EXAMINATIONS

April 2003

Advanced Certificate in Derivatives: Further Mathematics, Principles and Practice

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The examiners are mindful that a number of interpretations may be drawn from the syllabus and Core Reading. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

The report does not attempt to offer a specimen solution for each question — that is, a solution that a well prepared candidate might have produced in the time allowed. For most questions substantially more detail is given than would normally be necessary to obtain a clear pass. There can also be valid alternatives which would gain equal marks.

Mrs J Curtis

Chairman of the Board of Examiners

17 June 2003

Requires Page View for Q2 and Q3

QUESTION 1

Syllabus: (d)(i) & (ii)
Reading: Dowd Ch 3, 12 (reference to other sections e.g. Ch 10)

(i)

Positive Aspects

VaR will give a useful insight into the overall loss potential of the portfolio.

VaR is simple — one number for entire portfolio — if it's linear, no need for complicated method …
… and on a diverse portfolio, allows risks to be combined according to their size and relationships.

Variance/covariance method is tractable …
… easy to convert to different confidence levels and measurement periods (> 1 day).

Variance/covariance method includes correlations …
… so can be viewed as measuring diversification across the portfolio — therefore better than just adding up the separate risk numbers.

The VaR measure will change as the portfolio changes towards more/less volatile risks or as market gets more/less volatile. In this way gives a useful insight into the **changing** risk characteristics of the portfolio.

VaR is the only realistic approach to measuring the risk on arbitrage-style positions which have "basis" risk (where one instrument is purchased against a sold position in a very similar instrument).

VaR is a useful approach to recognising the diversification element of specific equity risk (as opposed to systematic or 'beta' risk).

Introduction of VaR important to impress/satisfy regulators, clients and even shareholders ….
… it could be used as a marketing tool as well as a management tool.

Negative Aspects

Need to map positions into standard format (equity, bonds etc) …
… not all risks can be expressed in this simple way.

Variance/covariance approach gives a poor insight into option risks – since the risk profile varies as options move in or out of the money …
… particularly close to expiry.

Volatilities and correlations are not stable, so that genuine loss levels could be much higher (or lower) than those output by the model
… correlations are especially unstable during big moves.

VaR is only one perspective on risk. Other perspectives are important if an overall risk picture is to be obtained.

Sole use of a VaR model will not give any insight into the nature or direction of the 'bets' taken at the level of portfolio diversification.

VaR is not a full maximum loss estimate:

- it only forecasts that losses will not exceed a particular level 95% of the time

- even if the model works well on a 1 day horizon basis, further losses may be taken over successive days before risks could be closed

- it does not allow for liquidity risk, so disposal prices in extreme market conditions may be at a discount, increasing forecast loss potential

- it does not allow for refinancing risk (e.g. the risk that additional borrowings will be required to satisfy futures variations margins, or increased collateral postings with counterparties).

The output of the VaR model is only as good as the assumptions made, which are frequently unstable and subject to sampling error …
… especially correlations.

The risk is in the tail of the distribution. Implicitly, a variance/covariance approach assumes a joint normal (or lognormal) statistical distribution for the risk variables involved. Most studies in fact see leptokurtic (or "fat tailed") distributions. This model fault will systematically underestimate economic risks present.

[*Another way of looking at this is to appeal to the Central Limit Theorem. However, this is not valid where the second moment of the various distributions used are not stable — as here.*]

A VaR model represents a statistical approach to an intrinsically non-statistical problem. The intervention of human beings in selecting the individual portfolio positions almost certainly creates a non-random bias to the distribution of possible outcomes and, probably affects the nature of the latter as well.

The VaR measure assumes that the portfolio remains unchanged over the chosen horizon. In this case this is 1 day so it may not appear to be a major issue. However, in most 'hedge fund' situations larger risk positions are frequently run on an intraday basis – then the VaR measure will tend to systematically underestimate overall risk.

(ii)

<u>Conclusion</u>

VaR is not a risk tool in common use for managing client funds, so would fall outside the normal industry standard practice (*though this on its own does not decide whether it is effective or not*).

Whilst introduction of VaR should be seen as a good thing, it is not a magic number giving the **only** valid perspective on portfolio risk.

It is only through multiple risk perspectives that managers can get a true insight into the risks present.

The claims made for the capabilities of VaR and for the variance/covariance approach in particular are overstated.

It would be very dangerous to adopt the variance/covariance measure as the only measurement of risk used by the business.

We should also, since client funds are involved, be aware of fiduciary and marketing obligations, i.e. client expectations of how we will manage risk.

Oppose as set out.

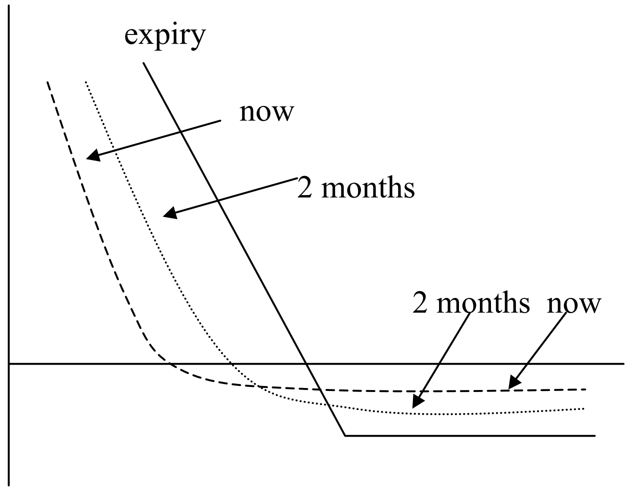
QUESTION 2

[*These diagrams are for illustration only.*]

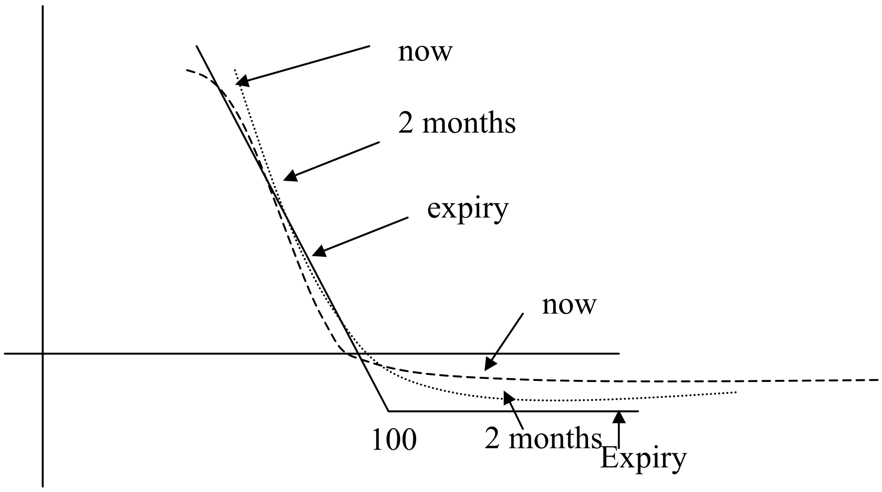
Syllabus: (a)(ii) 7
Reading: Hull 5th edition Chs 9, 14

(i) P&L diagrams

(a) European Put

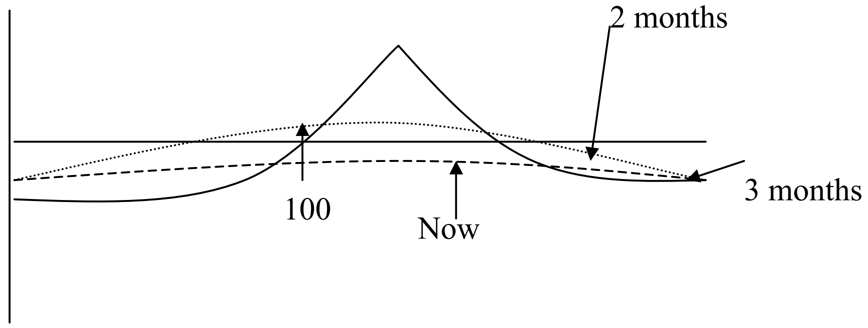


(b) American Put



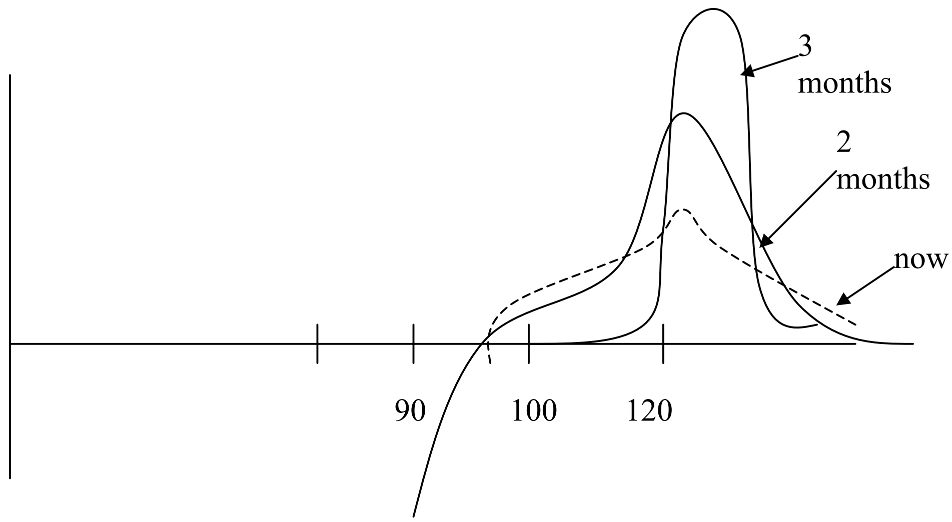
Because the option is American, the dotted lines cannot go to the left of the solid line.

(c) European + American Calls

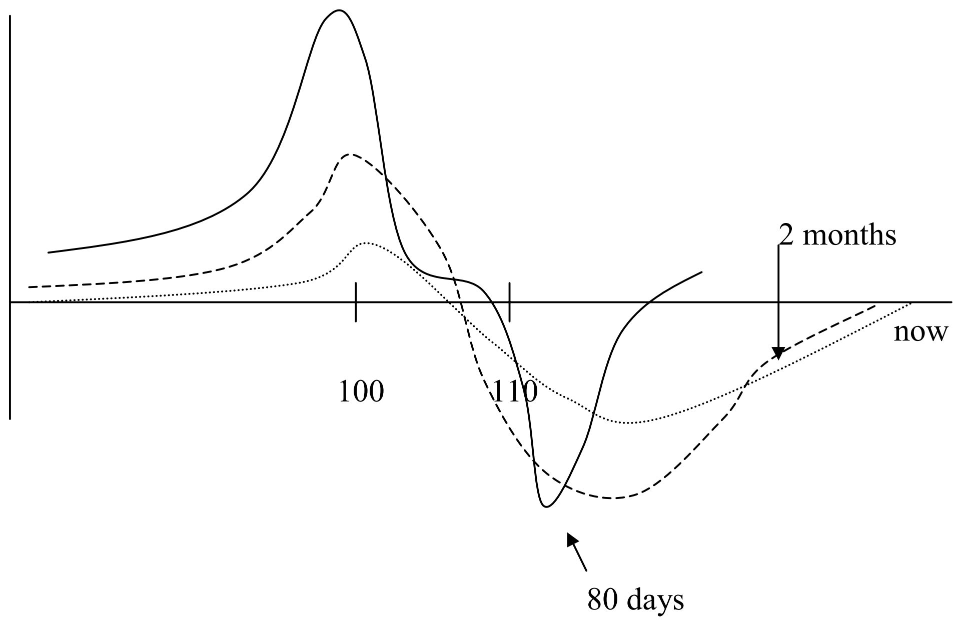


(ii) Gamma diagrams

(a) Knockout European Call



(b) Call Spread



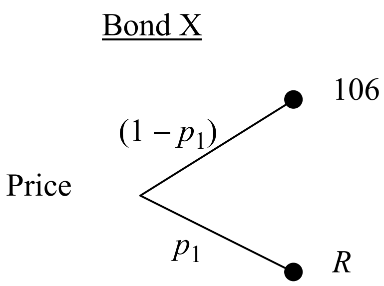
QUESTION 3
Syllabus: (a)(i) 6, (b)(i) 6
Reading: Hull Ch 23 + binomial maths

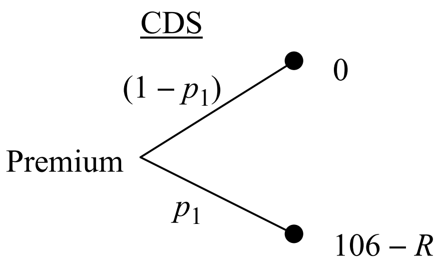
(i)

Bond X risk-free price = $106\ (1.0475)^{-1}$ $\quad = 101.1933$
Bond Y risk-free price = $5\ (1.0475)^{-1} + 105\ (1.0475)^{-2}$ $\quad = 100.4665$

(ii)

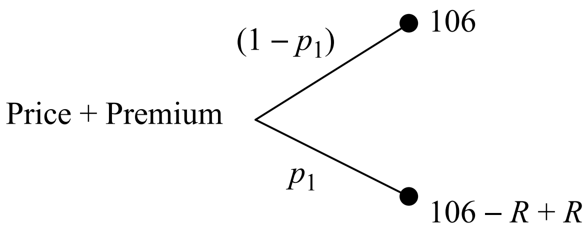
Let p_1 be the probability of default in year 1. The risk neutral trees are:





where R is the recovery amount per bond.

Consider the portfolio: "Buy Bond, Sell CDS":



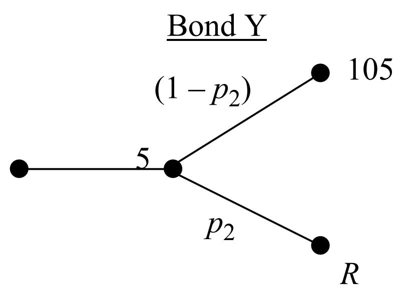
The outcome is independent of the path, i.e. is deterministic, so the portfolio must be risk neutral.

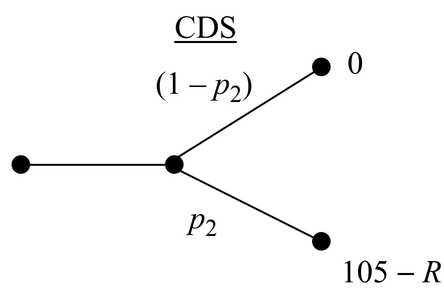
So $\qquad$ (Bond + CDS) = $106\ (1.0475)^{-1} = 101.1933$, as above.

Thus CDS premium = Discounted proceeds – Bond price
$\qquad\qquad\qquad\qquad$ = 1.693 (to 3dp)

(iii)

Let p_2 be the probability of default in year 2. (No default in year 1.)

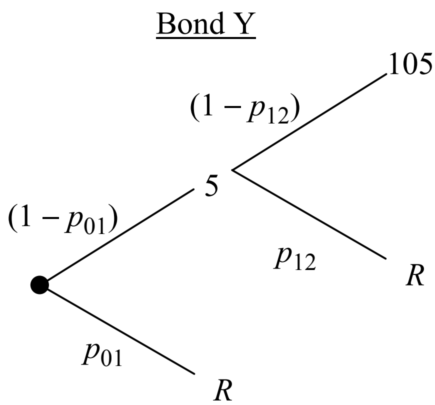


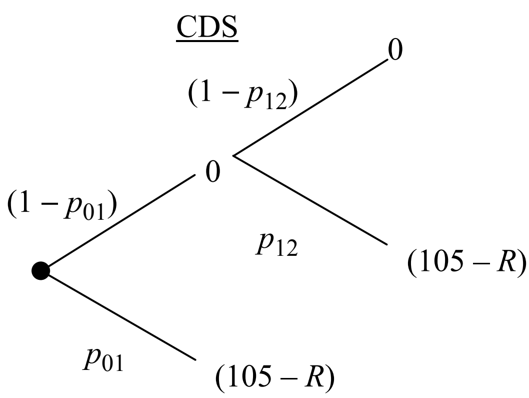


Then as before, Bond + CDS = 5 $(1.0475)^{-1}$ + 105 $(1.0475)^{-2}$ = 100.4665, since outcome is deterministic

and so here CDS = 2.467 (to 3dp)

(iv)





using v_{ij} to indicate the <u>risky</u> discount factor between time i and time j, it is easy to show that the Bond + CDS strategy is worth:

$$\text{Bond + CDS} = (1-p_{01})\,[105\, v_{02} + 5\, v_{01}] + p_{01}\,[105\, v_{01}]$$

or, moving to the risk neutral world (adding a ~ to the variables):

$$\text{Bond + CDS} = (1-\tilde{p}_{01})\,[105\,\tilde{v}_{02} + 5\,\tilde{v}_{01}] + \tilde{p}_{01}\,[105\,\tilde{v}_{01}]$$

Since we are valuing future cashflows, use the money-market curve:

$$\text{CDS} = (1-\tilde{p}_{01})\,100.4665 + \tilde{p}_{01}\,(100.2387) - \text{Bond}$$

So that we obtain:

CDS	$\tilde{p}_{01}$
2.467	0
2.444	0.1
2.353	0.5
2.239	1.0

<u>Reasoning</u>

The strategy is not risk neutral since it contains the unknown p_{01}. It is obvious that no strategy involving any combination of units of Bond Y and the CDS will achieve a Risk Neutral hedge. We could eliminate p_{01} only at the cost of introducing R.

[*In general terms, a risk neutral hedge is not possible because we are modelling a jump process with a variable jump size.*]

(v)

Consider the relationship between $\tilde{p}_{01}$ and $R\%$ (where now $R\%$ is the recovery as a percentage of the total price rather than an absolute value).

$$99.5 = [(1 - \tilde{p}_{01})\,106 + \tilde{p}_{01}\,106\,(R\%)]\ {}^{1}\!/_{1.0475}$$

so $$\tilde{p}_{01} = \frac{106 - 104.2263}{106(1 - R\%)}$$

$R\%$	$\tilde{p}_{01}$
0	0.0167
50	0.0335
100	∞

If $R\% = 100$, $\tilde{p}_{01}$ becomes infinite. There are two reasons:

- The default model is now meaningless, since $\tilde{p}_{01}$ drops out
- The bond is priced below its risk free equivalent price, so that if $R\% = 100$, i.e. you can recover all the proceeds with certainty, a risk free arbitrage is available.

(vi)

A sensible estimate of R depends on:

- The nature of the issuer (Are they a cashflow valuation entity? Do they have assets? etc.)
- The level of security offered by the bond documentation (i.e. the pricing of claim in default).

Without these details, and in the knowledge that historical recovery rates for senior unsecured bonds is around 50%, we will assume $R\% = 50$.

[*Note: candidates may choose another value if justified.*]

If $R\% = 50$, $\tilde{p}_{01} = 0.0335$

CDS (Y) = $0.9665 \times 100.4665 + 0.0335 \times 100.2387 - 98 = 2.459$

(vii)

Recommend hedging one unit of 2-year continuous trigger CDS on Bond Y by buying one unit of Bond X in year 1 (switch to Bond Y in year 2) and financing the cashflows, because:

- Bonds X, Y have full cross-default protection and the same documentation
- Bond X is cheaper than Bond Y in terms of risk neutral probability of default (from part (ii) this is 0.0302 for X)

but the better value of X is offset by the extra legal risk.

[*Note: Candidates also obtained marks for preferring to hedge with bond Y if they mentioned that they are wishing to avoid taking legal risk.*]

QUESTION 4
Syllabus: (a)(iv) & (v), (c)(iii)
Reading:GN1, GN25, Briggs et al Ch 4

[*There are a wide range of points to be made here. Some marks were also awarded for points not given here.*]

(i) Summary of GN25

The Insurance Companies Act 1982 (as amended in 1994) requires insurance companies to have systems of "sound and prudent management" encompassing all of the insurers' activities, including the use of derivatives. It also sets out criteria which must be fulfilled.

Guidance note GN1 (which is mandatory) also requires Appointed Actuaries to apprise themselves of whether systems exist for investments …
… and in particular whether suitable controls are in force concerning derivatives.

The Appointed Actuary needs to satisfy himself that suitable reserves or provisions exist …
… the sizes of which may be influenced by the quality of the management controls.

He (or she) should have a daily schedule showing the number and type of derivative transactions undertaken, the gross and net market exposures involved and the sensitivity of the portfolio to large market movements.

He has an ongoing responsibility to assess whether exposures are being satisfactorily monitored.

He needs to report to the Board immediately any sufficiently material breach of investment restrictions in relation to derivatives.

If the Appointed Actuary believes that the systems are deficient, or cannot form a judgement, he may need to make suitable allowance for this in quantifying the financial impact of the derivatives.

He must ensure that he this information is forthcoming. If the company fails to supply such information, he may need to qualify his certificate concerning the financial state of the company.

The materiality of derivative holdings in assessing such issues should be judged in the context of the exposure or risk of loss on a cautious basis, not necessarily on the value assigned to them in the valuation of assets or liabilities.

(ii) How the review might proceed

The Appointed Actuary should:

- Obtain copies of management objectives and policies covering derivatives for each fund. Be fully aware of how the derivatives guidelines relate to the overall investment guidelines.

- Check how these objectives and policies are monitored and enforced with respect to:

    defining the instruments that may be dealt;
    identifying credit and market risk limits on exposures or volumes;
    aggregating and netting exposures with non-derivative activities where appropriate;
    the principles behind the type of counterparties with which the organisation can deal;
    taking legal advice, particularly where there may be mutual obligation between parties.

- Ascertain how the company's use of derivatives satisfies statutory rules, prudential guidance or codes of practice published by relevant statutory authorities (e.g. "reduction in risk" and "efficient portfolio management").

- Ascertain that senior management with responsibility for control of derivative instruments are independent from the trading side and have sufficient understanding of the risks involved.

- Assess the degree to which the derivatives are dynamically hedged. Some types of derivative may need to be managed dynamically, so identify whether the management team is sufficiently large and skilled to undertake this work. A range of skills may be needed, both within the derivatives team and within more senior management.

- Assess the residual risks on hedged transactions. Derivatives or equivalent instruments may be bought solely to match some underlying liabilities. Even in these circumstances, however, risk exposures may still arise — e.g. counterparty risk or the possible purchase of too much or too little exposure — which need proper management.

- Review the legal agreements. For some types of derivatives, standardised forms of legal agreement exist. If these are not used then this may indicate deficiencies in the systems and controls in place.

- Review the tax treatment. If complex, obtain suitable tax advice.

- Where necessary and appropriate, obtain the advice of other professionals.

QUESTION 5

Syllabus: (b)(i) 1 & 2

Reading: Rebonato Ch 1 — also Hull Ch 4 and 5

(i) $$\frac{1}{(1.042)^5} = e^{-5r}$$

$\Rightarrow$ $r = 4.11\%$

(ii) Interpolate between 3 and 4 year rate = 3.8% [*other interpolation methods might be allowable*].

Value = $\dfrac{1}{(1.038)^{3.5}}$

Value = 87.76 per cent

(iii) Define $d_n = \dfrac{1}{[1+\text{zero}(n)]^n}$

Then if c_5 is the fixed coupon on the swap,

$$c_5\ (d_1 + d_2 + d_3 + d_4 + d_5) = 1 - d_5$$

using the standard compression formula for the floating side of a swap.

$d_1 = 0.96852$
$d_2 = 0.93622$
$d_3 = 0.89803$
$d_4 = 0.85645$
$d_5 = 0.81407$

Thus $c_5 = 4.156\%$

(iv) For the forward-starting swap, let the coupon be $c_{2/7}$.

We have:

$$c_{2/7}\ (d_3 + d_4 + d_5 + d_6 + d_7) = f_3 d_3 + f_4 d_4 + f_5 d_5 + f_6 d_6 + f_7 d_7$$

where f_n is the forward rate for time n, i.e. $f_n = (d_{n-1} / d_n) - 1$.

This formula simplifies to:

$$c_{2/7}\,(d_3 + d_4 + d_5 + d_6 + d_7) = d_2 - d_7$$

since $f_n d_n = d_{n-1} - d_n$ for each n.

Now

$$d_6 = 0.77454$$
$$d_7 = 0.73483$$

so $c_{2/7} = 0.20139 / 4.0779 = 4.939\%$.

(v) (a) 1-year CMS

The first payment at the end of year 1 is the 5-year swap rate at time 0, i.e. 4.156% (see part (iii)).

That will be the floating payment in 1 year's time, so that is the fixed payment also at that time.

[*This provided some easy marks for most candidates. CMS does not need a convexity adjustment because it has just one known payment.*]

(b) 2-year CMS

The first payment of the 2-year CMS is as for the 1-year CMS. The second payment is set at the end of year 1, and is paid at the end of year 2. This is the one which causes a problem, as we need to make a convexity adjustment.

The adjustment reflects the fact that the forward-setting swap is acting like a bond, which is convex in relation to interest rate movements. Any variation in interest rates between now and 1 year's time would increase the value of receiving the simple forward swap rate, so the receiver of the CMS rate must be compensated for this difference (to avoid arbitrage).

We need to find the expected swap rate at end of year 1 in a world that is forward risk neutral with respect to the 1-year zero coupon bond. This is the standard forward swap rate plus a convexity adjustment based on the joint evolution of the relevant interest and swap rates. [*No algebra was required to illustrate this.*]

Thus, as well as the term structure given in the question, we need to know the volatility of the forward swap rate, the volatility of the forward interest rate between years 1 and 2, and the correlation between these two rates.

QUESTION 6

Syllabus: (b)(i) 6 & 7

Reading: Rebonato Ch 8, 11, 13 — also Hull Ch 21

(i)

[*Rebonato Chapter 11*]

The rationale for a term structure yield curve model is to be able to price simultaneously options spread across the entire range of maturities in a yield curve. Exotic interest rate swaps and options include spread options, Bermudan swaptions and path-dependent options (knock-outs etc). These depend not just on the evolution of forward rates along the curve, but on the correlation between changes.

<u>Desirable</u> features are:

- the current swap (or bond) curve should be reproduced by the model
- easy to specify and calculate (on a suitable computer)
- easy to calibrate — for example, a log-normal expression of the model will help fit to cap prices, which are traded based on the standard log-normal Black model
- enough degrees of freedom (parameters) to make the model flexible to cope with any yield curve shape, but not overly flexible so there is instability between parameters from one day to the next
- volatility of rates of different maturity should be different, with shorter rates usually being more volatile
- imperfect correlation between forward rates, although this is only needed when pricing certain types of option where correlation has a big effect on the price (e.g. yield spread options, callable swaptions)
- interest rates cannot be negative
- reasonable dispersion of rates over time (due to the Brownian motion) — too large a probability of getting hugely high or low values will distort the model
- one possible way of allowing for this is to make the model "mean-reverting", that is, when rates go too high (or low), they tend to revert back to some central level

(ii)

(a) One-factor equilibrium models

A one-factor model creates a process for a variable, usually the short rate r, which leads to an evolution of rates over time.

Parameters (which can be time- and curve-dependent) govern the evolution. From this evolution, all present and future bonds and swap rates can be priced.

Single factor models are usually easy to use and calibrate, and can even in some simple cases lead to analytical solutions.

Equilibrium models are a particular class of models where a simplified form of an entire economy is described.

All securities and contingent claims are priced endogenously in equilibrium models. This gives a world of "absolute" prices, which may differ from real market prices.

In practice, the simplifications of the economy are so great that the resulting yield curve shapes are limited. This can lead to inaccurate pricing of securities.

(b) No-arbitrage models

No-arbitrage, or "arbitrage free", models are a class of models which allow recovery of market prices of one set of securities given prices of another set. This creates a world of "relative" prices.

In a non-"arbitrage free" model, securities could be priced using the model and then traded at a different price in the real world, leading to persistent profit. In simplest terms, no-arbitrage is the absence of a "free lunch".

No-arbitrage is very important in yield curve models, since most complex structures are limiting cases of simpler structures (such as swaps, caps, floors) and ideally the model should recover the prices of the latter exactly.

Also, hedging is done using the simpler structures, so the accounting process will not be distorted by imaginary gains and losses.

(iii)

[*Rebonato Chapters 8 & 13 and Hull Ch 21*]

For this part, only a description is required — candidates did not have to give numeric examples or complex algebra.

A trinomial tree is a discrete-time representation of the continuous rate process, with the stochastic process defined by branches which can have three states: up, down and mid-way between.

If the probabilities of going up, down and middle are p_u, p_d and p_m respectively, then $p_u + p_d + p_m = 1$ and all values are strictly positive.

The time horizon is split into constant intervals of time Δt, so the rates are compounded Δt-period rates.

Accuracy is improved with smaller time-steps, up to a point. In computer calculations you would normally use over 100 time steps per year.

The tree for the one-factor model is built in two stages. The method is very similar to that of Explicit Finite Differences.

[*The b(t) term in the rate process:*

$$dr = a\big(b(t) - r\big)dt + \sigma dz$$

is the market-fitting function, which is used to precisely price the zero-coupon bonds in the market. So firstly we build the tree with only mean-reverting features by setting b(t) to zero, then add back in the term structure.]

<u>Stage 1</u>

Firstly, set $b(t)$ to zero:

$$dr^* = -ar^*\,dt + \sigma dz$$

which gives a rate process r^* that is initially zero, and whose evolution is governed by a constant mean reversion towards zero.

There are three unknowns: p_u, p_d and Δr^* (since $p_m = 1 - p_u - p_d$).

It is convenient to set $\Delta r^* = \sigma\sqrt{3\Delta t}$, since this has been found to give the best numerical efficiency.

The other two equations come from the expectation and variance of Δr^* over the interval Δt, in which the tree must match the process.

There are three different forms of the mean and variance equations for the tree: when all the branches point upwards, when all point downwards, and when they straddle the line of unchanged rates.

There will be different probabilities for each situation.

There are also bounds (given by Hull & White in a paper on implementing their model) for the cross-over points at which the patterns must change for the probabilities to be always positive.

Since $E(\Delta r^*)^2$ contains the term $\sigma\Delta t$, the equations account for the volatility component.

[*A diagram would be helpful, but it is not necessary to give the branching equations or derive these bounds.*]

<u>Stage 2</u>

Now add back in the term structure, to move back from r^* to r.

Let $\alpha(t) = r(t) - r^*(t)$. Then:

$$d\alpha = a\left[b(t) - \alpha(t)\right]dt \qquad (**)$$

which is a <u>deterministic</u> mean-reverting process (i.e. varies over time but not stochastically), which can easily be calibrated to the term structure.

The best method of solving the differential equation in (**) is by forward induction along the Δt scale using Green's functions. A Green's function $Q_{i,j}$ is the present value of 1 unit of cash payable at node (i, j) on the tree.

Now, $Q_{0,0} = 1$ and α_0 $(= r)$ is set in terms of the price of a zero-coupon bond maturing at Δt.

Then $Q_{1,1}$, $Q_{1,0}$ and $Q_{1,-1}$ are set in terms of the tree probabilities p_u, p_d and p_m, $Q_{0,0}$ and α_0. There is a simple formula for α_1 based on the price of a zero-coupon bond maturing at $2\Delta t$ and these Q's.

And so on up to the time horizon.

[*More precise details are not required. Integration of (**) does not give exact enough values because we are approximating the continuous process.*]

(iv)

<u>Does HW fulfil the desirable features?</u>

The model is a suitable no-arbitrage full yield curve model, as it reproduces vanilla bond and swap prices exactly and prices options on these ...
... and has mean reversion and time-dependent parameters, so is flexible, behaves well and has little chance of producing negative rates.

The use of time-dependent mean reversion and volatility parameters can fit any current yield curve and forward volatility "hump" shape.

BUT

It only has one driving factor, so all forward rate moves have to be completely correlated — hence it cannot re-create complex yield curve changes ...

It is normal, not log-normal, so may be tricky to calibrate to cap prices …
… as at-the-money options are priced in the market with both normal and log-normal distributions. Any cap prices will include a range of at-the-money, out-of-the-money and in-the-money caplets.

Differences arise away from the current level of rates. These effects, if not disentangled, will lead HW to have an incorrect balance between its volatility and mean-reversion parameters. [*In some cases, caplet volatility which is sharply declining along the curve can lead to failure of the trinomial process further out in the tree.*]

The use of time-dependent mean reversion and volatility parameters leads to an implausible evolution of the future term structure of volatility.