{10.0}\times\frac{17.1}{8.5+5.5}=1+i\Rightarrow i=3.821\%\text{p.a.}$$

(iii) MWRR is higher since fund received a large (net) cash flow at a favourable time (i.e. just before the investment returns increased).

(iv) TWRR is more appropriate. Cash flows into and out of the fund are outside the control of the fund manager, and should not influence the level of bonus payable. TWRR is not distorted by amount and/or timing of cash flows whereas MWRR is.

The calculations in parts (i) and (ii) were generally well done but parts (iii) and (iv) were poorly answered (or not answered at all) even by many of the stronger candidates. In (iii) for example, candidates were expected to comment on the timing of the cashflows for this particular year.

7 (i) Let initial quarterly amount be X. Work in time units of one quarter. The effective rate of interest per time unit is

$$\frac{0.08}{4}=0.02\text{ (i.e.2\% per quarter)}$$

So

$$60{,}000=X\,a_{\overline{80}|}+100v^{16}a_{\overline{64}|}+100v^{32}a_{\overline{48}|}+100v^{48}a_{\overline{32}|}+100v^{64}a_{\overline{16}|}\quad\text{at }2\%$$

$$\left(\text{where } a_{\overline{64}|}^{2\%}=\frac{1-v^{64}}{0.02}=35.921415\right)$$

$$=39.7445X+2{,}616.695465+1{,}627.606705+907.1436682+382.3097071$$

$$\Rightarrow X=\frac{60{,}000-5{,}533.756}{39.7445}$$

$$=£1{,}370.41\text{ per quarter}$$

(ii) Interest paid at the end of the first quarter (i.e. on 1 October 1998) is

$$60{,}000\times0.02=£1{,}200$$

Hence, capital repaid on 1 October 1998 is

$$1370.41-1200=£170.41$$

Therefore, interest paid on 1 January 1999 is

$$(60000 - 170.41) \times 0.02 = 1196.59$$

$\Rightarrow$ capital repaid on 1 January 1999 is

$$1370.41 - 1196.59 = 173.82$$

(iii) Loan outstanding at 1 July 2011 (after repayment of instalment)

$$= 1670.41\, a_{\overline{12|}} + 1770.41\, v^{12}\, a_{\overline{16|}} \text{ at } 2\%$$

$$= 1670.41 \times 10.5753 + 1770.41 \times 0.78849 \times 13.5777$$

= £36,619

Candidates found this to be the most challenging question on the paper. The easiest method was to work in quarters with an effective rate of 2% per quarter. Where candidates worked using a year as the time period the most common error was to allow for an increase to payments of £100 pa when the increases were £400pa when they occurred. In part (i), the examiners were disappointed to see many attempts with incorrect and/or insufficient working end with the numerical answer that had been given in the question. A candidate who claims to have obtained a correct answer after making obvious errors in the working is not demonstrating the required level of skill and judgement and, indeed, is behaving unprofessionally.

Part (iii) was very poorly answered with surprisingly few candidates recognising the remaining loan was simply the present value of the last 28 payments.

8 (i) No, because the spread (convexity) of the liabilities would always be greater than the spread (convexity) of the assets then the 3rd Redington condition would never be satisfied.

(ii) Work in £millions

Let proceeds from four-year bond = X
Let proceeds from 20-year bond = Y

Require PV Assets = PV Liabilities

$$Xv^4 + Yv^{20} = 10v^3 + 20v^6 \qquad (1)$$

Require DMT Assets = DMT Liabilities

$$\Rightarrow 4Xv^4 + 20Yv^{20} = 30v^3 + 120v^6 \qquad (2)$$

(2) − 4 × (1)

$\Rightarrow 16Y\nu^{20} = 40\nu^6 - 10\nu^3$

$$\Rightarrow Y = \frac{40\nu^6 - 10\nu^3}{16\nu^{20}} = \frac{31.61258 - 8.88996}{7.30219} = £3.11175\text{m}$$

From (1):

$$X = \frac{10\nu^3 + 20\nu^6 - Y\nu^{20}}{\nu^4} = \frac{8.88996 + 15.80629 - 1.42016}{0.8548042} = £27.22973\text{m}$$

So amount to be invested in 4-year bond is

$X\nu4$=£23.27609m

And amount to be invested in 20-year bond is

$Y\nu^{20}$ = £1.42016m

Require Convexity of Assets > Convexity of Liabilities

$$\Rightarrow 20X\nu^6 + 420Y\nu^{22} > 120\nu^5 + 840\nu^8$$

LHS = 981.869 > 712.411 = RHS

Therefore condition is satisfied and so above strategy will immunise company against small changes in interest rates.

Or state that spread of assets (t = 4 to t = 20) is greater than spread of liabilities (t = 3 to t = 6).

Part (i) was poorly answered. In part (ii) many candidates correctly derived X and Y as the proceeds from the two bonds. However, only the better candidates recognised that the amounts to be invested (as required by the question) were therefore Xv^4 and Yv^{20}.

9 (i) PV of outgo (£000s)

$$105\left(1 + v^{\frac{1}{2}} + v\right) + 200v^{15} = 366.31 \qquad \text{at } 8\%$$

PV of income

$$\bar{a}_{\overline{1|}}\left[\begin{array}{l}20v+23v^2+26v^3+29v^4 \\ \qquad +29v^5 1.03\left(1+(1.03v)+(1.03v)^2+\ldots+(1.03v)^{24}\right)\end{array}\right]$$

$$=\bar{a}_{\overline{1|}}\left[20v+23v^2+26v^3+29v^4+29v^5 1.03\times\left(\frac{1-(1.03)^{25}v^{25}}{1-1.03v}\right)\right]$$

PV of income

$$=\bar{a}_{\overline{1|}}\{80.193+20.329\times 14.996\}=370.61$$

So NPV is 4.30 (=£4,300)

(ii) The NPV is very small. It is considerably less than the PV of the final year's income $\left(29\times(1.03)^{25}\times\bar{a}_{\overline{1|}}\times v^{29}=6.272\right)$; therefore the DPP must fall in the final year.

We know the DPP exists as the NPV > 0.

So DPP is $29+r$ where

$$366.31=\bar{a}_{\overline{1|}}\times\left\{80.193+20.329\times\left(\frac{1-(1.03)^{24}v^{24}}{1-1.03v}\right)\right\}$$

$$+29\times 1.03^{25}\times v^{29}\times\bar{a}_{\overline{r|}} \qquad \text{at } 8\%$$

$$\Rightarrow \quad 366.31=364.335+6.5169\bar{a}_{\overline{r|}}$$

$$\Rightarrow \quad \bar{a}_{\overline{r|}}=0.3031$$

$$\Rightarrow \quad v^r=0.97668 \quad \Rightarrow \quad r=0.307$$

So the DPP is 29.31.

This question tended to separate out the stronger and weaker candidates. The most common errors in part (i) were discounting for an extra year, not including the one-year annuity factor and incorrectly calculating the geometric progression. Many candidates also lost marks through poorly presented or illegible methods that were therefore difficult for the examiners to follow. Part (ii) was poorly attempted with few candidates completing the question.

10 (i)

$$E(1+i_t) = 1.06$$

$$Var(1+i_t) = 0.03^2 = 0.0009$$

$$\Rightarrow 1.06 = e^{\left(\mu+\frac{\sigma^2}{2}\right)} \qquad (1)$$

$$0.0009 = e^{\left(2\mu+\sigma^2\right)}\left(e^{\sigma^2}-1\right) \qquad (2)$$

$$\Rightarrow \frac{(2)}{(1)^2} = \frac{0.0009}{(1.06)^2} = e^{\sigma^2}-1$$

$$\Rightarrow \sigma^2 = Ln\left(\frac{0.0009}{(1.06)^2}+1\right)$$

$$= 0.000800676 \quad (\text{and } \sigma = 0.0282962)$$

$$\Rightarrow 1.06 = e^{\left(\mu+\frac{0.000800676}{2}\right)}$$

$$\therefore \mu = Ln(1.06) - \frac{0.000800676}{2}$$

$$= 0.0578686$$

(ii) (a) Working in £m. Assets would accumulate to $14 \times 1.04 = 14.56 < 15$

$\Rightarrow$ Probability = 1.00

(b) The guaranteed portion of the fund would accumulate to

$0.25 \times 14 \times 1.04 = 3.64.$

$\therefore$ non-guaranteed portion needs to accumulate to

$$15 - 3.64 = 11.36$$

$\therefore$ we require probability that

$$(0.75 \times 14)(1 + i_t) < 11.36$$

$$= \Pr(1 + i_t) < 1.081905$$

$$= \Pr(\ln(1 + i_t) < \ln 1.081905)$$

$$= \Pr\left(\frac{\ln(1 + i_t) - 0.0578686}{0.0282962} < \frac{\ln 1.081905 - 0.0578686}{0.0282962}\right)$$

$$= \Pr(Z < 0.7370169) \text{ where } Z \sim N(0,1).$$

$$= 0.77$$

(iii) (a) Return is fixed (= 4% p.a.) $\Rightarrow$ variance of return = 0

(b) Return from portfolio = $0.25 \times 0.04 + 0.75\, i_t$

$\therefore$ Variance of return = $0.75^2 Var(i_t)$

$$= 0.75^2 \times 0.0009 = 0.00050625$$

[In monetary terms the variance of return for (iii)(b) will be $(£14m)^2 \times 0.00050625 = £^2 99{,}225m$ which is equivalent to a standard deviation of £315,000]

This question was generally well answered by those candidates who had left enough time to fully attempt the question. In part (i) the common errors were equating the mean to 0.06 instead of 1.06 and using 0.03 as the variance instead of 0.03^2. Part (ii) was also well answered although many candidates quoted the probability of meeting liabilities when the probability of not meeting the liabilities was asked for. Part (iii) a) was answered well by the candidates who attempted it, while part b) was not answered well. In part (iii) answers given in terms of the annual return and in terms of the monetary amounts were both fully acceptable.

END OF EXAMINERS' REPORT