EXAMINATIONS

April 2006

Subject ST6 — Finance and Investment
Specialist Technical B
and
Certificate in Derivatives

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M Flaherty
Chairman of the Board of Examiners

June 2006

Comments

There were encouraging signs from this sitting that candidates were better prepared on both basic bookwork and in their understanding of applying the subject to numerical and practical problems.

Please note that the model solutions provided are indicative, i.e. adequate to achieve full marks but without covering ever possible correct response. Several points made by candidates were equally valid, and these also achieved the allocated marks.

Question 1

The algebra is part (i) was tackled successfully by almost all candidates. It was quite an easy step from there to identify the Normal distribution in the limit for part (ii)(a).

The final part eluded many candidates, however. Those who wrote down the zero coupon bond price as a product of discount factors involving a series of short rates were able to demonstrate that it had a log-Normal distribution. Since part (ii) only attracted five marks, a long derivation was clearly not required, though the examiners felt justified in giving credit to those candidates who sought to derive further parameters (i.e. mean and variance) for this distribution.

Question 2

The bookwork on Brownian motion in part (i) was well covered. Part (ii) was harder, and (if the suggestion was followed) required candidates to work with discrete increments to show that there was a mismatch between expectations of the two sides of the Newtonian differentiation. This led to a conclusion that a different form of calculus was required, which of course is the stochastic calculus of the Ito formula. Marks were not awarded to candidates who used the argument in reverse and derived the result from Ito.

A simple application of Brownian motion followed in part (iii), and this was well answered. Most candidates showed that all the criteria for part (i) were satisfied, and so proved Brownian motion neatly and quickly. Demonstrating a Normal distribution mean and variance for the combined distribution is an equally valid approach.

Question 3

Graphical questions come up regularly in ST6 as a way of demonstrating that the candidate has understood how option prices and sensitivities vary with different parameters. Previous reports have discussed the expectations examiners have when assessing such graphs. Part (i) required elementary manipulation of the derivative pricing formula and was well answered.

Part (ii) asked the candidate to explain how theta and gamma behaved for a basic at-the-money call or put. The resultant graphs were mostly accurate enough. The main points to demonstrate graphically were that at-the-money options have higher theta decay and gamma sensitivity as expiry approaches, since they retain significant (albeit declining) time premium, *and that out-of-the-money options expire worthless, so their gamma sensitivity disappears near expiry.*

Part (iii) asked for a practical assessment of reasons for hedging with options, and was probably simpler than several candidates thought. Candidates should note the model solution carefully. A hedging strategy with long and short option positions is designed to reduce risk — vega and gamma mainly (but not delta — that can be achieved without offsetting options).

Question 4

Most candidates made a reasonable attempt at this question on using historical data to assess implied volatility. Of course, the past is not always a reliable guide to the future, as recent spikes in oil, gas and precious metals have indicated.

Part (i) was bookwork and well covered. Part (ii) mostly brought out relevant comments on the validity of the data in certain circumstances, although some attempts were just too brief and sketchy to obtain full marks. In part (iii), an extension to a futures contract, examiners were looking for identification of issues such as splicing the time series, liquidity of the front and back contracts, and basis to the cash market. Some candidates could have picked up better marks here if, despite the pressure of time, they had paused for thought for a few moments.

Question 5

Part (i) gave candidates eight easy marks if they knew their bookwork. Indeed, part (ii) added two more marks for a definition of a martingale, so in all ten marks were available for bookwork. The remaining two marks were harder to earn, since the derivation of the required result required a neat (but quick) manipulation of conditional probabilities. Few managed it, but those that referred to the Tower Law were on the right track.

Question 6

Numerical questions are often good discriminators of ability in ST6. Generally, though, and very pleasingly, nearly everyone fared well in this question, making it the best answered in the paper. Candidates were at home with binomial trees, and were probably relieved to see this familiar technique being tested after its absence in both of the 2005 papers.

Parts (i) and (ii) involved two different methods of obtaining the same value and delta for an option. These presented few problems, and most managed to get their values to agree. Part (iii) was a trivial extension to an American option, and most were able to propose the correct adjustment to the tree, i.e. checking intrinsic values at each node. Many went on to predict the impact on the valuation, which for equity calls is nil.

Question 7

This question was not well answered. Several candidates did not attempt any part of it, or foundered early on before even deriving any formulae for the two types of forward.
Part (i) was simple, though the paper contained an unfortunate "trap". An incorrect value had been quoted in part (i)(b) for the value of the FRA at 6% rates. This was discovered in advance of assessing the scripts, so the examiners were able to take into account in their marks any candidates who appeared to struggle in their derivation. However, given the simple nature of the FRA pricing formula, most were in fact well able to derive the value given, even if they held suspicions about its correctness.

Parts (ii) and (iii) contained probably the hardest concepts in the paper, but those with a clear head were able at least to show the way to tackle the problems posed. The hedge ratio arises from the relative performance of the two legs of the hedge, and the profit or loss from the combined hedge position can be demonstrated by re-valuing the FRA and futures contracts at different interest rate levels. The calculations show that the sold FRA adds a (small) bond-like convexity to the hedge.

For part (iii)(a), use of Ito on the product process was well done, although some candidates failed to notice that the process was not Normal but log-Normal (i.e. with the variable s appearing on the RHS of the differential equation as a multiplier). There was almost universal silence about what to do with this derivation for part (iii)(b). Although detailed algebra was clearly not required here, the principle to note was that the forward rate itself and the discounting of value back to today form two correlated stochastic rate processes, enabling their combination (an FRA) to be valued.

Question 8

For this question, candidates should have been on home ground, with some simple interest rate yield curve calculations in part (i). Most started well, but did not always think clearly enough to derive part (i)(d). The key to success in all these yield curve questions is to set out precisely what cash flow applies at what date with what discounting — a very traditional actuarial compound interest approach, in fact.

In part (ii), the CMS product was clearly unfamiliar to many. However, part (ii)(a) was really fairly trivial, the 1-year CMS being the same as an ordinary swap, and part (ii)(b) required expressing general insights on convexity and the correlation of rates across the yield curve.

Part (iii) was not well answered, although probably there was a certain fatigue at the end of a long paper. Discussion of four key risk effects was required. Mostly, this part of the question could be answered from the reading material — any candidates who found it difficult should study the model solution, as this material is important and will undoubtedly surface again in another guise.

1 ***Syllabus section: (h) & (i)***

(i) [*The "spot" rate r_t in the question is, of course, the short rate.*]

$$ds_t = d(e^{at} r_t)$$

Hence $s_t = s_0 + ab\int_o^t e^{as} ds + \sigma \int_o^t e^{as} dB_s$

$$s_t = r_o + b(e^{at} - l) + \sigma \int_o^t e^{as} dB_s$$

So $\quad r_t = b + (r_o - b)e^{-at} + \sigma \int_o^t e^{a(s-t)} dB_s$

(ii) (a) It follows from (i) that r_t is Normal with

mean = $b + (r_o - b)e^{-at}$

variance = $\sigma^2 \int_{s=0}^{t} e^{2a(s-t)} ds = \sigma^2 \dfrac{(1 - e^{-2at})}{2a}$

So, as $t \to \infty$, $r_t \sim N(b, \sigma^2/2a)$

(b) A zero coupon bond of maturity T has value

$$B_t = \exp - RT = \prod_{i=1}^{RT/\Delta t} \exp - r_i \Delta t$$

[*alternatively, in the continuous form*: $B_T = \exp\left(-\int_0^T r_u du\right)$]

Since r_t is Normal, the distribution of B_T is Lognormal.

[*Although it was not specifically requested, or required, the examiners gave special recognition to a candidate who attempted to find the mean and variance of this distribution.*]

2 ***Syllabus section: (e)***

(i) The process $W = \{W_t : t \geq 0\}$ is a Brownian Motion under probability measure ***P***, if:

- W_t is continuous.
- $W_0 = 0$.
- $W_t \sim N(0, t)$, i.e. W_t is distributed under ***P*** as a Normal variable. **(*)**
- $W_{t+s} - W_s$ is distributed as Normal $N(0, t)$ under ***P***, and is independent of any filtration (or history) or the process up to time s.

[Note: (*) is usually given as a separate condition, although it can be derived from the other criteria. Candidates were not penalised for omitting it.]

(ii) (a) Divide the integral into n equal sections.

Then

$$\int_0^t W_s dW_s \approx \sum_{i=0}^{n} W\left(\frac{it}{n}\right)\left[W\left(\frac{(i+1)t}{n}\right) - W\left(\frac{it}{n}\right)\right]$$

The second term contained in […] is a typical increment for a Brownian motion, so the RHS has expectation is zero.

Hence, since the RHS terms have independent Normal distributions under Brownian motion, the expectation of the RHS is zero.

However, the expectation of W_t^2 is the variance of W_t (it has zero mean), which is t since the distribution is Normal.

Hence, given that $d(W_t^2)$ and $2W_t dW_t$ have different expectations, they cannot be equal.

[*Note: It is not a valid answer simply to quote Ito's Lemma here — the question is being asked to show the need for the Lemma.*]

(b) The significance is that Brownian motion needs a new form of differentiation that takes into account stochastic processes. This is often expressed as Ito's Lemma or formula.

(iii) Yes, the process $X_t = \rho W_t + \sqrt{1-\rho^2}\tilde{W}_t$ **is** a Brownian motion process.

This is because:

$$X_0 = 0 \text{ as } W_0 = \tilde{W}_0 = 0.$$

X_t is continuous because W_t and $\tilde{W}_t$ are continuous

As the Normal distribution is closed under addition,

$$X_t \sim [\rho^2 N(0,t) + (1-\rho^2)N(0,t)] = N(0,t).$$

The increment $X_{t+s} - X_s$ is the sum of an $N(0,\ t\rho^2)$ and an independent $N(0,\ t(1-\rho^2))$ variable, which is $N(0,\ t)$. Also, the increment is independent of both histories $\{W_u : u \le s\}$ and $\tilde{W}_u : u \le s\}$, hence it is also independent of the history $\{X_u : u \le s\}$.

[*Another valid approach, yielding the same result, is to derive the means and variances separately*.]

3 *Syllabus section: (f)(i)–(v)*

(i) Theta is the rate of change of option price with respect to time,

i.e. $\theta = \dfrac{\partial x}{\partial t}$

Gamma (Γ)is the rate of change of delta with respect to underlying stock, and delta (Δ) is the rate of change of the option price with respect to the underlying stock,

i.e. $\Gamma = \dfrac{\partial^2 x}{\partial s^2}$

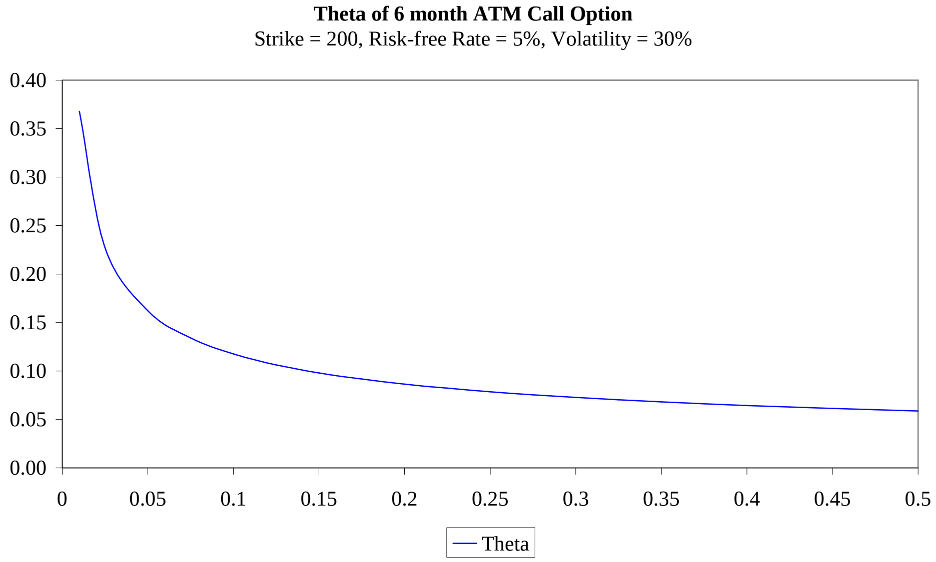
Now

$$\theta + rs\Delta + \tfrac{1}{2}\sigma^2 s^2 \Gamma = rx \quad \text{from the definition and the equation}$$

So, if $\Delta = 0$, as it is for a delta-neutral portfolio, then

$$\theta + \tfrac{1}{2}\sigma^2 s^2 \Gamma = rx$$

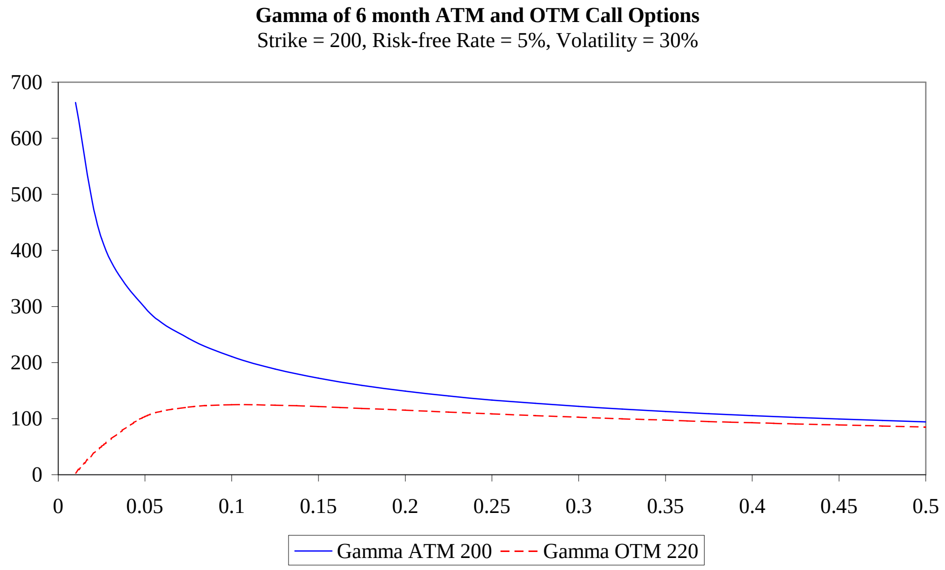
(ii) (a) *[The curve shown is <u>positive</u> time decay, i.e. $-\theta$.]*



The curve shows an exponential decay.

(b) & (c)

For at the money options the time value decays as time to maturity decreases, with the delta switching between 0 to 1, so the gamma has more effect; for the other options the gamma effects are similar at longer maturities, but since the option value decreases to zero approaching expiry both delta and gamma become insignificant.



(iii) The trader wants to hedge to reduce risk, particularly if he has taken on a large short position.

Taking positions in offsetting options can help reduce Vega and Gamma, even if they do not completely cancel them out.

The trader:

- may want to take some time to close out the risk from his client trades in the market place
- may need to mismatch hedges because the liquidity of the original particular deal was poor, so he would choose the more liquid options as a (partial) hedge
- may want to take spread positions in differing expiries or strikes as a position-taking in its own right, having 'inherited' one side of this from the client trade
- may not want to enter the market to show his exact business every time [½]
- may want to reduce currency risk

[Other points may be valid here.]

4 *Syllabus section: (f)(i)−(v)*

(i) Let S_i be the closing price of the commodity on day i.

Let $u_i = log\ {}^{S_i}\!/_{S_{i-1}}$

[ALTERNATIVE METHOD FOR CALCULATING u_i:

In practice in most markets, the log of the daily price ratios can be approximated as $\dfrac{S_i - S_{i-1}}{S_{i-1}}$. The difference is small when compared to statistical sampling error.]

Then daily variance = $\sigma^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}\left(u_i - \bar{u}\right)^2$, where n represents the number of historical data points.

and $\sigma_{\text{annual}} = \sigma_{\text{daily}}\sqrt{N}$, where N is the number of **business** days in a given year.

[Note: Use of calendar days is not correct, since this will over-estimate volatility over weekends.]

(ii) (a) It is important to match the data periods to the option periods.

Using 10 years of data reduces sampling error in your estimate provided volatility is stationary.

In practice it is not stationary, particularly over short periods.

Use enough data to reduce sampling error to a reasonable level whilst keeping the data relevant to the intended **future** forecast.

Historical supply/demand situation may have created a variability profile that is not relevant to the immediate future.

(b) The choice of data points for calculating volatility should match the timing of the delta hedging, which itself should match the assumption used in pricing.

Normally, closing prices are used as they are more stable & liquid, and conform to the usual pattern of hedging at the end of a day.

The market lacks liquidity at the opening and requires time to settle down — this may give a higher volatility estimate.

Rebalancing a hedge at the opening is not advisable, except as an emergency process, since the lower liquidity implies higher dealing costs.

(c) Jumps in the input data can be difficult to allow for. Black-Scholes assumes no jumps, i.e. all movements are samples from the same log-normal distribution.

It is good to be aware of significant distortions caused by one-off events.

Future jumps would theoretically need a jump diffusion model, but these are hard to build and difficult to parametrise.

The market allows for some future jump effects by adding a "smile" to the volatility of out-of-the-money options, particularly on puts.

(iii) Futures price series are not continuous over long periods, so they have to be spliced, since each contract will only trade for a maximum of (say) a year.

The front month futures prices is usually just as reliable as spot prices for estimating volatility, since the market is liquid (sometimes better), but if the contract is not the current front month, it may be less liquid and prone to data gaps and jumps.

Allow for convenience yield. There can be significant basis differences between futures and spot prices, hence additional factors can be incorporated into the volatility estimates.

5 *Syllabus section: (f)(i)–(v)*

(i) (a) **Stochastic process**

A stochastic process can be defined as a mathematical model designed to follow the progress over time of a random phenomenon such as the price of a stock.

In the simple one-period binomial model, the value of the process is S_0 at time 0. The stock price at the end of period 1 is a random variable S_1 capable of taking only two values, uS_0 and dS_0, and u and d are up- and down-ratios (this is a multiplicative process – alternatively, an arithmetic process could go to $S_0 + u$ or $S_0 + d$). The stock price process has only two sample paths: up to uS_0 or down to dS_0.

In period 2, each of these nodes has two further possibilities, and so on — this creates the tree effect.

(b) **Probability measure**

A probability measure $\boldsymbol{P}$ for a binomial stochastic process is specified simply by assigning a probability p to the up move and $1 - p$ to the down move. It describes how likely any up/down jump is at each node.

The values of p can depend on the time step, i.e. can be more complex with a different probability p_j at each point.

The choice of measure is related to how much the process drifts within its stochastic movement. A higher value of p for the up move means the process will tend to drift upwards more …

… a particular one being the risk-free probability measure.

(c) **Filtration**

A filtration is the history of the stock price movements up until a particular time on the binomial tree. The price process S_i, $0 \le i \le T$, generates a filtration F_i, $0 \le i \le T$, where F_i is the collection of all the events that depend only on $S_0, S_1, \ldots, S_i$.

The filtration fixes the history of choices and thus fixes the node.

A contingent claim on the tree is a function of the nodes at a claim time horizon T, i.e. it is a function of the filtration F_T.

(d) **Previsible process**

We will say that a process S_i, $0 \le i \le T$ is *previsible* if S_i depends only on the filtration F_{n-1}, i.e. up to the previous time step. That is to say, once its value is known at the previous time step, there is no uncertainty about its value at the next time step. (This is not true of most stochastic processes.)

A previsible process is a binomial tree process in its own right; but known one node in advance, for example, a bond due to redeem at step i. Previsible processes are useful to set up arbitrage strategies.

(ii) For S to be a martingale with respect to a measure $\boldsymbol{Q}$, it means that the future expected value at time j of the process S *under measure* $\boldsymbol{Q}$, conditional on its history up until time i is merely the process' value at time i.

Hence S is a $\boldsymbol{Q}$-*martingale* if:

$$\mathbf{E}_Q\left[S_j \middle| F_i\right] = S_i \quad \text{for all } i \le j$$

where F_i is the filtration up to time i.

It is onerous to check every single possible pair of combinations.

However, if we can show that the process S_i itself is identical to the conditional expectation process of its terminal value at time T, $\mathbf{E}_Q\left[S_T \middle| F_i\right]$, we have shown that it is a $\boldsymbol{Q}$-martingale.

This is because, if that know that $\mathbf{E}_Q\left[S_T \middle| F_i\right]$ is the same as S_i , then this implies that:
$\mathbf{E}_Q\left[S_j \middle| F_i\right] = \mathbf{E}_Q\left[\mathbf{E}_Q\left[S_T \middle| F_j\right] \middle| F_i\right]$ for any $j \geq i$ (substituting for S_j).

But $\mathbf{E}_Q\left[\mathbf{E}_Q\left[S_T \middle| F_j\right] \middle| F_i\right] = \mathbf{E}_Q\left[S_T \middle| F_i\right] = S_i$ (Tower Law of conditional probability), which gives the exact condition for a martingale.

[Note that the process S cannot be a martingale on its own, it has to be a ***Q****-martingale, i.e. a martingale with respect to the measure* ***Q****. The same process can be a martingale with respect to one measure and not to another.]*

6 *Syllabus section: (f)(i)–(v)*

(i) First, we are given risk-free rate $r = 0.03$ and semi-annual time step $\Delta t = 0.5$, and we will need the discount factor:

$$e^{-r\Delta t} = \mathbf{e}^{-0.015} = 0.98511$$

Strike $K = 35$, and stock price S follows binomial tree with up factor $u = 1.06$ and down factor $d = 0.96$.

Next, prepare the stock price tree for $t = 0$, 1 and 2:

		$S_{uu} = 39.326$
	$S_u = 37.1$	
$S = 35$		$S_{ud} = 35.616$
	$S_d = 33.6$	
		$S_{dd} = 32.256$

Calculate up probability p given by:

$$S = e^{-r\Delta t}\,[p\,.\,S_u + (1 - p)\,.\,S_d] \qquad (*)$$

so:

$$p = (e^{r\Delta t} - d) / (u - d) = 0.55113.$$

Then get values of the call at each node by discounting back through tree in the same method as equation (*) above, using the value of p just calculated, as follows:

$$\begin{array}{ccc} & & V_{uu} = 4.326 \\ & V_u = 2.6211 & \\ V = 1.5709 & & V_{ud} = 0.616 \\ & V_d = 0.3344 & \\ & & V_{dd} = 0 \end{array}$$

where, for example, $V_{uu} = \max(0, S_{uu} - \text{K}) = 39.326 - 35 = 4.326$ etc at $t = 1$ year, and $V_u = e^{-r\Delta t}\,[p\,.\,V_{uu} + (1-p)\,.\,V_{ud}]$ etc at $t = 0.5$ years.

For the delta, look at the nodes for $t = 0.5$ years:

$$\Delta \approx (V_u - V_d) / (S_u - S_d) = (2.6211 - 0.3344) / (37.1 - 33.6) = 0.6533.$$

(ii) Using part (i), start with the call price tree for the nodes at time $t = 1$ year:

$$\begin{array}{ccc} & & V_{uu} = 4.326 \\ & V_u & \\ V & & V_{ud} = 0.616 \\ & V_d & \\ & & V_{dd} = 0 \end{array}$$

"No arbitrage" implies setting up a portfolio (which will change from $t = 0.5$ to $t = 1$ year) of:

short one call option;
long Δ (delta) amount of equity

Start at $t = 0.5$, since we know the outcomes for $t = 1$, then work back to $t = 0$. All three stages here use the value of a riskless portfolio over a complete time step.

Up step at *t* = 0.5 years

$\Delta_u = +1$ (100% delta because option is exercised at both nodes),

so $\Pi_u(2) = 35$ and $\Pi_u(1) = \Pi_u(2)\, e^{-r\Delta t} = -34.4789$.

Hence $V_u = 37.1 \times 1 - 34.4789 = 2.6211$.

Down step at *t* = 0.5 years

$\Pi_d(2) = S_{du} \,.\, \Delta_d - V_{du} = S_{dd} \,.\, \Delta_d - V_{dd}$

$\Rightarrow 35.616\ \Delta_d - 0.616 = 32.256\ \Delta_d - 0$

$\Rightarrow \Delta_d = 0.18333$

$\Rightarrow \Pi_d(2) = 5.9136$

But $\Pi_d(2)$ has a value independent of the path up or down, so must have earned the risk-free rate from $t = 0.5$ to $t = 1$ year.

Hence $\Pi_d(1) = \Pi_d(2)\ e^{-r\Delta t} = 5.9136 \times 0.98511 = 5.8255$.

Thus $V_d = S_d \,.\, \Delta_d - \Pi_d(1) = 33.6 \times 0.1833 - 5.8255 = 0.3334$.

Initial step at *t* = 0

$\Pi(1) = S_u \,.\, \Delta - V_u = S_d \,.\, \Delta - V_d$

$\Rightarrow 37.1\ \Delta - 2.6211 = 33.6\ \Delta - 0.3334$

$\Rightarrow \Delta = 0.6536$

$\Rightarrow \Pi(1) = 21.6275$

But $\Pi(1)$ has a value independent of the path up or down, so must have earned the risk-free rate from $t = 0.5$ to $t = 1$ year.

Hence $\Pi(0) = \Pi(1)\ e^{-r\Delta t} = 21.6275 \times 0.98511 = 21.3055$.

Thus $V = S \,.\, \Delta - \Pi(0) = 35 \times 0.6536 + 21.3055 = 1.5705$.

These values agree with those in part (i), subject to rounding differences.

(iii) To value American options, we need to check whether exercise occurs at each intermediate node, as well as at the end, i.e. in this case at $t = 0.5$ and $t = 1$.

Multiply out as above, going from $t = 1$ backwards. Compare the option value at each node with the intrinsic (early exercise) value, and take higher before proceeding.

The practical impact of this is nil, since for equities an American call option will not be exercised early.

7 ***Syllabus section: (c)***

(i) *[The sign of the result is not important — could be a buyer or a seller.]*

(a) Value = $[7-6]\% \times \dfrac{1{,}000{,}000{,}000}{4} = 2{,}500{,}000$

(b) Value = $[7-6]\% \times \dfrac{1{,}000{,}000{,}000}{4} \times \dfrac{1}{\left(1+\dfrac{6\%}{4}\right)^{5.25 \times 4}}$

making the usual assumption (for this case) that interest rates quoted would be compounded quarterly

= 1,828,745

[*The examination paper quoted a value of 1,868,145, derived from a simplified formula using 6% as the annual rate for 5 years. The marking schedule shows above how the theoretically correct answer should be derived. The examiners took this fact into account when marking candidates' scripts, particularly where they felt a candidate's performance could have been affected by the error. The examiners also allowed other assumptions on interest rates, e.g. continuous compounding (value 1,854,363).*]

(ii) (a) There is likely to be a strong correlation between changes in forward rates and changes in spot zero coupon rates, so the trade should broadly work correctly as a hedge.

The trade also has positive convexity: the combined FRA and futures position behaves something like a bond due to the discounting term in the FRA.

(b) The hedge ratio based on values calculated in (i) is 2.5 / 1.828 = 1.36 which ~ 1.34 as given, allowing the emergence of profit at different rates between the contracts.

(c) Down move:

FRA value @ 6% = 1,828,745 (see above) × 13.40 = 24,505,181
FRA value @ 4% = 6,085,726 × 13.40 = 81,548,732

using 4% in the formula in part (i)(b) instead of 6%

Futures profit (which is linear) = $[6-4]\% \times 0.25 \times -10 \times 10^9$
= –50,000,000

Hence strategy profit = 81,548,732 – 24,505,181 – 50,000,000
= 7,043,551

Up move:

FRA value @ 6% = 1,828,745 (see above) × 13.40 = 24,505,181
FRA value @ 8% = –1,649,440 × 13.40 = –22,102,490

using 8% in the formula in part (i)(b) instead of 6%

Futures profit (which is linear) = [6 – 8]% × 0.25 × –10 × 10^9
= 50,000,000

Hence strategy profit = –22,102,490 – 24,505,181 + 50,000,000
= 3,392,329

(d) The hedge is not absolutely perfect, but there is clearly a convexity effect that favours the FRA when rates move, so the strategy profits on both an up and a down move.

(iii) (a)

$$d(s_1 s_2) = (s_2 s_1 \mu_1 + s_1 s_2 \mu_2 + \rho\sigma_1\sigma_2 s_1 s_2)dt + s_2 s_1 \sigma_1 dz_1 + s_1 s_2 \sigma_2 dz_2$$
$$= s_1 s_2(\mu_1 + \mu_2 + \rho\sigma_1\sigma_2)dt + s_1 s_2(\sigma_1 dz_1 + \sigma_2 dz_2)$$

(b) The value of the difference between future and FRA is the difference in payoffs discounted to today.

The "60–63" FRA settles against the discounted payment of the value of a 3-month loan at the maturity in 5 years, so its true price is equal to its expected value and does not need further adjustment.

The futures contract, however, settles in margin now any difference against the expected forward price, so there is an immediate benefit of changes in interest rates.

To find the additional value of the futures contract vs the FRA, we need to value the futures contract stochastically, taking into account the interaction of the forward rate **and** the discounting factor over 5 years.

One could therefore take two assets S_1 = a 5-year zero coupon bond, and S_2 = "60–63" forward rate, and integrate the formula in (a) above with respect to *t*.

Since we are using expected values, $\mu_1 = \mu_2 = 0$.

Assume values of volatilities σ_1 and σ_2 are given, plus the correlation (this is some further information required).

For the futures contract, its value is $\mathbf{E}_t[s_1 s_2] = s_1 s_2 \exp(\rho\sigma_1\sigma_2 t)$, where t = 5 years.

[On a risk neutral basis, the futures contract should stand at a value above that of the FRA (something like 0.15% — 0.20% on typical values).]

8 ***Syllabus section: (g) & (i)***

(i) (a) Interpolate between 2 and 3 year rate = 4.45% *[other interpolation methods might be allowable].*

$$\text{Value} = \frac{1}{(1.0445)^{2.5}} = 89.69 \text{ per cent}$$

(b) $$\frac{1}{(1.0455)^4} = \mathbf{e}^{-4r}$$

$\Rightarrow$ r = 4.45% continuously compounded

(c) Define $d_n = \dfrac{1}{[1+\text{zero}(n)]^n}$

Then if c_4 is the fixed coupon on the swap,

$$c_4\,(d_1 + d_2 + d_3 + d_4) = 1 - d_4$$

using the standard compression formula for the floating side of a swap.

$d_1 = 0.95923$
$d_2 = 0.91749$
$d_3 = 0.87630$
$d_4 = 0.83696$

Thus $c_4 = 4.542\%$

(d) For the forward-starting swap, let the coupon be $c_{2/6}$.

We have:

$$c_{2/6}\,(d_3 + d_4 + d_5 + d_6) = f_3 d_3 + f_4 d_4 + f_5 d_5 + f_6 d_6$$

where f_n is the forward rate for time n, i.e. $f_n = (d_{n-1} / d_n) - 1$.

This formula simplifies to:

$$c_{2/6}\,(d_3 + d_4 + d_5 + d_6) = d_2 - d_6$$

since $f_n d_n = d_{n-1} - d_n$ for each n.

Now $d_5 = 0.79862$ and $d_6 = 0.76132$

so $c_{2/6} = 0.15617 / 3.2732 = 4.771\%$.

(ii) (a) The first payment at the end of year 1 is the 4-year swap rate at time 0, i.e. 4.542% (see part (i) (c)).

That will be the floating payment in 1 year's time, so that is the fixed payment also at that time.

(b) [*No algebra is required to illustrate this answer.*]

The problem with using expected values is that it does not take into account the potential movement of rates between now and the time the swap coupon is calculated.

Any variation in interest rates between now and (say) 1 year's time would increase the value of receiving the simple forward swap rate, so the receiver of the CMS rate must be compensated for this difference, to avoid arbitrage.

The forward-setting swap is acting like a bond, which is convex in relation to interest rate movements, so an adjustment is required.

For example, we need to find the expected swap rate at end of year 1 in a world that is forward risk neutral with respect to the 1-year zero coupon bond.

This is the standard forward swap rate plus a convexity adjustment based on the joint evolution of the relevant interest and swap rates.

Thus, as well as the term structure given in the question, we need to know the volatility of the forward swap rate, the volatility of the forward interest rate between years 1 and 2, and the correlation between these two rates.

(iii) *[The following are examples of possible answers. Only four topics are required to answer the question. Other points might be equally valid. 1½ marks for each topic.]*

1. **Yield curve effects**

   Changes in the shape of the yield curve may cause different changes in the value of various instruments giving rise to an overall portfolio value change.

   Need to have more sensitivity points along the curve for complete hedging.

2. **Portfolio effects**

   The Black model treats each underlying instrument as being separate in itself.

   It cannot take into account the interaction (correlation) of instruments or the volatility of financing rates.

3. **Stress effects**

   A significant move in rates may cause the overall portfolio gamma and/or vega parameters to change dramatically, exposing to portfolio to risk.

   Hedges put on at one level may not work at very different levels.

4. **Time effects**

   Problems can arise with approximate hedging.

   Gamma on long dated options may be hedging gamma on short dated options, but this will not work as the short-dated options approach expiry.

5. **Long-dated options**

   Long-dated options on interest rate instruments have significant correlation with the financing rate, not captured by the Black model.

   Some form of interest rate model (e.g. with mean reversion) would be required to model these effects.

   To an extent this can be modelled by volatilities declining over time but for option maturities over, say, a year this can be a major factor.

6. **Calibration**

The volatilities for caps/floors and swaptions cannot be made consistent under the Black model.

A hedge where the vega of, say, caps is hedging the vega of, say, swaptions may not be effective.

END OF EXAMINERS' REPORT