EXAMINERS' REPORT

April 2010 Examinations

Subject CT1 — Financial Mathematics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

R D Muckart
Chairman of the Board of Examiners

July 2010

Comments

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Well-prepared candidates scored well across the whole paper and the examiners were pleased with the general standard of answers. However, questions that required an element of explanation or analysis were less well answered than those which just involved calculation. The comments below concentrate on areas where candidates could have improved their performance.

Q2.

A common error was to divide the nominal payments by the increase in the index factor (rather than multiplying).

Q3.

Many candidates made calculation errors in this question but may have scored more marks if their working had been clearer.

Q6.

Many candidates assumed that the accumulation in part (i) was for a single payment.

Q7.

The calculation was often performed well. In part (ii), many explanations were unclear and some candidates seemed confused between DMT and convexity although a correct explanation could involve either of these concepts.

Q9.

A common error was to assume that income only started after three years rather than 'starting from the beginning of the third year'.

Q10.

This question was answered well but examiners were surprised by the large number of candidates who used interpolation or other trial and error methods in part (ii) when the answer had been given in the question. The examiners recommend that students pay attention to the details given in the solutions to parts (iii) and (iv). For such questions, candidates should be looking critically at the figures given/calculated and making points specific to the scenario rather than just making general statements taken from the Core Reading.

1 (i) (a) Options – holder has the right but not the obligation to trade
Futures – both parties have agreed to the trade and are obliged to do so.

(b) Call Option – right but not the obligation to BUY specified asset at specified price at specified future date.

Put Option – right but not the obligation to SELL specified asset at specified price at specified future date.

(ii) $K = 60e^{0.06\times 3/12} - 2.80e^{0.06\times 1/12} = 60.90678 - 2.81404 = £58.09$

2 (i) Cash flows:

Issue price:	Jan 08	$-0.98\times 100,000$	$= -£98,000$
Interest payments:	July 08	$0.02\times 100,000\times \dfrac{112.1}{110.5}$	$= £2,028.96$
	Jan 09	$0.02\times 100,000\times \dfrac{115.7}{110.5}$	$= £2,094.12$
	July 09	$0.02\times 100,000\times \dfrac{119.1}{110.5}$	$= £2,155.66$
	Jan 10	$0.02\times 100,000\times \dfrac{123.2}{110.5}$	$= £2,229.86$
Capital redeemed:	Jan 10	$100,000\times \dfrac{123.2}{110.5}$	$= £111,493.21$

(ii) Equation of value is:

$$98000 = 2028.96v^{\frac{1}{2}} + 2094.12v + 2155.66v^{1\frac{1}{2}} + 2229.86v^2 + 111493.21v^2$$

At 11%, RHS = 97955.85 ≈ 98000

3 Purchase price = 0.45 × 1,000,000 = £450,000

PV of dividends = $50000 \times (1-0.2) \times \left[\left(v^2 + v^{2\frac{1}{2}}\right) + 1.03\left(v^3 + v^{3\frac{1}{2}}\right) + 1.03^2\left(v^4 + v^{4\frac{1}{2}}\right) + \cdots\right]$

$$= 40000\left(v^2 + v^{2\frac{1}{2}}\right)\left[1 + 1.03v + 1.03^2 v^2 + \cdots\right] \text{ @ } 8\%$$

$$= 40000 \times 1.68231 \times \left(\frac{1}{1 - 1.03/1.08}\right) = 1{,}453{,}516$$

$\Rightarrow$ NPV = 1,453,516 – 450,000 = £1,003,516

4 Let i = money yield

$\Rightarrow 1 + i = 1.0285714 \times 1.05 = 1.08 \quad \Rightarrow i = 8\%$ p.a.

Check whether CGT is payable: compare $i^{(2)}$ with $(1-t)g$

$$(1-t)g = 0.8 \times \frac{6}{105} = 0.04571$$

From tables, $i^{(2)} = 7.8461\% \;\Rightarrow i^{(2)} > (1-t)g$

$\Rightarrow$ CGT is payable

$$P = 0.8 \times 6a_{\overline{10|}}^{(2)} + 105v^{10} - 0.25(105 - P)v^{10} \text{ @ } 8\%$$

$$= \frac{0.8 \times 6a_{\overline{10|}}^{(2)} + 0.75 \times 105v^{10}}{1 - 0.25v^{10}}$$

$$= \frac{4.8 \times 1.019615 \times 6.7101 + 78.75 \times 0.46319}{1 - 0.25 \times 0.46319}$$

= £78.39

5 (i) Let P denote the current price (per £100 nominal) of the security.

Then, we have:

$$P = \frac{7}{1.044} + \frac{7}{1.044 \times 1.047} + \frac{7}{1.044 \times 1.047 \times 1.049} + \frac{107}{1.044 \times 1.047 \times 1.049 \times 1.05} = 108.0872$$

(ii) The gross redemption yield, i, is given by:

$$108.09 = 7 \times a_{\overline{4}|}^{i\%} + 100 \times v_{i\%}^{4}$$

Then, we have:

$$\left.\begin{array}{lll} i = 5\% & \Rightarrow & RHS = 107.0919 \\ i = 4.5\% & \Rightarrow & RHS = 108.9688 \end{array}\right\} \Rightarrow i \approx 0.045 + (0.05 - 0.045) \times \left(\frac{108.0872 - 108.9688}{107.0919 - 108.9688}\right) = 0.0473$$

(iii) The gross redemption yield represents a weighted average of the forward rates at each duration, weighted by the cash flow received at that time.

Thus, increasing the coupon rate will increase the weight applied to the cash flows at the early durations and, as the forward rates are lower at early durations, the gross redemption yield on a security with a higher coupon rate will be lower than above.

Note to markers: no marks for simply plugging 9% pa in, and providing no explanation for result.

6 (i) $E(1+i) = e^{\mu + \frac{1}{2}\sigma^2}$

$$= e^{0.05 + \frac{1}{2} \times 0.004}$$

$$= 1.0533757$$

$\therefore E[i] = 0.0533757$ since $E(1+i) = 1 + E(i)$

Let A be the accumulation at the end of 25 years of £3,000 paid annually in advance for 25 years.

Then $E[A] = 3000\ddot{s}_{\overline{25|}}$ at rate $j = 0.0533757$

$$= 3000\frac{\left((1+j)^{25}-1\right)}{j}\times(1+j)$$

$$= 3000\frac{\left(1.0533757^{25}-1\right)}{0.0533757}\times 1.0533757$$

$$= £158,036.43$$

(ii) Let the accumulation be S_{20}

S_{20} has a log-normal distribution with parameters 20μ and $20\sigma^2$

$\therefore E[S_{20}] = e^{20\mu+\frac{1}{2}\times 20\sigma^2}$

$\left\{\text{or}\,(1+j)^{20}\right\}$

$= \exp(20\times 0.05+10\times 0.004)$

$= e^{1.04} = 2.829217$

In $S_{20} \sim N\left(20\mu, 20\sigma^2\right)$

$\Rightarrow$ In $S_{20} \sim N(1, 0.08)$

$\Pr(S_{20} > 2.829217) = \Pr(\ln S_{20} > \ln 2.829217)$

$$= \Pr\left(Z > \frac{\ln 2.829217\text{-}1}{\sqrt{0.08}}\right) \quad \text{where} \quad Z \sim N(0,1)$$

$$= \Pr(Z > 0.14) = 1-\Phi(0.14)$$

$$= 1 - 0.55567$$

$$= 0.44433 \text{ i.e. } 44.4\%$$

7 (i) DMT of liabilities is given by:

$$\frac{1\times 1\times v_{7\%}+2\times(1.038835)\times v_{7\%}^{2}+3\times(1.038835)^{2}\times v_{7\%}^{3}+\ldots+40\times(1.038835)^{39}\times v_{7\%}^{40}}{1\times v_{7\%}+(1.038835)\times v_{7\%}^{2}+(1.038835)^{2}\times v_{7\%}^{3}+\ldots+(1.038835)^{39}\times v_{7\%}^{40}}$$

$$=\frac{(1.038835)^{-1}\times\left[\left(\frac{1.038835}{1.07}\right)+2\times\left(\frac{1.038835}{1.07}\right)^{2}+3\times\left(\frac{1.038835}{1.07}\right)^{3}+\ldots+40\times\left(\frac{1.038835}{1.07}\right)^{40}\right]}{(1.038835)^{-1}\times\left[\left(\frac{1.038835}{1.07}\right)+\left(\frac{1.038835}{1.07}\right)^{2}+\left(\frac{1.038835}{1.07}\right)^{3}+\ldots+\left(\frac{1.038835}{1.07}\right)^{40}\right]}$$

$$=\frac{v_{i^*}+2\times v_{i^*}^{2}+3\times v_{i^*}^{3}+\ldots+40\times v_{i^*}^{40}}{v_{i^*}+v_{i^*}^{2}+v_{i^*}^{3}+\ldots+v_{i^*}^{40}}$$

$$=\frac{(Ia)_{\overline{40}|}^{i^*}}{a_{\overline{40}|}^{i^*}}$$

where $v_{i^*}\equiv\frac{1}{1+i^*}=\frac{1.038835}{1.07}\Rightarrow i^*=\frac{1.07}{1.038835}-1=\frac{0.07-0.038835}{1.038835}=0.03$.

Hence, DMT of liabilities is:

$$\frac{(Ia)_{\overline{40}|}^{3\%}}{a_{\overline{40}|}^{3\%}}=\frac{384.8647}{23.1148}=16.65 \text{ years}$$

(Alternative method for DMT formula

$$DMT=\frac{v(1+2gv+3g^2v^2+\cdots+40g^{39}v^{39})}{v(1+gv+g^2v^2+\cdots+g^{39}v^{39})}=\frac{v(I\ddot{a})_{\overline{40}|}^{3\%}}{v\ddot{a}_{\overline{40}|}^{3\%}}=\frac{(I\ddot{a})_{\overline{40}|}^{3\%}}{\ddot{a}_{\overline{40}|}^{3\%}}=\frac{(Ia)_{\overline{40}|}^{3\%}}{a_{\overline{40}|}^{3\%}}$$

where $g=1.038835$.)

(ii) Even if the fund manager invested entirely in the 15-year zero-coupon bond, the DMT of the assets will be only 15 years (and, indeed, any other portfolio of securities will result in a lower DMT).

Thus, it is not possible to satisfy the second condition required for immunisation (i.e. DMT of assets = DMT of liabilities).

Hence, the fund cannot be immunised against small changes in the rate of interest.

(iii) The other problems with implementing an immunisation strategy in practice include:

- the approach requires a continuous re-structuring of the asset portfolio to ensure that the volatility of the assets remains equal to that of the liabilities over time
- for most institutional investors, the amounts and timings of the cash flows in respect of the liabilities are unlikely to be known with certainty
- institutional investor is only immunised for small changes in the rate of interest
- the yield curve is unlikely to be flat at all durations
- changes in the term structure of interest rates will not necessarily be in the form of a parallel shift in the curve (e.g. the shape of the curve can also change from time to time)

8 (i) Loan = $4500a_{\overline{20|}} + 150\left(Ia\right)_{\overline{20|}}$ at 9%

$\Rightarrow$ Loan = $4500 \times 9.1285 + 150 \times 70.9055$

$= 41,078.25 + 10,635.83 = 51,714.08$

(ii) Loan o/s after 9th year = $\left(4500 + 1350\right)a_{\overline{11|}} + 150\left(Ia\right)_{\overline{11|}}$ at 9%

Loan o/s = $5,850 \times 6.8052 + 150 \times 35.0533$

$= 39,810.42 + 5258.00 = 45,068.42$

Repayment = $6000 - 45,068.42 \times 0.09 = £1,943.84$

(Alternative solution to (ii)

(ii) Loan o/s after 9th year = $\left(4500 + 1350\right)a_{\overline{11|}} + 150\left(Ia\right)_{\overline{11|}}$ at 9%

$= 5,850 \times 6.8052 + 150 \times 35.0533 = 45,068.42$ as before

Loan o/s after 10th year = $\left(4500 + 1500\right)a_{\overline{10|}} + 150\left(Ia\right)_{\overline{10|}}$ at 9%

$= 6,000 \times 6.4177 + 150 \times 30.7904 = 43,124.76$

Repayment = $45,068.42 - 43,124.76 = £1,943.66$)

(iii) Last instalment = $4650 + 19 \times 150 = 7500$

Loan o/s = $7500 a_{\overline{1|}} = 7500v$

Interest = $7500 \times 0.91743 \times 0.09 = £619.27$

(iv) Total payments $= 20 \times 4650 + \frac{1}{2} \times 19 \times 20 \times 150$

$= 93,000 + 28,500 = 121,500$

Total interest = 121,500 – 51,714.08 = £69,785.92

9 (i) $\text{NPV} = -5 - 3v^{1/4} + 1.7\bar{a}_{\overline{15|}} v^2$ @10%

$\text{NPV} = -5 - 3 \times 0.976454 + 1.7 \times 0.82645 \times \frac{i}{\delta} a_{\overline{15|}}$ @10%

$= -5 - 2.929362 + 1.404965 \times 1.049206 \times 7.6061$

$= -7.929362 + 11.21213458$

$= 3.282772575$

NPV = £3.283m

(ii) DPP is $t + 2$ such that

$1.7\bar{a}_{\overline{t|}} v^2 = 5 + 3v^{\frac{1}{4}} \Rightarrow 1.474097708 a_{\overline{t|}} = 7.929362$ @10%

$\frac{1 - 1.1^{-t}}{0.1} = 5.379129 \Rightarrow 1 - 1.1^{-t} = 0.5379129$

$\Rightarrow 0.4620871 = 1.1^{-t} \Rightarrow \ln 0.4620871 = -t \ln 1.1$

$\Rightarrow t = 8.100$

$\therefore DPP = 10.1$ years

(iii) Accumulated profit 17 years from start of project:

$$= 1.7\bar{s}_{\overline{6.9}|7\%} = 1.7 \times \frac{\left(1.07^{6.9} - 1\right)}{\delta} \quad @\,7\%$$

$$= 1.7 \times \frac{\left(1.07^{6.9} - 1\right)}{0.067659}$$

$$= 1.7 \times 8.79346$$

$$= £14.95m$$

10 (i) The values of the funds before and after the cash injections are:

	Manager A		*Manager B*	
1 January 2007	120,000		100,000	
31 December 2007	130,000	140,000	140,000	150,000
31 December 2008	135,000	145,000	145,000	155,000
31 December 2009	180,000		150,000	

Thus, TWRR for Manager A is given by:

$$(1+i)^3 = \frac{130}{120} \times \frac{135}{140} \times \frac{180}{145} \Rightarrow i = 0.0905 \text{ or } 9.05\%$$

And, TWRR for Manager B is given by:

$$(1+i)^3 = \frac{140}{100} \times \frac{145}{150} \times \frac{150}{155} \Rightarrow i = 0.0941 \text{ or } 9.41\%$$

(ii) MWRR for Manager A is given by:

$$120 \times (1+i)^3 + 10 \times (1 \times i)^2 + 10 \times (1+i) = 180$$

Then, putting $i = 0.094$ gives $LHS = 180.03$ which is close enough to 180.

(iii) Both funds increased by 50% over the three year period and received the same cashflows at the same times.

Since the initial amount in fund B was lower, the cash inflows received represent a larger proportion of fund B and hence the money weighted return earned by fund B over the period will be lower, particularly since the returns were negative for the 2nd and 3rd years.

[Could also note that for fund B:

$$100\times(1+i)^3+10\times(1\times i)^2+10\times(1+i)=150$$

So by a proportional argument $120\times(1+i)^3+12\times(1\times i)^2+12\times(1+i)=180$

which when compared with the equation for fund A in (ii) clearly shows that the return for B is lower.]

(iv) The money weighted rate of return is higher for fund A, whilst the time weighted return is higher for fund B.

When comparing the performance of investment managers, the time weighted rate of return is generally better because it ignores the effects of cash inflows or outflows being made which are beyond the manager's control.

In this case, Manager A's best performance is in the final year, when the fund was at its largest, whilst Manager B's best performance was in the first year, where his fund was at its lowest.

Overall, it may be argued that Manager B has performed slightly better than Manager A since Manager B achieved the higher time weighted return.

11 (i) $t<5$

$$v(t) = e^{-\int_0^t (0.04+0.02s)ds}$$

$$= e^{-\left[0.04s+0.01s^2\right]_0^t}$$

$$= e^{-\left[0.04t+0.01t^2\right]}$$

$t\geq 5$

$$v(t) = e^{-\left\{\int_0^5 (0.04+0.02s)ds+\int_5^t 0.05ds\right\}}$$

$$= v(5)\times e^{-[0.05(t-5)]}$$

$$= e^{-0.45}\times e^{-[0.05(t-5)]} = e^{-[0.05t+0.2]}$$

(ii) (a) $PV = 1{,}000e^{-[0.05\times 17+0.2]} = e^{-1.05}$

$= 349.94$

(b) $1000\left(1+\dfrac{i^{(12)}}{12}\right)^{-204} = 349.94$

$\Rightarrow i^{(12)} = 6.1924\%$

(iii) $PV = \int_6^{10} e^{-0.45}e^{-[0.05t-0.25]}10e^{0.01t}dt$

$= 10e^{-0.2}\int_6^{10} e^{-0.04t}dt$

$= 10e^{-0.2}\left[-\dfrac{e^{-0.04t}}{0.04}\right]_6^{10}$

$= 8.18733\times 2.90769$

$= 23.806$

(Alternative Solution to (iii)

Accumulated value at time $t = 10$

$= \int_6^{10} 10e^{0.01t}\left(\exp\int_t^{10} 0.05ds\right)dt$

$= \int_6^{10} 10e^{0.01t}\left(\exp[0.05s]_t^{10}\right)dt$

$= \int_6^{10} 10e^{0.01t}e^{0.5-0.05t}dt = \int_6^{10} 10e^{0.5-0.04t}dt$

$= \left[\dfrac{10e^{0.5-0.04t}}{-0.04}\right]_6^{10} = -276.293 + 324.233 = 47.940$

Present value = $v(10)\times 47.940 = 0.63763e^{-[0.05\times 10-0.25]}\times 47.940 = 23.806$

END OF EXAMINERS' REPORT