INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2013 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

D C Bowie
Chairman of the Board of Examiners

December 2013

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the September 2013 paper

This paper proved to have some questions where the vast majority of candidates scored well and others where many candidates found challenging. Well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q1(ii), Q8(iv), Q10(iii) and Q11(v) were less well answered than those that just involved calculation. This is an area to which attention should be paid. Candidates should note that it is important to explain and show understanding of the concepts and not just mechanically go through calculations.

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 (i) (a) $d = \dfrac{0.045}{1.045} = 0.043062 = 4.3062\%$

(b) $\left(1 - d^{(12)}/_{12}\right) = (1.045)^{-1/12}$

$\therefore 1 - \dfrac{d^{(12)}}{12} = 0.99634$

$\therefore d^{(12)} = 0.043936$ or 4.3936%

(c) $\left(1 + \dfrac{i^{(4)}}{4}\right)^4 = 1.045$

$\therefore \left(1 + \dfrac{i^{(4)}}{4}\right) = 1.011065$

$\therefore i^{(4)} = 0.044260$ or 4.4260%

(d) $1.045^5 = 1.24618$

$\therefore$ five-yearly effective rate is 24.618%

(ii) The answer to (i)(b) is bigger than the answer to (i)(a) because the rate of discount convertible monthly is applied each month to a smaller (already discounted) sum of money. As such, in order to achieve the same total amount of discounting the rate has to be slightly more than one twelfth of the annual rate of discount. [An answer relating to the concept of interest payable in advance would also be acceptable].

The calculations were performed well but the quality of the explanations in part (ii) was often poor. A common error in (i)(d) was to state the answer as $i^{(1/5)}$ *rather than* $\dfrac{i^{(1/5)}}{1/5}$.

2 Present value of dividend = $0.1 \times 1.04^{-0.5} = 0.09806$

Value of forward is $(1.8 - 0.09806) \times 1.04^{0.75} = £1.75275$

3 (i) Work in half years.

$$P = 2a_{\overline{20}|} + 100v^{20} \text{ @} 1\tfrac{1}{2}\%$$
$$= 2\times 17.1686+100\times 0.74247$$
$$= £108.584$$

(ii)
$$P = (2\times 0.75a_{\overline{20}|} + 100v^{20})(1.015)^{91/182.5}$$
$$= (2\times0.75\times17.1686+100\times0.74247)\times (1.015)^{91/182.5}$$
$$= £100.7452$$

Part (i) was answered well although some candidates assumed an annual effective rate of 3%. In part (ii) many candidates did not deal with the 91 days elapsed duration – discounting instead of accumulating the 10-year bond price and/or assuming that 91 days equated to a quarter of a year.

4 (i) The investor pays a purchase price at outset.

The investor receives a series of coupon payments and a capital payment at maturity

The coupon and capital payments are linked to an index of prices (possibly with a time lag)

[Time lag does not have to be mentioned].

(ii) The investor pays a purchase price at outset

Shareholders are paid dividends. These are not fixed but declared out of profits.
Dividends may be expected to increase over time ….
….but may cease if the company fails.

There is a high degree of uncertainty with regard to future cash flows.

No maturity date
Would receive a sale price on the sale of the shares

Generally poorly answered with many candidates just writing down all characteristics they knew about these assets rather than concentrating on the cashflows. Many candidates omitted mention of the initial purchase price in each part.

5 (i) The return from the bonds issued by Country A is: $\frac{106}{101} - 1 = 0.049505$

The expected cash flows from the bonds from Country B are:

0.1 × 0 + 0.2 × 100 + 0.3 × 50 + 0.4 × 106 = 77.4

The price to provide the same expected return is P such that:

$$P = \frac{77.4}{1.049505} = €73.749$$

(ii) The gross redemption yield from the bond is such that:

$$73.749 \times (1+i) = 106$$
$$\therefore i = 43.731\%$$

(iii) The risk is higher for Country B's bond. Although the gross redemption yield is such that the expected returns are equal, the investor may want a higher expected return to compensate for the higher risk.

Many candidates had trouble with part (ii) not recognising that the gross redemption yield calculation will not include any allowance for default.

6 Divide the number of cars by 100 to obtain the share due to the pension fund

PV of income = $365 \times 400\, \bar{a}_{\overline{1|}} + 365 \times 500\, \bar{a}_{\overline{1|}}\, v \times 1.1 \times \left(1 + 1.01^2 v + 1.01^4 v^2 + ... + 1.01^{36} v^{18}\right)$

$$= 365 \times 400 \frac{i}{\delta} a_{\overline{1|}} + 365 \times 500 \frac{i}{\delta} a_{\overline{1|}}\, v \times 1.1 \times \left(\frac{1 - 1.01^{38} v^{19}}{1 - 1.01^2 v} \right)$$

$$= 365 \times 400 \times 1.039487 \times 0.92593$$
$$+ 365 \times 500 \times 1.039487 \times 0.92593 \times 0.92593 \times 1.1 \times \left(\frac{1 - 1.45953 \times 0.23171}{1 - 1.0201 \times 0.92593} \right)$$

= 140,523 + 178,907 × 11.93247

= 140,523 + 2,134,801 = 2,275,324 so NPV = £275,324

Candidates made a variety of errors in this question often ignoring one or more parts of the scenario (e.g. pension fund's 1% share of the project, the fact that daily vehicle numbers were given in the question, 1% increases in both vehicle numbers and tolls from the second year). Nevertheless, candidates who set out their workings clearly and logically often scored the majority of the available marks.

7 (i) $(1 + i_t) \sim$ lognormal (μ, σ^2)

$$\ln(1+i_t) \sim N(\mu,\sigma^2)$$

$$\ln\prod_{t=1}^{5}(1+i_t) = \ln(1+i_t)+\ln(1+i_t)+\ldots+\ln(1+i_t)$$

$$\sim N(5\mu,5\sigma^2) \text{ by independence}$$

$$\therefore \prod_{t=1}^{5}(1+i_t) \sim \text{ lognormal } (5\mu,5\sigma^2)$$

$$E(1+i_t) = \exp\left(\mu+\frac{\sigma^2}{2}\right) = 1.055$$

$$\text{Var}(1+i_t) = \exp(2\mu+\sigma^2)\left[\exp(\sigma^2)-1\right] = 0.04^2$$

$$\frac{0.04^2}{1.055^2} = \left[\exp\left(\sigma^2\right)-1\right] \therefore \sigma^2 = 0.0014365$$

$$\exp\left(\mu+\frac{0.0014360}{2}\right) = 1.055 \Rightarrow$$

$$\mu = \ln 1.055 - \frac{0.0014365}{2} = 0.052823$$

$5\mu = 0.264113$

$5\sigma^2 = 0.007182.$

Let S_5 be the accumulation of one unit after five years:

$$E(S_5) = \exp\left(5\times\mu+\frac{5\sigma^2}{2}\right) = \exp\left(0.264113+\frac{0.007182}{2}\right)$$

$$= 1.30696$$

$$\text{Var}(S_5) = \exp(2\times 5\mu+5\sigma^2)\left[\exp\left(5\sigma^2\right)-1\right]$$

$$= \exp(2 \times 0.264113 + 0.007182).(\exp 0.007182 - 1)$$

= exp 0.53541 (exp 0.007182 – 1)

= 0.012313

Mean value of the accumulation of premiums is

7,850,000 × 1.30696 = £10,259,636.

Standard deviation of the accumulated value of the premiums is

$$7,850,000 \times \sqrt{0.012313} = £871,061$$

Alternatively:

Let i_t be the (random) rate of interest in year t. Let S_5 be the accumulation of a single investment of 1 unit after five years:

$$E(S_5) = E\left[(1+i_1)(1+i_2)\ldots(1+i_5)\right]$$

$$E(S_5) = E[1+i_1]E[1+i_2]\ldots E[1+i_5] \text{ as } \{i_t\} \text{ are independent}$$

$$E[i_t] = 0.055$$

$$\therefore \quad E(S_5) = (1.055)^5 = 1.30696$$

$$E\left(S_5^2\right) = E\left[\left[(1+i_1)(1+i_2)\ldots(1+i_5)\right]^2\right]$$

$$= E(1+i_1)^2 E(1+i_2)^2 \ldots E(1+i_5)^2 \text{ (using independence)}$$

$$= E\left(1+2i_1+i_1^2\right)E\left(1+2i_2+i_2^2\right)\ldots E\left(1+2i_5+i_5^2\right)$$

$$= \left(1+2\times 0.055+0.04^2+0.055^2\right)^5$$

as $E\left[i_i^2\right] = V[i_t] + E[i_t]^2 = 0.04^2 + 0.055^2$

$$\therefore \text{ Var}[S_5] = \left(1+2\times 0.055+0.04^2+0.055^2\right)^5 - (1.055)^{10}$$

$$\therefore = 1.114625^5 - (1.055)^{10} = 0.0123128$$

Mean value of the accumulation of premiums is

$$7{,}850{,}000 \times 1.30696 = £10{,}259{,}636.$$

Standard deviation of the accumulated value of the premiums is

$$7{,}850{,}000 \times \sqrt{0.012313} = £871{,}061$$

(ii) If the company invested in fixed-interest securities, it would obtain a guaranteed accumulation of £7,850,000 $(1.04)^5$ = £9,550,725. In one sense, there is a 100% probability that a loss will be made and therefore the policy is unwise. The "risky" investment strategy leads to an expected profit. On the other hand, the standard deviation of the accumulation from the risky investment strategy is £871,061. Whilst there is a chance of an even greater profit from this strategy, there is also a chance of a more considerable loss than from investing in fixed-interest securities.

A poorly answered questions with many candidates not including enough derivation of the required results in part (i). Some candidates mixed their answers between the two methods given above e.g. they calculated μ and σ^2 for the log normal route, then used these in the alternative method for the mean and variance of i_t. Other candidates just used 0.055 and $.04^2$ as their values of μ and σ^2 .

8 (i) Purchase price of the annuity (working in half-years)

$$5{,}000 a_{\overline{50|}}^{(6)} \text{ calculated at } i = 2\%$$

$$= 5{,}000 \frac{i}{i^{(6)}} a_{\overline{50|}}$$

$$i = 0.02$$

$$i^{(6)} = 0.019835$$

$$a_{\overline{50|}} = 31.4236$$

$$\text{Purchase price} = 5{,}000 \times \frac{0.02}{0.019835} \times 31.4236$$

$$= £158{,}422$$

Individual needs to invest X such that: (working in years)

$$X\ 1.05^{20} = 158{,}422$$

$$1.05^{20} = 2.653297$$

$$\therefore X = \frac{158{,}422}{2.653297} = £59{,}708$$

(ii) Real return is j such that:

$$59{,}708 = \frac{158{,}422}{(1+j)^{20}} \times \frac{143}{340}$$

$$\therefore (i+j)^{20} = \frac{158{,}422}{59{,}708} \times \frac{143}{340}$$

$$=1.11595$$

$$\therefore j = 0.550\%$$

(iii) The amount of the capital gain is:

$$158{,}422 - 59{,}708 = 98{,}714$$

Tax = $0.25 \times 98{,}714 = 24{,}679$

Proceeds of investment = 133,744

Net real return is j' such that:

$$59{,}708 = \frac{133{,}744}{(1+j')^{20}} \times \frac{143}{340}$$

$$\therefore (1+j')^{20} = \frac{133{,}744}{59{,}708} \times \frac{143}{340}$$

$$= 0.942106$$

$$\therefore j' = -0.2977\%$$

(iv) The capital gains taxed has taxed the nominal gain, part of which is merely to compensate the investor for inflation. The tax has therefore reduced the real value of the investor's capital and led to a negative real return.

Parts (i) and (ii) were generally answered well but many candidates struggled with the calculation of the capital gain in part (iii) not recognising that this would be based on money values.

9 (i) PV is:

$$400v + 380v^2 + 360v^3 + \ldots + 120v^5$$

$$420\, a_{\overline{15}|} - 20\left(Ia\right)_{\overline{15}|} \qquad @\,4\%$$

= 420 × 11.1184 - 20 × 80.8539

= 4,669.728 – 1,617.078 = £3,052.65.

(ii) Interest component:

= 0.04 × 3,052.65 = £122.106

Capital component = 400 – 122.106
= £277.894

(iii) Seven repayments remain and the PV of the remaining payments is:

$$240v + 220v^2 + \ldots + 120\, v^7$$

$$260\, a_{\overline{7}|} - 20\left(Ia\right)_{\overline{7}|} \qquad @\,4\%$$

$$= 260 \times 6.0021 - 20 \times 23.0678 = £1,099.19$$

The loan is written down to: 0.5 × 1,099.19
= £549.595

The present value of the new repayment is:

$$\left(X - 2\right) a_{\overline{10}|} + 2(Ia)_{\overline{10}|} \qquad @\,8\%$$

$$\therefore 549.595 = \left(X - 2\right) 6.7101 + 2 \times 32.6869$$

$$\therefore X - 2 = \frac{549.595 - 2 \times 32.6869}{6.7101} = 72.163$$

$$\therefore X = £74.16$$

The best answered question on the paper although some candidates, when calculating the outstanding loan in part (iii), stated that the repayment in year 8 was £420. Some candidates also used the incorrect formula $Xa_{\overline{10}|} + 2(Ia)_{\overline{10}|}$ *for the repayment in part (iv).*

10 (i) (a) $e^{\int_0^7 0.05+0.002t\,dt}$

$= e^{\left[0.05t+\frac{0.002t^2}{2}\right]_0^7}$

$= \exp\left[0.05\times 7\right]+\dfrac{0.002\times 49}{2}$

= exp (0.399) = 1.490331

(b) $e^{\int_0^6 0.05+0.002t\,\mathrm{dt}}$

$= e^{\left[0.05\times 6+\frac{0.002\times 36}{2}\right]}$

= exp (0.336) = 1.399339

(c) $\dfrac{1.490331}{1.399339}=1.06503$

(ii) (a) Let spot rate = i_7

$(1+i_7)^7 = 1.490334$

$\Rightarrow i_7 = 5.8656\%$ p.a. effective

(b) $(1+i_6)^6 = 1.39934$

$\therefore i_6 = 5.7598\%$ p.a. effective

(c) From (i) (c) 6.503% per annum effective.

(iii) The forward rate is the rate of interest in the seventh year. The spot rate, in effect, is the rate of interest per annum averaged over the seven years (a form of geometric average). As the force of interest is rising the rate of interest in the seventh year must be higher than the rate averaged over the full seven year period.

(iv) $v(t) = e^{-\int_0^t 0.05+0.002s\ ds}$

$$= e^{-\left[0.05s+\frac{0.002s^2}{2}\right]_0^t}$$

$$= e^{-0.05t-0.001t^2}$$

We require

$$\int_3^{10} \rho(t)v(t)dt$$

$$= \int_3^{10} \frac{30\ e^{-0.01t}}{e^{-0.001t^2}}.e^{-0.05t}.e^{-0.001t^2}dt$$

$$30\int_3^{10} e^{-0.06t}\ dt$$

$$\frac{30}{-0.06}\left[e^{-0.06t}\right]_3^{10}$$

$$= \frac{30}{-0.06}\left[e^{-0.6}-e^{-0.18}\right]$$

$= -500(0.548812 - 0.83527)$

$=143.229$

The calculations were well-done but only the best candidates clearly explained their reasoning in part (iii).

11 (i) Present value of liabilities annuity

$10\ a_{\overline{40|}}$ at 4% $\quad a_{\overline{40|}} = 19.7928$

$= 10 \times 19.7928 = £197.928$m

(ii) Call 10 year security "security A" and five year security "security B".

We need to calculate the PV of £100 nominal for each of security A and security B

P.V of £100 nominal of A is:

$$5a_{\overline{10}|} + 100v^{10} \quad @4\%$$

$$a_{\overline{10}|} = 8.1109 \quad v^{10} = 0.67556$$

$\therefore$ PV = 5×8.1109 + 67.556 = 108.1105

P.V of £100 nominal of B is:

$$10\ a_{\overline{5}|} + 100v^{5} \quad @\,4\%$$

$$a_{\overline{5}|} = 4.4518 \quad v^{5} = 0.82193$$

$\therefore$ PV = 44.518 + 82.193 = 126.711

£98.964m, is invested in each security.

$\dfrac{98{,}964{,}000}{108.1105} \times 100$ per £100 nominal of A is bought.

= £91,539,674 nominal

$\dfrac{98{,}964{,}000}{126.711} \times 100$ per £100 nominal of B is bought

= £78,102,138 nominal

[other ways of expressing units are okay, but marks will be deducted if units are not correct]

(iii) Duration of the liabilities

$$= \frac{\sum tc_t \ \mathrm{v}^t}{\sum c_t \ \mathrm{v}^t}$$

Numerator $= \displaystyle\sum_{t=1}^{40} 10\ t\ v^t$ (in £ m)

$$= 10(Ia)_{\overline{40}|} = 10 \times 306.3231 = 3063.231 \quad \text{at 4\% p.a. effective}$$

$\therefore$ Duration = 3063.231/197.928 = 15.48 years

(iv) Numerator of duration is:

$$(5(Ia)_{\overline{10|}} + 10 \times 100\, v^{10}) \times 915,396.74$$

$$+(10(Ia)_{\overline{5|}} + 5 \times 100\, v^{5}) \times 781,021.38$$

Following the same reasoning as for the calculation of the duration of the annuity payments, adding the capital repayment and multiplying by the number of units of £100 nominal bought.

$$(Ia)_{\overline{10|}} = 41.9922$$

$$v^{10} = 0.67556$$

$$(Ia)_{\overline{5|}} = 13.0065$$

$$v^{5} = 0.82193$$

$$= (5 \times 41.9922 + 10 \times 100 \times 0.67556) \times 915,396.74$$

$$+(10 \times 13.0065 + 5 \times 100 \times 0.82193) \times 781,021.38$$

$$= 810,603,000 + 422,554,000$$

$$= 1,233,157,000$$

$\therefore$ Duration = 1,233,157,000/197,928,000
= 6.23 years

(v) The duration (and therefore the volatility) is greater for the liabilities than for the assets. As a result, when interest rates fall, the present value of the liabilities will rise by more than the present value of the assets and so a loss will be made.

Many candidates wrongly assumed that the same nominal amounts were bought of each asset rather than each asset amount having the same present value. This assumption made the calculations in part (ii) somewhat easier and the marks awarded in this part took this into account. Part (iii) was answered well. The explanations in part (v) were often poorly stated although time pressures at the end of the paper may have contributed to this.

END OF EXAMINERS' REPORT