INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2014 examinations

Subject CT3 – Probability and Mathematical Statistics Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context at the date the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners

November 2014

General comments on Subject CT3

Some of the questions in this paper permit alternative solutions from these presented in this report. All mathematically correct and valid alternative solutions or answers received credit as appropriate. Rounding errors were not penalised, unless excessive rounding led to significantly different answers. In cases where the same error was carried forward to later parts of the answer, candidates were only penalised once. In questions where comments were required, reasonable comments that were different from those provided in the solutions also received full credit where appropriate.

Comments on the September 2014 paper

The performance was generally satisfactory and the pass rate was in line with previous diets. Candidates that were sufficiently prepared were able to answer all questions and the best candidates scored close to full marks.

Questions that required precise mathematical derivations, and questions that covered topics that were not recently examined proved to be more challenging. Some fundamental topics in probability and statistics at this level, such as conditional probability and the likelihood function, were not well addressed by candidates who were inadequately prepared.

The comments on individual questions that follow cover important frequent errors, and specific parts that were not answered well.

1 Let $\left\{X_{1,} \ldots, X_n\right\}$ be the existing marks and $\left\{Y_1, \ldots, Y_n\right\}$ denote the transformed marks. Then $Y_i = 1.2X_i + 6$.

Median is $(n+1)/2$ th observation so same transformation applies to median.
New median = 49*1.2 + 6 = 64.8.

As for the median, each transformed quartile, $QY_i = 1.2QX_i + 6$. Then the new interquartile range is, $IQR_Y = QY_3 - QY_1 = 1.2QX_3 + 6 - (1.2QX_1 + 6) = 1.2IQR_X$.

New IQR = 19*1.2 = 22.8. [4]

Generally well answered.

2 Note that each customer has at least one contract, that is, $P[\text{Car} \cup \text{Home}] = 1$.

(i) $P\left[\text{Car}^C\right] = 1 - P[\text{Car}] = 0.2 = 20\%$ [1]

(ii) $P[\text{Car} \cap \text{Home}] = P[\text{Car}] + P[\text{Home}] - P[\text{Car} \cup \text{Home}]$ [1]

$= 0.8 + 0.7 - 1 = 0.5$

(iii) $P[\text{Home}|\text{Car}] = \dfrac{P[\text{Car} \cap \text{Home}]}{P[\text{Car}]} = \dfrac{0.5}{0.8} = 0.625$ [2]

(iv) $P\left[\text{Car}^C \cap \text{Home}\right] = P(\text{Home}) - P(\text{Car} \cap \text{Home}) = 0.7 - 0.5 = 0.2$

$P\left[\text{Car}^C|\text{Home}\right] = \dfrac{P\left[\text{Car}^C \cap \text{Home}\right]}{P[\text{Home}]} = \dfrac{0.2}{0.7} = 0.2857$ [2]

[Total 6]

Reasonably well done, with the exception of part (iv). Note that events are not independent here. Alternative ways to arrive at the correct answer were given full credit.

3 (i) $$E\left[e^{tS}|N=n\right]=E\left[\exp\left\{t\left(X_1+X_2+\ldots+X_N\right)\right\}|N=n\right]$$

$$=E\left[\exp\left\{t\left(X_1+X_2+\ldots+X_n\right)\right\}\right]=\prod_{i=1}^{n}E\left[\exp\left(tX_i\right)\right]=\left\{M_X\left(t\right)\right\}^n=\left(1-\frac{t}{\lambda}\right)^{-n}$$

$$\therefore M_S\left(t\right)=E\left(e^{tS}\right)$$

$$=E\left[E\left(e^{tS}|N\right)\right]$$

$$=E\left[\left\{M_X\left(t\right)\right\}^N\right]$$

$$=E\left[\exp\left\{N\log M_X\left(t\right)\right\}\right]$$

$$=M_N\left\{-\log\left(1-\frac{t}{\lambda}\right)\right\}$$

$$=\exp\left[\mu\left\{\left(1-\frac{t}{\lambda}\right)^{-1}-1\right\}\right]$$

[4]

(ii) $$E\left[X_i\right]=\frac{1}{0.025}=40,\ E\left[X_i^2\right]=\frac{1}{\left(0.025\right)^2}+40^2=3200$$

$$E\left[S\right]=\mu E\left[X_i\right]=100*40=4000$$

$$V\left[S\right]=\mu E\left[X_i^2\right]=100*3200=320,000$$

(OR

$$V\left[S\right]=E\left[N\right]V\left[X_i\right]+V\left[N\right]\left[E\left[X\right]\right]^2=100*\frac{1}{\left(0.025\right)^2}+100*40^2=320,000\text{)}$$

[3]
[Total 7]

Part (i) required careful and precise derivation of the result, and many candidates struggled with it. Answers in questions involving work with MGF expressions have also been problematic in the past – more practice and better understanding is needed.

4 If X is the total number of claims, with X_1 from group 1 (G1, with probability 2/3) and X_2 from group 2 (G2, with probability 1/3), we have

(i) $X_1 \sim$ Bin(4, 2/3) and $X_2 \sim$ Bin(2, 1/3) .

$$E(X) = E(X_1 + X_2) = E(X_1) + E(X_2)$$

$$= 4(2/3) + 2(1/3) = 10/3 = 3.333$$ [2]

(ii) $P(X = 1) = P(X_1 = 1, X_2 = 0) + P(X_1 = 0, X_2 = 1)$

$$= \binom{4}{1}(2/3)(1/3)^3 \times \binom{2}{0}(1/3)^0(2/3)^2 + \binom{4}{0}(2/3)^0(1/3)^4 \times \binom{2}{1}(1/3)^1(2/3)^1$$

$$= 4/81 = 0.0494$$ [2]

(iii) P(two randomly selected policies giving claims) =

P(both give claims | both from G1) * P(both from G1)
+P(both give claims | both from G2) * P(both from G2)
+ 2*P(both give claims | one from G1, one from G2) * P(one from G1, one from G2)

$$= \left(\frac{2}{3}\right)^2 \frac{4}{6}\frac{3}{5} + \left(\frac{1}{3}\right)^2 \frac{2}{6}\frac{1}{5} + 2\times\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\frac{4}{6}\frac{2}{5} = \frac{41}{135} = 0.3037$$ [4]

[Total 8]

Mixed performance. Parts (i) and (ii) were answered well, but there were many inadequate attempts in part (iii). In many cases candidates failed to see the different combinations resulting in the required event, while there were also problems in calculating the correct probability for each combination.

5 (a) $E[5X + 20Y] = 5*2 + 20*(-3) = 10 - 60 = -50$

(b) $Corr(X,Y) = \dfrac{1.6}{2} = 0.8$

(c) $E[XY] = Cov(X,Y) + E[X]E[Y] = 1.6 + 2*(-3) = -4.4$

(d) $V(X - Y) = V(X) + V(Y) - 2Cov(X,Y) = 4 + 1 - 3.2 = 1.8$

[4]

Generally very well answered. There were only a few problems with using the correct expression for the variance (taking into account the covariance).

6 (i) The two samples are from the same patients, so they are clearly not independent. [1]

(ii) First calculate differences d = measurement before – measurement after :

$$d: \quad 10 \quad 5 \quad 23 \quad 16 \quad 5 \quad 18 \quad -1 \quad 10$$

For these we have $\sum d = 86, \sum d^2 = 1,360$

giving $\overline{d} = 86/8 = 10.75$ and $sd(d) = \sqrt{(1360 - 86^2/8)/7} = 7.8876$

H_0: mean difference = 0 v H_1: mean difference > 0

$$t = \frac{\overline{d}}{sd(d)/\sqrt{n}} = \frac{10.75}{7.8876/\sqrt{8}} = 3.855$$

From tables, $t_7(0.005) = 3.499$ and $t_7(0.001) = 4.785$

Therefore, we have strong evidence against H_0 (P-value < 0.5%), and conclude that daily exercise has the effect of lowering blood pressure. [7]

[Total 8]

Mixed performance in part (ii). The question clearly indicates that a standard two-sample t test is not appropriate here, and candidates should recognise the need for a paired test. In some cases, although the correct test was identified, its application was wrong.

7 We denote by X_0 the number of policies with no claims, by X_1 the number of policies with one claim and by X_2 the number of policies with two claims per year. Let $X = X_0 + X_1 + X_2$

(i) Likelihood function

$$L(p) \propto (2p)^{X_0}\, p^{X_1}\, (0.25p)^{X_2}\, (1-3.25p)^{n-X}$$

Log-likelihood

$$l(p) = X_0 \log(2p) + X_1 \log(p) + X_2 \log(0.25p) + (n-X)\log(1-3.25p) + \text{constant}$$

$$\frac{dl}{dp} = \frac{X_0}{p} + \frac{X_1}{p} + \frac{X_2}{p} - \frac{3.25(n-X)}{1-3.25p} = \frac{X}{p} - \frac{3.25(n-X)}{1-3.25p}$$

$\frac{dl}{dp} = 0$ gives $X\left(1-3.25p\right)-3.25\left(n-X\right)p$ =0

$X - 3.25Xp - 3.25np + 3.25Xp = X - 3.25np = 0$

$\hat{p} = \frac{X}{3.25n}$

[Alternative solution:

Set $\theta = 3.25p$ to be the probability of at most two claims.

$L(\theta) \propto \theta^X (1-\theta)^{n-X}$ and $l(\theta) = X\,ln(\theta) + (n-X)\ln(1-\theta) + \text{constant}$

$\frac{dl}{d\theta} = \frac{X}{\theta} - \frac{(n-X)}{1-\theta}$ and setting equal to zero: $\hat{\theta} = \frac{X}{n}$.

Using the invariance property of the MLE we obtain:

$\hat{\theta} = 3.25\hat{p} \Rightarrow \hat{p} = \frac{X}{3.25n}$] [5]

(ii) $E[\hat{p}] = \frac{1}{3.25n}E[X]$, X has Binomial dist. with parameters *n* and $2p + p + 0.25p$

$E[X] = n(2p + p + 0.25p) = 3.25pn$

and therefore $E[\hat{p}] = p$ [3]

(iii) $X = 194$, $\hat{p} = \frac{194}{3.25 \times 200} = 0.2985$ [1]

(iv) The MLE in part (iii) takes the structure of the entire probability function into account while the estimator 58/200 only considers the number of policies with one claim. [2]

(v) No change required, since the MLE $\hat{p}$ turns out to dependent only on the total number of policies with less than three claims. [1]

(vi) χ^2-test [1]

[Total 13]

The later parts of the question were well answered. However there was a considerable number of poor answers in parts (i) and (ii). Part (i) particularly, deals with the likelihood concept which is fundamental in statistics. The setting does not refer explicitly to a usual *distribution, but involves a standard model, and candidates at this level need to make sure that they can work with the likelihood function in a variety of standard models.*

8 (i) Let $\{X_1,\ldots,X_{10}\}$ denote the sample at £14 and $\{Y_1,\ldots,Y_{10}\}$ the sample at £16.

$\Sigma x_i = 1216, \Sigma x_i^2 = 148220$

$$\Rightarrow \bar{x} = \frac{1216}{10} = 121.6, s_x = \sqrt{\frac{148220 - 121.6^2 * 10}{9}} = 6.275$$

$\Sigma y_i = 1061, \Sigma x_i^2 = 112863$

$$\Rightarrow \bar{y} = \frac{1061}{10} = 106.1, s_y = \sqrt{\frac{112863 - 106.1^2 * 10}{9}} = 5.685$$ [4]

(ii) $H_0 : \sigma_x^2 = \sigma_y^2, H_1 : \sigma_x^2 \neq \sigma_y^2$

Under H_0 $s_x^2 / s_y^2 \sim F_{9,9}$

$s_x^2 / s_y^2 = 6.275^2 / 5.685^2 = 1.22$

$F_{9,9;0.975} = \frac{1}{4.026} = 0.25, F_{9,9;0.025} = 4.026$ so we fail to reject H_0. [3]

(iii) Given (ii) we can assume that standard deviations are equal.

$$s_P^2 = \frac{1}{10+10-2}\left(9 * 6.275^2 + 9 * 5.685^2\right) = 35.847$$

$$\text{test statistic} = \frac{121.6 - 106.1}{s_P\sqrt{\frac{2}{10}}} = \frac{15.5}{5.987\sqrt{\frac{1}{5}}} = 5.789$$

test statistic $\sim t_{10+10-2} = t_{18} = 2.101$ at 2.5%.

So reject H_0: there is a significant difference between the means at 5% significance level. [4]

(iv) Difference in means = 14*121.6-16*106.1=4.8

$$s_P^2 = \frac{1}{10+10-2}\left(9*14^2*6.275^2 + 9*16^2*5.685^2\right) = 7995.7$$

Using t_{18} as before the confidence interval is

$$4.8 \pm 2.101*\sqrt{7995.7\left(\frac{1}{10}+\frac{1}{10}\right)} = \left(-79.22, 88.82\right)$$ [4]

(v) There is a significant lower attendance with the higher price but, as the confidence interval contains zero, no significant difference in revenues. Financially it doesn't matter which price the promoter chooses, but the lower price would get more people to see the show. [3]

[Total 18]

Parts (i) – (iii) were well answered. In part (iv) some candidates did not realise that the required CI referred to revenue. In the same part, there were also many errors in calculating the common variance correctly. In part (v) other sensible comments were also given credit as appropriate.

9 (i) The original values vary in scale among the 3 varieties, resulting in large differences in the variances of the 3 groups. This violates the ANOVA requirement that the error variance should not depend on the treatment concerned. [2]

(ii) The logarithm transformation gives very similar variances for the 3 groups, as opposed to the square root which still produces large differences. [1]

(iii) First calculate relevant sums:

$$\sum y_A = 2.4075\times 4 = 9.63,\ \sum y_A^2 = 3\times 0.2136 + 4\times 2.4075^2 = 23.825$$
$$\sum y_B = 4.725\times 4 = 18.9,\ \sum y_B^2 = 3\times 0.1892 + 4\times 4.725^2 = 89.870$$
$$\sum y_C = 6.4\times 4 = 25.6,\ \sum y_C^2 = 3\times 0.18 + 4\times 6.4^2 = 164.38$$

SST = 23.825 + 89.87 + 164.38 – $(9.63+18.9+25.6)^2$ / 12 = 33.9036
SSB = $(9.63^2+18.9^2+25.6^2)/4$ – $(9.63+18.9+25.6)^2$ / 12 = 32.1553
SSR = SST – SSB = 1.7483

ANOVA table:

Source of variation	*df*	*SS*	*MSS*
Between groups	2	32.1553	16.0777
Residual	9	1.7483	0.1943
Total	11	33.9036	

$$F = \frac{16.0777}{0.1943} = 82.75 \text{ on } 2, 9 \text{ df}$$

$F_{2,9}(1\%) = 8.022$, so P-value << 0.01

There is overwhelming evidence against the null hypothesis. We conclude that there are differences in the mean level of acidity of the three grape varieties.

[6]

(iv) The CIs are given by

$$\bar{y}_i \pm t_{9,0.975} \times \hat{\sigma} / \sqrt{n_i} \text{ with } t_{9,0.975} = 2.262 \quad \text{and } \hat{\sigma} = \sqrt{MSS_R} = 0.44079$$

For A: $2.4075 \pm 2.262 \times 0.44079 / 2$ i.e. $(1.909, \ 2.906)$
and on the original scale: $\left(e^{1.909}, e^{2.906}\right) = (6.75, \ 18.28)$

For B: $4.725 \pm 2.262 \times 0.44079 / 2$ i.e. $(4.226, \ 5.224)$
and on the original scale (68.44, 185.68)

For C: $6.4 \pm 2.262 \times 0.44079 / 2$ i.e. $(5.901, \ 6.899)$
and on the original scale (365.40, 990.28) [6]

(v) The CIs do not overlap. This agrees with the ANOVA conclusion, and in addition shows differences between all 3 pairs of means. [2]

[Total 17]

There were no problems with the ANOVA part of this question. However, the explanation in part (i) was often unclear. In part (iv) some candidates failed to transform back to the original scale.

10 (i)

$$S_{xx} = 3,355 - 8*19.875^2 = 194.875,$$
$$S_{yy} = 435 - 8*7.125^2 = 28.875,$$
$$S_{xy} = 1190 - 8*19.875*7.125 = 57.125$$

$$r = \frac{S_{xy}}{\sqrt{S_{xx} \ S_{yy}}} = 0.76153$$ [4]

(ii) $W = \dfrac{1}{2}\log\dfrac{1+r}{1-r}$ is normally distributed with mean $\dfrac{1}{2}\log\dfrac{1+\rho}{1-\rho}$ and standard deviation $1/\sqrt{n-3}$

Confidence interval for the mean of W: $W \pm 1.96/\sqrt{(n-3)}$

Using r from part (i), the estimated value of W is 0.999848.

This gives a confidence interval of

$$0.999848 \pm \frac{1.96}{\sqrt{5}} = [0.123309176,\ 1.87638647] \text{ for } W.$$

Since $r = \dfrac{e^{2W} - 1}{e^{2W} + 1}$ we obtain the C.I. for the true correlation ρ

$$\left[\frac{e^{2x0.123309176} - 1}{e^{2x0.123309176} + 1}, \frac{e^{2x1.87638647} - 1}{e^{2x1.87638647} + 1}\right] = [0.122688\ ,0.95417]$$ [6]

(iii) $Y_i = a + bX_i + \varepsilon_i$

$$\hat{b} = S_{xy} / S_{xx} = \frac{57.125}{194.875} = 0.293137$$

$$\hat{a} = \frac{1}{8}\left(\sum y_i - \hat{b}\sum x_i\right) = 1.29891$$ [3]

(iv) $R^2 = 0.76153^2 = 0.58$ [1]

(v) About 58% of the total variability of the response "cigarettes per day" is statistically explained by alcohol consumption. [1]

[Total 15]

Generally well answered with some problems in part (ii), which involves the more demanding (and less frequently examined) CI for the correlation coefficient.

END OF EXAMINERS' REPORT