JSS 17 (5) (1965) 407-421

AN APPLICATION OF COMPUTER TECHNIQUES TO MORTALITY GRADUATION

by

R. C. LLOYD

INTRODUCTION

1. S. BENJAMIN (*J.S.S.* 17, 93) advocates a re-examination of actuarial methods with a view to taking advantage of the capabilities of electronic computers. A somewhat specialized field of actuarial work where such a re-examination is profitable is mortality graduation by means of curve fitting. This field is a specialized one because, in practical actuarial work, graduation can usually be done adequately by straightforward means—for example, a graphic graduation of withdrawal rates for a pension scheme valuation. Full-scale curve fitting is usually reserved for the graduation of assured lives mortality experiences.

2. In the graduation of modern assured lives mortality experiences, where the range of ages is from about 15 to the end of life, it has been found that a mathematical formula usually must have at least 5 parameters for a reasonable fit. In these circumstances the conventional moments or summation method of fitting may fail and in practice a process of trial and error has to be adopted.

3. The effort required to complete a satisfactory graduation by such a trial and error process has to be experienced to be appreciated. Few actuaries or students have the opportunity of gaining practical experience in this field and a few remarks on the problems of manual methods may, therefore, not be out of place.

MANUAL METHODS

4. The following is one way in which the problem can be tackled. The first problem to be faced is the choice of a formula that will satisfactorily graduate the given data. The data are analysed at

least to the extent of deriving ungraduated values of q_x from quinquennial groups of data. The progression of these values may suggest one or more formulae that may be worth trying. If more than one formula has to be tried the work is greatly increased.

5. Assuming that the formula that has been selected is of the Perks or Beard type, an initial trial value for the parameter c will be selected—this value may be suggested by the ratios of ungraduated group q_{x+5}/q_x for the middle age range. With this value of c an attempt is then made to fit the middle age range upwards by a simplified formula consisting of those parameters that are important in that age range. The value of c will be varied until a reasonable fit is obtained. Then the rest of the data are brought in and the additional parameters fitted in one by one, at the same time making consequential adjustments to the parameters already fitted. This can be a very tricky process and it may well be that the value of c will also have to be changed, perhaps several times. It should by now have become clear whether or not a suitable formula has been chosen.

6. Up to this stage the work will probably have been done in quinquennial groups. The arithmetic can be reduced by using interest tables to produce values of powers of c. For example if $c = (1{\cdot}055)^2$, $5\frac{1}{2}\%$ interest tables can be used. The use of interest tables restricts the choice of c and, in general, variations of less than about $\frac{1}{4}\%$ in the value of c cannot be made.

7. When a graduation has been made that appears satisfactory, it is necessary to prepare a schedule based on individual ages, in order to test the graduation adequately. At this stage deficiencies in the graduation might appear that might require further adjustments to be made. For example, there might be too many consecutive deviations of the same sign.

8. The difficulties involved in the above processes should not be underestimated. The work involved can be so considerable as to lead to the acceptance of a graduation with one or more unsatisfactory features, merely because of the work involved in performing further trials.

THE COMPUTER APPROACH

9. While the computer's calculating ability can eliminate the tedium of manually testing the appropriateness of any given set of parameters, its use should not be limited to this function. The program should be made automatic by making the computer successively select new sets of parameters, test their effect and, by a process of successive approximation, eventually either arrive at a solution that satisfies some predetermined criteria or determine that such a solution does not exist. The program should if possible be designed to cater for a family of curves so that the selection of the best curve can be facilitated—the computer being made to try some or all of the various possibilities. Finally, the computer program should provide for the automatic calculation of the life table as well as some monetary functions, when a successful graduation has been obtained. This latter facility in the program is of particular advantage in judging the financial effect of different graduations and can aid the selection of the data to be used. For example, if several years' data are available, the effect of including or excluding the data for particular periods can readily be seen. For, once the program has been written, the computer should be able to perform a complete graduation and the calculation of the life table and monetary functions in a few minutes.

10. What then are the key requirements that must be met in order to write a suitable program? It must be possible

(*a*) to derive the values of a set of parameters for the next trial from a given 'information set' (a convenient name for certain given information);

(*b*) to specify quantitative criteria for judging whether or not a given trial has produced what is *prima facie* a successful graduation;

(*c*) to relate the results of the trial to the pre-set criteria in a way that can be used to alter the information set so that the next trial will be a closer approximation to a successful graduation.

THE APPLICATION

11. The family of curves selected for the application was

$$q_x = A + \frac{Bc^x}{Fc^{-hx} + 1 + Dc^x}$$

and the program caters for positive integral values of h from 1 to 9, although in practice h is not likely to exceed 3. When $h = 2$, the formula is Beard's formula, used to graduate the A 1949–52 experience. When $h = 1$ the formula has an obvious affinity with the generalized Perks form. The effect of increasing the value of h is to increase the slope of the curve in the section that lies between the late 30's and the middle age range. Thus the family of curves is suitable for graduating mortality experiences where

(*a*) it is desired to exhibit essentially level rates of mortality at the early ages,
(*b*) the middle age range mortality follows the Gompertz form, and
(*c*) there is a flattening of the mortality curve at the more advanced ages.

12. The program was developed for use in graduating the South African assured lives mortality experience and was used successfully in that connexion. It was felt, however, that for the purpose of this article it would be of more general interest to illustrate its effectiveness by using the $h = 2$ form to graduate the A 1949–52 2 years and over (all classes) data, as this was the formula actually used in the official graduation. The flexibility of the method is demonstrated by showing that minor variations in the pre-set criteria are all that are required in order to produce different graduations. Summarized results of two graduations are set out in §§20–23. One graduation aims at an overall approximate equivalence of actual and expected deaths, while the other graduation departs from strict adherence to the data to almost exactly the same degree as did the official graduation.

THE COMPUTER PROGRAM

13. A detailed description of the specification and operation of the program is given in the Appendix. Of particular note to those

who may not be familiar with computer techniques are the steps taken to ensure that the computer does not continue indefinitely when the problem is not capable of solution. Attention is drawn to the limitation placed on the number of trials (T) that may be made in a given run, and to the steps taken to abandon the run if the iteration process for solving the basic equation does not show reasonable signs of success at an early stage. Comment on the key points of the program follows.

14. The information set consists of an approximate value of the parameter A and approximations to the values of q_{40}, q_{60}, q_{80} and q_{100}. The initial values of the information set form part of the input to the computer, and need not be carefully chosen. From these 5 values, the program first derives the value of the parameter c and then calculates the values of B, D and F, so as to make the curve pass through the four selected points. The program arrives at further information sets that successively cause closer approximations to the final result.

15. The pre-set criteria for judging the success or otherwise of any trial consist of 5 pairs of limits. The age range is divided into 5 sections, viz. up to 30, 31–50, 51–70, 71–90 and 91 and over. Each pair of limits defines the lower and upper bounds of the ratio of expected to actual deaths in its particular age range. A trial is considered to be successful when, for each of the 5 sections of the age range, the ratio of expected to actual deaths is within the allowed limits. If one or more of the sectional ratios are outside the allowed limits the trials continue, unless trial T was the last trial completed.

16. The final link in the chain is a method of using the results of the trial to modify the information set so as to produce a closer approximation to a solution. The procedure adopted is a very simple one. Each of the sections of the age range is regarded as being primarily governed by one of the 5 items from the information set. Thus the parameter A governs the age range up to age 30, the value of q_{40} governs the age range 31–50 and so on. When a new trial is required, each item in the information set (even in the sections that were within limits) is modified by means of the relationship between the expected and actual deaths for the section it governs. The method of modification (except in the case of A)

assumes that a given change in the value of the item from the information set will produce a proportionate change in the value of the expected deaths for the appropriate section of the age range. This assumption is, of course, not strictly correct, for, apart from other considerations, any section of the age range cannot be regarded as being entirely independent of the others. However, the assumption is sufficiently accurate for practical purposes. The method of modification aims to make each sectional ratio of expected to actual deaths in the next trial equal to the mid-point of its range of limits. For example, suppose that the section governed by q_{40} had a ratio of expected to actual deaths of 1·04, whereas the limits were $1 \pm \cdot 01$. Then the new trial value of q_{40} would be the old one reduced in the ratio of 1/1·04.

17. There are two practical points worth noting on the choice of the closeness of the limits. The values of q are taken to 5 decimal places. Suppose that q_{40} is ·00200, then the smallest change that can be made is 1 in the last decimal place, or say $\frac{1}{2}$%. If the spread of the limits amounts to less than this, for example $1 \pm \cdot 002$, then it could happen in a particular case that the limit condition would never be satisfied. This is an eventuality that can readily be avoided when selecting the limits. The second practical point is that the limits would normally be set fairly close so as to ensure that expected and actual deaths do not differ by an appreciable amount. For example, if there were 10,000 deaths in a particular section, limits of $1 \pm \cdot 01$ could result in a discrepancy of 100 deaths, and limits of $1 \pm \cdot 001$ would be preferable. Even if as a matter of deliberate policy it was, for example, decided to overstate the deaths by 2% in a section, the limits would still be set close. With 10,000 deaths, appropriate limits might be $1 \cdot 02 \pm \cdot 001$.

18. During each trial the program sets up a comprehensive schedule of data and the schedule resulting from the final trial is printed out. The schedule has been designed to enable a number of conventional graduation tests, including the χ^2 test, to be easily applied. While the pre-set criteria adopted for this application are, if satisfied, *prima facie* evidence of a satisfactory graduation, it is still possible for the graduation to be defective in one way or another. For example, it may fail to pass a sectional χ^2 test. The

program also provides optionally for the calculation of life table values and monetary functions.

19. The whole process involves, therefore, feeding into the computer the program, the data, the information set, the pre-set criteria and a few miscellaneous items, and then waiting while the computer does in a few minutes what by manual methods might have taken days or even weeks.

RESULTS

20. In graduating the A1949–52 2 years and over (all classes) data the information set specified was:

$$A = \cdot 00115 \qquad P = \cdot 12000$$
$$M = \cdot 00200 \qquad Q = \cdot 43000$$
$$N = \cdot 01800$$

The pre-set criteria were:

	First graduation		Second graduation	
s	a_s	b_s	a_s	b_s
1	·9950	1·0050	·9800	·9900
2	·9970	1·0030	·9940	1·0000
3	·9995	1·0005	1·0105	1·0115
4	·9998	1·0002	1·0108	1·0112
5	·9997	1·0003	1·0534	1·0540

The pre-set criteria for the first graduation are centred on unity, while those for the second graduation are centred on ratios that fairly closely reproduce the distortion evident in the official graduation.

The miscellaneous input was:

$T = 16$,
$v = 1{\cdot}40$ (Refer *J.I.A.* 83, 34),
$y = 625$ (i.e. origin $62\frac{1}{2}$, as in official graduation),
$J = \cdot 0400$,
$H = 020$.

Also input were the exposed to risk and actual deaths for each age from 21 to 100 inclusive.

21. The first graduation was completed in 3 trials, and the second in 9 trials. The total computer time taken for the two

graduations, including the life table and monetary function calculations, was less than 15 minutes.

The values of the parameters were as follows:

	First graduation	Second graduation	Official graduation
c	1·109978	1·107733	1·107756 = $(1{\cdot}0525)^2$
A	·00113	·00111	·00111
B	·02165291	·02186937	·0218623
D	·03313705	·02732375	·0272978
F	·01554314	·01857066	·01846

22. The following table shows the differences between actual and expected deaths in quinquennial age groups for the two graduations as compared to those for the official graduation:

		First graduation		Second graduation		Official graduation	
	Actual	Actual – Expected		Actual – Expected		Actual – Expected	
Ages	deaths	+	–	+	–	+	–
21–25	338	19	—	24	—	25	—
26–30	821	—	17	—	3	—	4
31–35	1,239	—	90	—	73	—	73
36–40	2,602	30	—	62	—	62	—
41–45	4,890	—	38	—	7	—	7
46–50	8,489	95	—	77	—	69	—
51–55	11,562	314	—	196	—	193	—
56–60	13,026	—	99	—	272	—	271
61–65	11,826	86	—	—	44	—	41
66–70	8,442	—	305	—	373	—	371
71–75	8,953	—	35	—	85	—	85
76–80	8,763	—	160	—	223	—	223
81–85	6,601	196	—	106	—	106	—
86–90	3,431	—	6	—	102	—	103
91–95	1,129	28	—	—	26	—	26
96–100	174	—	28	—	44	—	44
	92,286	768	778	465	1252	455	1248
		–10		–787		–793	

The similarity between the second graduation and the official graduation is noteworthy.

23. The financial effects of the various graduations are briefly indicated by the following 4% values of $100P_x$.

Age	First graduation $100\,P_x$	Second graduation $100\,P_x$	Official graduation $100\,P_x$
20	·656	·656	·657
30	·987	·990	·990
40	1·572	1·579	1·579
50	2·618	2·632	2·632
60	4·500	4·525	4·525
70	8·064	8·117	8·117
80	14·881	15·104	15·105
90	26·494	27·640	27·650

For all practical purposes the second graduation can be regarded as identical with the official graduation. The first graduation, on the other hand, shows what would have resulted had the Committee decided to adhere closely to the data. This graduation adequately satisfies all the tests applied to it, including in particular the χ^2 test, the sectional χ^2 test, and the usual tests based on sign changes and standard errors.

GENERAL

24. It is perhaps worth recording that these two graduations were the only ones attempted on the given data. They are first trials—not final trials made after earlier unsuccessful attempts. The power and flexibility of the method is thus readily apparent. The specification of suitable limits, in itself a very simple process, enables the actuary either to adhere to the data, or to depart from it to the extent he chooses. The use of the program to construct hypothetical tables is, therefore, possible.

25. Other methods can no doubt be devised. Criteria might, for example, be based on χ^2. Furthermore, the choice of ages 40, 60, 80 and 100 in the information set is not essential. Ages 38, 57, 76 and 95 could just as well have been chosen, but then t would equal c^{19}, requiring obvious amendments to the program. The ages included in the various sections of the age range could also be varied, with consequent adjustment required in the program. However, for the graduation formulae selected, the method and the criteria chosen are, as has been demonstrated by the results, very effective.

26. Other graduation formulae might require variations in method. In some cases the expression for deriving the value of c becomes impossibly complicated, and the method has to be adapted to cater for successively improving assumed values of c.

27. Against the simplicity and value of using the computer must be set the effort required in devising a method and writing and testing the program. This effort is not, however, considered to be excessive in view of the results obtained, particularly as the program, once written and tested, is available for use as often as required. The extent of the programming effort in the present case can be judged from the fact that the program consisted of some 1500 single-address instructions, made up of about 400 for input/output subroutines, about 800 for the graduation process including setting up the schedule for printing and about 300 for the life table and monetary function calculations and setting them out for printing. The programming was done in machine code, but use was made of some standard subroutines. The effort would be considerably reduced if the program was written for a machine that had a suitable autocode system available.

ACKNOWLEDGEMENTS

28. I am indebted to Mrs B. Mittag, who wrote the program, and to Messrs M. Knox and S. Liprini, whose earlier programming assistance helped in the development of the method.

APPENDIX

Specification of program for fitting general formula:

$$q_x = A + \frac{Bc^x}{Fc^{-kx} + 1 + Dc^x}.$$

1. *Input.* The computer input via punched paper tape consists of the following:

Data

E_x = exposed to risk at each age $15 \leqslant x \leqslant 110$
θ_x = actual deaths at each age $15 \leqslant x \leqslant 110$

At each age the maximum values are 6 decimal digits plus 2 places of decimals for E_x, and 4 decimal digits for θ_x. Maximum total is

8 decimal digits plus 2 places of decimals for E_x, and 6 decimal digits for θ_x.

Information set

A' = 5-figure decimal fraction
= first approximation to the value of parameter A
$= \sum_{15}^{30} \theta_x \Big/ \sum_{15}^{30} E_x$, say.

M, N, P, Q are rough estimates (as 5-figure decimal fractions) of

$$q_{40},\quad q_{60},\quad q_{80}\quad \text{and}\quad q_{100}.$$

Pre-set criteria

Five pairs of numbers a_s, b_s where $a_s < b_s$. Each value of a_s or b_s is either a four-decimal fraction or unity plus a four-decimal fraction. Each value of s represents a section of the age range, as follows:

s	Age range
1	Up to 30 inclusive
2	31–50
3	51–70
4	71–90
5	91 and over

Each pair a_s, b_s defines respectively the lower and upper bounds of the permissible limits of the ratio of expected to actual deaths in the age range s.

Miscellaneous

T = maximum number of trials the computer may make in a given run.

v = factor for adjusting the standard error to allow for duplicate policies
= unity plus two decimal places.

y = origin
= any convenient 3-decimal digit number, which is interpreted as 2-decimal digits plus one place of decimals (to allow for ½ ages).

J = 4-decimal digit fraction
= rate of interest for Stage 5. If J is specified as 0, Stage 5 is omitted in that particular run.

H = 3-digit code specifying which formula or formulae are to be used in a given run, e.g. $H = 123$ means consecutively try $h = 1$, $h = 2$ and $h = 3$; $H = 020$ means only try $h = 2$.

2. *Output.* The computer output consists of a magnetic tape record of a schedule suitable for applying most of the more usual graduation tests, and, where Stage 5 applies, a magnetic tape record of the life table and monetary function calculations. A printed record is obtained by printing the results on an off-line printer. The details that appear on the schedule are given below together with explanatory notes for the programmer:

Column no.	Item	Programmer's notes
1	x (= age)	Assumed to be nearest exact on census date, thus q derived for $\frac{1}{2}$ ages
2	E_x	
3	θ_x	
4	$q_{x-\frac{1}{2}}$	Graduated value
5	Expected deaths	$E_x q_{x-\frac{1}{2}}$ rounded to 2 decimal places
6	Deviation +	Positive values of col. (3) − col. (5)
7	Deviation −	Negative values of col. (3) − col. (5)
8	Accumulated deviation +	Positive net values on summing cols. (6) and (7) together, starting at youngest age
9	Accumulated deviation −	Corresponding negative net values
10	Standard error squared	$v^2 E_x p_{x-\frac{1}{2}} q_{x-\frac{1}{2}}$ (2 decimal places)
11	Standardized deviation	Arithmetic value of col. (6) or col. (7) divided by square root of col. (10) (2 decimal places)
12	Chi squared	Square of col. (6) or col. (7) divided by col. (10) = col. (11) squared (2 decimal places)

Totals are provided for all columns except columns 1 and 4.

Stage 1—solving for parameters

3. The value of A' is subtracted from M, N, P and Q to give net values of m, n, p and q, which are then used to derive a value of c^{20} as follows. Writing t for c^{20}, we have

$$m(Ft^{-2h}+1+Dt^2) = Bt^2$$
$$n(Ft^{-3h}+1+Dt^3) = Bt^3$$
$$p(Ft^{-4h}+1+Dt^4) = Bt^4$$
$$q(Ft^{-5h}+1+Dt^5) = Bt^5.$$

By eliminating B, D and F we obtain the basic equation

$$t^{h+2}mn(q-p)-t(t^h+1)mq(p-n)+pq(n-m) = 0.$$

This is solved by iteration to obtain t.

4. The iteration process is carried out as follows: Let $L(t)$ represent the left-hand side of the equation. $L(6)$ is calculated and tested for sign. If it is negative, $L(7)$ is calculated; if positive, $L(5)$ is calculated, and so on until successive positive integers t_1 and t_2 are found such that $L(t_1)$ is negative and $L(t_2)$ is positive. The next step is to calculate

$$t_3 = \frac{t_1|L(t_2)|+t_2|L(t_1)|}{|L(t_2)|+|L(t_1)|},$$

and then to calculate $L(t_3)$ and test for sign. t_3 then replaces t_1 or t_2 depending on which one gives rise to the same sign in L. The process is continued with appropriate tests until successive values of t agree to a given number of decimal places. This value is the required value of c^{20}. If in the range $1 \leqslant t \leqslant 10$ a positive and a negative value of $L(t)$ are not found, the run is abandoned.

5. The 20th root of t is then found by starting with $u_1 = 1{\cdot}1$ and using the relationship

$$u_{n+1} = \frac{1}{20}\left[19u_n+\frac{c^{20}}{u_n^{19}}\right]$$

to obtain successive approximations. The square-root of c, required for half-age values of q_x is similarly calculated using the relationship

$$w_{n+1} = \frac{1}{2}\left[w_n+\frac{c}{w_n}\right].$$

6. The remaining parameters are derived as follows:

$$F = \left[\frac{t^{4h}(t-1)}{t^{h+1}-1}\right]\left[\frac{p(n-m)-t_m(p-n)}{m(p-n)t^{h+1}-p(n-m)}\right],$$

$$D = \frac{mt^{3h+1}-nt^{3h}+F(mt^{h+1}-n)}{(n-m)t^{3(h+1)}},$$

$$B = \frac{m[F+t^{2h}+Dt^{2(h+1)}]}{t^{2(h+1)}}.$$

7. In order to improve the look of the parameters, which might otherwise have too many zeroes after the decimal place, the program provides for an optional origin. This is not an essential step. In the run that produced the results set out in paragraphs 20 to 23 the origin was taken as $62\frac{1}{2}$. The specification of this origin causes the program automatically to adjust the calculated values of B and D by multiplying by $c^{62\frac{1}{2}}$ and of F by dividing by c^{125}.

Stage 2—preparation of schedule

8. The schedule is calculated one line (age) at a time, each line being output on to magnetic tape, when complete, to free the computing store. The number of the trial is indicated on the schedule as well as the values of all the parameters. Totals for the various columns are accumulated from line to line and, in particular, the sectional totals of expected and actual deaths are recorded. The ratios of expected to actual deaths are computed for the 5 sections (rounded to the nearest integer in the 4th decimal place) and compared with the appropriate limits a_s, b_s. If one or more sections have a ratio outside the allowed limits, the 5 pivotal values are all recalculated as follows:

Stage 3—recalculation of pivotal values

9. Let $(ED)_s$ = expected deaths in section s,
$(AD)_s$ = actual deaths in section s,
$(E_x)_s$ = exposed to risk in section s,
$(ab)_s$ = mid-point of the range a_s to b_s in section s.

10. The new trial value of A' is given by

$$\frac{(AD)_1(ab)_1-(ED)_1}{(E_x)_1}+\text{previous trial } A'.$$

The new trial values of M, N, P and Q are given by

$$(\text{previous trial } M, N, P \text{ or } Q)\times\frac{(AD)_s}{(ED)_s}\times(ab)_s.$$

All trial values so calculated are rounded to the nearest integer in the 5th decimal place.

Stage 4—repeated trials

11. The program now re-enters Stage 1 and trials continue until the *T*th trial is complete or until all 5 sections have ratios of expected to actual deaths within the limits a_8, b_8, whichever occurs first. Each schedule over-writes the previous one on magnetic tape, so that the last trial made is the only one recorded when the program terminates. Where more than one formula is being fitted in a given run, the program switches automatically to fitting the second formula after fitting (or being unsuccessful in fitting) the first, and so on. The results for each formula are allocated to separate areas of magnetic tape so that the over-writing process does not destroy valuable work previously done on fitting an earlier formula.

Stage 5—life table and monetary functions

12. Where the value of *J* is not 0, the program causes values of q_x, l_x and e_x to be calculated for each graduation, as well as values of D_x, N_x, A_x and $\ddot{a}_x$ at rate of interest *J*. These values are appropriately arranged on magnetic tape for subsequent printing.