Faculty of Actuaries **Institute of Actuaries**

REPORT OF THE BOARD OF EXAMINERS ON THE EXAMINATIONS HELD IN

April 2002

Subject — Advanced Certificate In Derivatives: Further Mathematics, Principles and Practice

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The examiners are mindful that a number of interpretations maybe drawn from the syllabus and Core Reading. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

The report does not attempt to offer a specimen solution for each question - that is, a solution that a well prepared candidate might have produced in the time allowed. For most questions substantially more detail is given than would normally be necessary to obtain a clear pass. There can also be valid alternatives which would gain equal marks.

K Forman

Chairman of the Board of Examiners

25 June 2002

QUESTION 1

(i)

Marks were also available for citing other pertinent extracts of PGN95/3 with a sensible attempt to apply them to the situation described in the question. Exact references are given below in italics, but were not required to achieve the marks.

(a) "Covered"

Requirement:

- Where a derivative involves an insurance company having to deliver certain assets under the contract, then the contract is said to be "covered" if the insurer holds the assets that are due to be delivered.
- The insurer must be sure that it is not exposing itself to any significant basis risk, i.e. it does not expect to suffer a significant loss as the result of acquiring or realising investments in order to satisfy its obligations under the derivative contract.
- But [*paragraph 6.7 of PGN95/3*] the question of cover does not arise in respect of contracts which can never impose obligations on the insurer (e.g. purchased options, warrants).

The swaption contract does not impose any obligation upon the insurer to deliver any assets, unless it exercises the option to swap ...

… and the insurance company would not exercise the swaption unless it were beneficial for it do so …

… so it is therefore not exposing itself to any significant basis risk in respect of the option component of the swaption.

Hence the requirement is satisfied.

A potential problem could arise if the insurance company needed to exercise the option, but then after exercising the option, wasn't holding the relevant floating rate bonds. To get round this, the insurance company could cash settle the derivative, and use the cash proceeds towards purchasing the underlying bonds it needs to match the annuity cash flows.

(b) "In Connection With"

Requirement:

- These combinations of free-standing derivatives pass the "in connection with test", provided that they are used for the purposes of efficient portfolio management or reduction of investment risks:
  - (a) a derivative which has the effect of an approved derivative contract used in connection with an admissible asset for such purposes
  - (b) a combination of derivatives which synthesise an admissible asset
  - (c) a combination of derivatives which synthesise a combination of an admissible asset and an approved derivative contract.

We have a combination of derivatives (an option and a swap) which effectively synthesise sterling fixed interest bonds.

Since sterling fixed interest corporate bonds are normally (subject to volume limits) admissible, it appears that arrangement (b) applies …

… assuming that the investment bank in question is also an approved counterparty.

Hence the derivatives satisfied this requirement.

[Note: It is sensible to apply the corporate bond admissibility volume limits even if the coupons being swapped are in respect of government bond securities – the counterparty here is the corporate investment bank in question.]

Alternatively [*Example P of PGN95/3*], it is legitimate to regard a purchased call option (which is effectively what the swaptions in question are) on an individual asset as being used "in connection with" cash or other readily realisable assets (which in our case are the floating rate bonds).

(c) Capable of Being Readily Closed Out

Requirement:

- The insurer must reasonably believe that the contract may be 'readily' closed out … [*Regulation 55(6)(b)*]
- … unless [*paragraph 3.4 of PGB95/3*], under realistic (or cautious) assumptions about mortality and other demographic factors, the insurance company does not realistically expect to have to close out the contract.
- By 'readily', it is meant that the insurer cannot reasonably foresee any circumstances in which, if it needs to close out all (or part) of the contract at a few days notice, it would not be able to do so.

It is not clear if the investment bank's willingness to quote 'close out' prices for limited volumes satisfies the requirement in full …

… so it might be argued that only this limited volume is admissible.

One could argue that any security, valued at market prices, can only realistically be traded in a limited size. But whilst this is true, the regulators have imposed special rules in PGN95/3 for derivatives which do not apply to 'traditional' assets.

But does the insurance company really need to close out the contract?

If it did not need the swapped income before the option exercise date, then it would simply not exercise the option without being able to settle in cash ...

… but if it did, it would be to match its liabilities relating to the deferred annuities, so it would not need to close out subsequently.

Hence it is probable that the derivatives satisfy this requirement.

In the unlikely event that the options are in the money but the insurance company no longer needs them to match its liabilities, and the contract does not have a cash settlement clause, then it would only exercise the option if it knew that it was able to immediately short the derived sterling fixed income stream and realise a profit. By shorting the income stream after exercising the option it is effectively closing out the contract.

(d) Admissible underlying assets

Requirement:

- A derivative contract based on a category of asset which is not admissible can never have an admissible value.
- However [*Paragraph 5.2 of PGN95/3*], there might be special cases where holdings of derivatives based on inadmissible assets represent a sensible method of hedging assets covering a specific liability. Under these circumstances, such derivatives can be admitted.

The derivatives under question are swaptions, which are derivatives and the underlying assets of these options are swaps, which are also derivatives. It is therefore questionable that the underlying assets of the options are admissible.

The case for the special concession would clearly be on the grounds that the use of the swaptions is a sensible method to hedge (or partially hedge) assets covering the specific liabilities relating to the deferred annuities and for managing these liabilities.

It is also possible that the regulator will accept the argument that a swaption is a derivative contract on interest rates in its own right. If this look-through basis is adopted, then it can be seen that the underlying assets are those whose cash flows are being swapped, if the option to swap is exercised.

Alternatively, it may be necessary to apply the full seven test criteria to the swap component first, to establish that this is admissible. If so, then this, being the underlying security of the option, means that the option satisfies the 'admissibility of the underlying' requirement.

Hence it is probable that the derivatives satisfy this requirement.

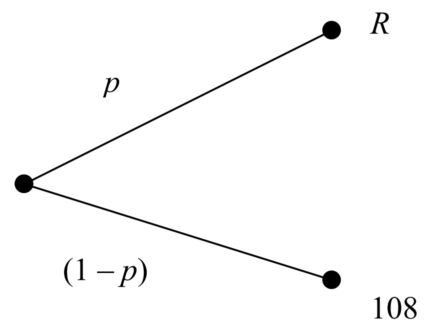
(ii)

The remaining three criteria for admissibility of derivatives in non-linked funds are that they must:

- be for the purposes of reduction of investment risks or efficient portfolio management [*Reg. 55(2)*]
- be listed on a regulated market or transacted with an approved counterparty [*Reg. 55(6)(a)*]
- have a prescribed pricing basis [*Reg. 55(8)*].

QUESTION 2

(i)



R = value of bond in default

P = probability of default

Value = $\frac{1}{1+j}$ {108 – p (108-R)}

where j = risk adjusted interest rate

(ii)

Two parts to the swap – (i) swapping the coupon, (ii) reducing to a par swap.

Swap part 1 will convert the 8% coupon into a floating rate margin, m_1.

Swap part 2 will convert the 5% price discount to par in a floating rate margin, m_2.

Swap part1: $8v$ = floating $(1 - v) + \frac{m_1}{2}(v^{1/2} + v)$

where $6.5v$ = floating $(1 - v)$ and $v = \frac{1}{(1.065)}$

=> $m_1 = 1.476\%$

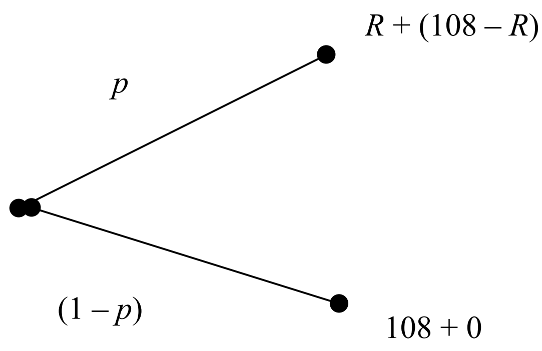
Swap part 2: $5 = \frac{m_2}{2}(v^{1/2} + v)$

=> $m_2 = 5.241\%$

Hence equivalent FRN coupon Libor = $m_1 + m_2 = 6.717\%$

(iii)

Consider the outcome of the package bond + default swap:



Since the payout is the same in either case, the value of the package is 108 / (1 + *i*) where *i* is the riskless interest rate (UK Government bond rate).

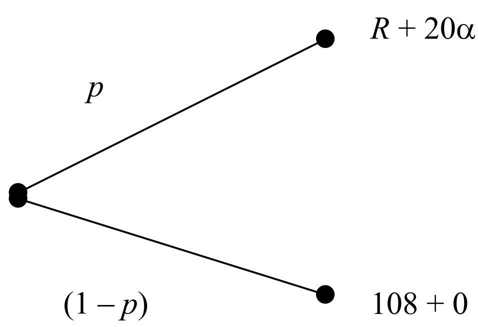
Thus, $\frac{108}{(1.06)}$ = 95 + default swap rate

=> default swap rate = 6.887% (of par)

[*Note: an argument can be made for using the swap curve (i.e. 6.5%) as the definition of the riskless rate. The swap rate is then 6.408%.*]

(iv)

Consider a portfolio consisting of 1 unit of bond and α units of the default option



The picture shows that the pay off at expiry is different between the two outcomes unless $R + 20\alpha = 108$. Since *R*, the recovery value, is unknown until after the default, one cannot calculate a constant value of α.

Thus, whilst one can estimate a sensible hedge amount α, the portfolio is not riskless – so a risk neutral value cannot be calculated.

Alternative: Another way of looking at this situation is that the bond price (i.e. 95) gives a view on the product of p and R. The Credit Default Option value is dependent only on p, thus the bond can only provide a riskless hedge if R is known for certain.

[*Not required, but for completeness*:

$$\text{option value} = \frac{1}{(1+j)} \ \{108 - p\,(108 - (R\text{-}20\alpha))\}.$$

To finance a risky portfolio, a compensatory interest rate – which will be higher than the risk neutral rate – should be used.]

(v)

Credit Default Option (CDO) value

= payout x probability of occurrence x discount factor

- The CDO information specifies that the payout is 20
- The probability of default could be estimated directly from the Examiners Guild forecast, i.e. 8%.
- Discount factor – use a risky interest rate

  One would expect this to be significantly higher than 6.5%.

  An argument could be made for using the value equation in (i) together with the 8% projected default rate and a possible recovery rate of 75%, i.e.

  $$(1 + j) = 105.36 / 95 \ \Rightarrow\ j = 10.90\%$$
- Thus, an ESTIMATE of value is:

  $$20 \text{ x } \frac{0.08}{1.109} \approx 1.4 \text{ to } 1.5$$

(vi)

Points for both (a) and (b):

- In both cases the trigger is more general than that provided by the bond in issue. Thus, a default probability forecast from the bond will understate the true default likelihood. Nevertheless, as stated the bond is the best estimation weapon available.
- For the same reason, an arbitrage neutral price and hedging strategy will not be available.

Points for (a):

- One should use the existing bond as the basis for estimating default likelihood. However, the short term likelihood will probably be different to that implied by a price of 95 for the one year bond.

- Since the bond is of longer maturity than the 6 month swap, the risks of a major error are reasonably low – other than the different definitions of default.

Points for (b):

- There is no obvious instrument which can be used to estimate default in the second year.

- Extrapolation (constant?) of the one year implied probability of default is possible but risk of error is very great.
- It may be 'cheating' but is there another market maker prepared to quote a price? One could price on that basis and hedge directly by offsetting the position with the other market-maker.

QUESTION 3

[*There are a lot of potentially valid answers not listed here, which can also receive marks.*]

(i)

<u>ERM definition</u>

The management of institutional risk as a single strategic concept covering all risk categories and business units.

<u>Risks covered</u> [*the examiners' were looking for most of these in order to award full marks*]

Market, credit, country exposure

Equity stock specific, liquidity, commodity.

Operational, system, legal/documentation, compliance, location.

<u>Advantages for bank</u>

- Defines parameters in which business units can take on risk => better decision making
- Protects core business from volatility of P&L due to unanticipated events.
- Focuses attention on weaknesses of procedure or controls, and attempts to quantify them so can influence behaviour through budgetary process.
- Adds discipline of data collection and analysis.
- Improves quality of risk-return analysis for all managers.
- Bland VaR numbers are of little use if major risks are outside the framework (e.g. UBS holding in LTCM in 1998).

- If ERM was in place, it would have prevented most of the major banking disasters of the last few years.
- Regulators are interested in better quality risk management, as it reduces systemic risk.
- Shareholders also interested, so can assess true company performance (although beware that risk reporting at this level can become a political exercise).
- Gives an overview on what is increasingly a complex, diverse business.

(ii)

<u>MAJOR PROJECTS</u>

<u>Management controls</u>

- Strategic project so needs management "buy-in" and setting of priorities
- Need to improve the monitoring process - extra staff and costs
- Monitoring independent from day-to-day businesses, otherwise unduly influenced
- Dialogue with regulators and internal audit/compliance
- Need to get correct report frequency (monthly, weekly, daily, even intra-day), distribution and design

<u>Decision-making processes</u>

Different management style and culture - not "seat-of-pants" but attending to detail and technical

- Define risk limits and escalation procedures
- Create an overall risk strategy
- Sharpen up budgetary process and project management

<u>System development</u>

- Requirement to set up centralised data warehouse, so data can be analysed by single group
- Putting data into common framework may require substantial manipulation of the data or even re-design of source systems
- Cleaning data where it is erroneous or incomplete (best to get source systems to fix these problems, but there are often conflicting priorities)
- Decisions on update frequency, holding of historical data etc
- Need to source/create new types of data - e.g. correlations.

<u>Analytical development</u>

- Need to define some sort of overall methodology - but theory has limitations in this area
- Need more skilled "quant" staff, separate from business lines, and these need to be data/system experts (e.g. to run Monte Carlo simulations)

- Some areas of risk very difficult to quantify
- Other areas have no methodology at all, and need grass roots investigation
- 80 -20 rule should apply (i.e. if you deliver 80% of what you need, that's enough to make a difference - the last 20% usually takes excessive time and effort)

<u>POTENTIAL IMPLEMENTATION PROBLEMS</u>

- Diverse entrenched management styles and cultures
- Sharing information - not easy cultural change
- Lack of skilled staff
- Danger of risk staff pursuing "red herrings" and getting out of touch with the running of the business
- Costs vs. benefits of ERM (costs will be incurred long before benefits appear)
- Slow to implement as it cuts so deep into the organisation
- Easy to lose sight of the goal amid huge complexity and numerous difficult compromise decisions
- Could be seen as unnecessary insurance - hard to get universal "buy-in"
- Operational risk changes can affect/delay decisions on projects
- Longer time to implement new product lines, especially complex (and potentially profitable) ones, because they have to fit into the risk framework first

and, indeed, everything that could go wrong probably will!

<u>Note on assumptions</u>

The question asks that the candidate record any assumptions he is making. These could include:

The question does not mention the risk appetite of the bank. Assume this is moderate.

No reference to international offices, so assume there are none.

(iii)

<u>Risk-adjusted performance</u>

[*This is only one example of the type of discussion expected.*]

Traditional bonus schemes act as incentives to take too much risk, as the payment is effectively a highly-geared option held by the trader.

If the trader's P&L is risk-adjusted, he cannot (in theory, at least) gain advantage from taking too much or little risk.

The reason for using risk numbers for performance measurement is that ideally one wants to reward the traders who makes the most return per "unit" of risk …

… but do not want to skew the measure so that risk-taking is completely discouraged.

The problem is what methodology to use, as it must encompass a wide range of products, some of which are more market-risk orientated, others more credit-risk.

No framework is perfect. Dowd favours the "Relative Sharpe Ratio" (RSR) approach, which starts by supposing a benchmark standard product, e.g. a 10-year bond …

… then define the incremental return *d* above (or below) this benchmark return. The risk-return ex-post Sharpe Ratio is then the ratio of the experienced return of *d* over the experienced standard deviation of *d*. (The ex-ante ratio can also be calculated if implied values can be assumed.)

This has advantages over other methods that use absolute values (such as RAROC - Risk-Adjusted Return On Capital) in that it avoids the folly of encouraging taking very little risk for a small but guaranteed return (e.g. invest in T-Bills), thereby getting an almost infinite RAROC.

The problem with RSR is in the choice of a benchmark, even if this is done separately for each trading area. Too aggressive or unrepresentative a benchmark might adversely affect the behaviour of the entire department.

Also, the introduction of credit risk adds a further dimension - time. Credit risk acts over a long period (up to 50 years for some swaps), so is incompatible with market risk which effectively increases linearly with time (or, at least, the variance does). It is not easy to calibrate incentives for taking on credit risk over different time periods.

Most efforts to date have been directed at controlling undue risk-taking in absolute terms (via limits and other stress testing).

Also, formulating reliable schemes for remunerating traders (without upsetting everyone) has proved difficult, so it would appear that the bonus is here to stay for a while longer …

Finally, at the very least the bank needs to ensure that the accounting system does not introduce distortions which encourage skewing of risk-taking. This is harder to achieve than it sounds, especially in the areas of cross-hedging and funding.

QUESTION 4

(i)

(a)

$$\text{FRA value} = [7\text{-}8] \text{ x } \frac{1{,}000{,}000{,}000}{400} \text{ x } \frac{1}{(1.07)^{2.5}}$$

$$= -2{,}110{,}963$$

(b)

$$\text{Futures value} = [7\text{-}8] \text{ x } \frac{1{,}000{,}000{,}000}{400} = -2.5 \text{ million}$$

(c)

These will be the same when interest rates are deterministic and not random variables.

In this situation the three month forward interest rate would be known and the price of both an FRA and Futures contract would be zero so that the discount value of the FRA would have no effect.

(ii)

(a)

5% case:

$$\text{Profit} = 11.84 \text{ x } \frac{(7-5)}{400} \text{ x } \{1 \text{ billion}\} \text{ x } \frac{1}{(1.05)^{2.5}} - 10 \text{ x } \frac{(7-5)}{400} \text{ x } \{1 \text{ billion}\} \text{ x } 1$$

$$= 2{,}402{,}071$$

9% case:

$$\text{Profit} = 11.84 \text{ x } \frac{(-2)}{400} \text{ x } \{1 \text{ billion}\} \text{ x } \frac{1}{(1.09)^{2.5}} - 10 \text{ x } \frac{(-2)}{400} \text{ x } \{1 \text{ billion}\} \text{ x } 1$$

$$= 2{,}273{,}953$$

(b)

The hedge ratio is 2.5 / 2.110963, which allows for the emergence of profit at different rates between the contracts.

There is likely to be a strong correlation between changes in forward rates and changes in spot zero coupon rates.

Thus, if FRA rates fall, the falling term interest rate will lead to a profit.

If FRA rates rise, the rising term interest rate will reduce less on the FRA leading again to a profit.

(iii)

(a)

$$d(s_1 s_2) = \{s_2 s_1 \mu_1 + s_1 s_2 \mu_2 + \rho_{12}\sigma_1\sigma_2 s_1 s_2\}dt + s_2 s_1 \sigma_1 dz_1 + s_1 s_2 \sigma_2 dz_2$$

$$\Rightarrow d(s_1 s_2) = s_1 s_2[\mu_1 + \mu_2 + \rho_{12}\sigma_1\sigma_2]\, dt + s_1 s_2(\sigma_1 dz_1 + \sigma_2 dz_2)$$

$$\Rightarrow d(s_1 s_2) = s_1 s_2[\mu_1 + \mu_2 + \rho_{12}\sigma_1\sigma_2]\, dt + s_1 s_2\, \sigma^* dz^*$$

where z^* is a new normal random variable.

(b)

Consider a 2½ zero coupon bond @ 7%:

$$\text{Bond value} = \frac{1}{(1.07)^{2.5}} = 0.8443851$$

Evaluating at a rate of 6.99% and taking the difference:

PV(100bp) = 0.02

[Prices in decimals. To 5 dp, PV(100bp) = 0.01973.]

Since there is no net investment in the futures or FRA contracts, $\mu_1 = \mu_2 = 0$.

$$d(s_1 s_2) = \rho_{12}\sigma_1\sigma_2(s_1 s_2)dt + s_1 s_2\,(\sigma_1 dz_1 + \sigma_2 dz_2)$$

Defining s_1 as a 2½ year zero coupon bond, s_2 as the FRA, it follows that:

$$E_t(s_1 s_2) = s_1 s_2 \exp(\rho_{12}\sigma_1\sigma_2 t)$$

with $\rho_{12}\sigma_1\sigma_2 t = 0.8915 \times [0.02 \times 0.2 \times 7] \times 0.25 \times 2.5$

and so $7.00\,[\exp(\rho_{12}\sigma_1\sigma_2 t) - 1] = 11$ bp (i.e. 0.11%).

Therefore, on a risk neutral basis the futures contract should stand at an equivalent of 11 bp below that of the FRA.

(iv)

This has value 0.11 x 10,000,000,000 / 400 = 2,750,000

The answers in (ii) (a) are smaller because, by rebalancing the trade in that case, the trader can make further profits in the period to expiry.

QUESTION 5

(i)

Other points could be made that earn marks.

Real world

The "real world" is the world which the investor inhabits, not the artificially constructed world where all investors are neutral as to whether they take risk or not.

The real world has different probabilities of future stochastic events occurring than the risk neutral world ...

... the drift on an asset is not the risk-free rate, rather it depends on other factors such as investor risk preferences and utility.

<u>Market price of risk</u>

The "market price of risk" (MPR) is a measure of the trade-off that an investor makes between risk and return in the real world.

Typically, in the real world, the drift μ_v on an asset V as seen by an investor is:

$$\mu_v = r + \lambda\sigma_v$$

where r is the risk-free rate, and σ_v is the volatility (standard deviation) of the asset's return.

$\lambda = 0$ in the risk-neutral world.

<u>MPR does not depend on asset values</u>

This measure can depend on the state variable, r, and changes over time, but is a general add-on for each investor and cannot depend on the particular asset under consideration. There are various ways to demonstrate this.

By algebra

For the short rate r as defined in the question, consider two assets $V_1 = V_1(r, t)$ and $V_2 = V_2(r, t)$, which depend only on r and t.

Using Ito's lemma (as ever) we can express the processes they follow as:

$$dV_1 = \mu_1 V_1 dt + \sigma_1 V_1 dz \quad \text{and} \quad dV_2 = \mu_2 V_2 dt + \sigma_2 V_2 dz$$

for some values of μ_1 and σ_1 etc (which will also possibly depend on r and t).

Now, consider the portfolio $\Pi = (\sigma_2 V_2)V_1 - (\sigma_1 V_1)V_2$, we have

$$d\Pi = (\mu_1\sigma_2 V_1 V_2 - \mu_2\sigma_1 V_1 V_2)dt$$

which is riskless, so must earn the risk-free rate over that period, i.e. $r\Pi dt$.

(*Actually, this should really be approximating with discrete time steps Δt.*)

Hence, substituting for Π and equating the dt terms:

$$r(\sigma_2 V_2 V_1 - \sigma_1 V_1 V_2) = (\mu_1\sigma_2 V_1 V_2 - \mu_2\sigma_1 V_1 V_2)$$

i.e. $\quad r\sigma_2 - r\sigma_1 = \mu_1\sigma_2 - \mu_2\sigma_1$

i.e. $\quad \dfrac{\mu_1 - r}{\sigma_1} = \dfrac{\mu_2 - r}{\sigma_2} = \lambda$, say

and hence λ can depend on r and t, but cannot be dependent on the process for any particular asset.

<u>OR</u>

By argument:

The short rate process r follows exponential Brownian motion, say under probability measure ***P***.

Changes between versions of the real world, and the risk-neutral world, only involve a change of probability measure.

But the CMG theorem states that a change of measure for r, from say $\boldsymbol{P}$ to $\boldsymbol{Q}$, will result solely in the introduction of a different drift term. This term is dependent only on the change of measure.

Under the real world $\boldsymbol{Q}$, any asset based on r must experience the same additional drift term, independent of the asset itself. The drift term is essentially the market price of risk, which therefore is the same for every asset being valued.

(ii)

There are pages and pages one could write about FD methods. The solution here, however, must be succinct, discuss general points but relate to interest-rate options. As the question suggests, it is not necessary to specify a particular model, but equally the candidate will not be penalised for doing so.

The FD method approximates the solution of the PDE by setting up a discrete grid of rate changes Δr and time step Δt.

These must be small enough to make the approximation accurate, but not so small that the number of steps is computationally intense (and rounding errors creep in).

Start by estimating some parameterisation of the particular interest rate model, either (a) by using current bond or swap prices as "market securities" or (b) by regressing on past data. (These parameters are e.g. mean reversion speed, volatility etc.) The former, (a), is preferable, as it starts with market values, so can be used immediately for arbitrage-free pricing. But (b) is useful to check that the model is stable from one day to the next.

Since we are performing this calculation in the real world, we also need to estimate the market price of risk (MPR). The MPR can be assessed in the same way as the parameters, except that it cannot be extracted from past data, as it is time-dependent.

Since everything is based on the single factor r, the entire initial yield curve can be constructed from $r(0)$, and thereafter at every node on the grid. Hence, in particular, the 5-year par bond price can be obtained at every node.

If the nodes are V_{ij}, where i is the number of the step in Δr and j is the number of the step in Δt, then e.g.:

$$\frac{dV}{dr} \approx \frac{V_{i+1,j} - V_{i-1,j}}{2\Delta r}, \quad \frac{d^2V}{dr^2} \approx \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{(\Delta r)^2}, \text{ and } \frac{dV}{dt} \approx \frac{V_{i,j+1} - V_{i,j}}{\Delta t}.$$

[*Hull and Rebonato use opposite node conventions - the above is consistent with Rebonato.*]

There are a number of possible types of FD method. Mostly common are Explicit FD and Implicit FD, but they can also be combined.

The Explicit FD sets V_{ij} and its first and second partial derivatives as a combination of $V_{i+1,j+1}$, $V_{i,j+1}$ and $V_{i-1,j+1}$. This behaves like a trinomial lattice. Indeed, it is really a lattice because the equations which govern the relationship between the nodes create three pseudo-probabilities, p_{up}, p_{mid} and p_{down}, which must be positive and sum to 1.

The Implicit FD reverses the Explicit FD arrangement, and sets V_{ij} as a combination of $V_{i+1,j-1}$, $V_{i,j-1}$ and $V_{i-1,j-1}$. This can only be solved implicitly, hence the name, but there are many methods (such as the Hopscotch method) which efficiently solve the resulting matrix of relationships at each time step.

Then, run the computer program backwards from $t = T$ to $t = 0$ to get the current value of the Call at the central node.

Could also transform variables using x = loge(r), which makes the grid easier to calculate.

Initial conditions: these actually occur at the option expiry, when the option value at each node is the exercise (intrinsic) value at that node.

For the initial conditions, you need to adjust the bond price to remove any coupons paid from $t = 0$ to T.

Boundary conditions: these are simply that the option is zero when the bond price is below a certain level, and equal to intrinsic value above a certain level. These levels need to be considered in relation to the specific problem. The only harm in mis-estimating them is that more calculations will need to be done at irrelevant nodes, which can make the matrix very large.

The American feature also creates a "free" boundary, which can be allowed for in the valuation by a straightforward comparison at each node, between node value (option value) and the early exercise value. If early exercise is optimal, replace the option value with the early exercise value.

Generally, you need to take care that:

- the method chosen is stable - implicit FD is always stable, but explicit FD can have fatal instability if the pseudo-probabilities are invalid (i.e. < 0 anywhere).
- the free boundary does not introduce discontinuities in the valuation.

(iii)

The FD method performs the integral directly by approximating continuous time and state variable values with small discrete steps.

The binomial lattice method also performs the integral, but more subtly, using an expectation over the risk-neutral probability measure.

Both methods integrate the PDE. Obvious for FD - for binomial as follows:

Show binomial does evaluate the integral

The definition of the binomial creates the risk-neutral probabilities of up and down moves - say this measure is $\boldsymbol{P}$.

Then, under $\boldsymbol{P}$, the expectation $\mathbf{E}_{\mathbf{P}}[V_t \mid s]$ is a Martingale, so equals V_s.

The binomial tree approximates this expectation directly using $\boldsymbol{P}$, so must provide an approximation of the expectation integral over each time step.

[*Rebonato p190 goes into more detail on this point.*]

Both methods are suitable for valuing derivatives, and can also easily obtain deltas, gammas and vegas.

FD is computationally intense, slow, fiddly, prone to instabilities ...

... however, lots of literature (especially from Physics) on how to make this method efficient and solve different boundary conditions.

Binomial is fast and efficient, intuitive, stable ...

... however, doesn't converge that fast, and becomes much more complex if the tree does not recombine, as tends to occur for certain processes.

In each case, the one-factor model is not ideal for an option on a 3-year bond, as correlation effects will occur between the 1-year discount rate for the option and the bond yield. The same would apply to any asset where correlation might affect the outcome.

Really a two-factor model is required for these situations - both FD and binomial can be so extended, but the calculation becomes very intense, particularly for FD.