Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

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Advanced Certificate in Derivatives: Further Mathematics, Principles and Practice

EXAMINERS' REPORT

1

(i)

1. BigBank could enter into a credit default swap with C Risk.
2. BigBank would pay C Risk a periodic[1] fee …
3. … typically expressed as a percentage of the notional amount of the loan (from BigBank to BigLoan)
4. in return for a payment by C Risk in the event that BigLoan defaults on …
5. … one or more of its (usually publicly quoted) debt instruments.
6. This would allow BigBank to lend to BigLoan because BigBank has transferred the credit risk of the loan to CRisk.
7. BigLoan need not be aware of the existence of the credit default swap and therefore there is no need to have a conversation with BigLoan about the fact that BigBank has reached the limit of its credit exposure to it.

(ii)

The periodic fee, payable by BigBank should be roughly equal the debt instrument's credit spread over government bonds of similar term and coupon.

Arbitrage arguments suggest that this approach ought to lead to a reasonable price.

(iii)

1. The approach assumes that CRisk's obligations under the contract have no credit risk.
2. The approach does not take account of the possibility that CRisk may default at the same time as BigLoan.
3. The approach assumes that, at the time of default, the no-default value of the bond is par.
4. The credit spread over government may represent a premium/discount for liquidity and/or the nature of the structure.
5. There may not be any quoted debt.

[1] A once-off fee is also acceptable as an answer.

(iv)

1. In the event of a credit default, credit default swaps are settled by either delivering the bonds for their par value or …
2. …by paying the difference between the par value and the market value of the bond.
3. Where settlement is by delivery of the bonds and there are a large number of credit default swaps based on the bond, this may drive up the price of the bonds in the event of a default.

2 (i) If the gamma of an option position is large and negative and the delta is zero, then the writer of the option will suffer a significant loss if there is a large movement — either up or down — in the price of the underlying asset.

(ii) $C = Se^{-q(T-t)}\,N(d_1) - Xe^{-r(T-t)}\,N(d_2)$

where $$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

and $$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\frac{\partial C}{\partial \sigma} = Se^{-q(T-t)}\,N'(d_1)\,\frac{\partial d_1}{\partial \sigma} - Xe^{-r(T-t)}\,N'(d_2)\,\frac{\partial d_2}{\partial \sigma}$$

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{T-t}$$

$$\frac{d_1^2}{2} - \frac{d_2^2}{2} = \ln\left(\frac{S}{X}\right) + (r-q)\,(T-t)$$

$$e^{\frac{d_1^2}{2}-\frac{d_2^2}{2}} = \frac{S}{X}e^{(r-q)(T-t)}$$

$$N'(X) = \frac{1}{\sqrt{2\pi}}e^{-\frac{X^2}{2}}$$

$\therefore$ $$XN'(d_2)\,e^{-r(T-t)} = SN'(d_1)\,e^{-q(T-t)}$$

So $$\frac{\partial C}{\partial \sigma} = Se^{-q(T-t)}\,N'(d_1)\left\{\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right\}$$

$$= Se^{-q(T-t)}\,N'(d_1)\,\sqrt{T-t}$$

$$\frac{\partial C}{\partial S} = e^{-q(T-t)} N(d_1) + Se^{-q(T-t)} N'(d_1)\frac{\partial d_1}{\partial S}$$

$$- Xe^{-r(T-t)} N'(d_2)\frac{\partial d_2}{\partial S}$$

Now $\dfrac{\partial d_2}{\partial S} = \dfrac{\partial d_1}{\partial S}$ and $XN'(d_2)\, e^{-r(T-t)} = SN'(d_1)\, e^{-q(T-t)}$

$$\therefore \quad \frac{\partial C}{\partial S} = e^{-q(T-t)} N(d_1)$$

$$\frac{\partial^2 C}{\partial S^2} = e^{-q(T-t)} N'(d_1)\frac{\partial d_1}{\partial S}$$

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{e^{-q(T-t)} N'(d_1)}{S\sigma\sqrt{T-t}}$$

(iii) Using the usual notation, let S, r, σ, q and t be the parameters common to both with the OTC option and the exchange traded option (ETO).

Let

(1) X and X^* be the strike prices of the OTC option and the ETO respectively.

(2) T and T^* the maturity date of the OTC option and ETO option respectively.

(3) d_1 and d_2 are calculated using the un-starred parameters.

(4) d_1^* and d_2^* are calculated using the starred parameters.

Let's say that w exchange traded options are brought for each OTC option written.

Consider a portfolio consisting of one OTC option and w ETOs.

The gamma of the portfolio is given by equation (1).

$$\text{Equation 1: } \frac{1}{S\sigma}\left\{\frac{N'(d_1)\, e^{-q(T-t)}}{\sqrt{T-t}} + \frac{wN'(d_1^*)\, e^{-q(T^*-t)}}{\sqrt{T^*-t}}\right\}$$

this must equal zero. So $w = \sqrt{\dfrac{T^*-t}{T-t}}\,\dfrac{N'(d_1)}{N'(d_1^*)}\,e^{-q(T-T^*)}$

the vega of the portfolio is given by equation 2.

$$\text{Equation 2: } S\left\{\sqrt{T-t}\,N'(d_1)\,e^{-q(T-t)} + w\sqrt{T^*-t}\,N'(d_1^*)\,e^{-q(T^*-t)}\right\}$$

setting the equation 2 to zero

$$w = -\sqrt{\frac{T-t}{T^*-t}}\,\frac{N'(d_1)}{N'(d_1^*)}\,e^{-q(T-T^*)}$$

The 2 values of w must be equal if the portfolio is to be both gamma and vega neutral.

Thus $\sqrt{\dfrac{T-t}{T^*-t}} = \sqrt{\dfrac{T^*-t}{T-t}}$

$\Rightarrow T - t = T^* - t$

$\Rightarrow T = T^*$

So the maturity date of the OTC option and ETO must be the same.

3 *[The report should have at least covered these points. If a candidate made a valid point not shown below, it could also have attracted marks.]*

Internal management

- Senior management must decide what sort of risks they want to take on, e.g. how much do they want to be involved in derivatives?
- There has to be congruence between the intended risk appetite and the actual risks taken on. Sometimes this means encouraging more risk-taking, other times less.
- Ensure effective delegation of risk control at both local and global level.
- Beware hidden risks, for example those that cannot yet be reported on the firm's risk systems. These are often the biggest risks, as they usually involve illiquid or unquantifiable positions.
- Everything should be reported even if it can't be risk-measured (and if not, management may wish to ask why the position was taken in the first place).

- Check the accounting methodology, particularly if business looks very profitable.

- Check the documentation, that it is industry standard and robust.

- Check the models. Don't always believe the rocket scientists — must have reasonability check e.g. beware if they claim to be making money if other firms' models are "wrong".

Regulatory framework (banking)

Regulation for derivatives in banks covered by SFA (Securities and Futures Authority), now part of FSA (Financial Services Authority).

Their principles include:

- Settlement and documentation departments to be separated from trading, so a trader cannot settle his own trades (else e.g. he could "bottom draw" bad trades).

- An *independent* risk management department to be set up within the bank, reporting to senior management and *not* the head of trading ...

- ... to measure and report both market and credit risk ...

- ... and verify the financial models used in the trading areas, that they are robustly applied and, if not industry standard, then soundly based theoretically.

- Traders must be set trading limits expressed in risk-related terms, and any exceptions must be reported daily.

- Daily risk reporting to senior managers, who must be qualified to understand the risks involved and aware of what action to take to mitigate them.

- Daily capital calculation, according to SFA rules, usually involving some sort of Value-at-Risk methodology and stress testing for large market moves.

- Adequate systems resources to be applied to valuation, P&L and risk reporting.

- Regular discussions with SFA and occasional visits.

Regulatory framework (insurance)

Regulation for derivatives in insurance subsidiaries is covered by DTI, based on the Insurance Companies Act 1982.

Their principles include:

- Prudential management controls must be in place, similar to the SFA's.
- Appointed Actuaries must ensure that suitable reserves or provisions exist ...
- ... and report on these at appropriate intervals — "continuous review" ...
- ... including a daily schedule of derivative transactions undertaken, exposures and sensitivities to large market movements.

Global vs. local considerations

Whenever businesses are managed in other locations, there is scope for unacceptable risk and unexpected losses, especially if a crisis occurs in one timezone whilst other timezones are closed.

Need to ensure adequate local supervision by trusted employees, who are empowered to intervene swiftly if the local situation demands, but who will also provide more general feedback to Head Office.

Comments on the specific businesses

(a) *Swaps and MM in New York and Tokyo:*

- The business described is a fairly standard Treasury operation.
- Documentation, settlement and accounting are crucial. If the business is "fairly large scale", these will need to be fully automated and robust.
- The key problem here is assessing **leverage** ...
- ... whether the positions are outright or arbitraged (hedged).
- A high-leverage operation will be risky ...
- ... but a low-leverage operation might not earn enough
- So look at return on capital employed ...
- ... for which Value-at-Risk would be useful here.
- Management will need to assess which parts of the business will require mark-to-market valuation and which will required accruals valuation. Usually, Swaps are market-to-market and MM instruments are accrued.

- The setting and monitoring of risk limits for proprietary traders is crucial.

- Risks should be approximately linear, as there are no options traded, so provided limits are adhered to there should be no surprises.

(b) *Equity options in the insurance business:*

- Extent of hedging of liabilities important — should in theory be exact. There should be little need to "churn" any of the positions.

- Biggest risk is credit risk with respect to investment banks — don't want to be too concentrated.

- Careful adherence to schedule of positions with regards exercise of options.

- Policy on managing lapses, i.e. closing out of options.

4 (i) Consider two portfolios A and B.

Portfolio A: One share

Portfolio B: An amount of cash equal to $Xe^{-r(T-t)}$ and an American call option on the share.

If the American call option is exercised early at time $\tau < T$ the value of portfolio B will be:

$S - X + Xe^{-r(T-\tau)}$
$S - X(1 - e^{-r(T-\tau)})$……………………**Z**

If $\tau < T$ and $r > 0$ then **Z** is always less than S.

So if the American call option is exercised early, portfolio B is always less than portfolio A.

If the call option is not exercised early, at time $t = T$, portfolio B is worth:

maximum$\{S_T$ and $X\}$

So portfolio B is always worth at least as much as portfolio A and sometimes worth more than portfolio A if the call option is not exercised early exercised at maturity.

So if the American call option is exercised early, portfolio B is always worth less than portfolio A.

Whereas if the American call option is held to maturity, portfolio B is worth at least as much as portfolio A and sometimes more than portfolio A.

(ii) Let the strike price, at time t, be X. Let the maturity date of the option be T.

The terminal strike price of the option is equal to $Xe^{h(T-t)}$.

Consider two portfolios A and B.

Portfolio A: One share

Portfolio B: An amount of cash equal to $Xe^{(h-r)(T-t)}$ and the American call option on the share where the strike price grows at a continuously compounded rate h. This amount of cash will grow to the terminal strike price

If the American call option is exercised early at time τ ($t < \tau < T$) the value of portfolio B will be:

$$S - Xe^{h(\tau-t)} + Xe^{h(T-t)}\, e^{-r(T-t)}\, e^{r(\tau-t)}$$

$$S - Xe^{h(\tau-t)}\,(1 - e^{-(r-h)(T-\tau)}) \text{ ....}\mathbf{Z}$$

If $\tau < T$, $r > 0$, $h > 0$ and $h < r$ then the expression **Z** is always less than S.

So if the American call option is exercised early, portfolio B is always worth less than portfolio A.

At time $t = T$, portfolio B is worth maximum$\{S_T$ and $Xe^{h(T-t)}\}$

So portfolio B is sometimes worth more than portfolio A if the call option is held to maturity.
So if the American call option is exercised early, portfolio B is always worth less than portfolio A.

Whereas if the American call option is held to maturity, portfolio B is worth at least as much as portfolio A and sometimes more than portfolio A.

5 (i)

[This solution gives a more detailed explanation than was required to have obtained full marks, as the question only asked the candidate to "write down" the formula with appropriate definitions. But it shows how the formula is compiled.]

Let d_t be the discount factors at time t. These can also be considered as the present values of unit payments made at time t, $t = 0, 1, 2$ etc. Of course, $d_0 = 1$.

A swaption is valued in two parts.

Firstly, the option to protect the interest rate of X% over T years is valued using the Black formula, assuming that the *forward* swap rate R_n is log-normally distributed with volatility σ. Let L be the nominal amount of the option.

The payoff for the interest rate option on its own at time T, when $R_n = R$, say, is:

$$L \,.\, \max(R - X, 0)$$

since the swaption gives the holder the right to receive R instead of X if the $R > X$, i.e. is a Call option on the interest rate.

so its present value, using the Black formula, is:

$$L \,.\, d_T \,.\, [R_n\, N(d_1) - X\, N(d_2)]$$

where as usual N is the cumulative Normal distribution, and d_1 and d_2 are given by:

$$d_1 = \frac{\ln(R_n / X) + \tfrac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{\ln(R_n / X) - \tfrac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

Secondly, this payoff is paid to the swaption holder on every payment of the swap, i.e. at times $T + 1$, $T + 2$, ... , $T + n$ (it is an annual-paying swap). A payment at each of these times has present value:

$$a(T, n) = d_{T+1} + d_{T+2} + \ldots + d_{T+n} = \sum_{i=1}^{n} d_{T+i}$$

so at time T it has value $\dfrac{a(T, n)}{d_T}$.

Putting the two parts together, the value of the swaption is:

$$L \,.\, a(T, n) \,.\, [R\, N(d_1) - X\, N(d_2)] \qquad (***)$$

with $a(T, n)$ defined as above is the "present value of a basis point" on the forward swap, i.e. discounted back to time 0.

[A continuous form of this equation, as used by Hull, is just as valid for this part of the question, but is not as useful for part (ii), which requires the explicit discount factors. In that case, a candidate would use e^{-rt} instead of d_t in the above.]

(ii) Using the notation of the formulae above:

$T = 2$, $n = 3$, $\sigma = 12\%$, and $a(2,3) = 0.830 + 0.779 + 0.731 = 2.340$

so the only difficulty is finding the current value of the forward 3-year swap rate. There are a variety of ways of doing this.

Probably the neatest is to recall that the ordinary n-year swap rate S_n is also the par coupon rate for the yield curve at time 0, so it satisfies the equation:

$$S_n = \frac{1 - d_n}{\sum_{i=1}^{n} d_i}$$

Hence if we transform the d_t to forward d_t, say d_t', we can use the same formula to calculate the *forward* swap rate. Let $d_i' = \frac{d_{T+i}}{d_T}$, then:

$$R_n = \frac{1 - d_n'}{\sum_{i=1}^{n} d_i'}$$

In numbers,

$R_n = (1 - 0.731 / 0.886) / (0.830 / 0.886 + 0.779 / 0.886 + 0.731 / 0.886)$
$= 0.0662393$, i.e. 6.62393%.

Substituting into the d_1 and d_2 formulae gives:

$$d_1 = [\ln(6.62393 / 6.5) + ½.2.(0.12)^2] / 0.12\sqrt{2} = 0.196144$$

$$d_2 = d_1 - 0.12\sqrt{2} = 0.026438$$

and using tables, $N(d_1) = 0.577751$ and $N(d_2) = 0.510546$.

so in formula (***) above, swaption value

$$= €10{,}000{,}000\ (2.34)\ [0.0662393 \times 0.577751 - 0.065 \times 0.510546]$$
$$= €118{,}973.$$

(iii) The clue was given above, viz. that the ordinary n-year swap rate S_n is also the par coupon rate. Or as Hull puts it, a swap can be regarded as an agreement to exchange a fixed rate bond for a floating rate bond.

Hence, if there is a reasonable correlation between the bond market (the question doesn't specify which bond market) and the swap market, the volatility of swap rates should be the same as the yield volatility of par coupon bonds.

Thus the volatility of swaptions and bond options should be very similar.

But differences will occur due to balance of buyers and sellers in each market, funding (repo) variations on bonds (affecting forward prices), and credit and liquidity considerations.

6 (i)

The Monte Carlo method uses a random number generator to provide potential values for the stochastic term of an interest-rate model. One set of random numbers gives one simulated path.

Method: turn time horizon into discrete time by small time steps Δt ...

... sample known probability distribution to give a path over all the time steps ...

... gather large number of paths, say 10,000, of equal probability ...

... value the derivative (option or whatever) on each path, and take the average.

Technically it can be thought of as a forward induction method of calculating $\hat{E}[e^{-\bar{r}T} f_T]$, the (risk-neutral) expectation of the current value of the time T payoff f_T.

Used to value complex derivatives that have no analytic solution, such as exotic options which are path dependent or have non-linear payoffs.

(ii)

Advantages

- Simple idea conceptually, albeit rather "brute force" in style.

- Extension of MC to multiple factors is simple, as the samples are taken e.g. from the multivariate Normal rather than the simple Normal distribution. For the same level of accuracy, the number of samples needed grows approximately linearly with the number of factors. In tree methods, it grows exponentially, so is much more costly.

- Binomial/trinomial trees have large numbers of very extreme (large and small) nodes which contribute almost nothing to the value of the derivative.

- MC time steps can easily be tailor-made for the problem. Trees and lattices are much less flexible.

- Increasing the number of time steps in a tree increases the number of calculations by the square of that number, whereas for MC it is roughly linear.

- Successful for valuing many path-dependent options, because the path is always generated for each sample.

Disadvantages

- Needs complex and generally large-scale computer program to work effectively.

- Convergence is slow. The variance of the set of independent random variables decreases with as $\frac{1}{n}$, where n is the number of sample paths, so accuracy (as measured by the standard error) is proportional to $\frac{1}{\sqrt{n}}$.

- Deltas and gammas are hard to obtain directly, due to compounding of the approximation errors for these second and third order quantities.

- Often hard to get unbiased enough random or quasi-random sequences, so results nearly always have some hidden bias from the true answer.

- American options are tricky or impossible to value under MC because the method does not enable one path to know about other paths, so as to compare values for the American feature. Trees and lattices do this readily.

- Intermediate values along the paths (for e.g. short rates or bond prices) are almost impossible to calculate in MC, certainly for common log-normal distributions. Trees and lattices are much better for this.

(iii)

[There are several possible methods — a few are listed below. The aim of all of these is to improve the effectiveness of the sample distribution, by giving the same coverage of the sample space with fewer sample paths.]

Antithetic Variables

For each sample path from the distribution, create another valid path by taking the opposite sign on each random element. This creates an "antithetic" path.

The odd moments of the sample distribution are therefore zero, which works well with non-skew distributions like the Normal distribution.

Moment Matching

The n sample paths are generated and stored, then adjusted by a scaling factor to ensure that the variance (second moment) of the sample exactly matches that of the initial distribution.

Together with antithetic variables, this produces a tighter fit to the probability distribution without adding further samples, so should be more accurate.

The fourth moment can also be matched in the same way if kurtosis is important.

Stratified Sampling

Divide the range of outcomes into bands according to probability, and then take more sample paths from those with the higher probability.

This does not affect the overall accuracy, but removes the need to sample those outcomes where there is a low contribution to the valuation.

Contravariates (or Control variates)

If an exotic derivative is being valued, there is probably an analytical expression available for a similar, more simple, derivative (e.g. if calculating an average-rate caplet, the formula for the plain vanilla caplet is known).

Then use MC to calculate the simple and exotic derivatives, and adjust the exotic value by the same amount as the simple value needs to adjusted to agree with the known formula.

Alternatively, the paths themselves can be adjusted (probably a few times) until the values agree.

Quasi-Random Sequences

The MC approach does not need random values to succeed — all it needs is to have representative enough paths that the approximation is not biased by the samples chosen.

There are some sequences which can be shown to be non-random, but which nevertheless will not bias the approximation. The advantage of this type of series is that its standard error is proportional to $\frac{1}{n}$, rather than $\frac{1}{\sqrt{n}}$ for the raw MC.

7 (i) [*The question does not ask for **all** the algebra, but some is helpful. Different approaches could also be valid here.*]

Consider the variable $y(t, T) = \ln P(t, T)$. Then, using Ito's lemma,

$$dy = d(\ln P) = \left[r - \frac{1}{2}\frac{\sigma^2}{P^2}\right]dt + \frac{\sigma}{P}dz = [r - \tfrac{1}{2}v^2]\,dt + v\,dz \qquad (*)$$

where $v = \sigma / P$ depends (like σ) on t, T and P.

But the time t forward rate spanning period T_1 to T_2 is given by:

$$f(t, T_1, T_2) = -\frac{y(t, T_2) - y(t, T_1)}{T_2 - T_1} \qquad (**)$$

and, substituting equation (*) into the differential of equation (**), gives an expression for $df(t, T_1, T_2)$ of the form [...] dt + [...] dz involving v, the normalised volatility ...

... but not r or P directly, i.e. only on these variables in so far as v depends on them.

Letting $T_2 \to T_1 = T$, the LHS of the above equation becomes the instantaneous forward rate $F(t, T)$:

$$F(t, T) = -\frac{\partial y(t, T)}{\partial T}$$

which leads to a risk-neutral process of the form [...] dt + [...] dz also for $dF(t, T)$ ...

... where the [...] terms again depend only on v and its partial derivatives with respect to P and T.

Therefore, once the volatility function v has been determined, this is enough to define both the drift (dt) and the random (dz) term ...

... and so there is an explicit link between the volatility of zero-coupon bond prices and the drifts of the forward rates.

(ii) **Benefits of two-factor HJM:**

- HJM is a generalised framework for modelling the yield curve evolution, enabling a wide range of valuations from one model, i.e. efficient and consistent.

- Two factors enable the model to describe greater variations in the evolution of the yield curve, since forward rates can be imperfectly correlated.

- Two factors also enable valuation of options which depend not just on the market level but also on the yield curve slope, i.e. include correlation between forward rates on the curve ⇒ valuing a wider range of options.

- HJM uses MC, which only needs twice the iterations of a single factor model, compared with square of that number for e.g. tree or lattice methods.

Choice of factors:

The method of principal components can be used to identify the main drivers of the yield curve ...

... which generally are the direction and the slope.

Implementation and calibration:

It is far from trivial to implement the HJM model based on market prices.

Would like to use prices of caps or swaptions to drive model parameters, but ...

... the problem is that these are valued in the market using the Black model, which is a log-normal distribution ...

... and specifying a log-normal volatility for forward rates in HJM severely restricts the forms of the HJM model that can be used.

Choice of correlation function is difficult — hard to be precise about it since so few reliable option values to base it on.

Swaptions and caplets imply certain correlation effects but, considering the calibration difficulties, are not likely to suggest a good correlation function.