EXAMINATION

13 April 2005 (pm)

Subject ST6 — Finance and Investment Specialist Technical B and Certificate in Derivatives

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You have 15 minutes at the start of the examination in which to read the questions. You are strongly encouraged to use this time for reading only, but notes may be made. You then have three hours to complete the paper.*
3. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
4. *Mark allocations are shown in brackets.*
5. *Attempt all 8 questions, beginning your answer to each question on a separate sheet.*
6. *Candidates should show calculations where this is appropriate.*

Graph paper is required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

NOTE: In this examination, you are never required to prove the use of an arbitrage-free methodology unless clearly stated in the question.

1 (i) List necessary and sufficient conditions for a stochastic process $\{W(t): t \geq 0\}$ to be a standard Brownian motion with respect to a probability distribution **P**. [2]

(ii) For the Brownian motion in (i), show that the random variables, $W(t_1)$, $W(t_2)$, ... , $W(t_n)$, have a multivariate Normal distribution with zero means and covariance matrix (v_{ij}), where $v_{ij} = \min\{t_i, t_j\}$. [3]

(iii) Let a stock price S follow a standard Wiener process with expected growth rate μ and volatility σ_S.

Show that, if the expected growth rate of S increases by $\beta\sigma_S$, for some constant β, then the expected growth rate of a function X of S (i.e. $X = f(S)$) increases by $\beta\sigma_X$, where σ_X is the volatility of X. [3]

[Total 8]

2 A purchased call option time spread is the simultaneous purchase of a call option of a given strike and maturity and a sale of a second call option of the same strike but shorter maturity.

(i) Discuss the conditions under which a trader or hedger might wish to purchase a FTSE call option time spread. [2]

(ii) A purchased call option time spread on the FTSE index is struck at the current index level of 4,500, with expiry dates three months and six months from today.

For this time spread, sketch the following variables against the index value, showing the situation now, two months from now and almost three months from now:

(a) Profit/Loss
(b) Gamma

You should assume for this part of the question that volatilities upon which option prices are devised remain constant in all situations. You may ignore financing costs in your diagrams. [8]

[Total 10]

3 (i) (a) Using prices from the following actively-traded instruments in the US market:

- money-market LIBOR rates, from overnight up to 1 year
- interest-rate futures for each 3-month period up to 8 years
- annual fixed-floating swaps for 2, 3, 5, 10, 20 and 30 years

Describe how you would construct a discount curve (zero coupon yield curve) for valuing US dollar interest rate swaps out to 30 years.

(b) Discuss any special features you might need to take into account. [5]

(ii) (a) Explain how you could use this discount curve to manage the market risk of a diverse portfolio of fixed interest instruments, including swaps and bonds but not options.

(b) Discuss the problems you might encounter. [5]
[Total 10]

4 (i) (a) Define the terms “forward contract” and “forward price”.

(b) Explain the concept of a forward interest rate in the context of forward prices.

(c) Describe a forward rate agreement.

[3]

(ii) Let Z_T be a normally distributed random variable with mean T_μ and variance $T\sigma^2$. Let S be a stock whose price at time 0 is given by S_0 such that the change in the logarithm of the price of S over a period T is Z_T.

Two parties enter into a forward agreement under which they agree to trade the stock after time T for a price of K.

Write down expressions for the following:

(a) The value of the contract at time T.

(b) The present value of the contract’s cash flows, discounted at the risk free rate of interest.

(c) The price of the stock at time T as a function of the current price S_0 and Z_T.

(d) The expected value $E[S_T]$ of the price of the stock at time T.

[6]

(iii) In the special case where $S_0 = 100$, the expected growth in the price of the stock is $g = 6\%$ (i.e. $E[S_1] = 106$), and the risk free rate of interest is $r = 4\%$ annually compounded, explain from first principles which of these values represents the arbitrage free one-year forward price of the contract:

- $S_0(1+r) = 104$ **or**
- $S_0(1+g) = E[S_1] = 106$ **or**
- $S_0(1+g)/(1+r) \approx 101.92$

[3]

[Total 12]

5 The interest rate curve for a particular economy is flat, in such a way that all zero coupon interest rates $i(t)$ are equal to some constant i, for all $t \geq 0$. An arbitrage free environment can be assumed, and transaction costs, taxes and payment conventions can be ignored.

(i) Derive the fixed payment c_1 on a standard five-year annual par swap, one which pays an annual fixed amount against an annual floating amount, set at the beginning of each period and paid in arrears. [3]

(ii) Show that the fixed payment c_2 payable twice per year on a five-year semi-annual par swap, otherwise on the same basis as in (i), is **not** $\frac{c_1}{2}$. [1]

(iii) Derive the fixed payment c_3 on a five-year annual par swap similar to that in (i) above, but for which the annual floating payment is set *in arrears* on the payment date itself. (This is known as a LIBOR-in-Arrears swap.) [3]

(iv) Derive the fixed payment c_4 on a five-year annual par swap similar to that in (i) above, but for which the annual floating payment, set at the beginning of the period and paid in arrears, is calculated to be the level of the five year par swap as calculated in (i) above. (This is known as a Constant Maturity Index swap.) [3]

(v) If the zero coupon interest curve is upwards sloping, i.e. $i(t_2) > i(t_1)$ for all $t_2 > t_1$, describe how the answer in sections (iii) and (iv) above will be affected. [2]

[Total 12]

6 A fund manager has a well diversified portfolio that tracks the performance of the S&P 500 and is worth \$275m. The current value of the S&P 500 is 1,100. The manager would like to buy portfolio insurance against a reduction of more than 5% in the value of the portfolio over the next year. The risk free rate is 5% p.a. The dividend yield on both the portfolio and the S&P 500 is 3%. The market implied volatility for the S&P 500 is currently 25% p.a.

(i) Calculate the cost of hedging the portfolio using European put options. [4]

(ii) Describe alternative strategies involving European call options which would have the same effect as the options in (i). [3]

(iii) Calculate the initial (delta) position if the manager sought to replicate the effect of the put options by investing part of the portfolio in risk-free securities. [2]

(iv) Calculate the initial number of futures contracts required if, instead of risk-free securities in (iii), the manager decided to use 9-month index futures contracts, each contract nominal being 250 times the index. [3]

[Total 12]

7 The Cox-Ingersoll-Ross (CIR) model is a one-factor interest-rate model of the form:

$$dr = \alpha(\mu - r)dt + \sigma\sqrt{r}dz$$

which leads to the solution at time t for the prices $B(t, T)$ of zero-coupon bonds of maturity T as follows:

$$B(t,T) = \exp\left[a(\tau) - b(\tau).r(t)\right]$$

where

$$a(\tau) = c_3 \ln\left\{\frac{c_1 \exp(c_2\tau)}{c_2[\exp(c_1\tau) - 1] + c_1}\right\} \text{ and } b(\tau) = \frac{\exp(c_1\tau) - 1}{c_2[\exp(c_1\tau) - 1] + c_1}$$

where $\tau = T - t$ and c_1, c_2 and c_3 are all positive constants ($c_1 > c_2$) containing terms in α, μ and σ.

(i) (a) List the key features that a good model of the entire interest rate yield curve should have.

(b) Discuss how well the CIR model matches your criteria.

(c) Describe in particular how the CIR model copes with two typical problems facing interest rate models: too wide a "dispersion" of rates over time due to future uncertainty, and allowing interest rates to become negative. [8]

(ii) By considering the behaviour of the logarithm of the bond price, prove that for large τ:

$$B(t,T) \approx \exp(-R\,\tau)$$

where $R = (c_1 - c_2)c_3$ is a constant long-term rate. [5]

(iii) The CIR model is a "no-arbitrage" model. Outline what is meant by this term, and the significance of such a property for an interest-rate model. [3]

[Total 16]

8 Let S_t be a stock price process which follows a geometric Brownian motion with parameters μ, σ^2, and with stochastic differential equation:

$$dS_t = \sigma S_t dW_t + (\mu + \tfrac{1}{2}\sigma^2) S_t dt$$

where W_t is a Brownian motion process.

Let B_t be a risk free asset whose price grows deterministically according to $B_t = e^{rt}$, and let $Z_t = B_t^{-1} S_t$ be the discounted stock price process.

Consider a dynamic portfolio (ϕ_t, ψ_t) consisting of ϕ_t units of S_t and ψ_t units of B_t, and let $X = f(S_T)$ be a path-independent claim on S_T.

(i) Derive the stochastic differential equation for Z_t. [5]

(ii) Explain what is meant by a self-financing and replicating strategy for X. [5]

(iii) (a) Explain how the Cameron-Martin-Girsanov (sometimes referred to as Girsanov's) theorem and the Martingale Representation theorem can be used to construct a replication strategy for X.

(b) Derive an expression for the stochastic differential equation for the value of the claim. [10]

[Total 20]

END OF PAPER