INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2013 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

D C Bowie
Chairman of the Board of Examiners

July 2013

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the April 2013 paper

This paper proved to be marginally more challenging than other recent papers and the general performance was of a slightly lower standard compared with the previous April exams. Well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q1(iii) and Q4(ii) were less well answered than those that just involved calculation. This is an area to which attention should be paid. Candidates should note that it is important to explain and show understanding of the concepts and not just mechanically go through calculations.

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 (i) TWRR, i, is given by:

$$\frac{1.9}{1.3}\times\frac{0.8}{1.9-0.9}=1+i\Rightarrow i=0.169 \text{ or } 16.9\% \text{ p.a.}$$

(ii) MWRR, i, is given by:

$$1.3\times(1+i)-0.9\times(1+i)^{\frac{3}{12}}=0.8$$

Then, we have:

$$\left.\begin{array}{lll} i=30\% & \Rightarrow & LHS=0.729 \\ i=40\% & \Rightarrow & LHS=0.841 \end{array}\right\}\Rightarrow i\approx 0.3+(0.4-0.3)\times\left(\frac{0.8-0.729}{0.841-0.729}\right)=0.36$$

or 36% p.a.

(iii) MWRR is higher as fund performs much better before the cash outflow than after. As the fund is smaller after 1 October 2012, the effect of the poor investment performance is less significant.

The calculations were performed well but the quality of the explanations in part (iii) was often poor. A common error was to cite the large withdrawal itself as the reason for the superior MWRR.

2 (i) (a) Options – holder has the right but not the obligation to trade.

Futures – both parties have agreed to the trade and are obliged to do so.

(b) Call Option – right but not the obligation to BUY specified asset in the future at specified price.

Put Option – right but not the obligation to SELL specified asset in the future at specified price.

(ii) Assume no arbitrage.

The present value of the dividends, I, is:

$$I=1.1v_{2.25\%}+1.1v_{2.5\%}^{2}=1.1\times(0.977995+0.951814)$$
$$=2.12279$$

Hence, forward price $F = (10.50-2.12279)\times 1.025^{2}$
$= £8.8013$

3 (i) $97 = 6\,a_{\overline{3}|} + 103v^3$

Try 9% RHS = 94.723
Try 8% RHS = 97.227

Interpolation gives

$$0.08 + \frac{97.227 - 97}{97.227 - 94.723} \times 0.01$$

$= 0.08091$

i.e. 8.09% p.a. (exact answer is 8.089%)

(ii) Let i_n = spot rate for term n

Then $97 = 109v_{i_1\%}$

$\Rightarrow i_1 = 12.371\%$ p.a.

$97 = 6v_{i_1\%} + 109v_{i_2\%}^2$

$109(1+i_2)^{-2} = 97 - \dfrac{6}{1.12371}$

$\Rightarrow i_2 = 9.049\%$ p.a.

Part (i) was generally well answered. Some candidates wasted time in (ii) through using linear interpolation to solve the yield for the one year bond.

4 (i) Maximum price payable by investor is given by:

$$P = 0.30 \times 1.05 \times v_{9\%} + 0.30 \times 1.05^2 \times v_{9\%}^2 + \ldots$$

$$= 0.30 \times \left(\frac{1.05}{1.09}\right) \times \left[1 + \left(\frac{1.05}{1.09}\right) + \left(\frac{1.05}{1.09}\right)^2 + \ldots\right]$$

$$= 0.30 \times \left(\frac{1.05}{1.09}\right) \times \frac{1}{1 - \frac{1.05}{1.09}}$$

$$= 0.30 \times \left(\frac{1.05}{1.09}\right) \times \frac{1.09}{0.09 - 0.05} = 0.30 \times \frac{1.05}{0.09 - 0.05}$$

$$= £7.875$$

(ii) (a) Increasing the expected rate of dividend growth, g , will increase the maximum price that the investor is prepared to pay to purchase the share since the dividend income is expected to be higher.

(b) An increase in the expected rate of future price inflation is likely to lead to an increase in both the expected rate of dividend growth (as nominal level of profits should increase in line with inflation) and the nominal return required from the investment (as the investor is likely to want to maintain the required real return).

Thus, the maximum price that the investor is prepared to pay will be (largely) unchanged – in fact, it will increase slightly due to $(1 + g)$ term in numerator.

(c) If the investor is more uncertain about the rate of future dividend growth (whilst the expected dividend growth is unchanged), then the required return, i, is likely to be increased to compensate for the increased uncertainty.

Thus, the maximum price that the investor is prepared to pay will reduce.

Part (i) was generally well answered although common errors included adding an extra 30 pence dividend at the start or to assume that the first dividend was payable immediately.

The examiners expected candidates to find part (ii) challenging and this was indeed the case with very few candidates scoring full marks. In (ii)(b) full marks were awarded for a reasoned argument that led to a final answer of either an increase or no change in the price. In general, some credit was given for valid reasoning even if the final conclusion was incorrect.

5 (i) Accumulated value at time 10 is:

$$100 \times \exp\left(\int_0^{10} \delta(t)\, dt\right) + 50 \times \exp\left(\int_7^{10} \delta(t)\, dt\right)$$

$$= 100 \times \exp\left(\int_0^{6} (0.1 - 0.005t)\, dt + \int_6^{10} 0.07 dt\right) + 50 \times \exp\left(\int_7^{10} 0.07 dt\right)$$

$$= 100 \times \exp\left(\left[0.1t - 0.0025t^2\right]_{t=0}^{t=6} + \left[0.07t\right]_{t=6}^{t=10}\right) + 50 \times \exp\left(\left[0.07t\right]_{t=7}^{t=10}\right)$$

$$= 100 \times \exp\left([0.6 - 0.09] + 0.28\right) + 50 \times \exp(0.21)$$

$$= 220.34 + 61.68$$

$$= £\,282.02$$

(ii) Present value at time 0 is:

$$= \int_{12}^{15} \rho(t)\, v(t)\, dt$$

$$= \int_{12}^{15} 50e^{0.05t} \times \exp\left(-\int_{0}^{t} \delta(s)\, ds\right) dt$$

$$= \int_{12}^{15} 50e^{0.05t} \times \exp\left(-\left[\int_{0}^{6} (0.1 - 0.005s)\, ds + \int_{6}^{t} 0.07 ds\right]\right) dt$$

$$= \int_{12}^{15} 50e^{0.05t} \times \exp\left(-\left[\left[0.1s - 0.0025s^2\right]_{s=0}^{s=6} + \left[0.07s\right]_{s=6}^{s=t}\right]\right) dt$$

$$= \int_{12}^{15} 50e^{0.05t} \times \exp\left(-\left[0.51 + (0.07t - 0.42)\right]\right) dt$$

$$= \int_{12}^{15} 50e^{0.05t} \times e^{-0.09 - 0.07t} dt$$

$$= 50e^{-0.09} \times \int_{12}^{15} e^{-0.02t} dt$$

$$= 50e^{-0.09} \times \left[\frac{e^{-0.02t}}{-0.02}\right]_{t=12}^{t=15}$$

$$= 2{,}500e^{-0.09} \times \left(e^{-0.24} - e^{-0.30}\right)$$

$$= £104.67$$

6 (i) $j = 0.05 \times 0.2 + 0.07 \times 0.6 + 0.09 \times 0.2$

$= 0.07$

$\Rightarrow$ mean accumulation $= 10{,}000 \times (1 + j)^{15}$

$= 10{,}000 \times (1.07)^{15}$

$= £27{,}590.32$

(ii) $s^2 = 0.05^2 \times 0.2 + 0.07^2 \times 0.6 + 0.09^2 \times 0.2 - 0.07^2$

$= 0.00506 - 0.00490$

$= 0.00016$

Var (accumulation) $= 10{,}000^2\{(1 + 2j + j^2 + s^2)^{15} - (1 + j)^{30}\}$

$= 10{,}000^2\ \{1.14506^{15} - 1.07^{30}\}$

$= 1{,}597{,}283.16$

SD (accumulation) $= \sqrt{1597283.16} = £1,263.84$

(iii) (a) By symmetry $j = 0.07$ (as in (i))

Hence, mean (accumulation) will be the same as in (i) (i.e. £27,590.32).

The spread of the yields around the mean is lower than in (i). Hence, the standard deviation of the accumulation will be lower than £1,263.84.

(b) Mean (accumulation) < £27,590.32 since the investment is being accumulated over a shorter period.

SD (accumulation) < £1,263.84 since investing over a shorter term than in (i) will lead to a narrower spread of possible accumulated amounts.

In part (i) some candidates misread the question and assumed the yield was fixed for the whole ten years rather than varying each year.

7 (i) Need $V_A(i) = V_L(i)$ with $i = 0.08$

$$V_L(i) = 6v^8 + 11v^{15}$$
$$V_A(\text{i}) = Xv^5 + Yv^{20}$$

Need $V_A'(i) = V_L'(i)$ with $i = 0.08$

$$V_L' = -48v^9 - 165v^{16}$$
$$V_A' = -5Xv^6 - 20Y\,v^{21}$$

Thus we have to solve simultaneous equations:

(a) $6v^8 + 11v^{15} = Xv^5 + Yv^{20}$

(b) $-48v^9 - 165v^{16} = -5Xv^6 - 20Y\,v^{21}$

Taking 5 times (a) + (1+*i*) times (b) we get

$$-18v^8 - 110v^{15} = -15Yv^{20}$$

$$\Rightarrow Y = \frac{18(1+i)^{12} + 110(1+i)^5}{15}$$

$$\Rightarrow Y = 13.79688$$

Substitute back in (a) to get $X = 5.50877$

Hence the values of the zero-coupon bonds are £5.50877 million and £13.79688 million.

(ii) We need to check that the third condition is satisfied:

$$V_A' = -5Xv^6 - 20Yv^{21}$$
$$\Rightarrow V_A'' = 30Xv^7 + 420Yv^{22}$$
$$\Rightarrow V_A''(0.08) = 30 \times 5.50877 \times 1.08^{-7} + 420 \times 13.79688 \times 1.08^{-22}$$
$$= 1162.31$$

$$V_L' = -48v^9 - 165v^{16}$$
$$\Rightarrow V_L'' = 432v^{10} + 2640v^{17}$$
$$\Rightarrow V_L''(0.08) = 432 \times 1.08^{-10} + 2640 \times 1.08^{-17}$$
$$= 913.61$$

Therefore $V_A''(0.08) > V_L''(0.08)$

Thus the third condition is satisfied.

[Or note that since the assets have terms of 5 years and 20 years and the liabilities have terms of 8 years and 15 years, the spread of assets around the mean term is greater than that of the liabilities. Hence, the convexity of assets is greater than the convexity of liabilities].

The best answered question on the paper.

8 (i) Work in £ millions

Let Discounted Payback Period from 1 January 2014 be n.

Then, considering project at the end of year n but before the outgo at the start of year n + 1

$$\begin{aligned}&-\ 19-9v^{1/2}-5v\\&-\ 6\times 9.5\left(v^2+1.04v^3+\ldots+(1.04)^{n-3}v^{n-1}\right)\\&+\ 6\times 12.6\left(v^3+1.04v^4+\ldots+(1.04)^{n-3}v^{n}\right)\geq 0\quad\text{at } 9\%\end{aligned}$$

Hence, $19+8.6204+4.5872\leq\left(75.6v^3-57v^2\right)\left(\dfrac{1-\left(\frac{1.04}{1.09}\right)^{n-2}}{1-\frac{1.04}{1.09}}\right)$

and RHS = $10.4013\times 21.8\times\left(1-\left(\dfrac{1.04}{1.09}\right)^{n-2}\right)$

Hence, $\dfrac{32.2076}{10.4013\times 21.8}\leq 1-\left(\dfrac{1.04}{1.09}\right)^{n-2}$

$$\Rightarrow\left(\frac{1.04}{1.09}\right)^{n-2}\geq 0.85796$$

$$\Rightarrow(n-2)\log\left(\frac{1.04}{1.09}\right)\geq\ \log\ 0.85796$$

$$\Rightarrow n-2\geq\frac{-0.06653}{-0.02039}=3.262$$

$$\Rightarrow n\geq 5.262$$

But sales are only made at the end of each calendar year.

$\Rightarrow$ DPP = 6 years

(ii) The DPP would be shorter using an effective rate of interest less than 9% p.a. This is because the income (in the form of car sales) does not commence until a few years have elapsed whereas the bulk of the outgo occurs in the early years. The effect of discounting means that using a lower rate of interest has a greater effect on the value of the income than on the value of the outgo (although both values increase). Hence the DPP becomes shorter.

In part (i), many candidates valued the total outgo for the whole production run and then attempted to find when the present value of income exceeded this. The working of many marginal candidates was difficult to follow and it was not clear to the examiners what the candidates were attempting to do.

9 (i) $\frac{D}{R}(1-t_1) = \frac{0.08}{1} \times 0.7 = 0.056 < \underset{6\%}{i}^{(2)} = 0.059126$

$\Rightarrow$ There is a capital gain and assume redeemed as late as possible.

Let $P =$ Price at 1/5/11 per £100 nominal

$$P = \left[0.7 \times 8\, a_{\overline{11}|}^{(2)} + 100v^{11} - 0.25(100-P)v^{11}\right] \times (1+i)^{4/12}$$

$$\Rightarrow P = 5.6 \times 1.014782 \times 7.8869 \times (1.06)^{4/12} + 75v^{10\,8/12} + 0.25Pv^{10\,8/12}$$

$$\Rightarrow P = \frac{45.6985 + 40.2839}{1 - 0.25v^{10\,8/12}}$$

$$= £99.319$$

(ii) $\frac{D}{R} = 0.08 > i_{7\%}^{(2)} = 0.068816$

$\Rightarrow$ Assume redeemed as soon as possible

$$\text{Sale Price per £100 nominal} = \left(8\, a_{\overline{4}|}^{(2)} + 100v^4\right) \times (1+i)^{3/12}$$

$$= (8 \times 1.017204 \times 3.3872 + 100 \times 0.76290) \times (1.07)^{3/12}$$

$$= £\,105.625$$

(iii) CGT is payable of $(105.625 - 99.319) \times 0.25$

$= £1.5765$

Equation of value:

$$99.319 = 0.7\times 4\times v^{2/12} + 0.7\times 4\times v^{8/12} + 0.7\times 4\times v^{12/12} + 0.7\times 4\times v^{18/12} + (105.625 - 1.5765)v^{111/12}$$

$$\Rightarrow 99.319 = (1+i)^{4/12}\times 5.6\, a_{\overline{2}|}^{(2)} + 104.0485v^{111/12}$$

At 8%, RHS is $1.08^{4/12}\times 5.6\times 1.019615\times 1.7833 + 104.0485v^{111/12}$
$= 100.226$

At 9% RHS is $(1.09)^{4/12}\times 5.6\times 1.022015\times 1.7591 + 104.0485v^{111/12}$
$= 98.568$

and since 98.568 < 99.319 < 100.226, the net yield is between 8% and 9% p.a.

Many candidates struggled with the four month adjustment in part (i). Common errors included:

- *ignoring the adjustment completely.*
- *discounting the present value of payments by four months rather than accumulating.*
- *adjusting the price at the end of the calculations (which does not allow for CGT correctly).*

In part (iii) some candidates wasted time by trying to solve the yield exactly rather than just show that 8% was too low and 9% too high.

10 (i) Original amount of loan is:

$$\begin{aligned} L &= 5{,}000v + 4{,}800v^2 + 4{,}600v^3 + \ldots + 1{,}200v^{20} \\ &= 5{,}200\times\left(v + v^2 + \ldots + v^{20}\right) - 200\times\left(v + 2v^2 + \ldots + 20v^{20}\right) \\ &= 5{,}200a_{\overline{20}|} - 200(Ia)_{\overline{20}|} \\ &= 5{,}200\times 13.5903 - 200\times 125.1550 \\ &= £45{,}638.56 \end{aligned}$$

(ii) Amount of 12th instalment is £2,800.

Loan o/s after 11th instalment is given by PV of future repayments:

$$\begin{aligned} L_{11} &= 2{,}800v + 2{,}600v^2 + 2{,}400v^3 + \ldots + 1{,}200v^9 \\ &= 3{,}000a_{\overline{9}|} - 200(Ia)_{\overline{9}|} \\ &= 3{,}000\times 7.4353 - 200\times 35.2366 \\ &= £15{,}258.58 \end{aligned}$$

Then, interest component of 12th instalment is: $0.04\times 15{,}258.58 = £610.34$.

Hence, capital repaid in 12th instalment is $2,800 - 610.34 = £2,189.66$.

(iii) (a) Then, after 12th instalment, loan o/s is

$15,258.58 - 2,189.66 = £13,068.92$.

This will be repaid by level instalments of £2,800.

Thus, remaining term of loan is n given by:

$$13,068.92 \le 2,800 \times a_{\overline{n}|}^{4\%} \Rightarrow a_{\overline{n}|}^{4\%} \ge 4.6675 \Rightarrow n = 6$$

i.e. remaining term is 6 years (i.e. loan is repaid by time 18)

(b) We need to find reduced final payment, R , such that:

$$13,068.92 = 2,800 \times a_{\overline{5}|}^{4\%} + Rv_{4\%}^{6} \Rightarrow 0.79031R$$
$$= 13,068.92 - 2,800 \times 4.4518 \Rightarrow R = £764.11$$

(c) Total amount of interest paid is given by:

$$5,000 + 4,800 + 4,600 + \ldots + 2,800 + 5 \times 2,800 + 764.11 - 45,638.56$$
$$= £15,925.55$$

In part (ii) the most common error was to not round n up, i.e. quoting a non-integer number of years for the revised loan. Part (iii) was answered poorly with candidates often not correctly allowing for the payments prior to the change in payment schedule.

END OF EXAMINERS' REPORT