Faculty of Actuaries **Institute of Actuaries**

EXAMINATIONS

April 2000

Advanced Certificate In Derivatives: Further Mathematics, Principles And Practice

EXAMINATION REPORT

1 (i) (a) The re-valued deferred annuity is given by

$$X = \begin{cases} A \text{ if } Q(n) \le A \\ Q(n) \text{ if } A \le Q(n) \le B \\ B \text{ if } B \le Q(n) \end{cases}$$

$$= \max(A, \min(Q(n), B)$$

where

$A = 1$(no decreases allowed)

$B = R(n) = 1.05^n$

(b) Looking at the benefit from the point of view of the deferred pensioner we can see that if there were no cap nor floor, his annualised annuity of £1 p.a. linked to the RPI would have increased to £$Q(n)$ per annum.

The impact of the cap and floor is as follows:

- If $Q(n)$ is less than the lower limit of A = £1 per annum (the amount of deferred annuity when the member left the scheme), then the member is entitled to £1 per annum. Thus the member effectively has a European put option which gives him the right to sell (at the vesting date of the pension i.e. time n) his pension annuity of £$Q(n)$ per annum for an annuity of £1 per annum.

- If $Q(n)$ is greater than the upper limit of $B = R(n) = 1.05^n$, then the member has written a European call option for the scheme. The scheme has the right (at the vesting date of the deferred annuity) to buy the member's pension annuity of $Q(n)$ per annum with an immediate annuity of $R(n)$ per annum.

Note that the total benefit to the policyholder is therefore the index-linked benefit plus the impact of the CAP and floor.

(ii) (a) Note that since a unit of investment in the RPI earns interest at force η, it follows that the present value of one unit deferred T years is $e^{-\eta T}$.

Also, since the uniform force of interest on fixed money investments is δ, the present values of the exercise prices A and B at a time T years before the exercise dates are $Ae^{-\delta T}$ and $Be^{-\delta T}$.

Using the Garman and Kohlhagen variation of the Black-Scholes formula, it follows that the value of the put option with exercise price A is:

$$V_A = Ae^{-\delta T}\,\Phi(f_1) + Q_0 e^{-\eta T}\,\Phi(f_2)$$

where

$$f_1 = \frac{\log(Ae^{-\delta T} / Q_0 e^{-\eta T})}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

$$f_2 = \frac{\log(Ae^{-\delta T} / Q_0 e^{-\eta T})}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}$$

Using the Garman and Kohlhagen variation of the Black-Scholes formula, it follows that the value of the call option with exercise price B is:

$$V_B = Q_0 e^{-\eta T}\,\Phi(d_1) - Be^{-\delta T}\,\Phi(f_2)$$

where

$$d_1 = \frac{\log(Q_0 e^{-\eta T} / Be^{-\delta T})}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

$$d_2 = \frac{\log(Q_0 e^{-\eta T} / Be^{-\delta T})}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}$$

The value of the whole benefit is:

$$\begin{cases} \text{value of 1 unit linked to the RPI} \\ +\ \text{value of the put option with exercise price } A(V_A) \\ -\ \text{value of the call option with exercise price } B(V_B) \end{cases}$$

This equals:

$$V_t = Q_0 e^{-\eta T} + V_A - V_B$$

$$= Q_0 e^{-\eta T} + Ae^{-\delta T}\,\Phi(f_1) - Q_0 e^{-\eta T}\,\Phi(f_2) - (Q_0 e^{-\eta T}\,\Phi(d_1) - Be^{-\delta T}\,\Phi(f_2))$$

$$= Q_0 e^{-\eta T}\,(1 - \Phi(d_1) - \Phi(f_2)) + Ae^{-\delta T}\,\Phi(f_1) + Be^{-\delta T}\,\Phi(d_2)$$

NOTE: To score high marks it was not necessary to know the Garman and Kohlhagen variation of the Black-Scholes formula by rote. Most of the marks were obtainable by stating that the result is derived by:

Writing down a Black-Scholes formula for each of the options V_A and V_B.

Subtract the formula for V_B from that for V_A and adding this expression to the value of one unit invested in the RPI.

(b) It can be seen that the total value V_{tor} is equal to $V_G + V_F$ where

$$V_G = Q_0 e^{-\eta T} (1 - \Phi(d_1) - \Phi(f_2))$$

$$V_F = Ae^{-\delta T} \Phi(f_1) + Be^{-\delta T} \Phi(d_2)$$

This says that the value of the liability can be replicated by the value of a portfolio of index linked gilts (equal to V_G) plus the value of a portfolio of fixed interest securities (equal to V_F). This is in fact the matching portfolio that results from the replicating strategies underlying the derivation of the Black-Scholes formulae in this case.

It can also be seen that the mean rate of inflation $\mu(t, Q(t))$ does not enter into the resulting formula. This is because if the hedging (i.e. replication) portfolio is maintained at all times, then the mean rate of inflation does not matter. It does not necessarily imply that η and δ are related through the expected force of inflation.

In practice, it is not possible to continuously and costlessly adjust the hedging portfolio, but in an approximate hedge can be maintained.

(c) Immediately after the member has left service:

- The simple assumption that inflation will always be higher than 5% so that the upper limit applies over-values the benefit.

- The simple approach of ignoring the CAP and floor and valuing the benefit as purely index-linked under-values the benefit.

- In both cases, the error of approximation increases with increasing volatility of the RPI.

Several years after the member has left the scheme and with only one year left during the deferral period:

- If past inflation has been low, the chance of reaching the 5% CAP over the whole period is small and the value of the benefit is almost fully index-linked. The hedging portfolio is almost entirely index-linked.

- If inflation has been high, the likelihood of the 5% CAP applying over the whole period is very high and the value of the benefit is very close to that obtained using a 5% deterministic interest rate. The hedging portfolio is almost entirely comprised of fixed interest stock.

- When inflation has been intermediate, the value of the portfolio is also intermediate.

As the outstanding term to go lengthens, the value of the benefit is less affected by actual past inflation and becomes more intermediate between being fully index-linked and fully deterministic.

2 Range forward contract

Let:

$$B = X_1^* = X_1^{-1} = \frac{1}{1.55} = 0.64516$$

$$A = X_2^* = X_2^{-1} = \frac{1}{1.65} = 0.60606$$

The payoff function for the range forward contract is:

$$P(S_{90}^*) = 1 \text{ million} \times \begin{cases} -S_{90}^* + X_2^* & \text{if } S_{90}^* < X_2^* \\ 0 & \text{if } X_2^* \le S_{90}^* \le X_1^* \\ -S_{90}^* + X_1^* & \text{if } S_{90}^* > X_1^* \end{cases}$$

The left half of the curve looks very much like a long put option whose payoff function (per $1 million) is $\max(-S_{90}^* + A; 0)$.

The right half of the curve looks very much like a short call option whose payoff function (per $1 million) is $\min(0; -S_{90}^* + B) = -\max(0; -S_{90}^* + B)$.

This suggests that the payoff function for the range forward contract is equivalent to the sum of the payoff functions for the long put (strike price A) and the short call (strike price B).

To prove that this is indeed the case, let $P_{RF}(S_{90}^*), P_{LP}(S_{90}^*)$ and $P_{SC}(S_{90}^*)$ denote the payoff functions (per \$1 million) for the range forward, long put and short call derivatives respectively.

For $S_{90}^* < A$

$$P_{RF}(S_{90}^*) = P_{LP}(S_{90}^*) = \max(-S_{90}^* + A; 0)$$

$$P_{SC}(S_{90}^*) = 0$$

and therefore $P_{RF}(S_{90}^*) = P_{LP}(S_{90}^*) + P_{SC}(S_{90}^*) = \max(-S_{90}^* + A; 0)$.

For $A \le S_{90}^* \le B$

$$P_{LP}(S_{90}^*) = \max(-S_{90}^* + A; 0) = 0$$

$$P_{SC}(S_{90}^*) = \min(-S_{90}^* + B; 0) = 0$$

$$P_{RF}(S_{90}^*) = 0$$

and therefore $P_{RF}(S_{90}^*) = P_{LP}(S_{90}^*) + P_{SC}(S_{90}^*) = 0$.

For $S_{90}^* > B$

$$P_{RF}(S_{90}^*) = P_{LP}(S_{90}^*) = \min(-S_{90}^* + B; 0)$$

$$P_{SC}(S_{90}^*) = 0$$

and therefore $P_{RF}(S_{90}^*) = P_{LP}(S_{90}^*) + P_{SC}(S_{90}^*) = \min(-S_{90}^* + B; 0)$.

Hence $P_{RF} = P_{LP} + P_{SC}$ for all values of S_{90}^*.

3 PGN 1995/3 says that in practice, the terms "efficient portfolio management" and "reduction in investment risks" *overlap*.

They are *alternative* conditions. This means that it is not necessary to decide whether a particular transaction is consistent with one of these terms rather than the other. Provided that a transaction is for the purposes either of reduction of investment risks or efficient portfolio management, the relevant requirement is satisfied.

PGN 1995/3 notes that the terms are also used in *other sectors*, and the interpretation given under PGN1995/3 does not necessarily correspond precisely with that of other regulators.

Reduction of Investment Risk

Reduction of investment risks is to be *interpreted broadly*. To qualify, a transaction *must* achieve the following:

(a) in any case where a group of assets is "*earmarked*" to match specific policyholder benefits where the policyholder bears an investment risk (notably in the case of linked liabilities), there must either be:

- a reduction in the risks to the company of mismatching of those assets and liabilities, while having a neutral or beneficial effect on the investment risks of the policyholder; or

- a reduction in the investment risks of the policyholder, while having a neutral or beneficial effect on the risks to the company of mismatching

(b) in any case where there is *no such earmarking* of assets, it must reduce the risks to the company of mismatching between its assets and liabilities *at large*.

In either case, exactly what constitutes a reduction in risks is *not very straightforward*. Most derivative contracts will leave the insurer worse off than if the contract had not been transacted under some foreseeable circumstances. The guidance does not insist that a contract aimed at "reduction of risks" can *never* leave the company or policyholder worse off, since practically nothing would qualify.

A much *less extreme view* is taken. A contract which brings benefit (to company or policyholder, as appropriate) under some circumstances while having *adverse consequences under other circumstances* can be said to be reducing investment risks if:

- any adverse consequences of the contract are *unforeseeable*; or
- the extent of any adverse consequences is *insignificant*, in partcular:
- small; and
- reasonable, given the benefits resulting under other circumstances

The obverse also applies. A derivative contract which has significant adverse consequences on investment risks cannot qualify as "reducing investment risks". It is *not a case of balancing* the advantages and the disadvantages; rather, *there must be no significant disadvantages*.

Efficient Portfolio Management

The *concept* of efficient portfolio management is related to the question of how an insurer manages its assets so as to fulfil its prudent adopted investment strategy.

When assessing whether a transaction caused a reduction of investment risks, the appropriate comparison was with a "do nothing" strategy. But when

considering efficient portfolio management, the right *comparison is with a non-derivative strategy* having broadly the same economic effect.
The fact that a derivative transaction increases asset risk (i.e. the risk that the value of the portfolio will drop) does not necessarily prevent it from being regarded as for the purposes of efficient portfolio management. The same *increase in asset risk* might well have been achievable by trading in the underlying assets.

However, a derivatives contract which gives rise to a *significant adverse consequence which could not result form a direct* and prudent strategy of investing in or disinvesting from (as the case may require) the assets underlying the transaction can never be consistent with efficient portfolio management.

It follows that use of derivatives which has the effect of *significantly gearing* the total investment return on the fund is incompatible with efficient portfolio management.

Subject to the over-riding condition that there must be no reasonably foreseeable significant adverse risk consequences arising from the use of derivatives (as opposed to arising from investing in or disinvesting from the underlying assets), a transaction is consistent with efficient portfolio management if, under normal circumstances, it will *assist* the company to make progress towards its investment *objectives* either: more quickly or more easily, more efficiently, more cheaply or more flexibly than can be achieved without the use of a transaction of that nature.

Where there are no material benefits from using derivatives other than *saving of tax*, then, provided that it is not reasonable to foresee that a tax advantage might be removed (or even reversed) with retrospective effect, it can legitimately be invoked to justify efficient portfolio management.

Further Guidance Applicable to both Reduction in Investment Risks and EPM

In assessing whether an adverse risk consequence arises, the company's *aggregate* investment risk position must be considered in all cases. Further, in any case where a group of assets is "*earmarked*" to match specific policyholder benefits where the policyholder bears an investment risk (notably in the case of linked liabilities), the consequences of the use of derivatives on the asset risk of the earmarked assets must also be taken into account.

A contract which decreases market risk but gave rise to a *significant* increase in counterparty risk is regarded by PGN1995/3 as having adverse risk consequences and therefore inconsistent with either reduction of investment risks or efficient portfolio management. What is "significant" in this context is a matter for professional judgement in the light of the circumstances of the company. In a *linked fund*, admissibility limits are irrelevant; it will probably be necessary for the company to set its own *benchmarks* as to "significance", which may reasonably vary to some extent according to the investment objectives of the fund.

The guidance also draws attention to the *continuing nature* of the test. A derivative which initially satisfies the "efficient portfolio management" or "reduction of risks" tests will not necessarily continue to do so. A derivative which fails the test cannot be admissible (or a permitted link) even if it passed the test at the outset. This possibility should therefore be kept under review.

4 1st coupon: $B(0.25,\ 0.75) = \dfrac{1 - e^{-0.1\times 0.5}}{0.1} = 0.4877$

2nd coupon and maturity proceeds: $B(0.25,\ 1.25) = \dfrac{1 - e^{-0.1\times 1}}{0.1} = 0.9516$

$P(0,\ t) = e^{-r(t).t} = e^{-(0.09+0.02t)t}$

$\dfrac{\partial P}{\partial t}\ (0,\ t) = -(0.09 + 0.04t)\ e^{-(0.09+0.02t)t}$

$$\ln A(t,\ T) = \ln\frac{P(0,T)}{P(0,t)} - B(t_t\ T)\,\frac{\partial P(0,t)}{\partial t} - \frac{1}{4a^3}\ \sigma^2(e^{-aT} - e^{-at})\ (e^{2at} - 1)$$

1st coupon: $A(0.25,\ 0.75) = 0.9926$

2nd coupon and maturity proceeds: $A(0.25,\ 1.25) = 0.9733$

To value the option we must find the interest rate which makes the bond price equal to the strike price of 100 at the maturity of the option.

So we must find the value of r such that:

$$6 \times 0.9926 \times e^{-0.4877r} + 106 \times 0.9733e^{-0.9516r} = 100$$

solving by trial and error r is 0.0943 or 9.43%

The option is the sum of two options on discount bonds.

1st option: option on a bond that pays 6 at time 0.75 with strike price $6 \times 0.9926e^{-0.4877\times 0.0943} = 5.688$

2nd option: option on a bond that pays 106 at time 1.25 and has strike price $106 \times 0.9733e^{-0.9516\times 0.943} = 94.3186$

Value of 1st put option: 0.01*

Value of 2nd put option: 0.43*

Value of put option on coupon securing bond 0.44.

* detailed workings

X	$P(0,\ T)$	$N(-h + \sigma_p)$	L	$P(0,\ S)$	$N(-h)$

1st option	5.68814	0.97653	0.674085024	6	0.92427096	0.67278
2nd option	94.3186	0.97653	0.674703944	106	0.86610425	0.61216

	1st option	*2nd option*
sigma p	0.006312543	0.007048901
h	–0.4477609102	–0.445890923

5 (i) $S = 1400$ $X = 1330$ $r = 0.06$ $\sigma = 0.3$

$T - t = 0.5$ $q = 0.03$

$$d_1 = \frac{\ln\left(\frac{1400}{1330}\right) + \left(0.06 - 0.03 + \frac{0.09}{2}\right) \times 0.5}{0.3\sqrt{0.5}} = 0.4185756$$

$$d_2 = d_1 - 0.3\sqrt{0.5}$$

$$= 0.20644357$$

$N(d_1) = 0.662237$ $N(d_2) = 0.581778$ $N(-d_1) = 0.337763$

$N(-d_2) = 0.418222$ Value of put option $Xe^{-r(T-t)}\,N(-d_2) - Se^{q(T-t)}\,N(-d_1)$

$= 73.96796$

$$\text{Total cost of insurance} = \frac{100{,}000{,}000}{1400} \times 73.96796 = 5.283\text{m}$$

(ii) The delta of one put option is $e^{-q(T-t)}\,(N(d_1) - 1)$
$= -0.3327$

33.27% of the portfolio (\$33.27m) should be sold initially and reinvested in risk free securities.

(iii) Synthetic put options don't work very well if the volatility of the index changes rapidly from that assumed in calculating delta or if the index jumps in value. If the index were to crash it is unlikely that the manager will be able to sell stock or index futures fast enough to protect her position. Exchange traded put options are likely to be much more effective in such circumstances. Dynamic hedging is not free. There are the bid-offer spreads on purchase and sales and market depth impacts to be paid. Similarly, there are bid/offer spreads to be when put options are purchased.

(iv) When the portfolio increases by ΔS, the index futures price increases by $\Delta Se^{+(r-q)T^*}$ where T^* is the time to maturity of the future contract.

Thus $e^{-(r-q)T^*}$ futures contracts have the same sensitivity to portfolio price movements as one unit of the portfolio.

The portfolio delta is –0.3376 representing a short position of $33.76m in the portfolio.

The spot short position required is:

$$\frac{33,270,000}{1,400} \text{ times the index.}$$

So the short position in index futures contracts should be:

$$\frac{33,270,000}{1,400} \times e^{-0.03\times 0.5} = 23,236 \text{ times the index.}$$

Each futures contract is for 250 × the index. So 93 contracts are required.

Marks were also given to candidates who eliminated all exposure to the market.

(v) From put-call parity we have

$$p = c - Se^{-q(T-t)} + Xe^{-r(T-t)}$$

so a put option can be synthesised by

- shorting $e^{-q(T-t)}$ of the index
- buying a call option and
- investing the balance at the risk-free rate of interest.

6 Credit Enhancement and Risk Reduction Strategies

While there are two approaches to credit enhancement and risk reduction it is important to remember that enforceable netting arrangements play a major role in reducing credit risk.

The two approaches to credit enhancement are:

- reduce the probability of default and
- reduce the underlying exposure

Credit enhancement by reducing the probability of default takes the form of third party letters of credit and guarantees to enhance creditworthiness. Although used less frequently, an independent third party may agree to guarantee a counterparty's performance in a derivatives transaction in return for a fee. Such guarantees are normally required to be an absolute and unconditional obligation and the guarantor will be required to waive ordinary rights of surety.

Where letters of credit are obtained from a bank acceptable to both parties the amount should be large enough to support the maximum potential exposure under the derivatives transaction.

Some derivatives dealers have enhanced their credit ratings by establishing special purpose vehicles or special operating subsidiaries with a credit rating that is higher than that of the parent. These vehicles are for OTC derivatives trading and usually carry a "AAA" rating. They are structured and operate in a way that removes the direct credit risk of the parent and provides enhanced credit protection for countperparties. The common features of "AAA" subsidiaries established to date can be conveniently listed under the following headings:

- capitalisation
- operating guidelines
- bankruptcy proceeding and
- transaction matching

Capitalisation

Credit rating agencies require that the capital of an "AAA" rated entity be able to withstand the most extreme stress scenario involving severe market movements. Thus many of the special purpose vehicles have a cushion of excess capital.

Operating Guidelines

The special purpose vehicles may operate in accordance with specific rules that aim to diversify credit exposure. It is vital that the AAA subsidiary is not combined with the parent in the case that the parent becomes insolvent. This concept of being legally separate from its parent is another important feature.

Transaction Matching

Transaction matching aims to ensure that the AAA subsidiary does not at any time have any open or unhedged positions.

The usual way of achieving this is to require that the AAA subsidiary enters into a mirror transaction with its parent for every transaction it enters into with a third party with the AAA subsidiary's exposure to the parent fully collateralised.

Sometimes the parent may be required to collateralise any excess exposures of the AAA subsidiary. In addition the AAA subsidiary can assign to the parent any transaction with a counterparty that has caused it to breach its operating guidelines. Such a breach might arise from a downgrade or default by one of the AAA subsidiary's counterparties.

Credit Enhancement through Reduction of Underlying Exposure

To offset exposure in derivatives transactions counterparties may post collateral. The idea being that under an enforceable collateral arrangement a counterparty which had pledged collateral and subsequently defaulted would forfeit that collateral to its non-defaulting counterparty in or towards payment of the defaulted obligation.

The most common form of collateral is cash and government securities. The security interest on the collateral usually has to be perfected with delivery mechanics and further assurances as demanded by the secured party.

When netting applies and is enforceable collateral is based on the net negative mark-to-market value. In the absence of netting collateral it posted on a gross bases.

Collateral arrangements can be bilateral or unilateral. Bilateral collateral arrangements require two-way flows of collateral. The counterparty with the negative mark-to-market value collateralises the exposure of the other party. Under unilateral collateral agreements one counterparty is required to deliver collateral to the other on trades in which it has a negative mark-to-market value. The other counterparty to the transaction is not required to post collateral. The latter type of collateral arrangement is used when one of the parties has a lower credit rating than the other.

Periodic cash settlement is another technique for credit enhancement through reducing underlying exposure. Under this arrangement, two counterparties agree to periodically send cash to cover any negative mark-to-market position that exists. The terms of the transaction are periodically reset to zero mark-to-market value at market rates, the party with the positive mark-to-market position taking ownership of the cash.

Collateral arrangements vary to suit the interests of the counterparties. For example, there may be an amount of up-front collateral required, the frequency of collateral calculations may vary from case to case or the obligation to post collateral may be triggered by an event such as a credit downgrade of upon reaching a specified threshold of exposure.
Triggers based on events like credit downgrades have the potential to create sudden and sizeable liquidity demands. In negotiating collateral provisions, a counterparty needs to consider its own and the other counterparty's ability to meet such potentially sudden and possible large liquidity demands.