INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2010 Examinations

Subject CT1 — Financial Mathematics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

T J Birse
Chairman of the Board of Examiners

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Comments

Please note that different answers may be obtained from those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown. Candidates also lose marks for not showing their working in a methodical manner which the examiner can follow. This can particularly affect candidates on the pass/fail borderline when the examiners have to make a judgement as to whether they can be sure that the candidate has communicated a sufficient command of the syllabus to be awarded a pass.

The general standard of answers was noticeably lower than in previous sessions and there were a significant number of very ill-prepared candidates. As in previous exams, questions that required an element of explanation or analysis were less well answered than those which just involved calculation.

Comments on individual questions, where relevant, can be found after the solution to each question. These comments concentrate on areas where candidates could have improved their performance.

1 Working in half years:

The present value of the security on 1st June would have been $\dfrac{3.5}{i^{(2)}}$

20 August is 80 days later so the present value is $\dfrac{3.5}{i^{(2)}}(1+i)^{80/365}$

Hence the price per £100 nominal is $\dfrac{3.5}{0.097618}(1.1)^{80/365} = £36.611$

2 (i) (a) Gross rate of return convertible half yearly is simply 4/110 = 0.03636 or 3.636%.

(b) Gross effective rate of return is $\left(1+\dfrac{0.03636}{2}\right)^2 - 1 = 0.03669$ or 3.669%

(ii) The net effective rate of return per half year is $0.75 \times \dfrac{0.03636}{2} = 0.013635$.

The net effective rate of return per annum is therefore:

$(1.013635)^2 - 1 = 0.02746$ or 2.746%.

A common error was to divide the nominal payments by the increase in the index factor (rather than multiplying).

3 (a) Let S_{20} be the accumulation of the unit investment after 20 years:

$$E(S_{20}) = E\left[(1+i_1)(1+i_2)\ldots(1+i_{20})\right]$$

$E(S_{20}) = E[1+i_1]E[1+i_2]\ldots E[1+i_{20}]$ as $\{i_t\}$ are independent

$$E[i_t] = j \quad \therefore \quad E(S_{20}) = (1+j)^{20} = 2$$

$$\Rightarrow j = 2^{1/20} - 1 = 3.5265\%$$

(b) The variance of the effective rate of return per annum is s^2 where

$$\mathrm{Var}\left[S_n\right] = \left(\left(1+j\right)^2 + s^2\right)^{20} - \left(1+j\right)^{40} = 0.6^2$$

$$s^2 = \left[0.6^2 + \left(\left(1+j\right)^{20}\right)^2\right]^{1/20} - \left(1+j\right)^2$$

$$= \left(0.6^2 + 2^2\right)^{1/20} - 2^{1/10} = 0.004628$$

Many candidates made calculation errors in this question but may have scored more marks if their working had been clearer.

4 (i) Assuming no arbitrage, buying the share is the same as buying the forward except that the cash does not have to be paid today and a dividend will be payable from the share.

Therefore, price of forward is:

$$700\left(1.05\right)^{1/12}\left(1.03\right)^{5/12} - 20\left(1.03\right)^{5/12}$$

= 711.562 – 20.248 = 691.314

(ii) The no arbitrage assumption means that we can compare the forward with the asset from which the forward is derived and for which we know the market price. As such we can calculate the price of the forward from this, without knowing the expected price at the time of settlement. [It could also be mentioned that the market price of the underlying asset does, of course, already incorporate expectations].

(iii) If it was not known with certainty that the dividend would be received we could not use a risk-free interest rate to link the cash flows involved with the purchase of the forward with all the cash flows from the underlying asset.

5 (a) Eurobonds

- Medium-to-long-term borrowing.
- Pay regular coupon payments and a capital payment at maturity.
- Issued by large corporations, governments or supranational organisations.
- Yields to maturity depend on the risk of the issuer.
- Issued and traded internationally (not in core reading).
- Often have novel features.
- Usually unsecured
- Issued in any currency
- Normally large issue size
- Free from regulation of any one government

(b) Convertible Securities

- Generally unsecured loan stocks.
- Can be converted into ordinary shares of the issuing company.
- Pay interest/coupons until conversion.
- Provide levels of income between that of fixed-interest securities and equities.
- Risk characteristics vary as the final date for convertibility approaches.
- Generally less volatility than in the underlying share price before conversion.
- Combine lower risk of debt securities with the potential for gains from equity investment.
- Security and marketability depend upon issuer
- Generally provide higher income than ordinary shares and lower income than conventional loan stock or preference shares

6 (i) Net present value (all figures in £m)

$$= -2,000 + 0.1 \times 200 \times v^3 \left(v^{0.5} + 1.1v^{1.5} + 1.1^2 v^{2.5} + \ldots + 1.1^5 v^{5.5} \right)$$
$$+ 0.2 \times 200 \times v^3 \left(v + 1.1v^2 + 1.1^2 v^3 + \ldots + 1.1^5 v^6 \right) + 3,500v^9$$

at 8% per annum effective.

$$= -2,000 + \frac{200}{1.1}\left(0.1v^{2.5} + 0.2v^3\right)\left(1.1v + (1.1v)^2 + (1.1v)^3 + \ldots + (1.1v)^6\right) + 3,500v^9$$

$$= -2,000 + \frac{200}{1.1}\left(0.1v^{2.5} + 0.2v^3\right) a'_{\overline{6|}} + 3,500v^9$$

where the annuity is evaluated at a rate of $\dfrac{0.08 - 0.1}{1 + 0.1} = -1.818\%$ per annum effective.

$$a'_{\overline{6|}} = \frac{1 - (1 - 0.018181)^{-6}}{-0.018181} = 6.4011$$

and so net present value is

$$-2,000 + \frac{200}{1.1}\left(0.1 \times 1.08^{-2.5} + 0.2 \times 1.08^{-3}\right) \times 6.4011 + 3,500 \times 1.08^{-9} = £31.66m$$

(ii) Accumulated profit at the time of sale is $31.66 \times 1.08^9 = £63.30m$

Many candidates assumed that the accumulation in part (i) was for a single payment.

7 (i) The present value of the assets is equal to the present value of the liabilities. The duration of the assets is equal to the duration of the liabilities.

The spread of the asset terms around the duration is greater than that for the liability terms (or, equivalently, convexity of assets is greater).

(ii) (a) Present value of liabilities (in £m)
$= 10a_{\overline{10}|}$ at 4% $= 10 \times 8.1109 = 81.109$

(b) Duration is equal to $\dfrac{10(Ia)_{\overline{10}|}}{10a_{\overline{10}|}}$ at 4% $= \dfrac{41.9922}{8.1109} = 5.1773$ years

(c) Let the amounts to be invested in the two zero coupon bonds be X and Y.

$$Xv^3 + Yv^{12} = 81.109 \qquad (1)$$

$$3Xv^3 + 12Yv^{12} = 419.922 \qquad (2)$$

(2) less 3 times (1) gives:

$$9Yv^{12} = 176.595$$

$$\Rightarrow Y = \frac{176.595}{9 \times 0.62460} = £31.415m$$

Substituting back into (1) gives:

$$X = \frac{(81.109 - 31.415 \times 0.62460)}{0.88900} = £69.164m$$

(iii) (a) In one year, the present value of the liabilities is:

$$10 + 10a_{\overline{9}|} \text{ at 5\%} = 10 + 10 \times 7.1078 = 81.078$$

Numerator of duration is $10 \times 0 + 10(Ia)_{\overline{9}|} = 332.347$

Duration of liabilities is therefore $\dfrac{332.347}{81.078} = 4.0991$ years

Present value of assets is:

$69.164 \times v^2 + 31.415 \times v^{11} = 69.164 \times 0.90703 + 31.415 \times 0.58468$
$= 81.101$

Duration of assets will be:

$$\frac{2\times 69.164\times v^2 + 11\times 31.415\times v^{11}}{81.101}$$

$$= \frac{2\times 69.164\times 0.90703 + 11\times 31.415\times 0.58468}{81.101} = 4.0383 \text{ years}$$

(b) One of the problems of immunisation is that there is a need to continually adjust portfolios. In this example, a change in the interest rate means that a portfolio that has a present value and duration equal to that of the liabilities at the outset does not have a present value and duration equal to that of the liabilities one year later.

The calculation was often performed well. In part (ii), many explanations were unclear and some candidates seemed confused between DMT and convexity although a correct explanation could involve either of these concepts.

8 (i) $t \le 20$:

$$v(t) = \exp\left(-\int_0^t 0.05 + 0.001s\,ds\right)$$

$$= \exp\left\{-\left[0.05s + \frac{0.001s^2}{2}\right]_0^t\right\}$$

$$= e^{-0.05t - 0.0005t^2}$$

$t > 20$:

$$v(t) = \exp\left\{-\left(\int_0^{20} \delta(s)\,ds + \int_{20}^t 0.05ds\right)\right\}$$

$$= v(20)\exp\left\{-\left[0.05s\right]_{20}^t\right\}$$

$$= e^{-1.2}e^{1-0.05t} = e^{-0.2-0.05t}$$

(ii) (a) PV = $100v(25) = 100e^{-0.2-0.05\times 25}$

$$= 100e^{-1.45} = £23.46$$

(b) $$100\left(1-\frac{d^{(4)}}{4}\right)^{4\times 25} = 100v(25) = 23.46$$

$$\Rightarrow d^{(4)} = 4\left(1-0.2346^{1/100}\right) = 0.05758$$

(iii) $$\text{PV} = \int_{20}^{25} 30e^{-0.015t}e^{-(0.2+0.05t)}dt$$

$$= 30e^{-0.2}\int_{20}^{25} e^{-0.065t}dt = \frac{30e^{-0.2}}{-0.065}\left[e^{-0.065t}\right]_{20}^{25}$$

$$= \frac{30e^{-0.2}}{-0.065}\left(e^{-1.625} - e^{-1.3}\right) = 28.575$$

$$\text{Accumulated value} = \frac{28.575}{v(25)} = 28.575e^{0.2+0.05\times 25} = 28.575e^{1.45} = 121.82$$

9 (i) The one-year spot rate of interest is simply 4% per annum effective.

For two-year spot rate of interest

First we need to find the price of the security, P:

$$P = 8a_{\overline{2}|} + 100v^2 \text{ at 3\% per annum effective.}$$

$$a_{\overline{2}|} = 1.91347 \quad v^2 = 0.942596$$

$$\Rightarrow P = 8\times 1.91347 + 100\times 0.942596 = 109.5673$$

Let the t-year spot rate of interest be it.

We already know that $i_1 = 4\%$. i_2 is such that:

$$109.56736 = \frac{8}{1.04} + \frac{108}{(1+i_2)^2}$$

$$\Rightarrow (1+i_2)^{-2} = 0.943287$$

$$\Rightarrow i_2 = 0.029623 \text{ or } 2.9623\%.$$

For three-year spot rate of interest we need to find the price of the security P:

$P = 8a_{\overline{3}|} + 100v^3$ at 3% per annum effective.

$a_{\overline{3}|} = 2.8286 \quad v^3 = 0.91514$

$\Rightarrow P = 8 \times 2.8286 + 100 \times 0.91514 = 114.1428$

i_3 is such that:

$$114.1428 = \frac{8}{1.04} + \frac{8}{(1.029623)^2} + \frac{108}{(1+i_3)^3}$$

$$\Rightarrow \frac{108}{(1+i_3)^3} = 114.1428 - 15.23860 = 98.9042$$

$\Rightarrow i_3 = 0.02976$ or 2.976%.

(ii) The one year forward rate of interest beginning at the present time is clearly 4%.

The forward rate for one year beginning in one year is $f_{1,1}$ such that:

$$1.04\left(1+ f_{1,1}\right) = 1.029623^2 \Rightarrow f_{1,1} = 0.01935 = 1.935\%.$$

The forward rate for one year beginning in two years is $f_{2,1}$ such that:

$$1.029623^2\left(1+ f_{2,1}\right) = 1.02976^3 \Rightarrow f_{2,1} = 0.03003 = 3.003\%.$$

The forward rate for two years beginning in one year is $f_{1,2}$ such that:

$$1.02976^3 = 1.04\left(1+ f_{1,2}\right)^2$$

$$\Rightarrow f_{1,2} = 0.02468 = 2.468\%$$

(iii) Let the t-year "spot rate of inflation" be e_t

For each term $\dfrac{(1+i_t)^t}{(1+e_t)^t} = 1.02^t \;\Rightarrow (1+e_t)^t = \left(\dfrac{1+i_t}{1.02}\right)^t$

$$(1+e_1) = \frac{1.04}{1.02} \Rightarrow e_1 = 1.96\%$$

and so the value of the retail price index after one year would be 101.96

$$(1+e_2)^2 = \left(\frac{1.029623}{1.02}\right)^2 \Rightarrow e_2 = 0.943\%$$

and so the value of the retail price index after two years would be $100(1.00943)^2 = 101.90$

$$(1+e_3)^3 = \left(\frac{1.02976}{1.02}\right)^3 \Rightarrow e_3 = 0.9569\%$$

and so the value of the retail price index after three years would be $100(1.009569)^3 = 102.90$

(iv) The "spot" rates of inflation or the price index values could be used.

Clearly the expected rate of inflation in the first year is 1.96%.

The expected rate of inflation in the second year is:

$$\frac{101.90 - 101.96}{101.96} = -0.06\%.$$

The expected rate of inflation in the third year is:

$$\frac{102.90 - 101.90}{101.90} = 0.98\%$$

A common error was to assume that income only started after three years rather than "starting from the beginning of the third year".

10 (i) The price of the securities might have fallen because interest rates have risen or because their risk has increased (for example credit risk).

(ii)

Date	*Market price of securities (£)*	*X* *No of securities held before purchases*	*X* *Market value of holdings before purchases (£)*	*Y* *No of securities held before purchases*	*Y* *Market value of holdings before purchases (£)*
1 April 2003	64	–	–	–	–
1 April 2004	65	100	6,500	100	6,500
1 April 2005	60	100	6,000	200	12,000
1 April 2006	65	1,100	71,500	300	19,500
1 April 2007	68	1,100	74,800	400	27,200
1 April 2008	70	1,100	77,000	500	35,000

(iii) (a) Money weighted rate of return is i where:

$$6,400(1+i)^5 + 60,000(1+i)^3 = 77,000$$

try i = 5% LHS = 77,625.70
try i = 4% LHS = 75,278.42

interpolation implies that

$$i = 0.05 - 0.01 \times \frac{77,625.70 - 77,000}{77,625.70 - 75,278.42} = 4.73\%$$

(*Note true answer is 4.736%*)

(b) Time weighted rate of return is i where using figures in above table:

$$(1+i)^5 = \frac{6,000}{6,400} \frac{77,000}{6,000 + 60,000} = 1.09375.$$

$$\Rightarrow i = 1.808\%$$

(iv) (a) Money weighted rate of return is i where:

$$6,400(1+i)^5 + 6,500(1+i)^4 + 6,000(1+i)^3 + 6,500(1+i)^2 + 6,800(1+i) = 35,000$$

Put in i = 4.73%; LHS = 37,026.95

Therefore the money weighted rate of return for Y is less to make LHS less.

(b) Time weighted rate of return for Y uses the figures in the above table:

$$(1+i)^5 = \frac{6,500}{6,400}\frac{12,000}{6,500+6,500}\frac{19,500}{12,000+6,000}\frac{27,200}{19,500+6,500}\frac{35,000}{27,200+6,800}$$

= 1.09375.

$\Rightarrow i = 1.808\%$

(*Student may reason that the TWRRs are the same and can be derived from the security prices in which case, time would be saved.*)

(v) The money weighted rate of return was higher for X than for Y because there was a much greater amount invested when the fund was performing well than when it was performing badly.

The money weighted rate of return for X (and probably for Y) was more than the time weighted rate of return because the latter measures the rate of return that would be achieved by having one unit of money in the fund from the outset for five years: both X and Y has less in the fund in the years it performed badly.

This question was answered well but examiners were surprised by the large number of candidates who used interpolation or other trial and error methods in part (ii) when the answer had been given in the question. The examiners recommend that students pay attention to the details given in the solutions to parts (iii) and (iv). For such questions, candidates should be looking critically at the figures given/calculated and making points specific to the scenario rather than just making general statements taken from the Core Reading.

END OF EXAMINERS' REPORT