EXAMINATIONS

April 2005

Subject ST6 — Finance and Investment
Specialist Technical B
and
Certificate in Derivatives

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

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Q1
This is a typical type of ST6 question on Brownian motion, and was generally well answered. Part (ii) followed easily from the definitions in part (i), but required setting out precisely to get full marks. Part (iii) was found straightforward by most candidates.

1 ***Syllabus: (e)(iv)***

(i) Process $W(t)$ in $t \geq 0$ is a **P**-Brownian motion **if and only if**:

(a) $W(t)$ is continuous and $W_0 = 0$

(b) $W(t) \sim N(0, t)$ under **P**

(c) $W(t) - W(s) \sim N(0, t - s)$, $s < t$, under **P** and is independent of its prior path up to time s.

(ii) $W(t_i)$ has a Normal distribution with zero means and variance t_i.

Hence all we need to prove is that $\text{cov}(W(s), W(t)) = \min\{s, t\}$.

For $s < t$,

$$E(W(s), W(t)) = E(W(s)^2 + W(s)[W(t) - (W(s)]) = E(W(s)^2) + 0$$

since $E(W(s)) = 0$ and W has independent increments.

So $\text{cov}(W(s), W(t)) = \text{var}(W(s)) = s$ as required.

(iii) Using Ito's Lemma on $X = f(S)$, from the dW term: $\sigma_X X = \frac{\partial X}{\partial S}\sigma_S S$

and from the dt term the drift is: $\frac{\partial X}{\partial S}\mu S + \frac{\partial X}{\partial t} + \frac{1}{2}\frac{\partial^2 X}{\partial S^2}\sigma_S{}^2 S^2$

So, if μ increases by $\beta\sigma_S$, then the drift of X increases by:

$$\frac{\partial X}{\partial S}\beta\sigma_S S = \beta\sigma_X X$$

Hence the growth rate increases by $\beta\sigma_X$, as required.

Q2

This question was not well answered, with only a few achieving above half-marks. Graphical questions should be expected from time to time, as with CiD, but they always seem to cause problems.

For part (i), few understood why a FTSE time spread might be purchased, and their subsequent guesses were seldom appropriate. If an option structure is unfamiliar, candidates need to apply their general knowledge of options and try to identify those aspects which are relevant to the question. In this case, that time value increases with volatility and option maturity are the key facts to focus on.

For part (ii), the main requirement is to demonstrate a knowledge of the basic shape of the curves. Even if the exact structure described were unfamiliar, this could have been worked out from the basic call option diagram that many reproduced somewhat hopefully. The fact that time value is greatest at-the-money means that there is a peak in the P&L curve at that point.

It should be noted that a perfect standard is not expected for the "sketch". Rather, demonstration of understanding is the important factor. Full marks can be obtained with a neat diagram that broadly shows the patterns required (as illustrated in the model solution). It is not necessary to draw precise lines or label axes accurately, but the diagrams must be clear and representative – for example, the peak of the P&L graph in part (ii)(a) must be at-the-money.

The gamma curve in part (ii)(b) was a tough challenge, although several candidates managed to sketch it correctly. Close attempts were given generous treatment.

2 ***Syllabus: (f)(vi)(ix)***

(i) Volatility is not constant across the curve, so the two options will behave differently in terms of vega, so this can be a volatility play.

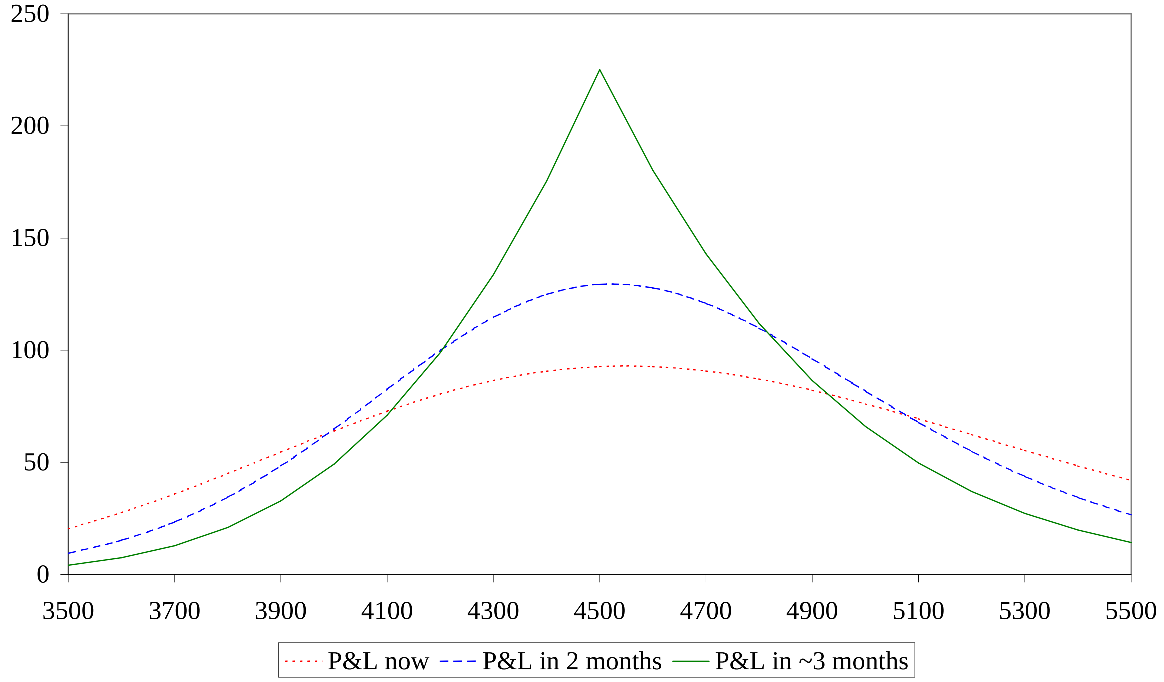
There may be a view that the volatility of the shorter maturity option would decline relative to the longer maturity options. This would occur if a quiet period was expected in the markets over the next few months, but that this lull would not last for the whole period of the longer option.

Another use is cost reduction — the higher time decay on the shorter dated option will reduce the cost of the longer dated option.

(ii) (a) **P&L**

The longer option will retain its time premium whilst the short one moves towards intrinsic value. This creates "curvy" expiry P&L lines on the graph.

P&L vs FTSE Price - 3 month / 6 month Time Spread
Strike = 4500, Risk-free Rate = 0%, Volatility = 25%

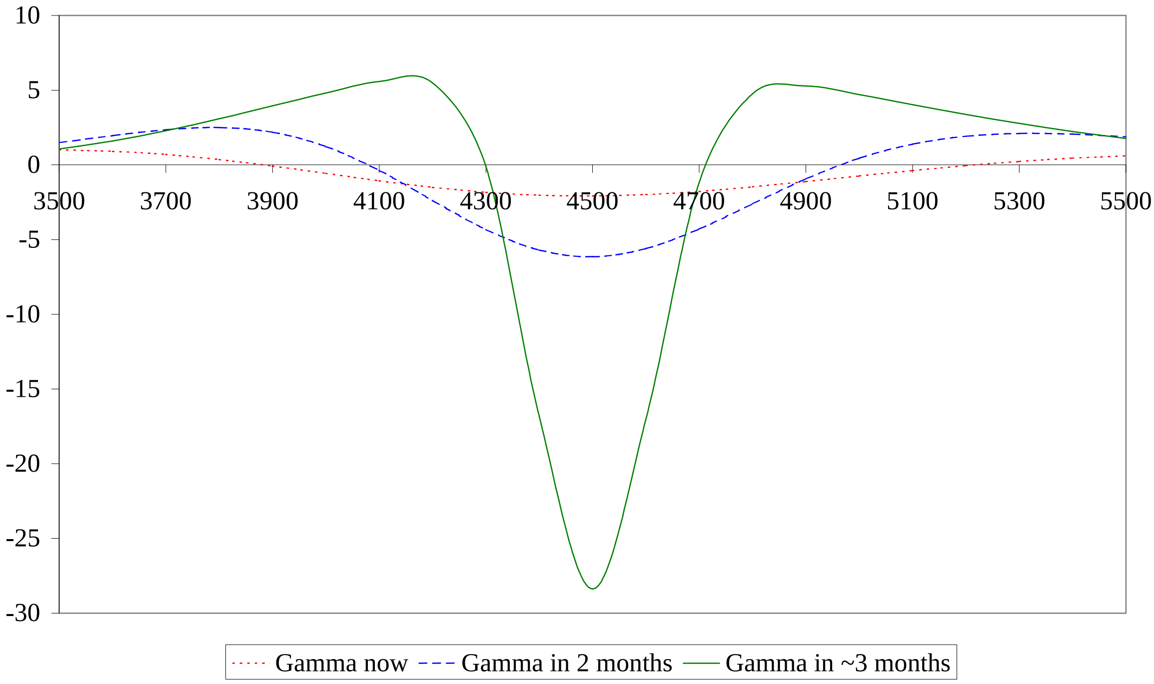


[Marks: ½ for sensible *x*-scale, 1 for each curve, *y* scale not important]

(b) **Gamma**

Gamma will increase at the money for the shorter option, so will result in a very short position at the money, but a long position elsewhere. [1]

Gamma vs FTSE Price - 3 month / 6 month Time Spread
Strike = 4500, Risk-free Rate = 0%, Volatility = 25%



Q3

This short question was designed to show ability to think about risk management and the problems of creating curves from market data. It is partly covered in the Core Reading, but there were extension elements that made it more than a bookwork question. Generally, there were some good attempts at certain parts, but few candidates managed a good response to the entire question.

For part (i)(a), bootstrapping was correctly identified by most, although fewer could correctly describe what bootstrapping meant in practice. Even fewer managed anything sensible for part (i)(b) – points such as interpolation and spanning were sought.

For part (ii), a description was required of rate perturbation to find risk sensitivities, and most candidates supplied this. However, part (ii)(b) eluded almost all, and this was disappointing, because the ability to analyse the practical effectiveness of any quantitative framework is an important attribute of an actuary. The most significant points are complexity and over-specification (inter-dependence).

3 ***Syllabus: (g)(iii)***

(i) (a) The most common method is the bootstrapping methodology.

Choose a set of spanning instruments. Normally this includes the early money-market rates up to the first interest rate future, say 1 week, 1 month, possibly 2 month. Then take all the interest-rate futures up to the limit of liquidity — usually this is a few years. Then use the swap rates from there on up to 30 years.

Interpolate between any missing swap rates so that at least an annual curve is produced: e.g. linearly, or use a cubic spline — the slope of the zero coupon curve should be zero at infinite duration and constant at zero duration.

Use the yield curve construction methodology to produce the discount factors. This is iterative, starting at time 1.

The method calculates the first period zero coupon rate from the first spanning instrument (usually a money-market rate). This gives the first period discount rate. The second spanning instrument is valued using the first and second period discount rates. As the first one is now known, the second period discount rate can be derived. The third instrument provides the third period discount rate, and so on.

Discount values for time t are present value of a payment of 1 paid at time t. Interpolate finely to get discount values for each day up to 30 years (10,000 numbers), so every future cash flow can be valued.
[max 3 marks]

(b) **Special features**

This method reproduces the prices of the spanning instruments (benchmarks) precisely, so is arbitrage-free.

Interpolation of rates for the swaps can affect the smoothness of the forward curve — something smooth more than just linear interpolation is really required here. Intermediate points would be best sourced from the market if liquid enough.

Longer Eurodollar futures have significant convexity, due to the fact they pay "up front" a payment calculated from a forward rate, so an adjustment is required especially out to the longer dated futures.

Other approaches to constructing a zero coupon curve are possible. If the bootstrap method is not used a better result may be achieved intellectually but care must be taken with its use to avoid arbitrage, because much of the market may be willing to trade against it.

[*Other points could be accepted.*]

(ii) (a) **Managing market risk**

For each of the spanning instruments used to construct the curve, perturb the yield by a small amount (1 basis point typically), thereby creating a new discount curve. Calculate the change in portfolio value from this perturbation. This gives a "delta" sensitivity to each instrument.

The market risk sensitivity of the portfolio is therefore summarised in terms of a hedge using the benchmark instruments. The risk can be neutralised by taking opposite positions in each of these instruments.

(b) **Problems**

There is no single number produced, so the results are hard to interpret and include hidden correlation effects — this is why Value-at-Risk has become so popular.

However, the information can be totalled to give a net figure, which would be the sensitivity to a 1bp parallel shift in the entire curve.

The results only apply for a small interval around the current market level, not further. There are no options, but even bonds and swaps have convexity effects.

The curve is over-specified by the benchmark instruments, so the hedge is not the most efficient to execute in terms of transaction costs.

Futures move along the curve, so do not have a fixed maturity. This can lead to strange jumps in sensitivities on roll-overs.

Government bonds trade at spreads to LIBOR, so bonds will not be properly valued using the LIBOR curve. This will make the risk calculations inaccurate, albeit not hugely so. There may also be a requirement to calculate spread risk (bond vs swap).

Also, different yield levels for bonds of the same approximate maturity but different coupon. Some of this reflects timing of the flows (i.e. a zero coupon effect) but some reflects the differential tax characteristics of market participants. Some government bond price differences still reflect supply / demand effects.

[*Other points could be accepted.*]

Q4

This basic question on forward prices was generally well answered, with at or near full marks for many. Dividends were not mentioned, as no explicit assumption was required – candidates correctly ignored them.
Part (i) was bookwork and caused no problems.
Part (ii) (a) and (b) were straightforward, (c) and (d) harder but still well answered. Although the model solution mentions both buyer and seller viewpoints, it was acceptable for the candidate to consider only one side.
Most candidates found part (iii) easy, although they needed to give a reason for their choice. The forward price was, of course, the first of the three alternatives, using the risk-free growth – an important result for derivative theory!

4 ***Syllabus: (c)(i)***

(i) (a) A forward contract is a contract in which two parties agree to trade at some point in the future. The forward price is the price at which they agree to trade.

(b) As interest rates represent the price of borrowing, forward interest rates represent the price of borrowing at a future date.

(c) A forward rate agreement (FRA) is an agreement that a certain interest rate will apply to a specified principal amount of money for a certain period of time.

(ii) (a) The value is $S_T - K$ from the point of view of the party obligated to buy the contract and $K - S_T$ from the counter-party's point of view.

(b) From the buyer's point of view, this is $(S_T - K)e^{-rT}$ (or $(K - S_T)e^{-rT}$ from the seller's viewpoint).

(c) We are told that $\log(S_T) = \log(S_0) + Z_T$ whence (on exponentiating both sides)

$$S_T = S_0 e^{Z_T}$$

As $Z_T \sim N(T\mu,\ T\sigma^2)$, it follows that e^{Z_T} is log-Normal, so using the "Formulae for Actuarial Examinations", we can immediately note that

$$E[e^{Z_T}] = \exp(T\mu + \frac{1}{2}T\sigma^2) .$$

(d) Further, as S_0 is a known constant, S_0 is independent of e^{Z_T} whence

$$E[S_T] = E[S_0 e^{Z_T}] = S_0 E[e^{Z_T}] = S_0 \exp(T\mu + \frac{1}{2}T\sigma^2)$$

(iii) The answer is A: $S_0(1+r) = 104$

If the price were greater than the lowest cost of borrowing, the seller, who is obligated to deliver the stock in one year's time, could borrow money know and buy the stock with the proceeds. In one year's time, he would receive the price *K* which is greater than the amount required to repay the loan, and hence the seller could make a riskless profit.

***Note**: If the seller is unable to borrow at the risk free rate, he could syndicate the opportunity with someone who can, and share the riskless profits. Hence an arbitrage free price must be* $K \le S_0(1+r)$.

If the price were less than the risk free cost of borrowing, the current holder of the stock could sell it now, realising a price of S_0, and invest the proceeds in risk free assets. In one year's time, the assets would have accumulated to $S_0(1+r)$ which would be greater than the agreed price, so he would then buy the stock back for an amount less than $S_0(1+r)$, thereby making a riskless profit. Hence an arbitrage free price must be $K \ge S_0(1+r)$.

As the forward price K is bounded above and below by $S_0(1+r)$, it must equal $S_0(1+r)$.

Q5
This question attempted to test simple algebraic manipulation of various types of interest-rate swap using compound interest. It was set up in a slightly artificial way, in that a flat yield curve of the type specified is not a common occurrence. However, the idea was that the yield curve was momentarily flat, rather than non-stochastic.
For parts (i) to (iv), to get full marks the candidate needed to simplify the final answer in each case, i.e. to get results like $c_1 = i$., not some longer expression involving v.
Parts (i) and (ii) were easy enough, but there were more problems with parts (iii) and (iv). In fact, these latter parts were really straightforward, and to some extent trivial, as some candidates were able to point out.
Part (v) required brief answers, but nevertheless the relationship between old and new (higher) coupons needed to be clearly expressed and a reason stated.

5 ***Syllabus: (g)(i)&(iii)***

(i) Value of fixed leg = $c\{v + v^2 + v^3 + v^4 + v^5\}$

$$= \frac{cv}{(1-v)}\ (1 - v^5)$$

$$v = \frac{1}{1+i} \qquad \therefore \qquad \frac{1}{v} - 1 = i$$

value of fixed leg = $c.\dfrac{(1-v^5)}{i}$

value of floating leg = $L_0 v + L_1 v^2 + \ldots L_4 v^5$

where $v^{k-1} = (1 + L_{k-1})v^k \qquad k = 1 \ldots 5$

i.e. $L_{k-1}v^k = v^{k-1} - v^k$

Thus, value of floating leg = $1 - v^5$

it follows that

$$\frac{c}{i}\ (1 - v^5) = 1 - v^5$$

$$c_1 = i$$

(ii) The semi annual interest rate j is given by

$$(1 + j/2)^2 = 1 + i$$

i.e. $j = 2\{(1 + i)^{½} - 1\}$

Using the same argument as above,

$$c_2 = \frac{j}{2} = \{(1 + c_1)^{½} - 1\}$$

(iii) The value of the floating leg is now given by

$$\text{FL} = L_1 v^1 + L_2 v^2 + L_3 v^3 + L_4 v^4 + L_5 v^5$$

where

$$v^k = (1 + L_k) v^{k+1}$$

i.e. $[L_k v^k] v = v^k - v^{k+1}$

$$L_k v^k = v^{k-1} - v^k \quad \text{as before}$$

It follows that $c_3 = c_1$

(iv) The value of the floating leg is now given by

$$\text{Fl} = s(5)_0 v^1 + s(5)_1 v^2 + \ldots s(5)_4 v^5$$

where $s(5)_t$ = the zero value 5 year swap coupon at time t.

Using the argument in (a) above, the coupon on a forward slant 5 year swap commencing "k" periods in the future is given by

$$\text{Fxd} = v^k \,.\, s_k(5) \,.\, [v + \ldots v^5]$$

$$= v^k s_k(5) \frac{(1 - v^5)}{i}$$

$$\text{Flt} = v^k (1 - v^5)$$

Thus, $s_k(5) = i = c_1$

it follows that $c_4 = c_1$

(v) The primary impact of the upwards sloping yield curve is that forward rates increase over time in relation to the curve.

Thus, c_3 new > c_3 old and
c_4 new > c_4 old

Q6

This question on portfolio hedging was surprisingly poorly answered overall, although part (i) was usually well attempted, computing the correct "insurance" cost using a version of Black Scholes adapted for a non-zero dividend yield.
Part (ii) required the candidate to apply Put-Call parity. Candidates clearly understood what was required here, and most tried to apply the concept of Put-Call parity, though few actually stated that they were doing this. What is more amazing was that a high proportion of candidates came up with the suggestion of selling Calls! In fact, a portfolio plus a purchased Put can equivalently be replaced with a purchased Call of the same strike, a stock sale and investment of the residual cash in risk-free bonds. This is very basic option theory.
Part (iii) involved a simple Put Delta calculation, although the wording of the question disguised this slightly – most understood what to do, though not all applied the necessary Delta formula correctly. A sense check on the Delta (somewhere close to minus 50% of the portfolio) would have helped avoid the worst mistakes.
Part (iv) turned the Delta into a futures equivalent. Very few answered this part correctly. Clearly, the numbers of contracts should be rounded to a whole number at the end.

6 ***Syllabus: (f)(vi)&(ix)***

(i) Fund = 275m, Index = 1,100. Therefore fund = \$250,000 times the index.

Fund falls 5% iff index falls 5% (to 1045).

So puts of \$250,000 times the index with strike 1,045 are required

Apply Black Scholes with

S_0 = 1,100; K = 1,045; T = 1

r = 5% (*risk free rate*); σ =25%; y = 3% (*dividend yield*)

Which leads to

[Note: in the following formula, candidates may equally calculate S_0 exp(-yT) and take it inside the logarithm instead of having an explicit term in y.]

$$d_1 = \frac{\ln(S_0/K) + (r - y + \sigma^2/2)T}{\sigma\sqrt{T}} = 0.41017$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.16017$$

$$\Phi(-d_1) = 0.34084$$

$$\Phi(-d_2) = 0.43657$$

Value of one put = $K\exp(-rT)\Phi(-d_2) - S_0\exp(-yT)\Phi(-d_1)$ = 69.9236

Total cost of insurance = \$250,000 $\times$ 69.9236 = \$17,481,000

(ii) From put-call parity (or by re-arranging the Black Scholes equations for valuing puts and calls) we have

$$V_{\text{put}} = V_{\text{call}} - S_0 e^{yT} + Ke^{-rT}$$

This shows that a put option can be created by shorting $e^{-qT}X$ the index, buying a call option (of same term and strike as the put) and investing the remainder in the risk free asset.

For fund manager under question, this involves:

Selling $\$275m \times \exp(-yT)$ = \$266,872,500 of stock.

Investing in \$250,000 X the S&P-500 index of calls with strike 1,045 and Term 1 year.

Investing the remainder in 1 year risk free zero coupon bonds yielding 5%.

(iii) The delta of one put option is

$$e^{-yT}(\Phi(d_1)-1) = e^{-0.03*1}(0.65916-1) = -0.33077$$

Hence 33.077% of the portfolio (i.e \$90.962m) needs to be sold (shorted) and the proceeds invested in risk free assets.

(iv) The delta of a nine month index futures contract is

$$e^{(r-y)T} = e^{(5\%-3\%)\times 0.75} = 1.0151.$$

The short position required is $\frac{90.962}{1,100} = 82,693$ times the index.

Hence a short position involving $\frac{82,693}{1.0151 * 250}$ = 325.85 (which rounds to 326) futures contracts is needed.

Q7

This question on the Cox-Ingersoll-Ross interest-rate model would also have been familiar to students of CT8, and was answered well overall.

Part (i) was bookwork. The candidate was asked to select key features in (a), then comment on them in (b) for the CIR model itself. All valid relevant points were rewarded, and the more obvious of these were generally well known. Nevertheless, it was surprising that so few achieved the full allocation of 8 marks. It is important in bookwork questions like this to make as many <u>distinct</u> (but relevant) points as possible.

Part (ii) was, in fact, fairly simple algebra in the limit as time increased (with a big hint as well!), but some candidates still lost their way. Part (iii) was bookwork and well answered.

7 ***Syllabus: (h)(i)***

(i) (a) **Desirable features** are:

- reasonable dispersion of rates over time (due to the Brownian motion), otherwise there will be too large a probability of getting an absurdly high or low value — market prices do not allow for these extremes [*another way of expressing this is to say that, when rates go too high or low, they tend to revert back to some middle level ("mean-reverting")*]

- no negative interest rates

- easy to use and calculate, especially when calibrating to market prices

- bond (and/or swap) prices should be reproduced by the model

- forward rates should be imperfectly correlated — although, this is only important when pricing certain types of option (e.g. yield spread options)

- volatility of rates of different maturity should be different, with generally shorter rates being the more volatile

(b) Good points for CIR are that it satisfies most of the desirable features above:

- it is easy to use (algebraic)
- it has time-dependent volatility

But:

- it has no de-correlation of forward rates

- it scores less well on the possible shapes of yield curve it can fit precisely (in fact, only simple upward sloping, simple downward sloping and single-humped are possible)

(c) **Over-wide dispersion** is contained by using the explicit mean-reversion parameter in the drift term.

μ is the long-term target for r and α is the speed (or force) of correction.

Negative rates are avoided by making the volatility term diminish as r approaches zero, so provided σ is not too large in relation to α and μ.

(ii) Let $R(\tau)$ be the τ–maturity spot rate for any given time t (dependency on t is dropped from the notation below as it does not affect the answer).

Then $B(\tau) = \exp(-R(\tau).\tau)$ (*)

or, equivalently, $R(\tau) = -\dfrac{1}{\tau}\ln B = -\dfrac{a-br}{\tau}$

Now, $a = c_3\left(\ln(c_1) + c_2\tau - \ln[c_2(\exp(c_1\tau)-1)+c_1]\right) \to c_3\left(c_2\tau - \ln[c_2(\exp(c_1\tau)]\right)$

as $\tau \to \infty$, since terms in τ dominate the constant terms, so

$$a \to c_3\left(c_2\tau - \ln c_2 - c_1\tau\right) \to c_3\left(c_2 - c_1\right)\tau$$

and, since $b \to \dfrac{1}{c_2}$ as $\tau \to \infty$ and, for a given t, $r(t)$ is a constant, so

$$\frac{-br}{\tau} \to 0 \text{ as } \tau \to \infty.$$

Hence $R(\tau) = -\dfrac{a-br}{\tau} \to c_3(c_1 - c_2)$ as $\tau \to \infty$, which is a (positive) constant, say R, and placing this into the formula (*) above gives the result.

(iii) **No-arbitrage** models are a class of models which allow recovery of market prices of one set of securities given prices of another set. This gives a world of "relative" pricing. In a non-arbitrage-free model, securities could be priced using the model and then traded at a different price in the real world, leading to persistent profit. In simplest terms, no-arbitrage is the absence of a "free lunch".

No-arbitrage is very important in yield curve models, since most complex structures are limiting cases of simpler structures (such as swaps, caps, floors) and hence ideally the model should recover the prices of the latter exactly. Also, hedging is done using the simpler structures, so the absence of no-arbitrage would mean the accounting process would be distorted by imaginary gains and losses.

Q8

This is a standard question relating to a well-known result in stochastic calculus. Most found the early parts easy, and only stumbled a little right at the end in getting the full 10 marks for parts (iii)(a) and (b).

Part (i) needed to be answered precisely to achieve full marks. The model solution is not the only way to derive the result but is broadly typical. It was important to follow the logic right through – a few candidates appeared simply to be reproducing a set of results they had remembered.

Part (ii) was straightforward bookwork. The definition of 'self-financing' is best described both verbally and algebraically.

In part (iii)(a), the candidate needed to demonstrate their knowledge of how to turn a non-martingale process into a martingale process, and to be comfortable with the use of this technique in derivatives theory.

Some candidates went into more detail when deriving the value of the claim in part (iii) (b) than does the model solution. In such cases, as the expected mark allocation was not made explicit, the examiners allocated proportionally more marks to (b) than (a) if the candidate had produced a good answer there.

8 ***Syllabus: (f)(i)&(iv)***

(i) As S_t follows GBM, $S_t = S_0 e^{\sigma W_t + \mu t}$

Let $L_t = \log(Z_t)$. Then

$$\begin{aligned} L_t &= \log(B_t^{-1} S_t) \\ &= \log(B_t^{-1}) + \log(S_t) \\ &= \log(e^{-rt}) + \log(S_0 e^{\sigma W_t + \mu t}) \\ &= -rt + \sigma W_t + \mu t. \end{aligned}$$

Hence

$$dL_t = \sigma dW_t + (\mu - r)dt.$$

Now as $Z_t = f(L_t) = \exp(L_t)$, we can apply Ito's formula to get:

$$dZ_t = \left(\sigma f'(L_t)\right) dW_t + \left((\mu - r) f'(L_t) + 0.5\sigma^2 f''(L_t)\right) dt$$

Substituting $f'(L_t) = f''(L_t) = \exp(L_t) = Z_t$ leads to

$$dZ_t = \sigma Z_t dW_t + Z_t(\mu - r + .5\sigma^2)dt$$

(ii) A portfolio is self-financing if and only if changes in its value depend only on changes in the prices of the assets constituting the portfolio.

Mathematically: If V_t denotes the value of the portfolio (ϕ_t, ψ_t), then the portfolio is self financing if and only if

$$dV_t = \phi_t dS_t + \psi_t dB_t$$

A replicating strategy for X is a strategy which involves investing in specifiable quantities (ϕ_t and ψ_t) of stock and risk free bonds, such that:

the portfolio of (ϕ_t, ψ_t) of stocks and bonds will be self-financing
the portfolio (ϕ_t, ψ_t) will have terminal value equal to the magnitude of the claim; i.e. $V_T = \phi_T S_T + \psi_T B_T = X$ which means that the portfolio's cash flows at claim exercise date match the cash flows under the claim.

When the underlying stock follows a continuous geometric Brownian motion process, there is an additional technical constraint for the strategy to work; viz:

$$\int_0^T \phi_t^2 \sigma^2 dt < \infty$$

(iii) (a) We have from (i) $dZ_t = \sigma Z_t dW_t + Z_t(\mu - r + .5\sigma^2)dt$. The drift of this process is $\mu - r + \frac{1}{2}\sigma^2$ which is non-zero. As the process is not driftless, it is not a Martingale.

The CMG theorem enables us to convert the process into a Martingale. It tells us that there exists a probability measure Q, equivalent to the measure P (defined by the probability distribution of W_t), such that Z is a Martingale.

To apply the theorem we set $\gamma_t = \mu - r + .5\sigma^2$ and verify that γ is a pre-visible process and that $E_P\left[\exp\left(\frac{1}{2}\int_0^T \gamma_t^2 dt\right)\right] < \infty$.

From the CMG theorem, the measure Q, that is equivalent to P is such that

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \gamma_t dW_t - \tfrac{1}{2}\int_0^T \gamma_t^2 dt\right)$$ and the Brownian motion

$\tilde{W}_t = W_t + \int_0^t \gamma_s ds$ is a Q-measure Brownian motion.

In differential form this is written as $d\tilde{W}_t = dW_t + \gamma dt$. Substituting

$dW_t = d\tilde{W}_t - \gamma dt$ into the SDE for Z, leads to $dZ_t = \sigma Z d\tilde{W}_t$.

This is the SDE for a driftless process under measure Q. Hence Z is a Martingale under measure Q.

The next step in constructing the replication strategy is to form the discounted expected claim process $E_t = E_Q[B_T^{-1} X \mid \mathbf{F}_t]$ and verify that this too is a Q-measure Martingale.

As both Z_t and E_t are Q-Martingales, the MRT gives us a pre-visible process ϕ_t such that $dE_t = \phi_t dZ_t$.
Note that in order to apply the MRT we need to verify that both Z and E are Q martingales and that the volatility of Z satisfies the additional condition that it is always (with probability one) non-zero.

The replication strategy then consists of holding ϕ_t units of stock and a volume

$\psi_t = E_t - \phi_t Z_t$ of risk free bonds

(b) The portfolio replicates the claim because the portfolio is self-financing (see below) and, at time T, when the claim falls due, the portfolios proceeds are

$$\begin{aligned}
&\phi_T S_T + \psi_T B_T \\
&= \phi_T S_T + (E_T - \phi_T Z_T) B_T \\
&= \phi_T S_T + (E_T - \phi_T B_T^{-1} S_T) B_T \\
&= E_T B_T \\
&= E[B_T^{-1} X \mid \mathbf{F}_T] B_T \\
&= X
\end{aligned}$$

i.e. the portfolio's proceeds will match the claim amount.

The value of the portfolio at any time t can be written

$$V_t = \phi_t S_t + \psi_t B_t = B_t E_t .$$

The stochastic differential equation for V_t is given by

$$dV_t = d(B_t E_t) = B_t dE_t + E_t dB_t \text{ (using the product rule)} \qquad (*)$$

To show that the portfolio is self-financing, we need to show that

$$dV_t = \phi_t dS_t + \psi_t dB_t \qquad (**)$$

This can be done, by substituting:

$E_t = \psi_t + \phi_t Z_t$ (by definition of ψ)

$dE_t = \phi_t dZ_t$ (from the MRT)

$d(B_t Z_t) = B_t dZ_t + Z_t dB_t$ (the product rule)

$S_t = B_t Z_t$ (by definition of Z)

into (*) and re-arranging the sde.

As the portfolio replicates the claim, the arbitrage-free condition requires that the value of the claim equals the value of the replicating strategy. Therefore, either (*) or (**) gives the stochastic differential equation for the value of the claim.

END OF EXAMINERS' REPORT