EXAMINATIONS

September 2006

Subject ST6 — Finance and Investment
Specialist Technical B
and
Certificate in Derivatives

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M Stocker
Chairman of the Board of Examiners

November 2006

Comments

Individual comments are shown after each question.

General comments

As in April 2006, there were encouraging signs from this sitting that candidates were well prepared on basic bookwork. However, there were still far too many sketchy scripts that answered bookwork questions adequately but whose solutions to questions requiring practical application lacked substance (or were not even attempted).

Please note that the model solutions provided are indicative, i.e. adequate to achieve full marks but without covering ever possible correct response. Several points made by candidates were equally valid, and these also achieved the allocated marks.

Comments on individual questions

Q1 *Both parts of this question were based on bookwork concerning arbitrage and replication theory. They were tackled successfully by almost all candidates.*

Q2 *This question invited candidates to draw sketches of the relationship between P&L and price for a straddle and a strangle, and then compare the two strategies. The terms "straddle" and "strangle" were defined in the question.*

Graphical questions come up regularly in ST6 as a way of demonstrating that the candidate has understood how option prices and sensitivities vary with different parameters. Previous reports have discussed the expectations examiners have when assessing such graphs.

Part (i) required straightforward sketching of the effects of adding a call and a put together, which most candidates achieved satisfactorily. However, candidates should be careful not to provide too casual a sketch, lest key points are lost. Most found the two characteristic bowl-shaped P&L graphs, but it was disappointing that several candidates could not see that the current valuation of the options at-the-money (price 400) should give zero P&L.

Part (ii) asked for a practical comparison of the two option strategies. A discussion was required of the non-directional nature of straddles/strangles, the need for large price moves to justify the cost of the options and the likely impact if volatility falls.

Q3 *Part (i) of this question looked at the adjustments needed to the basic Black-Scholes formula for various dividend-paying stocks and futures contracts. On the whole, it was well answered as these adjustments are standard bookwork.*

Parts (ii) and (iii) were more difficult. In part (ii), it was not necessary to quote exactly the formula given in the solutions, but the essential step was to show an appreciation that the correlation between the rate the payoff is based on and the rate used for discounting the option premium has an impact on pricing. In part (iii), the additional volatility of interest rates adds to the volatility used in Black Scholes.

Q4 *This was a hard question for most candidates, and few answered it well. (There were also different interpretations of how to apply the tax treatment — the model solution gives an alternative.)*

The case study here is actually a simplification of a real-life problem. In fact, although the problem might have seemed unfamiliar, the principles involved were straightforward. The question was essentially asking the candidate to value a put option, so in part (i) they needed to obtain the formula for the number of options and option payoff. Then, in part (ii), these should be put into the Black-Scholes formula, and finally in part (iii), a reference to put-call parity shows that call options plus cash could be used instead of stock plus puts.

It was disappointing that so few candidates produced more than cursory attempts, but possibly understandable given the limited time for the examination and the need in this question to think hard before understanding what was being asked.

Q5 *Attempts at this question were disappointing. The discussion of market risk factors required in part (i) was entirely based on bookwork from Unit 9 of the Core Reading, which is absolutely standard material. Many candidates simply did not write enough for the 9 marks available, particularly given that the question asked for responses "in detail".*

For part (ii), which asked for variances between two yield curve models, the points made were generally good, but not many candidates got this far. In this type of question, it is always best to mention several different points briefly rather than develop fewer points at length.

Q6 *This question was based on an application of discrete stochastic processes, and generally just required a steady head when manipulating algebra.*

Part (i) and (ii) were answered well, although several candidates hoped that they could get the marks by just re-quoting the question.

Part (iii) involved using the Central Limit Theorem to show that the set of discrete binomial variables X_n approached a Normal distribution in the continuous limit.

Part (iv) needed candidates to wade through a certain amount of algebraic spaghetti, but several made good attempts and these were rewarded.

Q7 *This question started with the concept of the "market price of risk", which is standard bookwork. Generally, for part (i), candidates failed to give a clear enough definition.*

Part (ii) was a simple application of the "market price of risk" formula given in part (i), and gave 3 rather easy marks to those candidates who realised its simplicity.

Part (iii) asked to develop partial differential equations for American options. It should have also been an easy question, given that it was simple bookwork for the most part. Note that American options only differ from European options in their *boundary conditions, not in their stochastic equation. The model solution also shows how to quote precise boundary conditions.*

Part (iv), asking to adjust the equation in (iii) for a different process, held no fears for those who kept their heads, but many had given up by this point. The key to questions of this type is to assess the impact on the drift and variance of the distributions.

Q8 *This question started in part (i) with some familiar interest rate calculations that were generally well executed.*

The derivation of the swaption formula in part (ii) was bookwork, and it was pleasing to see many cope well with this. Notice that an annuity value appears in the formula, since on exercise the difference between strike rate and actual rate can be considered effectively paid not once (as with a standard stock option) but several times, each time the swap pays a coupon.

For some reason, the attempts in part (iii) to put numbers to the formula from part (ii) were less successful, but several candidates achieved the correct answer for the swaption price. The final part (iv) asked for differences between swaption and bond option volatilities. These two markets are very similar, but it is surprising how few candidates mentioned this fact. (Although part (iv) was only a small section of a long question, it is recommended that future candidates study the model solution, as most of the responses here were well wide of the mark.)

1 ***Syllabus: (a), (b), (c), (d)***

(i) **Arbitrage** is a technique for making a risk-free profit, for example by taking advantage of two or more securities (or derivatives on those securities) being mispriced relative to each other.

If an arbitrage exists, there is a trading strategy that makes a riskless (or much reduced risk) profit by buying the cheap instrument and selling the expensive one, possibly in different markets, timezones or physical manifestations.

Some arbitrage opportunities disappear if tax or dealing costs are included.

Arbitrageurs seek to make profits from setting up arbitrage trades.

Their role is not entirely self-serving, since by their actions arbitrageurs keep cash and derivatives markets in line, hence justify the use of the "no arbitrage" principle for derivatives.

Significant amounts of capital are required to make arbitrage profits, which tend to be small relative to the size of the deals undertaken (since they are low risk), so arbitrage opportunities tend to be the province of global firms.

They add liquidity to markets.

(ii) (a) A portfolio of securities is said to be **self-financing** if and only if the change in its value occurs only as a result of changes in the prices of the securities, that is to say, there is no net inflow or outflow of cash.

Hence if (ϕ_t, ψ_t) is the portfolio of value V_t at time t, consisting of ϕ_t of the stock (price S_t) and ψ_t of the bond (price B_t), then:

(ϕ_t, ψ_t) is self-financing $\Leftrightarrow dV_t = \phi_t dS_t + \psi_t dB_t$

(b) Consider the claim X based on events up to time T.

A replicating strategy is a self-financing strategy (ϕ_t, ψ_t) whose variance is bounded up to time T, and which delivers the value of the claim at time T, that is,

$$V_T = \phi_T S_T + \psi_T B_T = X$$

Since the claim value is replicated at time T, and (ϕ_t, ψ_t) is self-financing, then

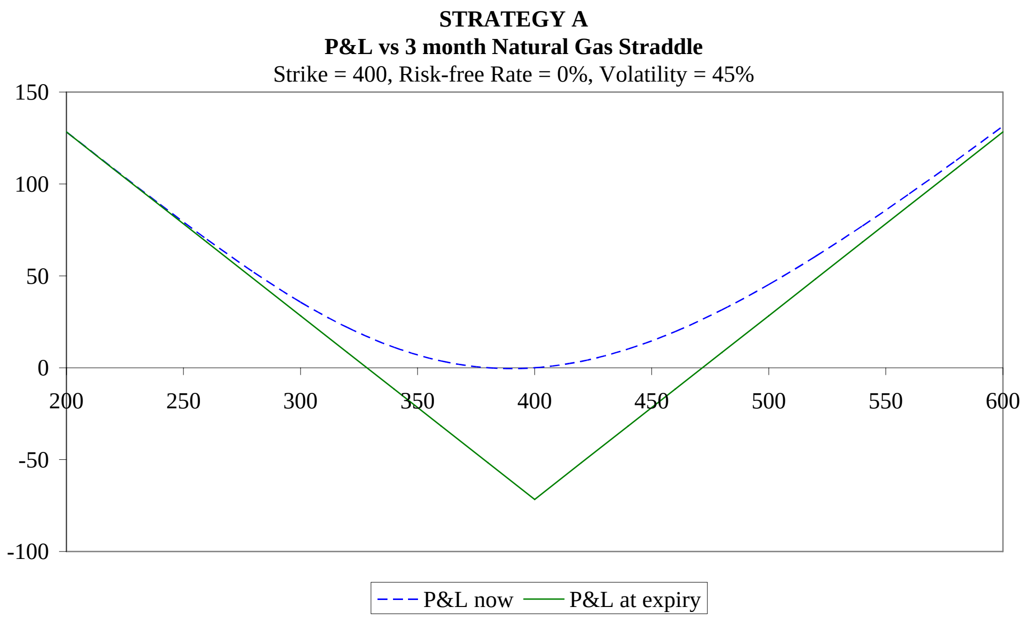
$$X = V_t = \phi_t S_t + \psi_t B_t \quad \text{for all } t < T$$

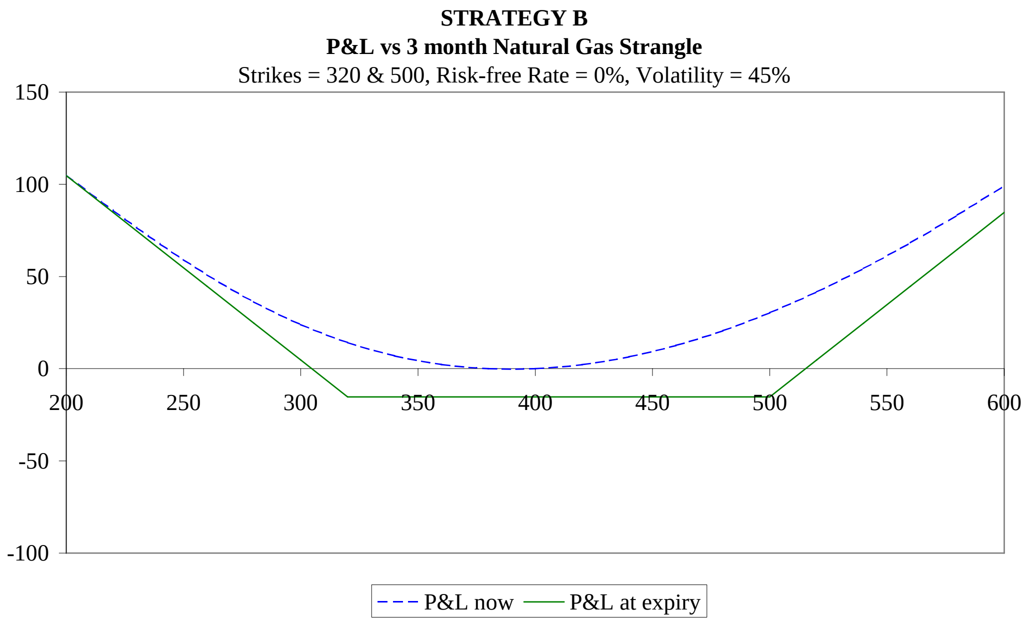
and this is especially true at $t = 0$, i.e. now.

Hence, if such a replicating strategy exists, we have found a way of pricing the claim.

2 *Syllabus: (f)(vi) on*

(i) [*These charts have been computer-drawn for clarity to illustrate the key features. The candidates' sketches are not required to be so precise.*]





[*The examples above have risk free rate r = 0. For r > 0, the dotted line would pass below the y-axis at the strike level.*]

(ii) *[The following points are typical of the ones that could be made.]*

Strategy A: Straddle

A long straddle trade is bullish on volatility, which is currently quite high.

The P&L is virtually linear in volatility — if it falls, A will lose unless the market trends from the current level by expiry time.

It is not directional, so whether the market rises or falls, A will profit.

The premium for a straddle is expensive (buying protection in both directions).

Numerically, given a 45% volatility and $t = \frac{1}{4}$ year, using a $\sigma\sqrt{t}$ rule, if the market moves by one standard deviation (s.d.), this will be around 22½%, i.e. a range of approx. ±90. So the strategy needs about a 1 s.d. move to succeed.

If the market is volatile, the trader can make delta-hedging profits (i.e. rebalancing the delta hedge several times before expiry) ...
... but if he/she is not delta-hedging there could be non-trending volatility that ends up around the current level, so the strategy might still lose.

Strategy B: Strangle

A long strangle trade is very bullish on volatility, i.e. needs a very big move from the current level at expiry to profit.

If volatility falls, B will lose, not as much as A in absolute terms, but in relative terms to the premium it will seem more significant.

It is not directional, so whether the market rises or falls, B will profit.

The premium for a strangle is less expensive than a straddle, but the trader is still buying protection in both directions.

If the market is volatile, the trader can make **some** delta-hedging profits although not as easily as A because the options are so far out-of-the-money.

B's real value is if the market moves **significantly** in either direction (numerically, this would have to be at least 1.5 s.d.'s).

3 ***Syllabus: (f)(vi) on***

(i) First we state the Black Scholes formula: Call price

$$C = S\,N(d_1) - Ke^{-rt}\,N(d_1 - \sigma\sqrt{t})$$

where S = current stock price, t = period to option expiry, r = continuous compounded risk-free rate, K = strike price, σ = constant volatility, and

$$N(x) = \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy \text{ and } d_1 = \frac{1}{\sigma\sqrt{t}}\left\{\ln\left(\frac{S}{Ke^{-rt}}\right) + \frac{1}{2}\sigma^2 t\right\}$$

[*This is just to set out the notation — no marks to be given.*]

(a) **Continuous dividend**

Let the continuous dividend rate be α.

This has the effect of reducing the growth of S by the dividend rate.

Hence change S in the formula above to $Se^{-\alpha t}$.

(b) **Single discrete dividend**

Let D be the dividend paid at time $\tau < t$.

Then F the forward price of S at time t is given by $F = Se^{rt} - De^{r(t-\tau)}$, and the volatility is that of the forward price.

Then

$$C = e^{-rt}\,[\,F\,N(d_1) - K\,N(d_1 - \sigma\sqrt{t})\,] \quad \text{with } d_1 = \frac{1}{\sigma\sqrt{t}}\left\{\ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2 t\right\}$$

Alternative: Replace S with $S - De^{-r\tau}$

(c) **Future (Black formula)**

Let F be the futures price at time t, so the volatility is that of the futures price.

Then $C = e^{-rt}$ [$F\ N(d_1) - K\ N(d_1 - \sigma\sqrt{t})$] with

$$d_1 = \frac{1}{\sigma\sqrt{t}}\left\{\ln\left(\frac{F}{K}\right) + \tfrac{1}{2}\sigma^2 t\right\} \text{ as in (b).}$$

Alternative: Replace S with Fe^{-rt}, where F is the futures price.

(ii) There is a positive correlation between changes in one year zero coupon bond interest rates and those of five year zero coupon bond interest rates.

Let the standard deviation of zero coupon interest rates of maturity t be given by θ_t.

Let the correlation between one year and five year zero coupon interest rates be ρ_{15}.

Then the "combined" volatility is given by

$$\sigma^2 t \to \{\theta_1^{\ 2} + \theta_5^{\ 2} - 2\theta_1\theta_5\rho_{15}\}t$$

The Black and Black-Scholes formulae may now be used.

(iii) The Black Formula in (i) (c) shows the relevant volatility is that of the forward bond price.

The zero coupon bond price $B = e^{-r(t).t}$, where $r(t)$ is a stochastic random variable that depends on time but is independent of the stock price process.

Let the standard deviation of zero coupon interest rates be θ.

Then log $B = -r(t)\ t$

var(log B) = $\theta^2 t^2$

The additional volatility is simply additive in variance terms.

Hence variance of the process changes from $\sigma^2 t$ to $\sigma^2 t + \theta^2 t^2$.

Thus, the original formula can be used with the transformation:

$$\sigma\sqrt{t} \to \left(\sigma^2 t + \theta^2 t^2\right)^{1/2}$$

4 *Syllabus: (f)(vi) on*

(i) Let

time T denote the time when the guarantees fall due = maturity of bonds
U_T represents the unitised funds under management at maturity
G represent the guarantee at maturity returning the original investment = U_0
c represent the annual management charge
τ the rate of income tax

A formula for the build of assets under management for one unit of index is:

$$U_T = U_0\left(1+\left(\frac{E_T}{E_0}-1\right)(1-\tau)\right)(1-c)^T = \hat{U}_0\left(1+\left(\frac{E_T}{E_0}-1\right)(1-\tau)\right)$$

where $\hat{U}_0 = U_0(1-c)^T$ is the index net of future fund charges.

The cost of the guarantee to the firm at time T is $\max(G-U_T,0)$, assuming that the fund charges can be used to offset the cost of the guarantee.

The payoff of N put options with strike X on the index is $N\max(X-E_T,0)$.

For the payoff to equal the guarantee cost cash flows, we require

$$N(X-E_T)^+ = (G-U_T)^+$$
$$\Rightarrow (NX-NE_T)^+ = (G-\hat{U}_0\frac{E_T}{E_0}(1-\tau)-\hat{U}_0(1-(1-\tau)))^+$$
$$\Rightarrow (NX-NE_T)^+ = (G-\hat{U}_0\tau-\hat{U}_0\frac{E_T}{E_0}(1-\tau))^+$$

where $^+$ indicates "max(... , 0)".

For the payoff of the puts to be exactly equal to the cost of the guarantee, this must apply (if possible) to all values of E_T.

So, equating terms in E_T and other terms we have:

$$NX = (G-\hat{U}_0\tau)^+ \text{ and}$$

$$-NE_T = -\hat{U}_0\frac{E_T}{E_0}(1-\tau)$$

whence

$$N = \frac{\hat{U}_0(1-\tau)}{E_0} = \frac{U_0}{E_0}(1-c)^T(1-\tau) \qquad (*)$$

and

$$X = \frac{(G-\hat{U}_0\tau)^+}{N} = \frac{(U_0-\hat{U}_0\tau)^+}{\hat{U}_0(1-\tau)}E_0 = \frac{1-(1-c)^T\tau}{(1-c)^T(1-\tau)}E_0 \quad (**)$$

[*Intuitively, this is logical, since the amount of options is the number of times the current index divides into the net fund invested, and the strike is the excess per option of the guarantee over the net fund invested.*]

Alternative method:

It was observed that a number of candidates interpreted from the note in the question that the amount subject to tax for a unit investment was $\left(\frac{E_T}{E_0}-1\right)$.

This note was written to assist the candidates but was capable of a slightly different interpretation from that intended.

Hence $U_T = U_0\frac{E_T}{E_0}(1-c)^T - U_0\left(\frac{E_T}{E_0}-1\right)\tau$.

This is equivalent to replacing each occurrence of τ with $(1-c)^T\tau$ in the above solution. This still gives a similar value to that in (ii) below, and gained full marks.

(ii) (a) To calculate the strike, substitute E_0 = 2,500, $G = U_0$ = £1.1 billion, and T = 4, τ = 0.25, c = 0.015 into (**) above:

So $\hat{U}_0 = U_0(1-c)^T$ = 1,035.47 million

and

$X = \frac{(U_0-\hat{U}_0\tau)^+}{\hat{U}_0(1-\tau)}E_0$ = (841.1 / 776.6) x 2,500 = 2,707.7

or around 2,700 as the question states.

(b) Substitute as for (a) but in (*), giving:

$$N = \frac{\hat{U}_0(1-\tau)}{E_0} = 776.6 \text{ million} / 2{,}500 = 310{,}600 \text{ options}$$

assuming each option is one times the index.

To get the price of a single put option, substitute also $\sigma = 20\%$, $r = 4\%$, $T = 4$ into the Black-Scholes formula:

$$V_P = Xe^{-rT}N(-d_2) - E_0 N(-d_1)$$

where

$$d_1 = \frac{\ln(E_0/X) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Hence

$d_1 = 0.4076$
$d_2 = 0.0076$

$N(-d_1) = 0.34178$
$N(-d_2) = 0.49697$

so $V_P = 2{,}700 \times 0.8521 \times 0.44983 - 2{,}500 \times 0.29941 = 289.0$ (in index points).

Hence the total cost of the options = 289 * 310,600 = 89.8 million.

(iii) The institution could buy a call option that delivered the future growth in the equity index after taxes ...
... plus a zero coupon deposit contract to deliver the guarantee.

[*Candidates also received marks for referring to put-call parity, or proposing dynamic replication either using the index + cash or index futures + cash.*]

5 ***Syllabus: (h) & (i)***

(i) [*This is a selection of points that can be made. Others may be relevant, provided that the solution offered focuses on the basic market risks.*]

Market risk is the risk that the value of a portfolio will fall due to an adverse change in the level or volatility of the market price of interest-rate instruments, currencies, commodities and equities.

The basic market risks in the portfolio are:

(1) **Directional price risk, or delta**

This is the change in the value of the portfolio due to changes in the prices of the underlying instruments — all other variables remaining constant.

For linear derivatives like futures, cash bonds and swaps, delta is a reasonably accurate forecast of market risk. (In fact, linearity does not exactly apply to ordinary bonds and swaps, but the error in assuming linearity is small.)

Delta (in conjunction with gamma and vega) also works well for vanilla options on futures and forwards, and delta hedging is a fairly straightforward technique.

Delta produces less reliable forecasts of market risk for exotic options, where the payoff can be very non-linear.

(2) **Convexity risk, or gamma**

This is the change in the delta arising from changes in the price of the underlying instruments — all other variables remaining constant.

Vanilla options have well behaved gamma profiles, so only at expiry can there be significant risk.

Exotic options have very variable gamma and vega (volatility) profiles so risk profile at the current market level is misleading. Some exotics are virtually riskless at current levels, but "blow up" somewhere far away (e.g. barrier options or cancellation options).

Stress tests or jump diffusion models need to supplement the "greeks" here.

(3) **Volatility risk, or vega**

This is the change in the value of the portfolio arising from changes in implied volatility of the underlying instrument — all other variables remaining constant.
Vega risk can only be hedged by taking positions in options (often traded options — but then there can be basis risk as described below).

(4) **Basis risk**

This is the change in the value of the portfolio arising from changes in correlated variables.

This is particularly important with options on futures versus options on cash markets or indices, which can get out of line with each other before expiry.

Hedged portfolios often experience basis risk, e.g. hedging the interest rate risk from a long position in a corporate bond using a government bond runs the risk.

(5) **Time-decay risk or theta**

This is the change in value of the portfolio arising from the passage of time — all other variables remaining constant.

Time decay is not usually hedged directly, but treated as an "accrual" cost of a portfolio. Instead, gamma and vega are hedged and there is a net effect on theta.

(6) **Interest rate risk or rho**

This is the change in the value of the portfolio arising from changes in interest rates used to discount future cash flows — all other variables remaining constant.

For currency options, which are based on currency pairs, there are two interest rates involved.

Other more minor issues relating to market risk are:

Structured bonds can mostly be decomposed into more basic securities, usually with a bond underlying. For example, an inverse floater can be transformed into a FRN plus twice the amount of a fixed-floating interest-rate swap. If there is significant optionality, however, this approach will not be adequate.

Credit spread risk is usually treated as a type of market risk present to measure the sensitivity to default risk in non-government bonds.

Liquidity risk is often linked with market risk — this relates to the difficulties of closing out positions in times of market stress.

Reset risk for cash flow resets in swaps or dividend estimation for equity indices.

(ii) Bond and interest-rate-derivatives traders want to be able to quote prices which are in line with prices being quoted by other traders …
… but the use of the Vasicek model may not be arbitrage free, since it will not be exactly inline with the full market prices …
… and Hull & White (HW) is better in that it allows the trader to price interest-rate linked contracts more accurately by reflecting the current *observed* term-structure of interest rates.
HW can be easily extended to include a time-varying but deterministic $\sigma(t)$, …
… which allows the model to be calibrated for all traded option prices as well as zero-coupon bond prices and so use the entire yield **and** volatility curves …
… but a disadvantage of this version of HW is that a simple calibration is harder to achieve than for Vasicek or simple HW
… and even this version of HW, with its extra flexibility, cannot cope with two- or three-factor problems (options based on more than one asset, e.g. spread options).
Otherwise Vasicek and HW are very similar …
… since $\mu(t)$ is deterministic the simple HW model is just as easy to set up and manipulate as the Vasicek model, …
… both HW and Vasicek suffer from the same drawback that interest rates might become negative (although the time-dependent parameters in HW can make this very unlikely), ...
… not all types of options are covered by either model as described above, …
… and models may break down under stress scenarios. (Stress testing is very important for complex option portfolios where financial effects are really only seen in the tails of the distribution. In stress situations, bid-offer spreads can widen, liquidity can dry up and firms can face extraordinarily large cash collateral demands.)

6 ***Syllabus: (f)(i)–(v)***

(i) $n = t / \delta t$

(ii) Let X_n be the number of up jumps, Y_n be the number of down jumps.

Then $X_n + Y_n = n$, so $X_n - Y_n = 2X_n - n$. (*)

Now by simple multiplication, $S_t = S_0 \exp\left(\mu(n\delta t) + \sigma\sqrt{\delta t}\,(X_n - Y_n)\right)$
and the answer follows from using (*) and the answer to part (i).

[***Alternative by induction***

Clearly true for n = 1, as this is identical to the formula given,

i.e. $\begin{cases} s\exp(\mu\delta t + \sigma\sqrt{\delta t}) & \text{if up} \\ s\exp(\mu\delta t - \sigma\sqrt{\delta t}) & \text{if down} \end{cases}$,

and can show that if true for n, then true for n + 1 using a variant of the solution above.]

(iii) X_n ~ Binomial with mean $n/2$ and variance $n/4$, using the definition of an up and down jump with equal probability.

Hence $\dfrac{2X_n - n}{\sqrt{n}}$ has mean 0 and variance 1.

By the Central Limit Theorem, this variable converges to N(0, 1).

So as $\delta t \to 0$ and $n \to \infty$, the distribution of S_t approaches log-Normal, as $\log(S_t)$ is Normal with mean $\log(S_0) + \mu t$ and variance $\sigma^2 t$.

(iv) Approximately $s\exp(\mu\delta t \pm \sigma\sqrt{\delta t}) = s(1+\mu\delta t \pm \sigma\sqrt{\delta t} + \tfrac{1}{2}\sigma^2\delta t)$ to order δt.

Given a continuously compounded risk-free rate r, the risk-free up probability is

$$q = \frac{s\exp(r\delta t) - s_{down}}{s_{up} - s_{down}} = \frac{[1+r\delta t] - [1+\mu\delta t - \sigma\sqrt{\delta t} - \tfrac{1}{2}\delta t\left(-2\mu\sigma\sqrt{\delta t} + \sigma^2\right)]}{2\sigma\sqrt{\delta t}}$$

$$= \tfrac{1}{2}\left(1 - \sqrt{\delta t}\left(\frac{\mu + \tfrac{1}{2}\sigma^2 - r}{\sigma}\right) + \mu\delta t\right)$$

to order δt, with the down probability $1 - q$.

7 *Syllabus: (e)*

(i) The market price of risk is a measure of the trade-off between risk and reward for different risky securities.

The market price of risk $\lambda = \dfrac{\mu - r}{\sigma}$ is the excess return over the risk-free rate per unit volatility, and holds true for all securities that have the same source of underlying uncertainty.

λ can depend on the underlying stochastic process w and time, but nothing else.

Hence if two securities have different expected returns, it is simple to calculate their relative volatilities, and vice versa.

(ii) *This uses the market price of risk.*

At $t = 10$, the volatility of G is $e^{-0.5} = 0.60653$ times the volatility of F.

However, the market price of risk must be the same for both F and G at $t = 10$, i.e. $\lambda = \dfrac{\mu - r}{\sigma} = \dfrac{\nu - r}{0.60653\sigma}$

so given that $\mu = 0.07$ and $r = 0.04$, $\nu = 0.60653 \times 0.03 + 0.04 = 0.0582$, i.e. 5.82%.

(iii) (a) **Differential equation**

We are given $df = \mu f dt + \sigma f dw$

Let the call option claim be $x(f, t)$.

Stochastic process from Ito's Lemma is:

$$dx = \left(\frac{\partial x}{\partial f}\mu f + \frac{\partial x}{\partial t} + \frac{1}{2}\frac{\partial^2 x}{\partial f^2}\sigma^2 f^2 \right) dt + \frac{\partial x}{\partial f}\sigma f dw$$

Using the replication argument, construct a portfolio π consisting of one unit of derivative and α units of stock

$$\pi = x + \alpha f$$

Over a small time interval,

$$\Delta\pi = \Delta x + \alpha \Delta f$$

hence, using the stochastic process above in its discrete version:

$$\Delta\pi = \Delta t \left(\frac{\partial x}{\partial f}\mu f + \alpha\mu f + \frac{\partial x}{\partial t} + \frac{1}{2}\frac{\partial^2 x}{\partial f^2}\sigma^2 f^2 \right) + \Delta w \left(\frac{\partial x}{\partial f}\sigma f + \alpha\sigma f \right)$$

Thus, if α is chosen to be $\alpha = -\frac{\partial x}{\partial f}$, then

$$\Delta\pi = \Delta t \left(\frac{\partial x}{\partial t} + \frac{1}{2}\frac{\partial^2 x}{\partial f^2}\sigma^2 f^2 \right)$$

Since the portfolio is riskless, it will earn the riskless rate of return, i.e. $\Delta\pi = r\pi\Delta t$.

Thus,

$$r\left[x - \frac{\partial x}{\partial f} f \right] = \frac{\partial x}{\partial t} + \frac{1}{2}\frac{\partial^2 x}{\partial f^2}\sigma^2 f^2$$

i.e. $rx = \frac{\partial x}{\partial t} + rf\frac{\partial x}{\partial f} + \frac{1}{2}\sigma^2 f^2 \frac{\partial^2 x}{\partial f^2}$

(b) **Boundary conditions**

$x \geq 0$ for $0 \leq t \leq T$
$x = \max(f^2 - K, 0)$ at $t = T$
$x \geq f^2 - K$ for $0 \leq t \leq T$ (i.e. the American feature)

(iv) The differences in drift between F and G are not relevant, since drift has dropped out of the differential equation.

Since the volatility element is deterministic (i.e. not stochastic), the same differential equation of value is valid ...
... but with the previous volatility replaced by the dampened one.

In algebraic terms, this is: $rx = \frac{\partial x}{\partial t} + rg\frac{\partial x}{\partial g} + \frac{1}{2}\sigma^2 g^2 e^{-2\beta t}\frac{\partial^2 x}{\partial g^2}$.

8 *Syllabus: (g)*

(i) (a) Let d_t be the discount factors at time t. These can also be considered as the present values of unit payments made at time t, t = 0, 1, 2 etc. Of course, $d_0 = 1$.

Forward rate at time t is $f_t = \frac{d_{t-1}}{d_t} - 1$ (converted to a percentage)

Using the data in the question, we get the following annual rates in %:

$f_1 = 4.167$, $f_2 = 4.348$, $f_3 = 4.664$

(b) An n-year swap fixed coupon rate S_n is also the par coupon rate for the yield curve at time 0, so it satisfies the equation:

$$S_n = \frac{1 - d_n}{\sum_{i=1}^{n} d_i} \quad \text{(converted to a percentage)}$$

[*The algebraic formula is not essential — alternatives approaches can also obtain this mark.*]

Hence for $n = 3$, annual swap fixed rate
= (1 – 0.879) / (0.96 + 0.92 + 0.879) = 0.0438565, i.e. 4.386% to 3dp.

(c) For the forward swap, if we transform the d_t to *forward* d_t, say d_t', we can use the same formula to calculate the *forward* swap rate. Let $d_i' = \dfrac{d_{T+i}}{d_T}$, then:

$$R_n = \frac{1 - d_n'}{\sum_{i=1}^{n} d_i'} \quad \text{(converted to a percentage)}$$

Hence for $n = 3$ and $T = 2$, forward annual swap rate
= (1 – 0.804 / 0.92) / (0.879 / 0.92 + 0.84 / 0.92 + 0.804 / 0.92)
= (0.92 – 0.804) / (0.879 + 0.84 + 0.804) = 0.0459770,
i.e. 4.598% to 3dp.

(ii) [*A continuous form of this equation can be developed, and is just as valid for this part of the question, but is not as useful for part (i), which uses the given discount factors. In that case, the candidate would use* e^{-rt} *instead of* d_t *in what follows.*]

A swaption is valued in two parts.

Firstly, the option to protect the interest rate of X% over T years is valued using the Black formula, assuming that the *forward* swap rate R_n is log-normally distributed with volatility σ. Let L be the nominal amount of the option.

The payoff for the interest rate option on its own at time T, when $R_n = R$, say, is:

$$L \,.\, \max(R - X, 0)$$

since the swaption gives the holder the right to receive R instead of X if the $R > X$, i.e. is a Call option on the interest rate.

So its present value, using the Black formula, is:

$$L \,.\, d_T \,.\, [R_n N(d_1) - X N(d_2)]$$

where as usual N is the cumulative Normal distribution, and d_1 and d_2 are given by:

$$d_1 = \frac{\ln(R_n / X) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{\ln(R_n/X) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}\ .$$

Secondly, this payoff is paid to the swaption holder on every payment of the swap, i.e. at times $T + 1$, $T + 2$, ... , $T + n$ (it is an annual-paying swap). A payment at each of these times has present value:

$$a(T,n) = d_{T+1} + d_{T+2} + \ \ldots\ + d_{T+n} = \sum_{i=1}^{n} d_{T+i}$$

so at time T it has value $\dfrac{a(T,n)}{d_T}$.

Putting the two parts together, the value of the swaption is:

$$L \cdot a(T, n) \cdot [R\, N(d_1) - X\, N(d_2)] \qquad (***)$$

with $a(T, n)$ defined as above is the "present value of a basis point" on the forward swap, i.e. discounted back to time 0.

(iii) Using the notation of the formulae above:

$T = 2$, $n = 3$, $\sigma = 15\%$, and $a(2,3) = 0.879 + 0.84 + 0.804 = 2.523$

The forward swap rate was calculated in part (i) (c), namely 4.598%.

Substituting into the d_1 and d_2 formulae gives:

$d_1 = [\ln(4.598 / 4.5) + ½.2.(0.15)^2] / 0.15\sqrt{2} = 0.20763$

$d_2 = d_1 - 0.15\sqrt{2} = -0.00450$

and using tables, $N(d_1) = 0.58224$ and $N(d_2) = 0.49820$.

So in formula (***) above, swaption value

= €50,000,000 (2.523) [0.04598 × 0.58224 – 0.045 × 0.49820]
= €549,055.

(iv) The ordinary n-year swap rate S_n is also the par coupon rate (a swap can be regarded as an agreement to exchange a fixed rate bond for a floating rate bond).

Hence, if there is a reasonable correlation between the bond market (the question doesn't specify which bond market) and the swap market, the volatility of swap rates should be the same as the volatility of par coupon bonds.

The bond market will also trade off a different curve. Swaps are traded on an interbank curve, whereas bonds are traded on either a government or corporate curve, depending on the bonds in question. Effects on option pricing may not be material, however.

Thus the volatility of swaptions and bond options should be very similar.

But differences will also occur due to balance of buyers and sellers in each market, funding (repo) variations on bonds (affecting forward prices), and liquidity considerations.

END OF EXAMINERS' REPORT