INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2011 examinations

Subject CT3 — Probability and Mathematical Statistics Core Technical

Purpose of Examiners' Reports

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and who are using past papers as a revision aid, and also those who have previously failed the subject. The Examiners are charged by Council with examining the published syllabus. Although Examiners have access to the Core Reading, which is designed to interpret the syllabus, the Examiners are not required to examine the content of Core Reading. Notwithstanding that, the questions set, and the following comments, will generally be based on Core Reading.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report. Other valid approaches are always given appropriate credit; where there is a commonly used alternative approach, this is also noted in the report. For essay-style questions, and particularly the open-ended questions in the later subjects, this report contains all the points for which the Examiners awarded marks. This is much more than a model solution – it would be impossible to write down all the points in the report in the time allowed for the question.

T J Birse
Chairman of the Board of Examiners

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General comments on Subject CT3

All valid alternative solutions receive credit as appropriate. Rounding errors are not penalised, unless if excessive rounding has led to significantly different answers. In cases where the same error is carried forward to later parts of the question, candidates are not penalised twice. In questions where comments are required, reasonable comments that are different than those provided in the solutions also receive credit.

Comments on the September 2011 paper

In general the paper was not answered as well as in recent diets. However, the overall performance was satisfactory with a number of candidates achieving notably high scores. Questions 8 and 9 were on topics frequently present in CT3 papers but perhaps examined from a slightly different angle, and as a result were answered less well. The comments on individual questions that follow concern specific parts that candidates answered poorly and important frequent errors.

1 (i) mean $\bar{x} = 4.61$ or £4,610.

(ii) mean of the whole 100 = $\dfrac{20(4610)+80(5025)}{100} = \dfrac{494200}{100} = £4,942$

Generally very well answered.

2 Required moment $= \dfrac{1}{\sum f} \sum_x fx^3$

$$= \frac{1}{50}\left(15\times 0^3 + 20\times 1^3 + 10\times 2^3 + 5\times 3^3\right) = \frac{235}{50} = 4.7$$

Many candidates calculated the third central moment (around the sample mean), rather than the moment around zero as required in the question.

3 Sample proportion = 49/130

Upper 1% normal percentage point = 2.326

98% CI is
$49/130 \pm 2.326*[(49/130)(81/130)/130]^{1/2}$ i.e. 0.3769 ± 0.0989 i.e. (0.278, 0.476)

Answers here were generally satisfactory. Some candidates erroneously computed CIs based on men and women separately.

4 $V[X] = E[V(X|Y)] + V[E(X|Y)]$
$= E[Y^2] + V[2Y] = (100 + 200^2) + 4*100 = 40{,}500$

Generally very well answered.

5 (i) $P[Y = 2] = 0.25 + 0.05 = 0.3$

(ii) $P[X = 0] = 0.75$ and

$P[\{X = 0\} \cap \{Y = 2\}] = 0.25 \neq 0.225 = 0.3 * 0.75 = P[X = 0] * P[Y = 2]$

Therefore X and Y are not independent.

(Any other joint probability can be used.)

(iii) The probability function is

r	1	2	4	8	9	18
$P(R = r)$	0.25	0.05	0.3	0.1	0.2	0.1

(iv) $E[R] = 0.2 * 9 + 0.3 * 4 + 0.25 * 1 + 0.1 * 18 + 0.1 * 8 + 0.05 * 2$
$= 1.8 + 1.2 + 0.25 + 1.8 + 0.8 + 0.1 = 5.95$

In part (ii) notice that one example of P(XY) = P(X)P(Y) is not sufficient for showing independence (it needs to hold for all cases). Also, some candidates failed to provide the probability function in (iii).

6 (i) $p = \Pr(N \geq 1) = 1 - \Pr(N = 0) = 1 - e^{-\lambda}$.

(ii) (a) $X \sim \text{Bin}(20,p)$

(b) $L(p) \propto p^X (1-p)^{20-X}$

$\Rightarrow l(p) = \log(L(p)) = X \log p + (20 - X)\log(1-p)$

and $l'(p) = 0 \Rightarrow \dfrac{X}{\hat{p}} - \dfrac{20-X}{1-\hat{p}} = 0 \Rightarrow X - 20\hat{p} = 0 \Rightarrow \hat{p} = \dfrac{X}{20}$

(and $l''(p) = -\dfrac{X}{\hat{p}^2} - \dfrac{20-X}{(1-\hat{p})^2} \leq 0$)

(iii) $\hat{p} = \dfrac{5}{20} = 0.25$

Then, using the invariance property of the MLE:

$\hat{p} = 1 - e^{-\hat{\lambda}} \Rightarrow \hat{\lambda} = -\log(1-\hat{p}) = -\log(0.75) = 0.288$

Likelihood function now is:

(iv) $L(\lambda) \propto P(X = 0)^{15} \times P(X = 1)^4 \times P(X = 2)$
$\propto \left(e^{-\lambda}\right)^{15} \left(\lambda e^{-\lambda}\right)^4 \left(\lambda^2 e^{-\lambda}\right)$

$\Rightarrow l(\lambda) \propto -15\lambda + 4\log\lambda - 4\lambda + 2\log\lambda - \lambda = -20\lambda + 6\log\lambda$

and $l'(\lambda) = 0 \Rightarrow -20 + \dfrac{6}{\hat{\lambda}} = 0 \Rightarrow \hat{\lambda} = 0.3$

(Also $l''(\lambda) = -\dfrac{6}{\hat{\lambda}^2} < 0$, hence max)

Notice that part (ii)(b) requires the use of the binomial distribution from (ii)(a). In part (iii) the invariance property must be used and mentioned for full credit.

7 (i) $F = 0.213/0.0106 = 20.094$ and at the 5% significance level, $F_{3,20}(0.05) = 3.098$.
Since $F = 20.094 > 3.098$, there is strong evidence against the null hypothesis, and we conclude that there are differences in the mean amounts paid out by the companies.

(ii) The variance of the residuals seems to be similar for the four companies; this is consistent with the assumption of constant variance in the response variable. Also there are no obvious patterns or outliers. The analysis seems valid.

(iii) (a) $LSD = t_{20}(0.025)\sqrt{\sigma^2\left(\frac{1}{6}+\frac{1}{6}\right)}$

$= 2.086\ \sqrt{0.0106/3} = 0.124$

(b) The four company (treatment) means are:

$$\bar{y}_{1\bullet} = \frac{17.810}{6} = 2.968,\ \ \bar{y}_{2\bullet} = \frac{18.940}{6} = 3.157,\ \ \bar{y}_{3\bullet} = \frac{17.715}{6} = 2.953$$

$$\bar{y}_{4\bullet} = \frac{16.185}{6} = 2.698$$

which are given in order and underlined as

$$\bar{y}_{4\bullet} < \underline{\bar{y}_{3\bullet} < \bar{y}_{1\bullet}} < \bar{y}_{2\bullet}$$

Amounts paid out by companies 2 and 4 are significantly different from those paid out by the other two companies. Company 4 seems to pay out significantly lower amounts, with Company 2 paying significantly higher.

Parts (i) and (ii) were generally well answered. In part (iii) many candidates did not use the correct formula for LSD and then performed pair-wise comparisons using the wrong statistic.

8 (i) $E[\hat{\lambda}] = \frac{1}{n}\sum_i E[X_i] = \lambda$

$V[\hat{\lambda}] = \frac{1}{n^2}\sum_i V[X_i] = \frac{1}{n}\lambda$ (using independence of X_i)

(ii) $P[0.2 \le \hat{\lambda} \le 0.3] = P[2 \le 10\hat{\lambda} \le 3]$

$= F(3; \lambda = 2.5) - F(1; \lambda = 2.5) = 0.75758 - 0.28730 = 0.47028$

(iii) (a) $10\hat{\lambda} = \sum_{i=1}^{10} X_i \sim N(2.5, 2.5)$ approximately.

With continuity correction:

$P[0.2 \le \hat{\lambda} \le 0.3] = P[2 \le 10\hat{\lambda} \le 3] \approx P[2 - 0.5 \le 10\hat{\lambda} \le 3 + 0.5]$

$$= P\left[\frac{1.5-2.5}{\sqrt{2.5}} \le Z \le \frac{3.5-2.5}{\sqrt{2.5}}\right] = 2 * F_Z\left(\frac{1}{\sqrt{2.5}}\right) - 1$$

$= 2* F_Z\,(0.63246) - 1 = 2 * 0.73565 - 1 = 0.4713$

(b) $10\hat{\lambda} = \sum_{i=1}^{10} X_i = Y \approx N(2.5, 2.5)$ approximately.

Without continuity correction:

$$P[0.2 \le \hat{\lambda} \le 0.3] = P[2 \le 10\hat{\lambda} \le 3] \approx P\left[\frac{2-2.5}{\sqrt{2.5}} \le Z \le \frac{3-2.5}{\sqrt{2.5}}\right]$$

$$= 2*F_Z\left(\frac{0.5}{\sqrt{2.5}}\right) - 1 = 2*F_Z(0.32) - 1 = 2*0.62552 - 1 = 0.2510$$

(iv) When compared to the exact probability in (ii) the results in (iii) (a) and (b) show that the continuity correction reduces the approximation error significantly for this small sample size.

(v)

$$P[0.2 \le \hat{\lambda} \le 0.3] \approx P\left[\frac{0.2-0.25}{\sqrt{0.25/n}} \le z \le \frac{0.3-0.25}{\sqrt{0.25/n}}\right] = 2 * F_Z\left(\frac{0.05}{\sqrt{0.25/n}}\right) - 1 = 0.95$$

$$\frac{1.95}{2} = F(z), \text{ then } z = 1.96 = \frac{0.05}{\sqrt{0.25}}\sqrt{n}, \text{ and } \sqrt{n} = 1.96\frac{0.5}{0.05} = 1.96, \text{ and } n \approx 384$$

(vi) Using the normal approximation we find:

$$0.27 \pm z_{0.975}\sqrt{\frac{\hat{\lambda}}{n}} = 0.27 \pm 1.96\frac{\sqrt{0.27}}{20} = 0.27 \pm 0.05092 = [021908, 0.32092]$$

In part (i) independence must be mentioned for full marks in the derivation of the variance. In (ii) most candidates either went straight to a normal approximation, or incorrectly calculated the Poisson probability. In part (iii) many candidates applied the continuity correction wrongly.

9 (i) H_0: no association exists v. H_1: association exists

	men	*women*	
for	138	130	268
against	62	70	132
	200	200	400

Under H_0: expected frequencies: 134 134
66 66

O – E: 4 –4
–4 4

$$\chi^2 = 4^2\left(\frac{1}{134} + \frac{1}{134} + \frac{1}{66} + \frac{1}{66}\right) = 0.724$$

P-value = $P(\chi^2_1 > 0.724) = 0.395$

No evidence against H_0 – we conclude that no association exists between gender and attitude to proposal X.

[*Note:* using Yates' correction (not in the Core Reading)
P-value = $P(\chi^2_1 > 0.554) = 0.457$]

(ii) (a) For England:

P-value = $P(\chi_1^2 > 6.653) = 0.010$

Evidence against H_0 – we reject it at the 1% level of testing and conclude that an association exists between gender and attitude to proposal X in England.

For Wales:

P-value = $P(\chi_1^2 > 1.333) = 0.248$

No evidence against H_0 – we conclude that there is no association between gender and attitude to proposal X in Wales.

(b) England: there is evidence of an association – 82% of men and only 66% of women support proposal X – these proportions are significantly different.

Wales: there is no evidence of an association – 56% of men and 64% of women support proposal X – these proportions are not significantly different.

The effects are in different directions and cancel out to some extent when the data are combined: now there is no evidence of an association – overall 69% of men and 65% of women support proposal X – these proportions are not significantly different.

The combined data give a misleading message – they hide the effect of the factor "country" and fail to reveal that there <u>is</u> an association in England.

(iii) (a) The χ^2 value doubles to 6.664

P-value = $P(\chi^2_2 > 6.664) = 0.0357$

Conclusion: reject "no association" at the 3.6% level of testing and conclude that an association does exist.

(b) Comment: having more data with the same proportions provides strong enough evidence to justify claiming that an association exists.

Caution required with the null and alternative hypotheses in (i) – some candidates got these wrong. Also, the associated degrees of freedom were wrongly given in some cases. Part (ii)(b) required comments on the results, but very few candidates did this. Part (iii) was not well answered either.

10 (i) (a) $S_{xx} = 420 - 58^2/10 = 83.6$, $S_{yy} = 217 - 41^2/10 = 48.9$,
$S_{xy} = 202 - 58*41/10 = -35.8$

$SS_T = 48.9$, $SS_{REG} = (-35.8)^2/83.6 = 15.3306$
$\Rightarrow R^2 = 15.3306/48.9 = 0.3135$ (or 31.4%)

[OR using $R^2 = S_{xy}{}^2/S_{xx}S_{yy}$]

(b) Fitted line $y = \hat{\alpha} + \hat{\beta}x$:

$\hat{\beta} = -35.8/83.6 = -0.42823$, $\hat{\alpha} = 4.1 - (-0.42823 * 5.8) = 6.58373$

Fitted line is $y = 6.5837 - 0.4282x$

(ii) $\hat{\sigma}^2 = (48.9 - 15.3306)/8 = 4.1962$

$s.e.(\hat{\beta}) = (4.1962/83.6)^{1/2} = 0.2240$

95% confidence interval for β is $\hat{\beta} \pm t_8 * s.e.(\hat{\beta})$
i.e. $-0.42823 \pm 2.306*0.2240$

i.e. $(-0.945, 0.088)$

(iii) (a) At $x = 9$, $\hat{y} = 6.5837 - 0.4282 * 9 = 2.7299$ i.e. 2.730

(b) $s.e.^2 = \left(\dfrac{1}{10} + \dfrac{(9-5.8)^2}{83.6}\right)4.1962 = 0.93360 \Rightarrow s.e. = 0.9662$

(iv) Addition of new observation makes data more randomly scattered. The strength of the linear relationship is reduced from "weak" to "almost nothing".

Generally well answered. Some problems were encountered in part (iii)(b), where the wrong formula was used.

END OF EXAMINER'S REPORT