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Taking the lid of Least Squares Monte Carlo

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Introduction & Background

Introduction

- Proxy models are simplified functions that
- Represent liabilities and/or assets
- Can very quickly be revalued under revised risk driver values
- Are calibrated to results from detailed actuarial models

Proxy models are used for rapid updates of the balance sheet for:

- Regular solvency monitoring
- Capital calculation with Monte Carlo simulations

Various forms of proxy models are in use with the most popular approach in the UK being the use of polynomials.

Example with two risks, x =(x_1, x_2), up to quadratic terms:

$$f(x_1, x_2) = c + \underbrace{a_1 x_1 + a_2 x_2}_{\text{Linear relationship}} + \underbrace{a_{12} x_1 x_2}_{\text{Interaction between risks}} + \underbrace{a_{11} x_1^2 + a_{22} x_2^2}_{\text{Non-linear relationship}}$$

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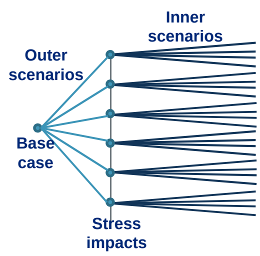
Introduction



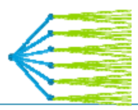
The proxy modelling calibration inputs consist of two items:

– Stressed scenarios ('outer scenarios')
– Balance sheet in stressed scenarios

Inner scenarios:

For deterministically valued business, there is one inner scenario.

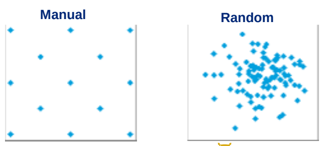
For stochastically valued business, there could be thousands of inner scenarios.



Outer scenarios:

These can be prepared in different ways

- Manually selected, often according to a rule
- Randomly generated
- A combination of manual and random



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Proxy modelling errors

Measuring the error

- There are different measures for errors. The most

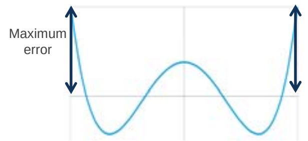
RMSE: Root mean square error

Commonly used.

The RMSE is minimised using **regression** techniques (i.e. minimising **least squares**).

No firm relation between RMSE and SCR: RMSE is an average error. The error in the SCR estimate may be larger because not all points have an equal impact on the SCR.

Under certain conditions, the error function using least squares is equivalent to a **Legendre** polynomial:

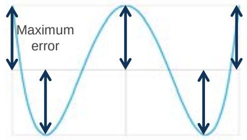


Maximum absolute error

Minimising the maximum absolute error ('**minimax**') is difficult to compute.

Firm relationship between maximum error and SCR: The error on the SCR cannot exceed the maximum error of the proxy model. Hence, minimax is theoretically a better objective to optimise than the RMSE.

Under certain conditions, the error function using minimax is a equivalent to a **Chebyshev** polynomial:



Regression is often preferred to a minimax fit as it is easier to compute but it is difficult to translate a RMSE into error bounds for the SCR.

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The proxy modelling process

Risk factors and calibration simulations

- Which risk factors / balance sheet states should be used?
- How to draw from a risk neutral return distribution starting at t=1? How many RN sims per point?
- Type of risk factors and calibration distribution of calibration points
- Validation of calibration simulations

Specification of Regression

- Regression method (e.g. least squares with basis, non-parametric methods, ...)
- Transformation of risk factors for regression (z.B. ln or N(d)) as enlargement of basis function space
- Criterion for selection of regression terms from a number of basis functions
- Control variates

Testing procedures and quality criteria

- Statistical assessment, analysis of residuals
- Plausibility of results
- Choice of Out-of-sample test points
- Quantitative quality criteria

Choice of software

- Licensed software vs. in-house development
- Integration in the overall SCR calculation / more general process
- Definition of interfaces
- Automation of the process

Proxy modelling errors – A taxonomy

- There are three types of errors with proxy models. These are summarised below including considerations how these can be minimised and reduced.
- When using specific methods, it is important to be aware of the nature of the errors in order to improve the fit.

	Description	Considerations
Sampling error (inner scenarios)	Error from randomness in fitting to stress results based on Monte Carlo scenarios (i.e. options and guarantees)	• Observed for all stochastically valued business • Can be reduced by running more simulations • LSMC can correct for this error • Can be measured using regression techniques
Fitting error	Error from using a fitting technique that does not produce the optimal fit (i.e. does not achieve minimax) for the choice of basis functions	• Present in all proxy models that are not fitted to minimise the maximum error • Can be reduced by using more appropriate stress tests for calibration and/or minimax fit • Cannot be measured using regression techniques
Spanning error	Error in the best fit to a given choice of basis functions (i.e. fit cannot be improved by altering the fitting technique or outer scenarios)	• Present in all proxy models • Can be reduced by increasing the polynomial order or use of terms that capture the liability behaviour more closely (e.g. replicating portfolios and implied parameter Black Scholes) • Cannot be measured using regression techniques

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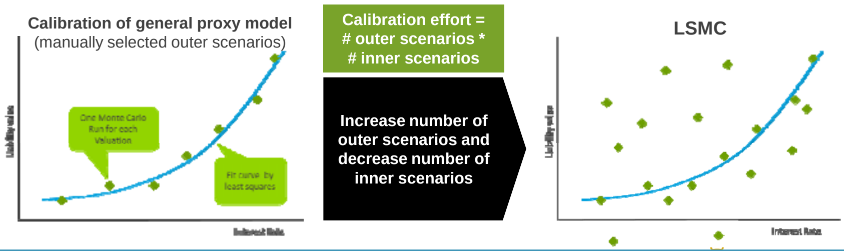
Proxy modelling – LSMC

LSMC introduction

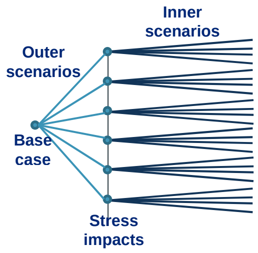
- Run time of detailed actuarial models is a challenge, particularly for stochastically modelled business.
- LSMC makes better use of outer scenarios which results in lower run time of detailed actuarial models for calibration than the traditional curve fitting approach to stress tests.
- However, LSMC requires more onerous validation as none of the calibration stress test results help in assessing the goodness-of-fit (see illustration below).
- For given functions, the fits can be equally good for stress tests with least squares and LSMC.



LSMC is based on polynomials and suffers the same challenges as the traditional polynomial curve fit to stress tests.

LSMC in more detail

LSMC
- uses polynomials
- uses randomly generated outer scenarios
- is applied to stochastically valued business (i.e. for cost of guarantee)



Inner scenarios must be generated using a market consistent ESG reflecting the market conditions implied by the outer scenarios.

Outer scenarios must be expressed in terms of the risk drivers X1, X2, .. used in the proxy model.

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LSMC in more detail - Process

Define framework

Determine approach and parameters e.g.
- ✓ risk drivers and distributions
- ✓ number of inner scenarios
- ✓ validation scenarios

Produce outer scenarios

- ✓ Sample from multivariate distribution
- ✓ Combined stresses for market and non-market risks

Run ESG

- ✓ Calibrate and run ESG for each market risk outer scenario
- ✓ Calibrate and run ESG for each validation scenario

Run asset and liability models

- ✓ Set up and run full asset and liability models with ESG files for each outer scenario
- ✓ Run full asset and liability model for each validation scenario

Fit formulae

- ✓ Use simple least squares regression to fit polynomial formula using results of each outer scenario

Test fit

- ✓ Validate results against validation runs

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LSMC in more detail

The strength of polynomials is that they are a generic family of curves:

- Can be used for lots of classes of business.
- Can be used when not all the factors impacting the business are transparent.
- Increasing the order of polynomials will ultimately result in convergence to true function (see below) BUT a very high order (with complexity and run time implications) may be required for the fit to be good enough.

However, there are limitations in practice:

- Option behaviour is poorly described by polynomials.
- This can be seen on the second derivative of the option value:
- The second derivative of an option price is a bell shaped curve.
- The second derivative of a polynomial is still a polynomial.

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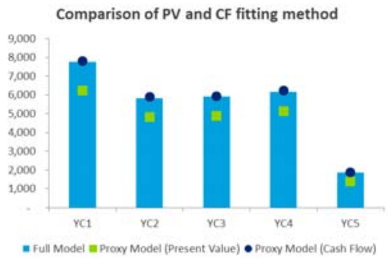
Illustration

Refining proxy models - Illustration

- A simple step for annuities is to model cashflows directly rather than fitting a polynomial to the BEL – this can improve results considerably.
- The following example looks at fitted and actual results under five different yield curve stresses (YC1 to YC5).



Challenge: Replication of historical yield curves with limited number of yield risk factors.

Present value vs. cash flows

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Illustration

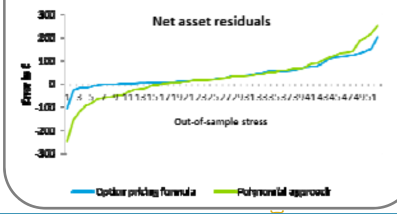
Refining proxy models - Illustration

Fitting to the implied parameters of a Black-Scholes formula can be used as an approach to using polynomial basis functions.

- This only applies to options and guarantees (e.g. With-Profits business).
- The approach has the advantage of matching the behaviour of the liability value whilst being able to use low order polynomials.
- It makes validation easier as it is possible to check inputs as well as the formula for reasonableness.
- To use the full benefit of this approach, it should be applied at a granular level (e.g. grouped model points).

GAO example

- The chart below shows a comparison of polynomial and implied parameter Black-Scholes fitting.
- 52 scenarios at the 1-in-200 net asset stress were tested for both approaches.
- The implied parameter Black-Scholes' errors are about 50% smaller on average.



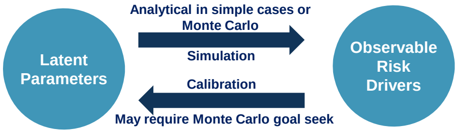
Option pricing formulae (such as Black-Scholes) can allow for option-type behaviour and reduce spanning error.

LSMC – The calibration problem

The ESG needs to be recalibrated for each outer scenario. There are two options:

– Use of latent parameters: The outer scenarios consist of simulated ESG parameters. No recalibration required as all required parameter can be read off the outer scenarios. However, the outer scenarios must be calibrated to ESG parameters: What is a 1-in-200 move of the mean reversion rate?

– Use of observable market variables: The outer scenarios consist of yield curves, volatilities, etc., and hence a recalibration is required. This approach is more intuitive for plausibility checks and better communication.



Outer scenarios:

– A popular choice for generating the outer scenarios is the uniform distribution. However, our research showed that outer scenarios need to be generated in a specific way to give a better fit:

- The wider the range of the outer scenarios, the better the fitting is.
- Having more outer scenarios at the edges of the interested range gives a better fit, i.e. the use of the uniform distribution is not optimal.
- The quantity to fit drives the choice of the range of the outer scenarios.

– **The selection of outer scenario distributions for LSMC is still often a matter for subjectivity.**

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LSMC in more detail

– LSMC uses linear regression to fit a formula. Linear regression is a well researched area and a number of statistical techniques are available to test a regression fit.

– However, statistical techniques applied to proxy model fits should be handled with care.

– Statistical goodness-of-fit measures can be misleading, understating the fitting error.

– Regression techniques assume that residuals are independent and identically distributed. But they are not:

- Spanning error, which occurs because the underlying function to fit is not a polynomial. The first picture shows random errors, the second spanning error.
- Volatility stress tests lead by construction to residuals that are more volatile than stress results without volatility stress.

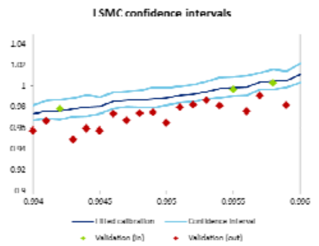
It is tempting to look at confidence intervals when assessing the error of a LSMC fit.

Example: Put option exposed to interest rate level, equity level and equity volatility risk

The following chart shows the 20 scenarios around the fitted 99.5th percentile. For each scenario the following quantities are shown:

✓ Fitted result (dark blue line)
✓ True result (green and red diamonds)
✓ Confidence intervals (light blue lines)

***Observation**: The use of confidence intervals failed in this example as all bar three validation scenarios lie outside the confidence intervals.*



Statistical methods to assess the goodness-of-fit fail as the assumptions to apply these tests are not met. Users potentially get false comfort from good fitting measures.

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Using Linear Regression to Fit Proxy Functions

$$Y=\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{pmatrix}; B=\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ B_{21} & B_{22} & & \\ B_{31} & B_{32} & & \\ \vdots & & \ddots & \\ B_{n1} & & & B_{np} \end{pmatrix}; b=\begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix}$$

$$Y = B\lambda + N(0, \sigma^2 I_n)$$

$$\hat{\lambda} = (B^T B)^{-1} B^T Y$$

$$\mathbf{Var}(\hat{\lambda}) = \sigma^2 (B^T B)^{-1}$$

$$\hat{\sigma}^2 = \frac{1}{n-p}(Y - B\hat{\lambda})^T (Y - B\hat{\lambda})$$

$$(n-p)\frac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-p}$$

$$\mathbf{Var}(b\hat{\lambda}) = \sigma^2 b(B^T B)^{-1} b^T$$

$$\frac{b\hat{\lambda} - b\lambda}{\hat{\sigma}\sqrt{b(B^T B)^{-1} b^T}} \sim \text{student } T_{n-p}$$

n = # fitting scenarios
p = # basis functions
Y = BOF in each fitting scenario
B = basis functions in each fitting scenario
Want to estimate BOF(b)

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Risk factor Selection

A fundamental problem given the large amount of risk drivers available to select as a part of the proxy modelling process (and not unique to LSMC) is selecting the underlying polynomial for use in the intended algorithm. There are many ways in which the underlying terms are selected or de-selected, the most common are listed below:

–

– Use of Akaike Information Criteria (AIC): The AIC is the estimate of a constant plus the relative distance between unknown true likelihood function of the data and the fitted likelihood function of the model.

AIC = 2k - 2ln(L)

– Use of Bayesian Information Criteria (BIC): The BIC is an estimate of a function of the posterior probability of a model being true under a certain Bayesian setup. BIC penalises model complexity more heavily.

BIC = kln(n)-2ln(L)

Where k is the number of free parameters, L is the Maximum Likelihood Function of the model. A lower value of AIC and BIC is preferred.

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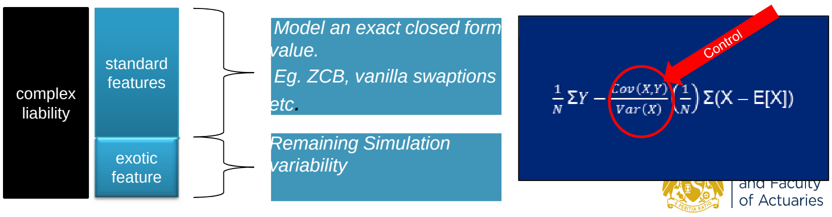
Optimisation

We can optimise our LSMC estimate in a number of ways, with the most common being listed below:

- Add increasing number of risk factors, cross terms, and increase the power and order of the polynomial. Whilst this does provide increasing effectiveness at diminishing rates of information, there may be certain scenarios for which the LSMC does not provide a very good fit nonetheless.
- Attempt to minimise the maximum error within a tolerable level versus the least squares, although there may be increased error around the biting/critical scenario as a result.
- Use Control Variates to segregate known simulation errors from the total monte carlo simulation error. Using Control Variates reduces the overall variance of the polynomial estimate proxy function and hence leads to more stable results with reduced confidence intervals.

- An example of using Control variates and how to calculate them:



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LSMC in more detail – ID the formula shape

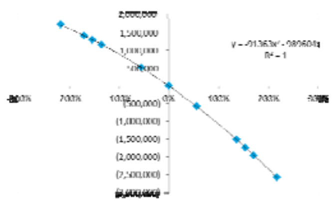
- There are two main approaches for identifying the polynomial terms:
- **Manually** identified by looking at selected stress impacts.
- **Automatically** identified using statistical techniques.

Manual

Specific outer scenarios are defined in order to identify the shape by risk and risk pairs:

- Polynomial shapes are identified in isolation for each risk driver by looking at single risk stresses.
- Transformation of risk drivers are applied if they improve the fit to lower order polynomials.
- Then cross terms are fitted.

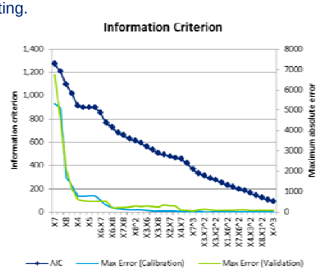
Cannot be used with randomly generated outer scenarios.



Automatic

The most common approaches are step-wise optimisations:

- At each step in these algorithms, possible additional formula terms are tested and the term that most improves the fit is selected.
- The algorithm can be forwards, backwards or both ways selecting.



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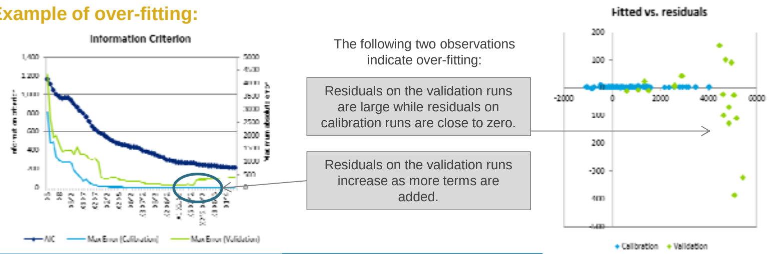
LSMC in more detail

- Step-wise regression techniques can be used to automate the selection of formula terms but these techniques should be applied with care:
- Assumptions underlying the algorithms do not apply, so the fits may not be optimal.
- Over-fitting can be a problem.
- Confidence intervals generated by these tools can be invalid, particularly for LSMC fits, due to assumptions not being met.
- Significant out-of-sample testing must still be done to judge the success of the fit.
- The techniques do not consider transforming risk drivers to improve the fit.
- **Example of over-fitting:**



We believe that optimisation techniques can be useful in automating and speeding up fitting processes, but they should be used with care and combined with manual investigations.

Comparison of methods

- Curve fitting to stress tests and LSMC are both using polynomial basis functions and are therefore subject to spanning error.
- The two approaches can get the same answer when the outer scenarios are bunched in a small number of fitting scenarios.
- LSMC can use higher order polynomials without significantly increasing the run time for creating the fitting inputs.
- With curve fitting to stress tests, the fitted curve can be compared against balance sheet impacts of outer scenarios to provide additional visual comparison of the fit while balance sheet impacts from LSMC processes have too big a sampling error to be used.
- – Curve fits to stress tests can be derived using existing models without coding changes while the use of LSMC requires changes to liability models to use the scenarios generated for the fitting.
- The generation of fitting scenarios for LSMC (outer scenarios followed by inner scenarios) is complex and requires a tool.

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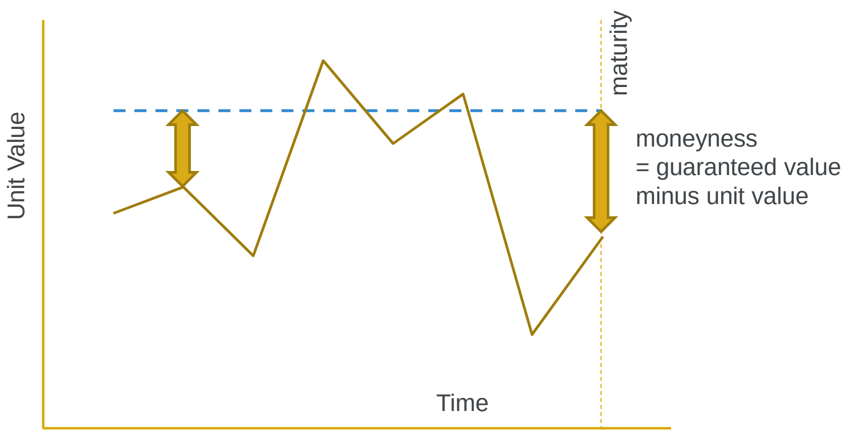
LSMC – An example

Example Maturity Guarantee



Bachelier's (1900) Option Pricing Model

- Random Walk Model
  - Assume future moneyness = current moneyness + $N(0,s^2)$
- Option price is the (conditional)expected future moneyness
  - Function of current moneyness and of the volatility s
- Contemporary option pricing models are more complicated than this

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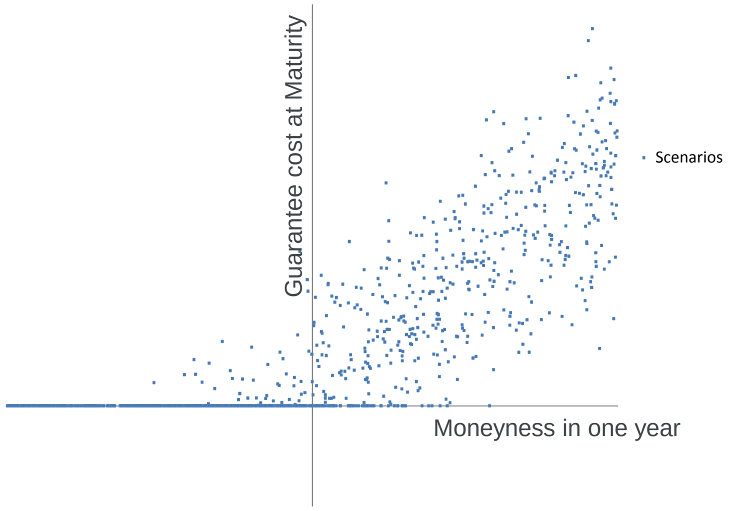
Can we Avoid Nested Scenarios?



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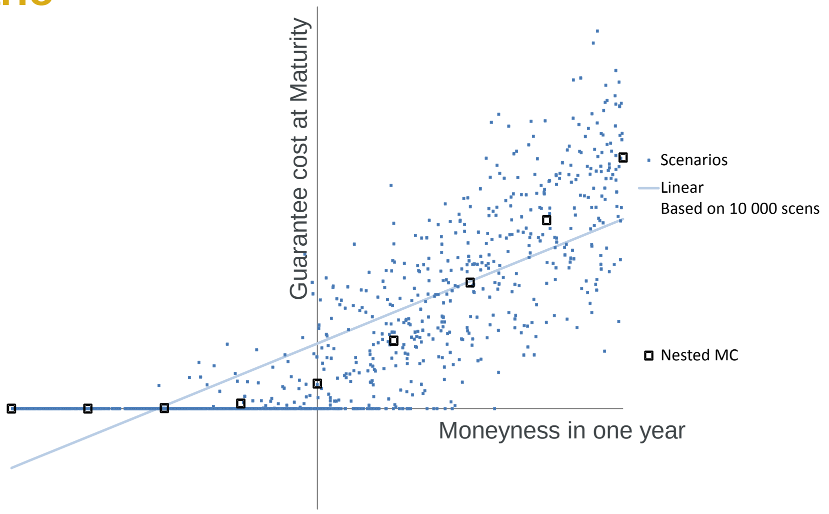
Linear fit compared to Nested Monte Carlo



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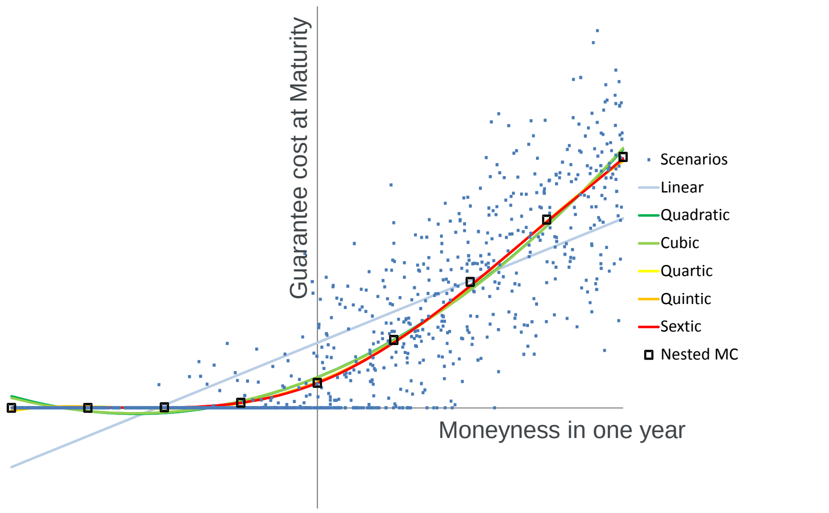
Higher Order Polynomials



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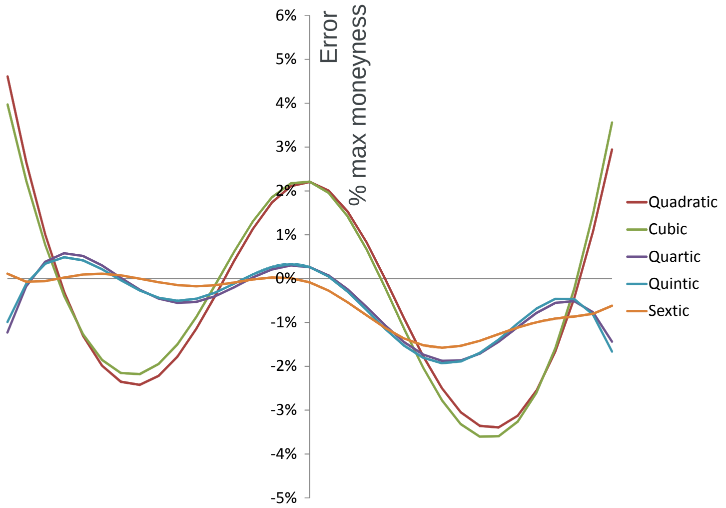
Errors do not look like Random Noise!



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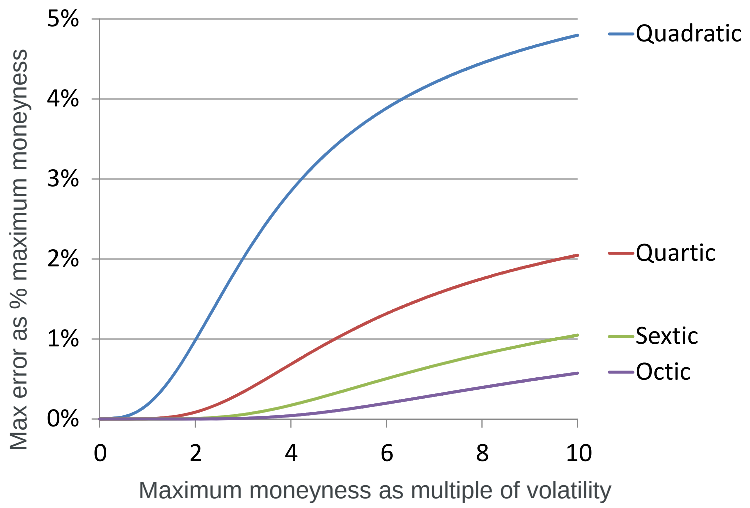
1D Spanning Error: Accuracy Potential



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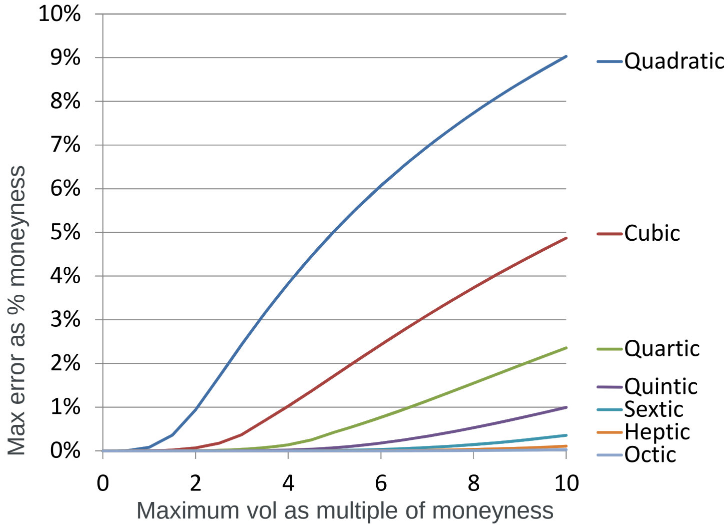
What About Varying Volatility?



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LSMC – Conclusion

The final word isn't written yet…

Least squares with polynomial basis functions (common today):

Advantages

- ✓ Polynomials are easy to understand and simple to implement
- ✓ Extremely fast calculation (solution of a linear system of equations)
- ✓ Can get the same answer as curve fitting to stress tests when the outer scenarios are bunched in a small number of fitting scenarios
- ✓ Can user higher order polynomials due to the richness of the outer scenarios
- ✓ Useful toolkit to have but not a panacea
- ✓ Error estimates combine inner scenario sampling error and estimation of basis function loadings

Disadvantages

- × Requires a tool that automates the recalibration of the ESG to each outer scenario
- × Requires changes to liability models to use the simulations generated for the fitting
- × More extensive validation
- × Model error! (True liability function may not be well-approximated by polynomials)
- × Liability estimate at every calibration point influences estimate at every other, no matter how distant
- × Difficult to apply inner scenario validation tools (arbitrage-free and market-consistent tests, calibration tests) for each outer scenario

Methodology improvements in this area are likely, so it is desirable to have the option to upgrade the basis functions without changing the rest of the process.

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Possible alternatives in the future

Fully non-parametric regression methods

Advantages

- ✓ Very successful for many applications in other fields
- ✓ Appears to work well when used for proxy models for a few risk factors
- ✓ Possible to extract BOF-properties such as smoothness of the BOF function
- ✓ Ability to validate the noise in the model

Disadvantages

- × High dimensions and scarcity of calibration points an even greater challenge
- × Mathematics becomes considerably more complex
- × No clear method to identify the optimal non-parametric method

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Questions Comments

Expressions of individual views by members of the Institute and Faculty of Actuaries and its staff are encouraged.

The views expressed in this presentation are those of the presenter.

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