Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

September 2007

Subject ST6 — Finance and Investment Specialist Technical B

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

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QUESTION 1

(i)

(a) A filtration is the history of values of X up until a particular time. The process X_i, $0 \le i \le T$, generates a filtration $\mathbf{F}_i$, $0 \le i \le T$, where $\mathbf{F}_i$ is the collection of all the events that depend only on $X_0, X_1, \ldots, X_i$.

(b) The measure $\mathbf{Q}$ is a set of probabilities, covering all possible future paths, that governs the evolution of the process X.

$\mathbf{Q}$ is binomial if at each time step i and value X_i, there are only two possible values that X_{i+1} can take at the next time step.

The binomial stochastic process is specified by assigning a probability q to the up move and $1 - q$ to the down move. It describes how likely any up/down jump is at each node.

(c) The conditional expectation $Y = \mathrm{E}_{\mathbf{Q}}(X_T \mid \mathbf{F}_i)$ is the expected value of the variable X given the history up to time i under measure $\mathbf{Q}$.

The conditional expectation Y is itself a stochastic process.

(d) A process ϕ_i, $0 \le i \le T$ is *previsible* if ϕ_i depends only on the filtration $\mathbf{F}_{i-1}$, i.e. up to the previous time step. That is to say, once its value is known at the previous time step, there is no uncertainty about its value at the next time step. (This is not true of most stochastic processes.)

(ii)

Consider the process $M_i = \mathrm{E}_{\mathbf{Q}}(X_T \mid \mathbf{F}_i)$.

The Tower Law of conditional probability states that:

$$\mathrm{E}_{\mathbf{Q}}\left[\mathrm{E}_{\mathbf{Q}}(X_T \mid \mathbf{F}_j) \middle| \mathbf{F}_i\right] = \mathrm{E}_{\mathbf{Q}}(X_T \mid \mathbf{F}_i) \quad \text{for all } i \le j$$

i.e. conditioning on paths up to time j and then conditioning those values on paths up to an earlier time i is equivalent to just conditioning on paths up to time i directly, since every path from i to T corresponds to a path that goes from i to j and then j to T.

Substituting, $\mathrm{E}_{\mathbf{Q}}(M_j \mid \mathbf{F}_i) = M_i \quad \text{for all } i \le j$.

This is the exact condition for M to be a martingale.

[*There may be other equally valid ways of expressing this solution. For example, we could equally also consider the discounted process* $M_i = \mathrm{E}_{\mathbf{Q}}(B_T^{-1} X_T \mid \mathbf{F}_i)$ *where* B_T *is the price of a zero coupon bond maturing at time T. The text given above is typical of the argument that should be used.*]

(iii)

(a) Let M be a process that is a martingale under binomial measure **Q**.

The Binomial Representation Theorem states that, for every other process N that is a martingale under measure **Q**, there exists a previsible process $\phi = \phi_i$ such that:

$$N_i = N_0 + \sum_{k=1}^{i} \phi_k (M_k - M_{k-1})$$

or, equivalently, $\Delta N_i = \phi_i \Delta M_i$.

[*Essentially this means that there is really at most one martingale process for a given measure, since all other martingales can be represented in terms of that process.*]

(b) To see how this could be used to price contingent claims on a stock, consider a stock with price S and discounted price Z (i.e. $Z_i = B_t^{-1} S_t$ for discount bond price B_t).

Provided we can find a measure **Q** that makes Z a martingale, then for any contingent claim X_T on S, the process $C_i = \mathrm{E}_{\mathbf{Q}}(B_T^{-1} X_T \mid \mathbf{F}_i)$, which represents the present value of the claim, is a martingale ...

... and further we can find a previsible tool ϕ from which to construct it from Z ...

... and so we can price C and hence the claim.

QUESTION 2

(i)

The Black-Scholes formula is: $V = S\,\mathrm{N}(d_1) - Ke^{-rT}\,\mathrm{N}(d_2)$

where $d_1 = \dfrac{\log_e\left[S/Ke^{-rT}\right] + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$,

N(x) is the Cumulative Normal distribution,

S = (spot) asset price, K = strike level,

r = continuously compounded annualised interest rate for the period

T = period in years, and σ = annualised standard deviation of logarithmic returns.

Then

Probability of exceeding strike = $\Pr\left[S(T) > K\right] = \mathrm{N}(d_2)$.

(ii)

Using the formula from (i) above with T = ¼ year,

$$d_2 = \frac{\left[\log_e\left(100/120\right) + 0.06.\tfrac{1}{4} - \tfrac{1}{2}\sigma^2.\tfrac{1}{4}\right]}{\sigma\sqrt{\tfrac{1}{4}}} = [-0.18232 + 0.015 - \sigma^2 / 8] / [\sigma / 2]$$

For each of the equities, set σ as per the question, then:

Pr [$S_1(T) > 1.2S_1(0)$] = N(–1.724) = 0.042

Pr [$S_2(T) > 1.2S_2(0)$] = N(–1.402) = 0.080

Pr [$S_3(T) > 1.2S_3(0)$] = N(–1.191) = 0.117

(iii)

Let P_k = probability that stock k fulfils the payout condition.

Probability of basket option payout = $P_1 \cup P_2 \cup P_3$

The events are independent, so payout probability $P = 1 - (1 - P_1)(1 - P_2)(1 - P_3)$

= 1 – 0.958 x 0.920 x 0.883 = 0.222

Thus option value = Pe^{-rT} per £1 million

= 0.222 exp(–0.015) x 1,000,000

= £218,700

(iv)

Consider what would happen if all three equities were 100% correlated, i.e. $\rho_{ij} = 1$. Then the probability of the investment product being in the money would be the highest of the three individual probabilities, i.e. 0.117, and the most volatile of the equities would decide the value of the product.

For $1 > \rho_{ij} > 0$ for $i \neq j$, the probability is in between this and the value in (iii).

Hence, if the equities are positively correlated, the probability of reaching the payoff target is reduced, and so the value of the product is reduced.

QUESTION 3

(i)

(a) In the Black-Scholes formula adapted for currencies (i.e. the Garman-Kohlhagen derivation), European Call and Put prices are respectively:

$$C = S\exp(-r_1T)N(d_1) - K\exp(-r_2T)N(d_2)$$

$$P = K\exp(-r_2T)N(-d_2) - S\exp(-r_1T)N(-d_1)$$

where S = currency value, K = strike, T = expiry time, r_1 = risk-free rate in foreign currency, r_2 = risk-free rate in local currency. N(..) is the cumulative Normal distribution, and the values in the brackets (..) are the adapted Black-Scholes parameters d_1 and d_2.

Hence

$$P - C = K\exp(-r_2T) - S\exp(-r_1T)$$

$$\Rightarrow \quad S = K\exp((r_1 - r_2)T) - (P - C)\exp(r_1T)$$

since $N(-d_1) = 1 - N(d_1)$ and $N(-d_2) = 1 - N(d_2)$.

[*An alternative valid method is to show that the payoff at time T of a portfolio of underlying currency amount S plus the Put is the same as the payoff of cash (strike amount) plus the Call. With an identical underlying stochastic process, their present values are the same.*]

(b) In the examples given, $S = 1.90$, $K = 1.80$, $T = 1$, $r_1 = r_2 = 0.055$.

Hence $K - (P - C)\exp(0.055) = 1.80 - (-0.09465 \times 1.05654) = 1.80 + 0.10 = 1.90 = S$

(ii)

The implied volatilities are different because of a skew (or smile) effect.

There are several reasons why this can occur.

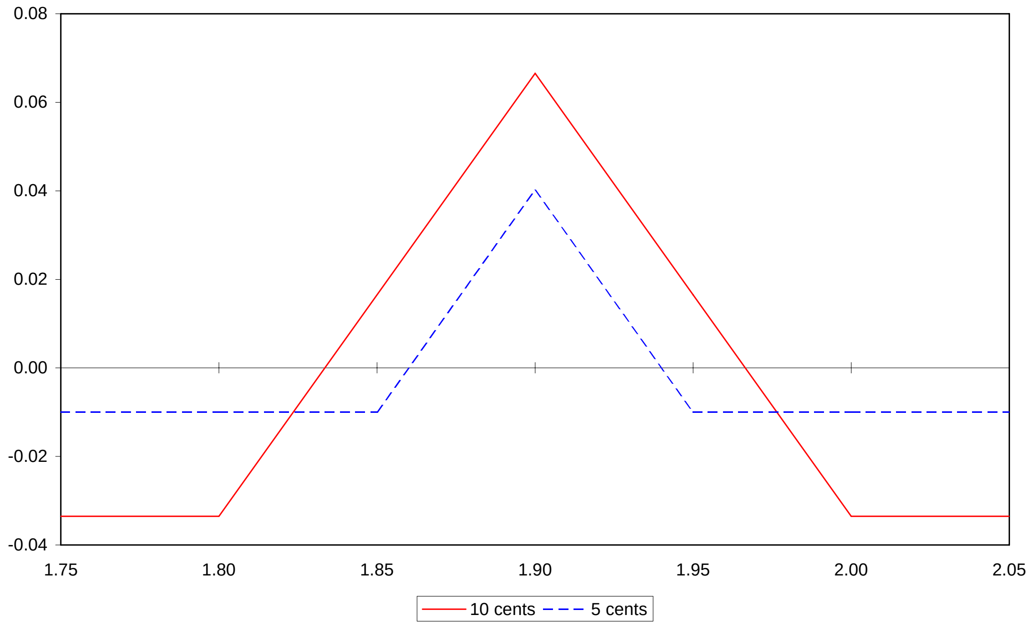
- The implied volatility of options with strikes further out-of-the-money is often higher, reflecting the gearing effect to the seller of dealing in small premiums (small upside = premium, unlimited downside).
- Where the market has a tendency to move rapidly or jump in a particular direction, the implied volatility of options in that direction is often higher, reflecting the demand for protection against those moves and the potential jump diffusion effects that are not priced into the Black-Scholes model.
- The price process is not exactly log normal (e.g. has fatter tail).
- There can be supply/demand effects for particular option strikes.

(iii)

The payoff structure of the butterfly spread is:

Range of underlying	**Delta at expiry**	**Payoff at expiry**
Below K_1	zero	$K_1 - K$
Between K_1 and K	1 x underlying	underlying – K
Between K and K_2	– 1 x underlying	K – underlying
Above K_2	zero	$K - K_2$

To obtain the Profit/Loss on the spread, the premium paid for the spread should be deducted. For the currency option example in the question, we can graph the payoff at expiry.



From this diagram or from the payoff table, it can be seen that the butterfly spread starts like a long Call position, at K would flatten out, i.e. like a Call spread, but in fact then turns down like a reverse (sold) Call spread.

Hence it is equivalent to purchasing a Call spread with strikes K_1 and K, and selling a Call spread with strikes K and K_2.

[*The chart and table above are typical of the expiry diagram that can be drawn to illustrate option strategies. Neither is required to answer the question, but the candidate must show enough understanding to gain all the marks. Sketches with no explanation are not sufficient.*]

(iv)

(a) 10 cents premium = (537.9 + 537.9) – 232.6 – 178.3 = 664.9 (or $0.0665)

5 cents premium = (537.9 + 537.9) – 354.5 – 320.8 = 400.5 (or $0.0401)

(b) The answer lies in determining in what market conditions a butterfly spread performs well.

Basically, the owner has sold volatility at the central point and bought it at the out-of-the-money points.

The intention is to gain income now from the sold at-the-money straddle, but if that were done outright the naked position has unlimited downside risk.

They will benefit from a period of consolidation around the current level ...

... but are protected if there is an extreme move.

Since implied volatility has risen recently, if the market calms this will decline, and the butterfly (which rises in value as volatility falls) will give a profit.

The dealer may be taking a view on the likely trading range for the currency.

(c) The 5 cents spread has a low purchase price and limits the downside well, but gives only a modest income when the market is quiet.

The 10 cents spread has a higher purchase price, limits the downside less well, but gives a better income when the market is quiet.

If recent volatility is expected to continue, and the market to move outside its current range around $1.90, neither spread is worth buying, although the 5 cents one is less risky.

If the recent volatility is expected to decline, this is a good trade. The 10 cents spread has better upside potential – it's revenue maximum is higher.

QUESTION 4

(i)

A portfolio is self-financing if and only if changes in its value depend only on changes in the prices of the assets constituting the portfolio, i.e. there is no external cash amounts entering or leaving the portfolio.

Algebraically: If V_t denotes the value of the portfolio (ϕ_t, ψ_t) of two variables S_t and B_t respectively, then the portfolio is self financing if and only if

$$dV_t = \phi_t dS_t + \psi_t dB_t$$

(ii)

We are given

$$dS_t = \mu S_t dt + \sigma S_t dw_t$$
$$dB_t = rB_t dt$$

Applying Ito's Lemma in the real world:

$$d\left(\frac{B_t}{S_t}\right) = \frac{dB_t}{S_t} - \frac{B_t}{S_t^2} dS_t + \tfrac{1}{2}\frac{2B_t}{S_t^3} dS_t^2$$

where $dS_t^2 = \sigma^2 S_t^2 dt$.

This, on substitution, reduces to

$$d\left(\frac{B_t}{S_t}\right) = \frac{B_t}{S_t}(r - \mu + \sigma^2)dt - \frac{B_t}{S_t}\sigma\, dw_t$$

Hence $\dfrac{B_t}{S_t}$ is a martingale if it is driftless, i.e. if $\mu = \sigma^2 + r$.

[*Note: A valid alternative approach is to use the Equivalent Martingale Measure result (see e.g. Hull 6th Edition p595), which tells us that if the Market Price of Risk* λ *is set equal to the volatility of the numeraire asset (in this case S), then f / S will be a martingale for all choices of f, in particular f = B. This gives* $\lambda = (\mu - r)/\sigma = \sigma$*, or* $\mu = \sigma^2 + r$ *as required. Also, the question referred to the "risk neutral" probability measure where, in fact, the real world calculation was intended. Candidates who tried to convert to a risk neutral measure using the market price of risk were given appropriate recognition.*]

(iii)

Let V_t be the value of the portfolio, i.e. $V_t = \phi_t S_t + \psi_t B_t$.

Since $\phi_t = S_t$, then $V_t = S_t^{\ 2} + \psi_t B_t$

and hence $dV_t = 2S_t dS_t + \frac{1}{2} 2\, dS_t^2 + \psi_t dB_t + B_t d\psi_t$ (A)

where $dS_t^2 = \sigma^2 S_t^2 dt$.

However, a self financing portfolio must be such that $dV_t = \phi_t dS_t + \psi_t dB_t$

so $dV_t = S_t dS_t + \psi_t dB_t$ (B)

From equations (A) and (B):

$$B_t d\psi_t = -S_t dS_t - \sigma^2 S_t^2 dt = (\mu + \sigma^2) S_t^2 dt - \sigma S_t^2 dw$$
$$= -(2\sigma^2 + r) S_t^2 dt - \sigma S_t^2 dw$$

i.e. $d\psi_t = -\dfrac{S_t^2}{B_t}\left[(2\sigma^2 + r)dt + \sigma dw\right]$

and ψ_t is the integral of this from 0 to t (since $\psi_0 = 0$).

QUESTION 5

(i)

[*There are other valid ways of expressing these definitions.*]

(a) "Arbitrage free" (or "no-arbitrage") models are a class of models that allow recovery of market prices of a set of securities given prices of another set. This gives a world of 'relative' pricing.

In a non-arbitrage-free model, securities could be priced using the model and then traded at a different price in the real world, leading to persistent arbitrage profits. In its simplest terms, "arbitrage free" means the absence of a 'free lunch'.

(b) No-arbitrage is very important in yield curve models, since most complex structures are limiting cases of simpler structures (such as swaps, caps, floors) and hence ideally the model should recover the prices of the latter exactly.

Also, hedging is carried out using positions in the simpler structures, so the absence of no-arbitrage would mean the accounting process would be distorted by imaginary gains and losses.

(ii)

(a)

$d(\ln P) = \frac{1}{P}dP - \frac{1}{2}\frac{1}{P^2}(dP)^2$ since P is based on a stochastic process.

By Ito's Lemma:

$$dP = \left[\frac{\partial P}{\partial t} + \lambda(\mu - r)\frac{\partial P}{\partial r} + \tfrac{1}{2}(\sigma\sqrt{r})^2\frac{\partial^2 P}{\partial r^2}\right]dt + \frac{\partial P}{\partial r}\sigma\sqrt{r}\,dw_t$$

and $(dP)^2 = \left[\left(\frac{\partial P}{\partial r}\right)^2(\sigma\sqrt{r})^2\right]dt$.

Hence

$$d(\ln P) = \frac{1}{P}\left[\frac{\partial P}{\partial t} + \lambda(\mu - r)\frac{\partial P}{\partial r} + \tfrac{1}{2}(\sigma\sqrt{r})^2\frac{\partial^2 P}{\partial r^2} - \tfrac{1}{2}(\sigma\sqrt{r})^2\frac{1}{P}\left(\frac{\partial P}{\partial r}\right)^2\right]dt + \frac{\partial P}{\partial r}\sigma\sqrt{r}\,dw_t$$

In general, $d(\ln P)$ follows a process $d(\ln P) = m\,dt + v\,dw_t$, with $m\,dt$ as the expectation.

But we are given that $\frac{\partial P}{\partial t} + (\sigma - \lambda r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 r\frac{\partial^2 P}{\partial r^2} - rP = 0$. (*)

Hence, comparing terms in dt:

$$\begin{aligned} m &= \frac{1}{P}\left[\frac{\partial P}{\partial t} + \lambda(\mu - r)\frac{\partial P}{\partial r} + \tfrac{1}{2}(\sigma\sqrt{r})^2\frac{\partial^2 P}{\partial r^2} - \tfrac{1}{2}(\sigma\sqrt{r})^2\frac{1}{P}\left(\frac{\partial P}{\partial r}\right)^2\right] \\ &= \frac{1}{P}\left[\frac{\partial P}{\partial t} + (\lambda\mu - \sigma + \sigma - \lambda r)\frac{\partial P}{\partial r} + \tfrac{1}{2}\sigma^2 r\frac{\partial^2 P}{\partial r^2} - \tfrac{1}{2}\sigma^2 r\frac{1}{P}\left(\frac{\partial P}{\partial r}\right)^2\right] \\ &= (\lambda\mu - \sigma)\frac{1}{P}\frac{\partial P}{\partial r} + r - \tfrac{1}{2}\sigma^2 r\frac{1}{P^2}\left(\frac{\partial P}{\partial r}\right)^2 \end{aligned}$$

(b)

We are asked to try a solution to the PDE of the form

$$P(t,T) = \exp(A(t,T) - B(t,T)r)$$

Then

$$\frac{\partial P}{\partial t} = P\left(\frac{dA}{dt} - r\frac{dB}{dt}\right)$$

so substituting into equation (*) we obtain

$$P\left[\frac{dA}{dt} - r\frac{dB}{dt} - (\sigma - \lambda r)B + \tfrac{1}{2}\sigma^2 rB^2 - r\right] = 0$$

This must be true for all r, hence this implies:

$$-\frac{dB}{dt}+\lambda B+\tfrac{1}{2}\sigma^2 B^2-1=0$$

(iii)

Both Vasicek and CIR are models of the short rate r.

The question asks us to assess the advantages of CIR over Vasicek as a model of the yield curve. To answer this, we show how well CIR and Vasicek satisfy the main features required from an interest rate model:

1. Easy to use and calculate, especially when calibrating to market prices

   <u>CIR</u>: has an algebraic solution but less tractable than Vasicek ✓

   <u>Vasicek</u>: as CIR but easier to use ✓

2. Prices of basic market instruments (swaps, caps, floors etc) should be reproduced by the model without arbitrage opportunities

   <u>CIR</u>: most simple shapes of yield curve can be modelled due to time-dependent (strictly rate-dependent) volatility ✓

   <u>Vasicek:</u> simpler volatility, so less scope ✗

   [*There is a trade off for having analytical tractability. CIR scores less well on the possible shapes of yield curve it can fit precisely than more advanced models – it can only obtain simple upward sloping, simple downward sloping and single-humped curves.*]

3. Interest rates remain positive

   <u>CIR</u>: negative rates are avoided by making the volatility term decline as r approaches zero, so provided σ is not too large in relation to α and μ ✓

   <u>Vasicek:</u> can have negative rates ✗

4. Rates disperse reasonably of rates over time, so don't want too large a probability of getting an absurdly high or low value, i.e. when rates go too high or low, they tend to revert back to some middle level $\Rightarrow$ mean-reversion

   CIR: mean reversion parameter in the drift term ✓

   [*μ is the long-term target for r and α is the speed (or force) of correction.*]

   Vasicek: simple volatility parameter gives limited fitting of yield curve shapes ✗

QUESTION 6

(i)

For the kth caplet in a cap (k = 1, 2, ... n), let the principal amount be L, the strike rate R_k and time to expiry t_k. [*Usually $R_k = R$ for all k, i.e. the same strike applies to all the caplets, but this is not essential*.]

Denote the underlying reference interest rate for this caplet as r_k , the rate that applies between times t_k and t_{k+1}, and let F_k be the equivalent forward rate.

Let the volatility of each r_k be σ_k.

Let $\delta_k = t_{k+1} - t_k$.

Then:

Value at t_k of payoff of kth caplet at time t_{k+1} = $\dfrac{L\delta_k}{(1+r_k\delta_k)}\max(r_k - R_k, 0)$.

The Black model states that:

Price of kth caplet at time 0 = $L\delta_k P(0,t_{k+1})\left[F_k N(d_1) - R_k N(d_2)\right]$

where $d_1 = \dfrac{\ln\left(F_k/R_k\right) + \frac{1}{2}\sigma_k^2 t_k}{\sigma_k\sqrt{t_k}}$, $d_2 = d_1 - \sigma_k\sqrt{t_k}$,

and $P(0, t_{k+1})$ is the price of a zero-coupon bond paying 1 at time t_{k+1}.

(ii)

Strike rate R_k = 0.06 (6%), σ_k = 0.16, t_k = 0.5 and $\delta_k = t_{k+1} - t_k$ = 0.08333 (1 month).

Assuming 5.35% flat continuous zero rates

$\Rightarrow$ P(0, t_k) = exp(–0.0535 x 0.5) = 0.97360

P(0, t_{k+1}) = exp(–0.0535 x 0.58333) = 0.96927

Hence the forward rate F_k = [P(0, t_k) / P(0, t_{k+1}) – 1] / δ_k = 5.361%.

Then $d_1 = \dfrac{\ln\left(F_k/R_k\right) + \frac{1}{2}\sigma_k^2 t_k}{\sigma_k\sqrt{t_k}}$ = (–0.11261 + 0.00640) / 0.11314 = –0.93876

and $d_2 = d_1 - \sigma_k\sqrt{t_k}$ = –0.93876 – 0.11314 = –1.05190

Hence binary caplet price

= 1,000,000 x P(0, t_{k+1}) x N(d_2)

= 1,000,000 x 0.96927 x 0.14642 = 141,920.

[*Candidates might have been unable to interpret the term 'binary' and calculated a full caplet price, even though this requires more effort! The answer, which was allowed, is:*

Caplet price:

= 1,000,000 x 0.08333 x 0.96927 [0.05361 x N(–0.93876) – 0.06 x N(–1.05190)]

= 807772.5 [0.05361 x 0.17392 – 0.06 x 0.14642] = 43.55

If candidates further used 5.35% as the forward rate, the answers become:

$d_1 = -0.95692$, $d_2 = -1.07006$, $N(d_1) = 0.169304$, $N(d_2) = 0.142296$

and the caplet price = 42.00, but they should lose a mark for this.]

(iii)

"Flat" volatilities (vols) use a single implied volatility for each cap, which itself is a collection of caplets, but different rates for each cap tenor. This is how cap vols are usually quoted in the market.

They are simpler to use, and there is an advantage in keeping prices in line with market convention.

"Spot" vols use a different implied volatility rate for each caplet depending on expiry. This is how traders like to see their volatility surface, as it enables them to spot any pricing anomalies or arbitrages.

Traders also can directly compare spot vols with those from options on the short interest rate futures (e.g. Eurodollar, Euribor, Eurosterling).

Traders can convert flat to spot vols. They take the flat vols, interpolating to ensure that they have a complete sequence of tenors, then bootstrap caplet implied volatilities from each successive cap one at a time. This requires the choice of a common strike price (usually at-the-money).

Both curves often show a hump in the short-dated vols, which is thought to be due to greater trading activity taking place in the tenors just after the very short ones, which are controlled by the central banks.

It is not clear whether there is a best approach – the spot vols approach should be theoretically more correct, but flat vols are more stable and heavy supply/demand for specific cap tenors sometimes means they are more representative of market prices.

QUESTION 7

(i)

(a)

Forward exchange rates:

year 1 1.28 x exp(rate1 – rate2) = 1.28 x exp(6% – 4%)

= 1.28 x 1.0202 = 1.30586

year 2 1.30586 x 1.0202 = 1.33224

year 3 1.33224 x 1.0202 = 1.35915

(b)

Current FX rate = 1.28

Therefore, $200 million is equivalent to 200 / 1.28 = 156.25 million Euros.

This gives the two bond amounts.

In year 1, net $ payment = 156,250,000 x 0.045 x 1.30586 – 200,000,000 x 0.0575

= 9,181,828 – 11,500,000 = –2,318,172

Discounted back to today this is –2,318,172 x exp(–0.06) = –2,183,172

In year 2, net $ payment = 156,250,000 x 0.045 x 1.33224 – 200,000,000 x 0.0575

= 9,367,313 – 11,500,000 = –2,132,687

Discounted back to today this is –2,132,687 x exp(–0.06 x 2) = –1,891,524

In year 3, net $ payment = 156,250,000 x 1.045 x 1.35915 – 200,000,000 x 1.0575

= 221,923,711 – 211,500,000 = +10,423,711

Discounted back to today this is 10,423,711 x exp(–0.06 x 3) = 8,706,615

$\Rightarrow$ Value of swap = –2,183,172 –1,891,524 + 8,706,615

= 4,631,919 or $4.632 million.

[*Some candidates may prefer to express this as the difference between a 3-year US bond, price 197.690, and a 3-year Euro bond, price 158.064, which is 202.322 in $.*]

(ii)

If the swap is cancelled part way through, the number of coupons and the timing of the repayment of the principal is uncertain

Hence this effect needs to be valued using option theory ...

... however, the option is complicated because there are interest rate movements and currency movements to consider ...

... as well as the correlation between those factors.

So a more complex multi-factor model would be needed to value the swap.

(iii)

(a) The Monte Carlo (MC) method uses a random number generator to provide potential values for the stochastic term of an interest-rate model.

One set of random numbers gives one simulated path; the aggregate gives the distribution.

Used to value complex derivatives that have no analytic solution, such as exotic options which are path dependent or have non-linear payoffs.

Method: turn time horizon into discrete time by small time steps Δt ...

... sample known probability distribution to give a path over all the time steps ...

... gather large number of paths, say 10,000, of equal probability ...

... value the derivative (option or whatever) on each path, and take the average.

(b)

The MC set up needs at least three risk factors: one for the currency rate and two for the local and foreign interest rates.

It may need more factors since the underlying options are complex yield curve options.

The paths must be compared with the payoff of the complex option.

This problem is very hard – complicated to set up the payoff function and hard to get the correct interaction (correlations etc) between the interest rates and currency.

(iv) [Note: candidates were asked to describe only <u>two</u> methods.]

[*There are several possible methods to choose from – the most common are listed below. The idea behind all of them is to give the same representative coverage of the sample space but with fewer sample paths.*]

(A) <u>Moment Matching</u>

The n sample paths are generated and stored, then adjusted by a scaling factor to ensure that the variance (second moment) of the sample exactly matches that of the initial distribution.

This produces a tighter fit to the probability distribution without adding further samples, so should be more accurate.

(B) <u>Antithetic Variables</u>

For each sample path from the distribution, create another valid path by taking the opposite sign on each random element. This creates an "antithetic" path.

The odd moments of the sample distribution are therefore zero, which works well with non-skew distributions like the Normal distribution.

(C) Quasi-Random Sequences

The MC approach does not actually need truly random values to succeed – all it needs is to have representative enough paths that the approximation is not biased by the samples chosen.

There are some sequences which can be shown to be non-random, but which nevertheless will not bias the approximation. The advantage of this type of series is that its standard error is proportional to $\frac{1}{n}$, rather than $\frac{1}{\sqrt{n}}$ for the raw MC.

(D) Stratified Sampling

Divide the range of outcomes into bands according to probability, and then take more sample paths from those with the higher probability.

This does not affect the overall accuracy, but removes the need to sample all those outcomes where there is a low contribution to the valuation.

(E) Contravariates (or Control variates)

If an exotic derivative is being valued, there is probably an analytical expression available for a similar, more simple, derivative (e.g. if calculating an average-rate caplet, the formula for the plain vanilla caplet is known).

Hence an approach is to use MC to calculate the simple and exotic derivatives, and adjust the exotic value by the same ratio as the vanilla value is changed to agree with the known formula.

(F) Importance Sampling

A lot of values in the MC iteration will have little impact on the value of the option. This method chooses those values that have maximum impact, without compromising the randomness of the data.

QUESTION 8

(i)

(a) Value at risk (VaR) for a portfolio is a number which indicates the maximum loss which the portfolio can sustain over a given timeframe (1 day) for a given confidence level (95% certainty).

It is normally calculated by taking into account:

- future price movements (volatility)
- the interdependencies of the portfolio constituents (correlations).

VaR is simple and tractable – one number for entire portfolio – and if the risk is linear, there is no need for complicated methods.

VaR allows different correlated risk sources (e.g. equities, interest risk at different maturities, currencies, etc, etc) to be aggregated together on a consistent basis.

(b) Limitations of VaR as a risk measure

- VaR is not directional, so gives no indication whether there will be a loss or profit.
- VaR is not a maximum possible loss figure, and the tail of the loss distribution can be extreme.
- VaR is essentially a linear measure of risk, i.e. assumes the loss is always proportional to the exposure, which is true except for option-type instruments. VaR can be adjusted for these convex securities, but not if the convexity is too pronounced (e.g. very "out-of-the-money" or exotic options).
- VaR hides structural mismatches, such as gross interest rate maturity mismatches.
- VaR depends on stable (and meaningful) relationships between risk elements (i.e. stable covariance / correlation). This is rarely true on large market moves.
- VaR will give poorer predictions on large moves (due to conversions) or where significant holdings in convertible bonds are held.

(ii)

Reasons for measuring credit risk on derivatives

Swaps and options have replacement value. If the counterparty defaults, the life office will need to replace that value.

Derivatives change value with market movements, so the counterparty risk is dynamic.

Use of a single measure of credit risk (e.g. Credit VaR) to show overall risk appetite.

Method

The method is to assess the replacement cost of the derivative using a forward projection – either simply, using a volatility estimate – or in a more complex way using a proper stochastic model of the assets.

The derivative value is made up of the current mark-to-market (MTM) – e.g. swap present value, or in the case of purchased options, the option premium – plus a potential future exposure (PFE) due to future adverse market movements.

The peak exposure is usually measured against some notional limit.

In the simple method, options are assumed to be exercised, but a full stochastic process would assess the exercise conditions.

(iii)

[*These points are examples. Other suitable reports could be compiled. It is important that the candidate mentions <u>relevant</u> analysis for the business, not just any risk controls.*]

The mismatch needs to be analysed more thoroughly, with additional market risk sensitivity analysis to supplement VaR.

Additional analysis/reports that could be added are as follows:

- Structural asset class mismatch report.
- Business analysis by term, product and currency.
- Credit risk limit report – see part (ii).
- Concentration risk and/or reinsurance.
- Maturity mismatch for interest rate risk (shown either as Gap Report, basis point sensitivities or equivalent positions in standardised instruments).
- Volatility report for the key drivers of the overall VaR number, together with analysis of trends and commentary.
- Analysis of changes in asset/liability management and solvency position.
- Exception report on instruments with option characteristics if these exceed "x"% of portfolio or "y"% of VaR number.
- Stress tests of parameters – e.g. annuity rates, equity prices, credit.

END OF EXAMINERS' REPORT