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ACTUARIAL NOTE ON THE CALCULATION OF ISOLATED (MAKEHAM) JOINT ANNUITY VALUES

by

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It is sometimes overlooked that any Makeham joint-life annuity value at any arbitrary interest rate may be calculated to any required degree of accuracy by using the first few terms of one of two alternative series expansions. Moreover, tables (Pagurova, 1961) are now available that eliminate the need for this series evaluation in a great number of cases.

This observation is of much wider application than may appear at first sight. It is well known that almost any 30-year range of ages in an arbitrary mortality table can be successfully regraduated using a Makeham formula. Since a life annuity deferred for 30 years has a very small value we may thus quickly arrive at a good approximation to a non-Makeham joint-life annuity by:

(i) choosing three ages suitably extending over the next 20-30 years (e.g. ages 45, 57, 69 for three joint-lives aged 40, 50 and 60);

(ii) calculating the three Makeham constants from the corresponding tabular values of p_x; and

(iii) utilising the above-mentioned series expansions or table.

This suggestion has previously been made and illustrated by Lefrancq (1906), Smid (1938) and Fletcher (1944).

Mathematical Basis

Suppose that, according to Makeham table j ($j=1, 2, \ldots m$), the force of mortality at age z_j is

$$\mu_{z_j}=A_j+B_jc^{z_j}\equiv A_j+B_je^{\gamma z_j} \quad \gamma=\log_e c.$$

where $B_j>0$ and $c>1$, but A_j may be positive, zero (Gompertz) or negative. An alternative notation is

$$-\log p_{z_j}=a_j+b_jc^{z_j}$$

where

$$a_j\equiv A_j \log e$$

$$b_j\equiv B_j(c-1)(\log e)^2/\log c$$

and the logarithms are to base 10.

Using the former notation the probability of (z_j) surviving t years is

$$ {}_tp_{z_j}=\exp\left\{-\int_0^t \mu_{z_j+\tau}d\tau\right\}=\exp\left\{-A_jt-\frac{B_j}{\gamma}(e^{\gamma t}-1)e^{\gamma z_j}\right\} $$

and the probability of all m lives surviving t years is

$$ {}_tp_{z_1z_2\,\ldots\,z_m}=\prod_{j=1}^{m}{}_tp_{z_j}=\exp\left\{-t\sum_{j=1}^{m}A_j-\frac{e^{\gamma t}-1}{\gamma}\sum_{j=1}^{m}B_je^{\gamma z_j}\right\}. $$

Hence the joint continuous annuity value on these m lives at force of interest δ is

$$ \bar{a}_{z_1z_2\,\ldots\,z_m}=\int_0^\infty e^{-\delta t}\,{}_tp_{z_1z_2\,\ldots\,z_m}dt $$

$$ =\int_0^\infty \exp\left\{-t\left(\delta+\sum_{j=1}^{m}A_j\right)-\frac{e^{\gamma t}-1}{\gamma}\sum_{j=1}^{m}B_je^{\gamma z_j}\right\}dt $$

$$ =\frac{e^x}{\gamma}\int_1^\infty e^{-xu}u^{-\nu}du $$

where $\nu=\gamma^{-1}\left(\delta+\sum_{j=1}^{m}A_j\right)+1\equiv\alpha+1$ say

$$ x=\gamma^{-1}\sum_{j=1}^{m}B_je^{\gamma z_j} $$

and $e^{\gamma t}=u$ so that $\gamma u dt=du$

$$ =\gamma^{-1}e^xE_\nu(x)\quad x>0 \tag{1} $$

where $E_\nu(x)$ is called the generalised exponential integral.

By writing $\int_1^\infty$ as $\int_0^\infty-\int_0^1$ and expanding the exponential in the subtractive term we obtain, after formal term by term integration (α not zero nor a positive integer),

$$ E_\nu(x)=E_{1+\alpha}(x)=x^\alpha\Gamma(-\alpha)+\frac{1}{\alpha}+\frac{x}{1-\alpha}-\frac{x^2}{(2-\alpha)2\,!}+\frac{x^3}{(3-\alpha)3\,!}-\ldots \tag{2} $$

$$ =-\frac{x^\alpha}{\alpha}\Gamma(1-\alpha)+\frac{1}{\alpha}M(-\alpha,\,1-\alpha,\,-x) $$

where M is the confluent hypergeometric function (Rushton, 1954)

$$ M(\beta,\gamma,x)=1+\frac{\beta}{\gamma}\frac{x}{1\,!}+\frac{\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2\,!}+\frac{\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3\,!}+\cdots $$

(γ not equal to zero or a negative integer) and satisfies Kummer's relation

$$ M(\beta,\gamma,x)=e^xM(\gamma-\beta,\gamma,-x). $$

This allows (2) to be written as an infinite series of positive terms.

On the other hand, if we integrate $E_\nu(x)$ by parts we obtain, for large x,

$$E_{1+\alpha}(x) \sim \frac{e^{-x}}{x}\left\{1-\frac{1+\alpha}{x}+\frac{(1+\alpha)(2+\alpha)}{x^2}-\frac{(1+\alpha)(2+\alpha)(3+\alpha)}{x^3}+\cdots\right\} \quad (3)$$

Although this series diverges for all values of x, nevertheless the error made by breaking off the calculation at the kth term cannot exceed the absolute value of the $(k+1)$th term. This latter comment also applies to the convergent series in (2).

COMPUTATIONAL PROBLEMS

Now in general $\alpha=\gamma^{-1}\left(\delta+\sum_{j=1}^{m}A_j\right)$ will be a positive fraction. When the A_j's are negative and δ is very small (or zero), α may be negative; exceptionally α might exceed unity. These two unusual cases may be treated expeditiously by means of the recurrence relation.

$$\nu\{e^xE_{\nu+1}(x)\}+x\{e^xE_\nu(x)\}=1 \quad (4)$$

We need thus consider only the function

$$e^xE_{1+\alpha}(x) \qquad 0<\alpha<1$$

Turning now to the likely range of values of x, we may assume $\gamma\approx{\cdot}1$ (values outside the range $\cdot07<\gamma<\cdot115$ are most unlikely). However, the range of B-values in modern mortality tables is rather wide, being as low as ·00002 (Barten and Schlaeger, 1955) and as high as ·1 (Wegmüller *et al.*, 1952). Even though Makeham tables are now usually applied only at ages over 60 (say) a very low value of B will nevertheless occasionally result in values of x as low as ·08. On the other hand, with larger values of B and ages in the 70's or 80's, even a single-life annuity may involve values of x of the order of 500 and in the case of joint life annuities correspondingly more. We must therefore be prepared for a wide variation in the values of x encountered in practice.

Now Pagurova's (*loc. cit.*) Table III provides values of $e^xE_\alpha(x)$ to seven significant figures for the ranges

$$x={\cdot}01(0{\cdot}01)\,7{\cdot}00({\cdot}05)\,12{\cdot}0(0{\cdot}1)\,20\,; \quad \alpha=0({\cdot}1)1$$

Referring to relation (4), it is noted that actuarial usage would generally require the computation of

$$e^xE_{1+\alpha}(x)=\frac{1-xe^xE_\alpha(x)}{\alpha} \quad 0<\alpha<1$$

If $\bar{a}_{z_1 z_2 \ldots z(m)}$ is to be calculated correct to three decimals from relation (1) the foregoing function requires calculation correct to four decimal places. This in turn implies a five-decimal computation of $xe^x E_\alpha(x)$.

Considering, first, variations in the x-direction it is found that, provided $x \geqslant 1{\cdot}65$, linear interpolation in Pagurova's table will result in five-decimal accuracy. Second differences give this degree of accuracy for $x \geqslant {\cdot}3$. Furthermore, when $x \leqslant 1$ use of relation (2) through the term in x^6 implies an error of less than a unit in the fourth place of decimals. When $x > 20$, which is beyond the limit of Pagurova's table, the error in $e^x E_{1+\alpha}(x)$ obtained by using relation (3) through the term in x^{-3} is less than a unit in the fourth decimal place.

Interpolation in the α-direction is more laborious. Second differences are needed for all values of $x \geqslant 1$ if five-decimal accuracy is to be obtained in $xe^x E_\alpha(x)$. On the other hand, provided $x \geqslant 6$, linear interpolation will provide four decimal accuracy in $xe^x E_\alpha(x)$ and thus at least two correct decimals in $\bar{a}$.

In summary, then, we recommend the use of relation (2) *for* $x < 1$, *relation* (3) *for* $x > 20$, *and Pagurova's tables for* $1 \leqslant x \leqslant 20$. *While linear interpolation will be sufficient in the* x*-direction when* $x \geqslant 1{\cdot}65$, *second differences are advisable for lower values of* x *and when interpolating for* α.

Numerical Illustration

In the Jenkins-Lew annuity basis (*T.S.A.* Vol. 1, 1949, pp. 369-466) the Makeham constants for males of 60 and over are

$$A = 4 \times 10^{-3} \quad (\log e)^{-1} b = 3{\cdot}1 \times 10^{-5} \quad \log c = {\cdot}043$$

so that

$$B = 2{\cdot}94906 \times 10^{-5} \quad \text{and} \quad \gamma = {\cdot}0990112$$

It is required to calculate $\bar{a}_{75:75}$ at 3% correct to three decimal places.

We have

$$\alpha = \frac{\delta + 2A}{\gamma} = {\cdot}37934 \quad \text{and} \quad x = \frac{2Be^{75\gamma}}{\gamma} = 1{\cdot}00008$$

and require $\gamma^{-1} e^x E_{1+\alpha}(x)$ where

$$e^x E_{1+\alpha}(x) = \alpha^{-1}\{1 - xe^x E_\alpha(x)\}.$$

Using Pagurova's Table III we find, using a three-point Lagrangean formula and Kelley (1948),

α	$e^x E_\alpha(x)$
·3	·84264
·4	·79851
·5	·75787

and thus $e^x E_{.37934}(x) = {\cdot}80734$.

Hence $\bar{a}_{75:75} = {\cdot}50772/{\cdot}0990112 = 5{\cdot}128$.

On the other hand relation (2) gives

$$e^x E_{1+\alpha}(x) = -\frac{1{\cdot}00003}{{\cdot}37934} \times \frac{{\cdot}89601}{{\cdot}62066} + 2{\cdot}6362 + 1{\cdot}6113 - {\cdot}3086$$
$$+ {\cdot}0636 - {\cdot}0115 + {\cdot}0018 - {\cdot}0002$$
$$\doteq {\cdot}1868 \text{ (}{\cdot}18678 \text{ to five decimal places)}$$

where Γ (1·62066) was obtained by linear interpolation in Pearson and Hartley's (1954) table of $\log \Gamma(y)$ and use was made of the relation

$$\Gamma(y) = (y-1)\Gamma(y-1).$$

Thus finally $\quad \bar{a}_{75:75} = e^{1{\cdot}0008} \times {\cdot}1868/{\cdot}0990112 = 5{\cdot}129$

Review of Literature

It is interesting to note that relation (1) was discovered by Makeham himself (1873) who published a five-decimal table of

$$\log e^{x+1} E_{1+\alpha}(x)$$

for $\alpha = 0{\cdot}0\ ({\cdot}1 \log_e 10)\ 0{\cdot}9$ and $\log\left(\frac{x}{\log_e 10}\right) = -4{\cdot}0\ (0{\cdot}1)\ 0{\cdot}0$

with tabulated differences to facilitate interpolation. Shortly thereafter, Emory McClintock (1874), derived the positive series expansion which, as mentioned above, is obtainable from (2), and illustrated its use numerically. The alternating series (2) first appeared in actuarial literature in Gram (1904), and an interesting asymptotic series of positive terms involving factorials of the form $\{(1+x)(2+x)(3+x)\ldots\}^{-1}$ was ascribed to Laudi by Blaschke (1903). However, expansion (3) has not been encountered in actuarial literature.

Besides Makeham's original table there are two published four-decimal tables of $e^x E_{1+\alpha}(x)$. These are:

(i) Belt's (1907) table with $\alpha = {\cdot}340\ ({\cdot}005)\ {\cdot}525$ and $\gamma x = {\cdot}0060\ ({\cdot}0005)\ {\cdot}0200$, which was reproduced by Lefrancq (1906); and

(ii) Thalmann's (1931) table with $a = \cdot 000\ (\cdot 025)\ \cdot 975$ and $x = \cdot 0020\ (\cdot 0002)\ \cdot 0050\ (\cdot 0005)\ \cdot 010\ (\cdot 001)\ \cdot 020\ (\cdot 003)\ \cdot 050\ (\cdot 005)\ \cdot 100\ (\cdot 015)\ \cdot 250\ (\cdot 025)\ \cdot 50\ (\cdot 05)\ 1{\cdot}0\ (\cdot 1)\ 2{\cdot}0\ (\cdot 2)\ 3{\cdot}0$, which was republished by Franckx (1939).

The authors of both these tables intended them to be used with linear interpolation in both directions.

Note that Pearson's (1934) incomplete-gamma table could be used to obtain $E_{1+\alpha}(x)$ from the relation $(0 < a < 1)$

$$E_{1+\alpha}(x) = \frac{1}{\alpha}\left\{e^{-x} - x^{\alpha}\Gamma(1-a) + x^{\alpha}\Gamma(1-a)\mathrm{I}\left(\frac{x}{1-a}, -a\right)\right\}$$

where $\mathrm{I}(u, p)$ is tabulated by Pearson to seven decimal places for $-p = \cdot 00\ (\cdot 05)\ 1{\cdot}00$ and for values of u ranging in steps of $\cdot 1$ from zero to the value at which $\mathrm{I}(u, p)$ becomes unity. However, this table would be laborious in its application to annuity calculations.

It follows immediately from relation (1) that any **Makeham** joint-life annuity can be written in terms of a " universal " table of single-life annuity values $\bar{a}_z$ tabulated for all ages z and suitably close-ranging values of the force of interest. In fact, if primed parameters refer to the " universal " table,

$$e^{x}E_{1+\alpha}(x) = \gamma\bar{a}_{z_1 z_2 \cdots z_m} = \gamma'\bar{a}_{z'} \tag{5}$$

$$a = \frac{\delta + \sum_{j=1}^{m} A_j}{\gamma} = \frac{\delta' + A'}{\gamma'} \tag{6}$$

$$x = \frac{\sum_{j=1}^{m} B_j e^{\gamma z_j}}{\gamma} = \frac{B' e^{\gamma' z'}}{\gamma'} \tag{7}$$

These relations have been rediscovered on a number of occasions during the first thirty years of this century [Blaschke (1903), Gram (1904), Achard (1912), Whitney (1912), Dubois (1927) and King (1931)].

Three such " universal " tables have been published :

(i) Blaschke (*loc. cit.*) whose Makeham table had the constants $A = \cdot 0064414692$, $B = \cdot 000084335$ and $\gamma = \cdot 092981$ and who tabulated $\bar{a}_z$ for $z = 25\ (1)\ 99$ and $100i = 0{\cdot}1\ (0{\cdot}1)\ 5{\cdot}5$;

(ii) Gram (*loc. cit.*)* who set $\gamma = \cdot 1$, $B = 10^{-4}$ and introduced a parameter $s = 10(A + \delta)$ with tabulated range $s = 0{\cdot}00\ (\cdot 02)\ 1{\cdot}00$ The tabular ages were $z = 21\ (1)\ 70$;

* Gram's table was republished in Lefrancq (1906) and Jorgensen (1913).

(iii) Whitney (*loc. cit.*) who utilised Hunter's American Experience table with A=·003296862, B=·000032063, and γ=·1054494. His tabulation was of a_z, instead of $\bar{a}_z$, and his range: z=0 (1) 120, i=·02 (·01) ·07.

It will be noticed from (6) that equal increments in the argument δ' imply equal increments in $\gamma'\alpha$, while the relation $e^{\delta}=1+i$ shows that equal increments in i result in decreasing increments in δ'. Furthermore, relation (7) indicates that a unit increase in z' implies a geometrical series of x-values, the common ratio being $e^{\gamma'}=c'$. We may thus summarise the ranges of the foregoing "universal" tables in the following exhibit. Neither these tables nor those of Belt and Thalmann mentioned above are as appropriate for wide actuarial application as the methods and table proposed in this note.

Table	α-range	Range of increment in α	x-range	Incremental ratio of x
Blaschke	·08003 to ·64510	·01074 to ·01020	·00927 to 9·023	1·09744
Gram	0 to 1·0	·02	·00817 to 1·097	1·10517
Whitney	·21906 to ·67289	·09252 to ·08904	·00030 to 95·168	1·11121

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