EXAMINATIONS

April 2007

Subject ST6 — Finance and Investment
Specialist Technical B

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M A Stocker

Chairman of the Board of Examiners

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QUESTION 1

Syllabus section: (h) (i) – (iii)

Core reading: Units 8 – 10

(i)

(a) The two measures **P** and **Q** are equivalent if and only if

1. They operate on the same sample space; and
2. They agree on what is possible (and what is impossible)

Alternative answers

If A is any event in the common sample space on which **P** and **Q** operate, then **P** and **Q** are equivalent if and only if $P(A) > 0 \Leftrightarrow Q(A) > 0$.

OR Quote the Radon-Nikodym Theorem involving $\frac{d\mathbf{Q}}{d\mathbf{P}}$.

(b) The continuous process W_t is a Brownian motion under measure **P**

if and only if

- $W_0 = 0$
- $W_t \sim N(0,t)$ under **P** for all $t > 0$.
- The increment $W_{t+s} - W_s \sim N(0, t)$ under **P** for all $t > 0$, and is independent of the filtration $\mathbf{F}_s$, the history of the process from time 0 to time s.

(ii)

The C-M-G theorem states that, if W_t is a **P**-Brownian motion and γ_t is a bounded **F**-pre-visible process

then there exists a measure **Q** equivalent to **P** such that

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = \exp\left(-\int_0^T \gamma_t dW_t - \tfrac{1}{2}\int_0^T \gamma_t{}^2 dt \right)$$

and

$$\tilde{W}_t = W_t + \int_0^t \gamma_s ds \quad \text{is a } \mathbf{Q}\text{-Brownian motion.}$$

(iii)

(a) From C-M-G, with $\gamma = \frac{\mu}{\sigma}$, there exists a measure **Q** equivalent to **P**,

such that $\tilde{W}_t = W_t + \frac{\mu}{\sigma} t$ is a **Q**-Brownian motion.

Hence, under **Q**, $X_t = \sigma \tilde{W}_t$, a scaled Brownian motion, i.e. X has no drift under **Q**.

(b) $\mathbf{E_P}(X_t) = \mu t$ under **P**, since $\mathbf{E_P}(W_t) = 0$, and $\mathbf{E_Q}(X_t) = 0$ under **Q**.

$\mathbf{E_P}(X_t{}^2) = \mu^2 t^2 + \sigma^2 t$ under **P**, and $\mathbf{E_Q}(X_t{}^2) = \sigma^2 t$ under **Q**.

Hence the variance under **P** and **Q** are both the same, namely $\sigma^2 t$,

so the volatility in both cases is σ.

(c) The Radon-Nikodym derivative valued at time 0 up to time *t* is:

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_0^t = \exp\left(-\int_0^t \gamma_s dW_s - \tfrac{1}{2}\int_0^t \gamma_s{}^2 ds\right) = \exp\left(-\frac{\mu}{\sigma}W_t - \frac{\mu^2}{2\sigma^2}t\right)$$

[*Candidates may equally interpret the question as valuing the derivative at time t up to final time T; in this case the formula would be:*

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_t^T = \exp\left(-\int_t^T \gamma_s dW_s - \tfrac{1}{2}\int_t^T \gamma_s{}^2 ds\right) = \exp\left(-\frac{\mu}{\sigma}(W_T - W_t) - \frac{\mu^2}{2\sigma^2}(T-t)\right).]$$

QUESTION 2

Syllabus section: (h) (i)-(iii)

Core reading: Units 8 – 10

(i)

Firstly, $X_0 = 0$.

Secondly, $E(X_t) = 0$ since $E(W_t) = 0$, and $\text{var}(X_t) = \frac{1}{a}at = t$

Hence $X_t \sim N(0, t)$.

Thirdly, $E(X_{s+t} - X_s) = 0$ for $t > 0$, and

$$\text{var}(X_{s+t} - X_s) = \frac{1}{a}\text{var}(W_{as+at} - W_{as}) = \frac{1}{a}at = t$$

Hence $X_{s+t} - X_s \sim N(0, t)$ and is independent of the filtration up to time *s*, $\mathbf{F}_s$.

Thus X_t must be a Brownian motion.

(ii)

$$\frac{\partial Y_t}{\partial t} = -\tfrac{1}{2}b^2 Y_t, \quad \frac{\partial Y_t}{\partial W_t} = bY_t, \quad \frac{\partial^2 Y_t}{\partial W_t^2} = b^2 Y_t$$

Using Ito's Lemma gives:

$$dY_t = (-\tfrac{1}{2}b^2 Y_t + \tfrac{1}{2}b^2 Y_t)dt + bY_t dW_t = bY_t dW_t$$

Y_t is driftless (no *dt* term), hence it is a martingale.

(iii)

$$\frac{\partial Z_t}{\partial t} = -3W_t, \quad \frac{\partial Z_t}{\partial W_t} = 3W_t^2 - 3t, \quad \frac{\partial^2 Z_t}{\partial W_t^2} = 6W_t$$

Using Ito's Lemma gives:

$$dZ_t = (-3W_t + \tfrac{1}{2}6W_t)dt + (3W_t^2 - 3t)dW_t = 3(W_t^2 - t)dW_t$$

Z_t is driftless (no dt term), hence it is a martingale.

QUESTION 3

Syllabus section: (g) and (i)

Core reading: Units 7 and 12

(i)

(a) The delta of an option is the change in option price for a given change in underlying asset value.

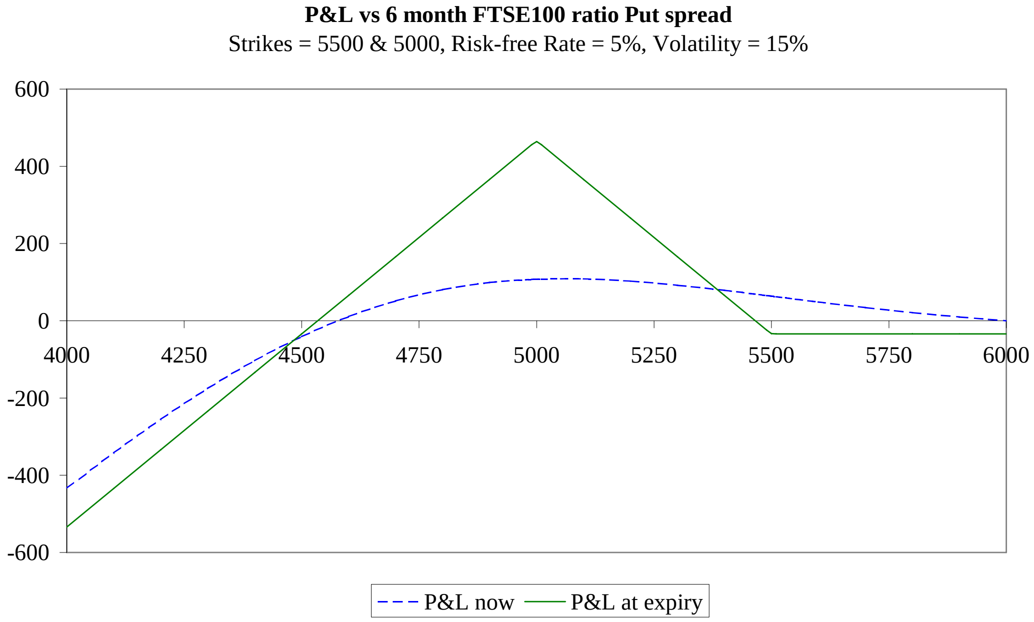
The net delta of the portfolio is the aggregate delta of all options in a portfolio added together.

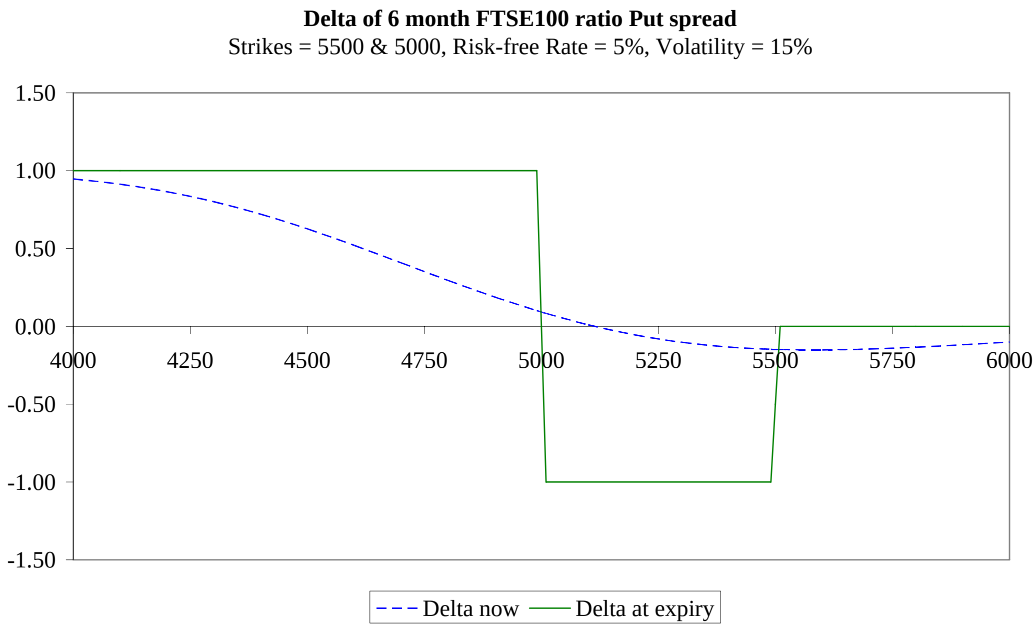
In symbols, this is $\Delta = \sum_{i=1}^{n} X_i \frac{\partial V_i}{\partial S}$ where X_i and V_i are the amount and value of the ith option respectively, and S is the price of the underlying asset.

(b) A delta-neutral hedge of the portfolio is taking an opposite position in the underlying asset that offsets the delta of the portfolio. It is called delta-neutral because the portfolio of derivatives and underlying then has a net zero delta. This is usually undertaken dynamically to ensure that the portfolio remains delta-neutral.

(ii)

(a) & (b)





[*The solutions shown above are more accurate than expected in the examination – they are computer drawn for clarity. Note that the dotted Profit/Loss (P&L) line goes to the left of the expiry line as S decreases. This is because risk free rate r is not zero, hence there is a time shift towards expiry. It would be acceptable to assume r = 0 and have the dotted line approach the expiry line asymptotically.*]

(iii)

The Black Scholes model assumes all volatility inputs are constant and there are no jumps or hedging costs. However, this is not in line with observed fact. Hence the implied volatility curve is not flat.

The skew in favour of low price Put options could arise from several possible causes:

- the skew indicates that the returns from equity indices are not exactly log-normal, with a larger tail on the downside or a risk of jumps;
- traders are reflecting the fact that low price options are more risky to sell – this is a supply and demand effect;
- rehedging costs are assumed to be zero in Black-Scholes, but if large moves occur then there might need to be significant re-balancing;
- the equity market has a tendency to fall with faster speed than it rallies – the reason for the skew is therefore sometimes given as "crashophobia";
- firms who issue equity are leveraged, so the lower the equity price, the more risky the stock becomes.

QUESTION 4

Syllabus: (h)(iv)-(ix), (i)

Core reading: Units 10 – 13

(i)

From the assumptions, we are given that $dC = mCdt + vCdz$, with the usual Wiener process dz.

Setting $G = \ln C$ and using Ito's Lemma, we have:

$$dG = (m - \tfrac{1}{2}v^2)dt + vdz$$

which is a generalised Wiener distribution with drift $m - \tfrac{1}{2}v^2$ and volatility v, both constants.

Hence the change in ln C from t to T is Normally distributed with mean $(m - \tfrac{1}{2}v^2)(T - t)$ and variance $v^2(T - t)$, i.e.:

$$\ln C_T - \ln C_t \sim N\left((m - \tfrac{1}{2}v^2)(T - t), v^2(T - t)\right)$$

or

$$\ln C_T \sim N\left(\ln C_t + (m - \tfrac{1}{2}v^2)(T - t), v^2(T - t)\right)$$

This is a log-normal distribution as required.

[*Candidates may equally choose to set t = 0 in the above.*]

(ii)

From (i), at time t, the expected value of C_T is

$$E_t(\ln C_T) = \ln C_t + (m - \tfrac{1}{2}v^2)(T - t)$$

In a risk-neutral world, we replace the return on C_t by the risk-free rate, hence:

$$E_t(\ln C_T) = \ln C_t + (r - \tfrac{1}{2}v^2)(T-t) \quad (\textit{risk-neutral})$$

Hence the value H_t of COLTS at time t is:

$$H_t = e^{-r(T-t)}\left[\ln C_t + (r - \tfrac{1}{2}v^2)(T-t)\right]$$

[*Candidates who allowed for proportional storage costs, say u, received an additional credit. The formulae above, and also in (iii) below, change r to r + u.*]

(iii)

The result can be obtain by substitution. Consider:

$$\frac{\partial H}{\partial t} = re^{-r(T-t)}\left[\ln C_t + (r - \tfrac{1}{2}v^2)(T-t)\right] - e^{-r(T-t)}(r - \tfrac{1}{2}v^2)$$

$$\frac{\partial H}{\partial C} = \frac{1}{C}e^{-r(T-t)}, \quad \frac{\partial^2 H}{\partial C^2} = -\frac{1}{C^2}e^{-r(T-t)}$$

Using these in the Black-Scholes-Merton partial differential equation:

$$\begin{aligned}\text{LHS} &= e^{-r(T-t)}\left[r\ln C_t + r(r - \tfrac{1}{2}v^2)(T-t) - (r - \tfrac{1}{2}v^2) + r - \tfrac{1}{2}v^2\right] \\ &= re^{-r(T-t)}\left[\ln C_t + (r - \tfrac{1}{2}v^2)(T-t)\right] \\ &= rH_t \\ &= \text{RHS}\end{aligned}$$

as required.

QUESTION 5

Syllabus: (h)(iv)-(ix), (i)

Core reading: Units 10 – 13

(i)

[*The solution may seem long for this part, but it is mostly standard bookwork.*]

(a)

The FD method approximates the solution of the PDE by setting up a discrete rectangular grid of price changes ΔS and time steps Δt spanning all possible outcomes of the stock evolution over the time 0 to T.

These steps must be small enough to make the approximation accurate, but not so small that the number of steps is computationally intense (leading to rounding errors).

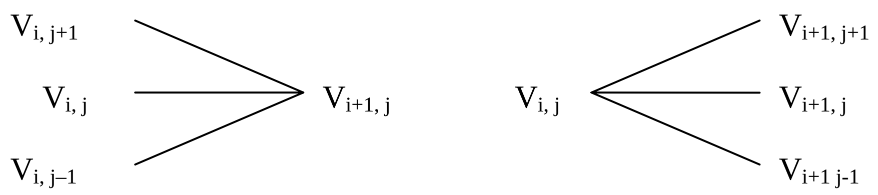
Let the steps in time go from i = 1 to N, and in stock price from j = 1 to M, where $T = N\Delta t$, $S_{\max} - S_{\min} = M\Delta S$ ($S_{\max}$ / $S_{\min}$ are the highest / lowest prices in the grid).

The approach is to approximate the differential terms $\frac{\partial V}{\partial S}, \frac{\partial V}{\partial t}$ etc with values from neighbouring nodes.

The two types of approach are Implicit FD, which approximates a difference by taking values at the nearest previous time step, and Explicit FD, which does the same but for the next time step.

The approaches can be summarised in a diagram:

Implicit FD **Explicit FD**



The implicit method is stable and robust, and always converges …

… but can only be solved implicitly, hence the name – there are many methods (such as the Hopscotch method) which efficiently solve the resulting matrix of relationships at each time step.

The explicit method is easy to compute, functionally the same as the trinomial tree …

… but can introduce instabilities if the "pseudo-probabilities" created by the three branches are invalid anywhere (i.e. < 0 or > 1).

(b)

In the example given, the option is a 1-year American Put, so $T = 1$. Also don't forget that σ depends on t and hence changes each time step.

<u>Initial and boundary conditions</u>

Initial conditions occur at the option expiry:

$$V_{N,j} = \max(K - j\Delta S,\ 0) \ \text{ for } j = 1, 2, 3, \ldots M$$

Boundary conditions occur at $S = S_{\min}$ and $S = S_{\max}$:

$$V_{i,0} = K, \quad V_{i,M} = 0$$

[*Candidates who choose the <u>explicit</u> method should state that boundary conditions are not required.*]

The American feature creates a "free" boundary – at each node, compare node value (option value) with the early exercise value. If early exercise is optimal, replace the option value with the early exercise value.

Alternative 1 – Implicit FD

Set up the following difference equations:

$$\frac{\partial V}{\partial S} \approx \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta S}$$

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{(\Delta S)^2}$$

$$\text{and } \frac{\partial V}{\partial t} \approx \frac{V_{i+1,j} - V_{i,j}}{\Delta t}.$$

Substitute into the original PDE and solve by running backwards from $t = T$ to get the current value of the Put option at the central node at $t = 0$. (A solving algorithm will be required.)

OR

Alternative 2 – Explicit FD

Set up the following difference equations:

$$\frac{\partial V}{\partial S} \approx \frac{V_{i+1,j+1} - V_{i+1,j-1}}{2\Delta S}$$

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{i+1,j+1} - 2V_{i+1,j} + V_{i+1,j-1}}{(\Delta S)^2}$$

$$\text{and } \frac{\partial V}{\partial t} \approx \frac{V_{i+1,j} - V_{i,j}}{\Delta t}.$$

Substitute into the original PDE and solve by running backwards from $t = T$ to get the current value of the Put option at the central node at $t = 0$.

(ii)

(a)

The FD method performs the integral directly by approximating continuous time and state variable values with small discrete steps.

This is a discrete time approximation of a continuous time integral.

The binomial tree method performs an expectation using the risk-neutral probability measure.

The binomial tree creates the risk-neutral probabilities of up and down moves – say this measure is ***P***.

Then, under ***P***, the expectation $\mathbf{E}_{\mathbf{P}}[V_t \mid \mathbf{F}_s]$ is a Martingale, so equals V_s.

The binomial tree approximates this expectation in discrete time using ***P***, so must provide an approximation of the integral that creates the continuous time expectation.

(b)

Both methods are suitable for valuing stock options, provided they are not strongly path dependent.

FD is computationally intense, slow, difficult to program, prone to instabilities ...

... however, there is a lot of literature (especially from Physics) on how to make this method efficient and solve different boundary conditions.

Binomial is fast and efficient, intuitive, stable ...

... however, it doesn't converge that fast (oscillates)

... and becomes much more complex if the tree does not recombine, as tends to occur for certain processes.

QUESTION 6

> ***Syllabus: (l) and (m)***
>
> ***Core reading: Units 15 and 16***

General

In summary, the report should explain that the three areas targeted are all fields where advanced technical understanding is essential, and therefore caution is urged.

In the case of inflation and longevity risk, there is little underlying liquidity in the market with which to effect hedging, so it may not be possible to offset some positions completely.

In the case of credit derivatives, the market is very broad but some of the products are extremely complex, and even the largest trading houses do not always fully understand the interactions that can occur.

Unlike interest rate or FX risk, credit risk is usually very granular (i.e. specific to a counterparty or bond) and so hedges are often at best imperfect and may in certain cases not even work at all.

Additional risk factors

For credit derivatives, several new risk factors are introduced:

- Default risk (the risk that a name defaults on which protection has been sold – this is similar to existing issuer risk, but is exacerbated by the gearing available with credit derivatives)
- Event risk (the risk that an event occurs unexpectedly that triggers a payment)
- Basis risk (the risk that the credit defaults swap diverges from the underlying credit)
- Correlation risk (the risk that two or more issuers have correlation between defaults or creditworthiness – higher correlation reduces the benefits of diversification)
- Operational risk (related to handling of trade confirmations, settlement of payments, monitoring of defaults etc.)

- Reputational or legal risks (e.g. mis-selling of inappropriate complex products)

For inflation swaps, the new risk factor is inflation risk (the risk that the inflation index, e.g. RPI, evolves differently from that anticipated). Under a certain scenario, inflation could spiral out of control and result in heavy losses for those banks who have sold inflation protection.

For longevity hedges, the new risk factor is mortality risk (the risk that mortality rates are lower than expected, i.e. people live longer). Pension funds are sensitive to this risk because pensions will be more expensive if people live longer. Hence pension funds wish to buy protection. We will assume that the bank has a life insurance subsidiary that can sell this protection.

There is also liquidity risk for these two classes (the risk that the market is unable to supply sufficient capacity to effect the hedges required to maintain a neutral position).

<u>Hedging approach</u>

It is important with any business to be able to reduce the sensitivity to the risk factors that the bank does not wish to have exposure to.

For credit derivatives:

- Default risk is controlled within a suitably sophisticated credit exposure manager by the impositions of limits and thresholds. Offsets will be allowed in certain well-defined situations when justified (the simplest being a credit default swap purchased to hedge an underlying single name exposure).
- Event risk is controlled by ensuring that ISDA agreements are signed with protection buyers and sellers, and also by introducing adequate systems and controls.
- Correlation risk can be controlled by diversification.

For both inflation swaps and longevity hedging, OTC hedges are hard to find in the market, so the following could be used:

- Buy and sell the nearest equivalent products, but this could lead to big mismatch positions.
- Adopt natural hedges by taking on products with opposite characteristics – e.g. annuities and term assurance offset in relation to longevity risk.
- Reinsure or securitise (e.g. issue an inflation-linked bond)
- Accept as a business risk and don't hedge (but this will limit positions).
- Diversify by selling in different markets, jurisdictions, etc.

<u>Control aspects</u>

Suggest limiting exposure whilst unfamiliar with the products.

There must be adequate separation between trading and support functions that act as controllers, e.g. those responsible for daily marking of the books, risk management, settlement etc. This also applies to the building of models and model validation.

It is very important that proper risk monitoring takes place.

For example, for credit derivatives a series of market risk controls should be available – credit spread value-at-risk, credit spread sensitivities by rating grade and asset class, basis risk sensitivities, etc. These should be refreshed daily.

The mechanism whereby offsets of issuer risk take place should be agreed and put into credit policy.

There is also a need to reflect the residual risk to the counterparty who has sold or bought protection.

Also need control of concentrations to avoid building up a large position in one credit.

Stress testing against extreme scenarios is very important for credit derivatives.

Modelling problems

The model building effort requires strict standards to ensure the process is well controlled, any model assumptions, deficiencies and limitations are clearly known, restrictions on use are carefully adhered to and change control procedures are observed when models are updated.

Models for single name credit default swaps are well understood and effective, so should cause no problems.

There is still no market standard model for correlation based products such as Nth to default or index tranches – a number have been used, but it is still individual trader views that govern selection. These models can be very complex.

This imposes a concern that the model building and independent validation are performed by experts with appropriate experience and specialist knowledge.

Modelling for inflation swaps is simple, bootstrapping off the inflation-linked swap or bond market; however, there can be problems building models when this data is unavailable, e.g. in markets where there are few such swaps or bonds.

Lack of data for credit derivatives or parameters for other models (e.g. longevity).

QUESTION 7

Syllabus: (k)

Core reading: Unit 14

(i)

The value at time t of the forward rate spanning period T_1 to T_2 is given by:

$$f(t,T_1,T_2) = -\frac{\ln P(t,T_2) - \ln P(t,T_1)}{T_2 - T_1} \qquad (*)$$

The value at time t instantaneous forward rate of maturity T is given by:

$$F(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}$$

which is equivalent to lim $f(t, T, T')$ as $T' \to T$.

(ii)

Using Ito's lemma,

$$d(\ln P) = (r - \tfrac{1}{2}v^2)dt + vdz$$

so inserting this in the formula from (*) above for T_1 and T_2 gives:

$$df(t,T_1,T_2) = \tfrac{1}{2}\frac{v(t,T_2,\Omega)^2 - v(t,T_1,\Omega)^2}{T_2 - T_1}dt - \frac{v(t,T_2,\Omega) - v(t,T_1,\Omega)}{T_2 - T_1}dz$$

Letting $T_2 \to T_1 = T$, the LHS of the above equation becomes the instantaneous forward rate $F(t, \mathrm{T})$.

Hence

$$dF(t,T) = \tfrac{1}{2}\frac{\partial\left[v(t,T,\Omega)^2\right]}{\partial T}dt - \frac{\partial v(t,T,\Omega)}{\partial T}dz$$

which depends only on v and its partial derivatives with respect to T.

(iii)

(a) <u>HJM vs Hull-White (HW) vs LIBOR Market Model (LMM)</u>

HW is a simple yield curve model. HJM and LMM are more complicated.

Short rates are the driving risk factor calculated in HW. In HJM, it is the instantaneous forward rates, and in the LMM it is market forward rates.

The assumptions for the processes mean that in each model the driving factor behaves log-normally. [*Although the models are technically incompatible, in most practical cases the different assumptions are not hugely significant (i.e. do not lead to large variations in pricing).*]

HJM and LMM are very similar. They are equally valid theoretically, but LMM is easier to fit to caplet prices, which are derived from market forward rates (whereas instantaneous forward rates are not observable).

Generally, HJM and LMM take longer and are more complex to implement than HW.

It is far from trivial to implement the HJM model based on market prices.

One would like to use prices of caps or swaptions to drive model parameters, but ...

... the problem is that these are valued in the market using the Black model, which is a log-normal distribution ...

... and specifying a log-normal volatility for forward rates in HJM severely restricts the forms of the HJM model that can be used.

Both HJM and LMM are good fits because volatility parameter is very flexible, unlike HW. However, whenever model parameters are allowed to depend on several variables, this uses up lot of information, so the ease of calibration may obscure the fact that the models show little "insight" or long-term stability.

LMM adds the insight of "instantaneous volatility", which enables modelling of the volatility function, especially for skew effects (using Constant Elasticity of Variance models).

Non-recombining ("bushy") trees are required for HJM and LMM, which are messy. HW uses trinomial trees, which are much easier to implement and test.

Usually HJM and LMM use the MC method.

(b) <u>Note on one or more factors</u>

In one-factor models, correlation is implicit and the evolution of certain yield curve shapes cannot be reproduced.

Also can't simultaneously price swaptions and caplets.

One benefit of more factors is that hedging improves – difficult with only one factor.

Whether one or more factors are required depends on what you are valuing. Two or more factors are needed when there are significant correlation related impacts: e.g. spread options, interest rate securities with FX effects etc.

Swaptions and caplets imply certain correlation effects but, considering the calibration difficulties, are not likely to suggest a good correlation function.

HJM and LMM are usually used in their two- or three-factor version.

MC is used for HJM and LMM, and is good with more factors – calculations scale as n as opposed to n^2 in trees.

HW is also suitable for 2-factors but uses a recombining tree approach.

QUESTION 8

Syllabus: (j)

Core reading: Units 5 and 13

(i)

Let g_j be the coupon of bond j, and B_t^j its price.

The bond formula is $B_t^j = g_j \sum_{k=1}^{t} d_k + 100 d_t$, with B_t^j and g_j in %.

Hence $d_t = \dfrac{B_t^j - g_j \sum_{k=1}^{t-1} d_k}{100 + g_j}$, which can be solved iteratively, starting at $t = 1$.

So

$d_1 = 99.876 / 105 = 0.95120$

$d_2 = (99.970 - 5.25\ d_1) / 105.25 = 0.90239$

$d_3 = (101.882 - 6\ (d_1 + d_2)) / 106 = 0.85623$

but those first three values were given in the table.

The fourth and fifth year discount factors are:

$d_4 = (98.072 - 4.75\ (d_1 + d_2 + d_3)) / 104.75 = 0.81337$

$d_5 = (96.880 - 4.5\ (d_1 + d_2 + d_3 + d_4)) / 104.5 = 0.77537$

[*Note that redemption yields were not required.*]

(ii)

(a) At outset, the value of the fixed payments on a swap equals the value of the floating ones.

Hence the fixed coupon for a swap of length t is $S_t = \dfrac{1-d_t}{\sum_{k=1}^{t} d_k}$.

For a 5-year swap:

Value of fixed coupons = $d_1 + d_2 + d_3 + d_4 + d_5$ = 4.29856

Value of floating coupons = 1 – d_5 = 0.22463

Hence swap fixed coupon = 0.22463 / 4.29856 = 0.05226, or 5.226 in %.

(b) The bond prices are for Government bonds, but swaps are priced by banks who have a lower credit rating.

Hence there might be a credit margin added to the rates in the marketplace, so a swap might trade at a slightly higher rate. (The difference will be small, though.)

Large demand for government bonds (e.g. pension funds) might depress their yields.

(iii)

(a) Forward rates give us an indication of market expectations of interest rate movements.

The forward rate from time t – 1 to t is $f(t-1,t) = \dfrac{d_{t-1}}{d_t} - 1$ using annual compounding.

The forward rates are thus: 5.130, 5.410, 5.390, 5.270 and 4.901. This clearly shows a trend up to year 2, then down again.

[*Candidates might decide only to calculate forward rates at t = 2 and t = 5, or continuously compounded rates – these are acceptable. The continuously compounded rates are 5.003, 5.269, 5.250, 5.136 and 4.784.*]

(b) Holders of bonds must accept reinvestment risk ...

... the current redemption yield might be obtained now, but if rates fall the investor will not be able to achieve such a good rate in the future once the original bond has redeemed. [½]

[*Bond prices are sensitive to changing interest rates – however, the question was not specifically asking about this.*]

(iv)

[*The strike price of 100 was omitted from the question, so if candidates used a different strike (e.g. at the money = 96.88) they were not penalised. The solution given here applies for a strike price of 100.*]

Use Black-Scholes on the value of the bond less the present value of any coupons received before the option expires. Hence:

S = 96.88 – 4.5 d_1 = 92.59959

[*Alternatively, use the forward price* $F = Se^r$ *= 97.347 and* $\Phi\left[\dfrac{\ln(F/K)+\frac{1}{2}\sigma^2}{\sigma}\right]$ *for* N_1 *etc.*]

$K = 100$ [*see above – a different value could be chosen by the candidate*]

$T = 1$, volatility $\sigma = 0.06$, $r = 0.05$

$$N_1 = \Phi\left[\frac{\ln(S/K)+r+\frac{1}{2}\sigma^2}{\sigma}\right] = \Phi[(-0.07689 + 0.00180 + 0.05) / 0.06]$$

$$= \Phi[-0.418091] = 0.33794$$

$$N_2 = \Phi\left[\frac{\ln(S/K)+r-\frac{1}{2}\sigma^2}{\sigma}\right] = \Phi[(-0.07689 - 0.00180 + 0.05) / 0.06]$$

$$= \Phi[-0.47809] = 0.31629$$

Hence Call price = $S N_1 - Ke^{-r} N_2$ = 31.293105 – 30.086435 = 1.20667

(v)

In creating a forward bond price, the Black model assumes that the risk-free rate is constant and independent of the underlying bond price.

However, the 5-year bond price is correlated to some extent with the one-year interest rate, i.e. their movements are not independent.

Hence a more representative pricing model would be a 2-factor model including correlated 1-year and 5-year risk factors.

The model would need to measure a suitable forward volatility and choose an appropriate correlation – neither of these is easy to find.

The log-normal assumption for bond price movements (i.e. geometric Brownian motion) might be inappropriate.

END OF EXAMINERS' REPORT