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THE ANALYSIS OF VARIANCE OF MORTALITY RATES

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THE statistics of the *Continuous Mortality Investigation: Assured Lives* 1924–38 (see *J.I.A.* Vol. LXXI, pp. 259 et seq. and pp. 409 et seq.) afford an opportunity for an application of the statistical method of Analysis of Variance. This method is an extension of the well-known χ^2 method and applies to multiple classifications. It has not yet, to our knowledge, been used for the examination of mortality statistics. We suggest, however, that it can with advantage be applied to such questions as the comparison of mortality rates, which was dealt with on pp. 259–61 of the *Journal, loc. cit.*

This note describes an investigation into the material for the years 1934–38, age groups $25\frac{1}{2}$–$29\frac{1}{2}$ to $70\frac{1}{2}$–$74\frac{1}{2}$, classified according to class (whole-life or endowment assurance), participation in profits (with or without profits), and medical examination (medical or non-medical). Furthermore, the rates for durations 3 and 4 are compared with those for durations 5 and over. Thus there are $2\times2\times2\times2=16$ subgroups.

Like many other statistical methods, the Analysis of Variance gives evidence for or against the acceptance of a hypothesis by assuming that it is true and by then calculating the probability that certain tests would give the same or a larger numerical result than the one actually arrived at from the material under consideration. If this probability is smaller than, say, 5% (1%), then the discrepancy from the hypothesis is judged significant (highly significant).*

In the present case our initial hypothesis assumes that the basic probabilities depend on the age group only and that differences between the rates of the 16 subgroups are merely due to random fluctuations. This hypothesis can be tested by the formula†

$$\psi^2=\sum_0 \frac{\theta^2_{cdmp}}{E_{cdmp}}-\frac{\left[\sum\limits_0 \theta_{cdmp}\right]^2}{\sum\limits_0 E_{cdmp}}, \tag{1}$$

where $\sum\limits_0$ means summation over all subgroups. θ and E denote deaths and exposed to risk respectively and the subscripts are explained as follows:

Attribute	Symbol	Value of symbol 1	Value of symbol 2
Class of assurance	c	Life	Endowment assurance
Participation in profits	p	With profits	Without profits
Medical examination	m	Medical	Non-medical
Duration	d	3 and 4	5 and over

The ratio θ_{cdmp}/E_{cdmp} will subsequently be denoted by Q_{cdmp}.

The probability of getting this or some higher value for ψ^2, if the hypothesis is correct, can be found by entering the tables for the χ^2 distribution

* H. L. Seal, in *Tests of a mortality table graduation*, *J.I.A.* Vol. LXXI, pp. 5 *et seq.* calls the hypothesis 'of doubtful improbability' or 'improbable' respectively.

† For the theoretical basis of this and the subsequent formulae and for the limits of their applicability see S. Vajda, *The algebraic analysis of contingency tables*, *J. Roy. Statist. Soc.* Vol. CVI (1943).

(cf. Seal's Table A, *loc. cit.*) at $\chi^2 = \psi^2/\sigma^2$ and for 15 degrees of freedom. Here σ^2 is the variance (square of the standard deviation) which is assumed to be the same in each subgroup and can be estimated by

$$\frac{\sum_0 \theta - \sum_0 (\theta^2/\mathrm{E})}{\sum_0 \mathrm{E} - 16}.$$

(This tentative assumption will itself be tested by the outcome of the calculations. If these point to large discrepancies between the probabilities, the above estimate is of no further value. But if the probabilities appear to be the same in all subgroups, then this estimate is sound.)

If the initial hypothesis is disproved, then we must find the attributes which are, singly or in combination, responsible for this fact. This can be done by testing each single attribute; for instance, the class-effect (c) will be tested by

$$\psi^2 = \frac{\left[\sum_0 \mathrm{Q}_{cdmp} (-1)^c\right]^2}{\sum_0 \frac{1}{\mathrm{E}_{cdmp}}} = \frac{\left[\sum_d \sum_m \sum_p (\mathrm{Q}_{2dmp} - \mathrm{Q}_{1dmp})\right]^2}{\sum_0 \frac{1}{\mathrm{E}_{cdmp}}}. \tag{2}$$

Analogous formulae hold for the (d), (m) and (p) effects. These simple effects do not, however, exhaust all the possibilities. It might be the case that the whole-life mortality is substantially higher than the endowment assurance mortality for medically examined lives, but that the opposite is true for non-medical assurances. Such a fact would be called a (cm) interaction. It would be tested by a formula of the same type as (2), namely,

$$\psi^2 = \frac{\left[\sum_0 \mathrm{Q}_{cdmp} (-1)^{c+m}\right]^2}{\sum_0 \frac{1}{\mathrm{E}_{cdmp}}}.$$

Interactions of higher order are, for instance, (cmp) with

$$\psi^2 = \frac{\left[\sum_0 \mathrm{Q}_{cdmp} (-1)^{c+m+p}\right]^2}{\sum_0 \frac{1}{\mathrm{E}_{cdmp}}},$$

and finally ($cdmp$) with

$$\psi^2 = \frac{\left[\sum_0 \mathrm{Q}_{cdmp} (-1)^{c+d+m+p}\right]^2}{\sum_0 \frac{1}{\mathrm{E}_{cdmp}}}.$$

The number of degrees of freedom for every formula of this type is 1.

It would be rather inconvenient if it were necessary to examine all these 15 effects and interactions individually and it is satisfactory to have formulae to test combinations of them directly. Thus

$$\psi^2 = \sum_0 \frac{\theta^2_{cdmp}}{\mathrm{E}_{cdmp}} - \sum_c \sum_d \sum_m \frac{\sum_p \theta^2_{cdmp}}{\sum_p \mathrm{E}_{cdmp}} \tag{3}$$

tests the significance of the aggregate of all effects and interactions excluding

(c), (d), (m) and their interactions of every order. In other words, this is the aggregate of (p) and all interactions containing p. It has 8 degrees of freedom, and, if it proves non-significant, then any apparent difference between with-profit and non-profit mortality rates can be ignored.

Now our calculations are naturally meant to lead to conclusions not only for separate age groups, but for the life table as a whole. Hence we make use of the additive property of the χ^2 distribution by adding the values of χ^2 and the degrees of freedom of the separate age groups. The result will be judged by entering again the χ^2 tables. Thus any isolated effect in one age group will lose its significance if it is not borne out by others.

Table 1 shows the first stages of the calculations. We find that, on the whole, some effects certainly exist, but that the total for the aggregate involving p lies between the 5 % and the 1 % points. We find, furthermore, that this is due to one relatively high value for age group $40\frac{1}{2}$–$44\frac{1}{2}$. Now it must be remembered that any heterogeneity, e.g. an uncommonly high number of duplicate policies, may increase the real variance above its estimate σ^2, and that the inclusion of different ages in the same group and the combination of results of different years may have a similar effect, as may also the lumping together of material coming from different offices. It appears therefore that, on the whole, participation in profits has no appreciable effect on the mortality.

This conclusion does not contradict the statement on p. 260 of the *Journal*, *loc. cit.*, which refers to 'older ages', i.e. those over 70, whereas our investigation stopped short at 75.

We may then combine any two subgroups which differ only in respect of p and have to deal with only eight subgroups for each age group. The summations implied in the formulae will then extend to eight terms only.

The outcome of the calculations regarding effects and interactions for all age groups is given in Table 2. The results are somewhat surprising inasmuch as they show a value for (d) smaller than those for (c) and (m), and rather high interactions. The meaning of this is not easy to guess. A closer scrutiny reveals that the features can be traced to the sporadic occurrence of particularly high rates. They are most likely due to duplicate policies and we must therefore introduce some suitable adjustment.

After some lengthy trials the best method of dealing with this difficulty was thought to be the replacement of the particularly high rates by the average rate of the remaining seven subgroups and this was done in the following five cases:

Age group	$25\frac{1}{2}$–$29\frac{1}{2}$	$45\frac{1}{2}$–$49\frac{1}{2}$	$55\frac{1}{2}$–$59\frac{1}{2}$	$60\frac{1}{2}$–$64\frac{1}{2}$	$70\frac{1}{2}$–$74\frac{1}{2}$
Subgroup	Life, Medical, Durations 5 & over	Life, Non-medical, Durations 5 & over	Life, Non-medical, Durations 3 & 4		
Original rate ‰	2·3717	7·3139	17·642	33·694	100·775
Replaced by ‰	1·8426	5·6039	14·527	23·268	61·770

The results of the computations based on the latter rates are entered in brackets in Table 2. The following features are now brought out:

(i) The duration is certainly highly significant and so is, to a lesser degree, the class. (As regards the latter, this agrees with the conclusion on p. 260 of the *Journal*, *loc. cit.*)

(ii) On the other hand the mortality rates for medical and for non-medical business do not seem to be substantially different. (The opposite conclusion mentioned in the *Journal* refers to durations 0–2.)

(iii) There are no significant interactions.

It may be well to mention that the warning given in the *Journal* concerning any conclusion on these matters refers with equal force to our procedure. Moreover, it is obvious that no method dealing with the statistics that emerge can of itself reveal any weakness of the data, e.g. inclusion of some cases where the life assured is dead, etc. It must also be borne in mind that we have replaced certain values and thus reduced the number of degrees of freedom by a fraction of 1 in each case.

Our procedure is certainly open to criticism and any better method of dealing with the question of duplicates will be welcomed. It should, however, be clear that the satisfactory result of our calculations is not a necessary consequence of the procedure adopted. If it were, the method would obviously be valueless.

Further calculations which we have carried out seem to show that the examination of earlier periods would not essentially alter the above statements. The 1924–29 statistics lend themselves to a further classification in respect of the year of experience, which necessitates more complicated formulae, involving 5 degrees of freedom for this attribute alone. The results of some calculations regarding this period are given in the upper part of Table 3, which is similar to Table 3 in R. H. Daw's paper *On the validity of statistical tests of the graduation of a mortality table*. It will be seen that the effect of participation in profits is again small, but that the same is not true about the class of policy or the year of experience. The effect of medical examination appears negligible.

We have therefore combined medical and non-medical business and have applied the same tests again, the results of which are given in the second part of the Table. The result is much the same as before, and the same holds if we examine medical business only (see lower part of the Table).

These conclusions do not yet take account of duplicates. If, on this account, the estimate of the standard deviation is increased by 50%, the significance of years of experience disappears, but this does not hold for the class effect. The results are given in brackets in Table 3.

As a final remark concerning duplicate policies it may be pointed out that their incidence is certainly not the same for all ages and factors. As an illustration it can be mentioned that for the age group 46–50, non-medical, Life, without profits, 1926, durations 5 and over, the deaths are more than four times the expected number and that three different ages contribute to this result.

Table 1

Age group	$\sum_{0} \frac{\theta^2_{cdmp}}{E_{cdmp}}$	$\frac{\left[\sum_{0} \theta\right]^2}{\sum_{0} E}$	$1000\sigma^2$	Formula (1) χ^2	*	$\sum_{c}\sum_{d}\sum_{m} \frac{\left[\sum_{p} \theta\right]^2}{\sum_{p} E}$	Formula (3) χ^2	*
25½–29½	3·004	2·966	1·855	20·70	n.	2·986	9·82	n.
30½–34½	5·713	5·677	2·099	16·91	n.	5·702	4·84	n.
35½–39½	10·017	9·975	2·718	15·57	n.	10·002	5·75	n.
40½–44½	19·304	19·164	3·789	36·97	h.s.	19·234	18·49	s.
45½–49½	41·941	41·709	5·604	41·43	h.s.	41·884	10·06	n.
50½–54½	96·019	95·594	8·923	47·60	h.s.	95·909	12·34	n.
55½–59½	191·231	189·707	14·322	106·41	h.s.	191·020	14·78	n.
60½–64½	296·667	294·003	22·751	117·12	h.s.	296·530	6·00	n.
65½–69½	447·607	444·390	36·781	87·46	h.s.	447·137	12·76	n.
70½–74½	738·667	736·670	57·972	34·45	h.s.	737·997	11·56	n.
Total (Degrees of freedom)				524·62 (150)	h.s.		106·40 (80)	s.
Total excluding age group 40½–44½ (Degrees of freedom)							87·91 (72)	n.

* n. = not significant (probability larger than 5 %).
s. = significant (probability between 1 % and 5 %).
h.s. = highly significant (probability less than 1 %).

Table 2. Values of χ^2 for effects and interactions

Age group	(*c*)	(*d*)	(*m*)	(*cd*)	(*cm*)	(*dm*)	(*cdm*)
25½–29½	·13	·52	3·15	·03	5·45	1·78	·26
	(1·11)	(·00)	(1·19)	(·75)	(2·73)	(·42)	(·03)
30½–34½	·69	3·02	·57	·15	2·96	·74	·11
35½–39½	·30	3·58	·36	·56	·06	·46	·00
40½–44½	1·59	·04	·36	2·64	·88	4·43	2·49
45½–49½	1·22	12·76	2·13	5·05	·68	·85	8·04
	(·09)	(4·73)	(·00)	(·72)	(5·00)	(·23)	(2·06)
50½–54½	8·24	3·42	1·77	·44	·63	·10	·48
55½–59½	11·78	3·80	13·12	·17	·79	3·54	·00
	(5·64)	(9·05)	(6·58)	(2·17)	(·03)	(·68)	(1·18)
60½–64½	22·32	·58	21·00	2·38	3·57	12·12	4·27
	(6·05)	(9·17)	(5·37)	(·53)	(·14)	(1·48)	(·04)
65½–69½	1·46	12·07	·64	·00	3·16	·20	·55
70½–74½	9·12	·02	1·39	4·13	6·15	1·88	4·37
	(2·59)	(2·36)	(·05)	(·39)	(1·14)	(·00)	(·46)
Total	56·85 h.s.	39·81 h.s.	44·49 h.s.	15·55 n.	24·33 h.s.	26·10 h.s.	20·57 s.
Rectified total	27·76 h.s.	47·44 h.s.	16·89 n.	8·35 n.	16·73 n.	8·74 n.	7·40 n.

The figures in brackets and the 'rectified totals' refer to the rates as corrected for duplicate policies. Each entry in the body of the Table carries 1 degree of freedom. n., s., h.s. have the same meanings as in Table 1.

Table 3. Mortality of Assured Lives 1924–29: durations 5 and over

Values of χ^2

Age group	26–30	36–40	46–50	56–60	66–70
All data					
Life—Endowment assurance	·05 n.	·16 n.	2·48 n.	5·94 s. (2·55 n.)	1·19 n.
With profits—Without profits	·10 n.	·01 n.	1·03 n.	·76 n.	3·79 n.
Medical—Non-medical	·18 n.	·75 n.	·44 n.	·02 n.	·41 n.
Years of experience	9·58 n.	2·84 n.	16·41 h.s. (7·03 n.)	13·97 s. (5·99 n.)	8·18 n.
Medical and non-medical combined					
Life—Endowment assurance	·20 n.	·83 n.	13·09 h.s. (5·61 s.)	35·73 h.s. (15·31 h.s.)	13·36 h.s. (5·73 s.)
With profits—Without profits	·92 n.	·07 n.	1·43 n.	2·15 n.	6·78 h.s. (2·91 n.)
Years of experience	7·92 n.	12·61 s. (5·43 n.)	18·49 h.s. (7·92 n.)	13·17 s. (5·64 n.)	8·19 n.
Medical only					
Life—Endowment assurance	·07 n.	1·71 n.	13·15 h.s. (5·64 s.)	31·37 h.s. (13·44 h.s.)	12·76 h.s. (5·47 s.)
With profits—Without profits	2·94 n.	·02 n.	3·04 n.	3·03 n.	6·06 s. (2·60 n.)
Years of experience	3·34 n.	12·38 s. (5·31 n.)	20·22 h.s. (8·67 n.)	11·55 s. (4·95 n.)	1·72 n.

The figures in brackets refer to a 50% increase in the estimated standard deviation.
n., s., h.s. have the same meanings as in Table 1.