JIA 84 (1958) 0001-0036

INSTITUTE OF ACTUARIES

THE APPLICATION OF ELEMENTARY LINEAR PROGRAMMING TO APPROXIMATE VALUATION

By S. BENJAMIN, B.A., F.I.A. and
C. W. BENNETT, D.S.C., F.I.A.
of the Prudential Assurance Company Ltd.

[Submitted to the Institute, 30 September 1957]

It is a common experience in many fields of investigation to find that the standard methods which have been developed and used in a given field prove inadequate in a particular application. In these circumstances it is often found that no further progress can be made along lines hitherto considered conventional, because some fundamental question has not been explicitly posed and answered by the standard methods. In life assurance a feature of existing methods of approximate valuation, which has not received prominence, is their inability to measure satisfactorily the size of the possible error involved. A wide variety of approximate methods have been in use, and some have been found to give consistently 'good' results, but the question of what is meant by 'good' has been left unanswered. The background of this paper is that the authors, faced with this situation when tackling an essentially practical problem, were virtually forced to break away from traditional methods and seek a new tool.

1·2. It was considered desirable to reduce the valuation records of some of the tables in the Industrial Branch, in particular endowment assurances maturing at a fixed age, e.g. 65 n.b.d. For each maturity age current practice used a dual tabulation with a separate class book for each year of entry, and within the book a separate page for each age at entry. The data tabulated were number of policies, sum assured, and office premium, and the clerical work throughout the year which this involved was considerable having regard to the absolute amount of the net liability involved. The problem therefore was to select a method of grouping which, in conjunction with an approximate method of valuation, would give satisfactory estimates.

Now the accuracy of an estimate depends upon

(*a*) the method of grouping adopted,
(*b*) the amount of basic data tabulated,
(*c*) whether or not the chosen method of approximation gives the 'best' estimate in the given conditions.

In the course of the paper these three components are briefly examined and by an elementary application of linear programming a criterion is indicated for judging them. M. G. Kendall, in his Alfred Watson Memorial lecture (*J.I.A.* **82**, 221), stated that linear programming 'deals with the problem of allocating resources in some optimal way...'. The less general and more formal definition given below may be found in *The Theory of Games and Linear Programming*, by S. Vajda, to which the reader is referred for more advanced study of the subject.

1·3. The technique of linear programming is relatively modern and there is no reference to its potential application to life office valuations in *J.I.A.* The purpose of the first part of this paper is, therefore, two-fold:

(1) It may serve as an elementary introduction to a subject which may well demand increasing attention with the advent of electronic computers.

(2) Since no really satisfactory approximate method appears to have been published for the valuation of Industrial Branch endowment assurances maturing at a fixed age, e.g. 65 n.b.d., it develops the technique of linear programming using a block of such business as an example.

2·1. The term Linear Programming (L.P.) describes the solution of the following type of problem:

Let a set of k equations, or constraints, be given involving the n non-negative variables $x_1, x_2, \ldots, x_n$, where $k<n$,

$$\left.\begin{aligned} a_1x_1+a_2x_2+\ldots+a_nx_n&=A,\\ b_1x_1+b_2x_2+\ldots+b_nx_n&=B,\\ \cdots\cdots\cdots\cdots\cdots&\cdots\\ k_1x_1+k_2x_2+\ldots+k_nx_n&=K.\end{aligned}\right\} \qquad 2\text{(i)}$$

It is required to find values of the variables which satisfy the constraints and make the value of a given linear form (L.F.)

$$m_1x_1+m_2x_2+\ldots+m_nx_n=M \qquad 2\text{(ii)}$$

as small (or large) as possible, i.e. we are required to minimize or maximize the L.F. In general, we speak of 'optimizing' whenever we mean minimizing or maximizing or both. It is assumed that the k constraints are not linearly dependent; otherwise one or more of them could be omitted. If the L.F. is linearly dependent on the left-hand sides of the constraints the maximum and minimum of the L.F. are equal and a unique value of M will be obtained.

2·2. Applying this definition to life assurance valuations (Industrial Branch) consider the situation in which it is required to estimate the total reserve value from a grouped tabulation of the basic data. $A, B, \ldots, K$ represent the grouped valuation data (e.g. A may be total sums assured and B total premiums, etc.), $x_1, x_2, \ldots, x_n$ the sums assured at each age and $m_1, m_2, \ldots, m_n$ the reserve values per £1 sum assured, M being the total reserve value. Before proceeding further we need the following definitions.

We define a 'feasible distribution' as any hypothetical full tabulation of data which, if grouped in the given manner, would lead to the same given information, i.e. any distribution $x_1, x_2, \ldots, x_n$ which would satisfy all k constraints. Thus the original distribution is one of a set of feasible distributions. A 'feasible value' is defined as any (total reserve) value which would arise from a full valuation of a feasible distribution (whether or not it could also arise from a non-feasible distribution). Thus a feasible distribution $x_1, x_2, \ldots, x_n$ substituted in the L.F. will produce a feasible value of M.

In the definition of L.P. there were k constraints and n variables and we define a 'basic feasible distribution' as a feasible distribution which contains $(n-k)$ zero elements and k non-zero elements. Where such distributions also optimize the L.F. they are called 'optimum basic feasible distributions'.

3·1. Consider a block of endowment assurances issued out of the Industrial Branch at ages 17–49 n.b.d. and due to mature at age 65 n.b.d. Assume that the classification is by year of issue and that for a particular duration t at the date of valuation the only information given is the total sum assured in force $\mathcal{S}'$, and let

$$S_x = \text{sum assured in force for age at entry } x \text{ n.b.d.},$$

so that $\sum_x S_x = \mathcal{S}'$, and let

$$V_x = \text{net liability factor per £1 sum assured for age at entry } x \text{ n.b.d.},$$

so that $\sum_x S_x V_x = \mathcal{V}$ the total net liability. Formally we have one linear constraint, viz.

$$S_{17} + S_{18} + \ldots + S_{49} = \mathcal{S}'.$$

We require the maximum and minimum of the linear form (L.F.)

$$V_{17} S_{17} + V_{18} S_{18} + \ldots + V_{49} S_{49} = \mathcal{V},$$

i.e. we wish to 'optimize' $\mathcal{V}$ for non-negative values of the variables $S_{17} \ldots S_{49}$.

3·2. Numerically, as at the valuation date, year of assurance 9, the sum assured in force $\mathcal{S}'$ was £204,866. The valuation factors $V_x = {}_{9\frac{1}{2}}V_{x-\frac{1}{2}:\overline{65-x}|}$ were based on E.L.T. No. 10 (Males) $2\frac{1}{4}\%$ and by inspection the largest value of V_x was ·535 at age 49 and the lowest ·128 at age 17. The distribution and the solution were therefore:

Maximum

$$\left.\begin{array}{ll} & S_{49} = 204{,}866 \\ \text{all other} & S_x = \qquad 0 \end{array}\right\} \quad \text{whence} \quad \mathcal{V} = 204{,}866\, V_{49} = 109{,}603.$$

Minimum

$$\left.\begin{array}{ll} & S_{17} = 204{,}866 \\ \text{all other} & S_x = \qquad 0 \end{array}\right\} \quad \text{whence} \quad \mathcal{V} = 204{,}866\, V_{17} = 26{,}223.$$

Since we shall wish to compare different ranges in different circumstances we need a standardizing measure of zero dimension. The range $(\alpha;\ \beta)$, where $\alpha < \beta$, will be written as

$$\pm\theta\% \text{ standardized,} \quad \text{where} \quad \theta = 100\left(\frac{\beta-\alpha}{\beta+\alpha}\right).$$

Thus in the above example the range (26,223; 109,603) becomes $\pm 61{\cdot}4\%$ standardized, which is too wide for practical use, and it is clear that insufficient information has been made available.

4·1. Consider now the above example with the added information that the total office premiums in force corresponding to $\mathcal{S}'$ were $\mathcal{P}'$.

Let P_x = office rate of premium per £1 sum assured for age at entry x n.b.d. Formally we have two linear constraints, viz.

$$S_{17} + S_{18} + \ldots + S_{49} = \mathcal{S}', \qquad 4\,\text{(i)}$$

$$P_{17} S_{17} + P_{18} S_{18} + \ldots + P_{49} S_{49} = \mathcal{P}'. \qquad 4\,\text{(ii)}$$

We require the maximum and minimum of the L.F.

$$V_{17} S_{17} + V_{18} S_{18} + \ldots + V_{49} S_{49} = \mathcal{V} \qquad 4\,\text{(iii)}$$

for non-negative values of the variables $S_{17} \ldots S_{49}$. Standard methods of solution, where more than two constraints are involved, normally require an onerous arithmetical routine, but fortunately where only two linear constraints are imposed the following simple graphical method is available.

Construction I

4·2. Divide 4 (i), 4 (ii) and 4 (iii) by the constant $\mathcal{S}'$.

Putting $S_x/\mathcal{S}' = W_x$, we have

$$W_{17} + W_{18} + \ldots + W_{49} = 1, \qquad 4\,(\text{iv})$$

$$P_{17}W_{17} + P_{18}W_{18} + \ldots + P_{49}W_{49} = \frac{\mathcal{P}'}{\mathcal{S}'}, \qquad 4\,(\text{v})$$

and it is required to find the maximum and minimum of the L.F.

$$V_{17}W_{17} + V_{18}W_{18} + \ldots + V_{49}W_{49} = \frac{\mathcal{V}}{\mathcal{S}'}, \qquad 4\,(\text{vi})$$

i.e. we are required to optimize the L.F. for non-negative values of the variables

$$W_{17}, W_{18}, \ldots, W_{49}.$$

Plot the 33 points (P_{17}, V_{17}) $(P_{18}, V_{18}) \ldots (P_{49}, V_{49})$ in two-dimensional rectangular Cartesian co-ordinates. (In this and subsequent constructions, we ignore, for simplicity, the case in which more than two points are collinear.)

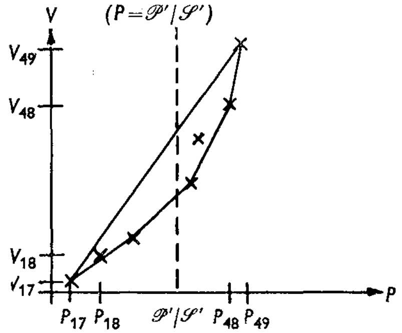

Fig. 1

A useful physical analogy to this problem may now be used by thinking of the $W_{17} \ldots W_{49}$ as a set of variable positive weights which must add up to unity. If these weights are placed at the points $(P_{17}, V_{17}) \ldots (P_{49}, V_{49})$ we may interpret 4 (v) and 4 (vi) as saying that the resulting centre of gravity is to be at the point $(\mathcal{P}'/\mathcal{S}', \mathcal{V}/\mathcal{S}')$. Now $\mathcal{P}'/\mathcal{S}'$ is, of course, a constant and therefore the centre of gravity is restricted to a vertical line, the equation of which is $P = \mathcal{P}'/\mathcal{S}'$. The problem is thus to find the two distributions of the weights which will give the 'highest' and 'lowest' centre of gravity under these conditions.

The required optimizing distributions of the weights are fairly evident, but an insight into more general problems of this nature may be obtained by proceeding indirectly. Consider the infinity of all possible distributions of the weights at the points (P_x, V_x) without the operation of constraint 4 (v). Their resulting centres of gravity can take up any position within a certain region, and it is clear that the 33 distributions which place all the weight at each of the points (P_x, V_x) in turn are specific distributions and that their centres of gravity, i.e. the points (P_x, V_x), must themselves be inside the region or on its boundary. Next assume that the total weight is distributed between any two of the points. The centre of gravity of this distribution can lie anywhere between these two points on the line joining them. If now a third point be added it is clear that the centre of gravity of the three points can lie anywhere in the triangle formed by the three points. By continuing this process it follows that the region of all possible centres of gravity possesses the general property that if any two points (centres of gravity) belong to it then all points on the line joining them and lying between them will also belong to the region. This in fact defines what is known as a convex region. It is characteristic of an L.P. problem that its solution, which seeks to optimize a given L.F., is to be found on the boundary of an associated convex region.

In the given case the boundary can easily be found. By inspection choose any pair of the original points (P_x, V_x) such that if the line joining them were extended in both directions all the other points would lie entirely to one side of it. The association of any of these other points (with positive weights) with the two given points cannot produce a centre of gravity on the opposite side of the line, and therefore the segment lying between the two chosen points is a part of the boundary of the region. The original 33 points cannot produce more than 33 such lines (there will usually be fewer) and it will be found that the segments form a closed convex polygon which is the required boundary. Thus all the vertices of the convex polygon are some (or all) of the points (P_x, V_x), the remaining points (centres of gravity) lying inside the polygon or on an edge.

Evidently the solution to the original problem is given by the two points where the vertical line $P = \mathscr{P}'/\mathscr{S}'$ cuts the edges of the convex polygon.

For an accurate solution we make use of the fact that if the line $P = \mathscr{P}'/\mathscr{S}'$ cuts the polygon for, e.g., a maximum in the edge which joins the two vertices (P_i, V_i), (P_j, V_j)—which we find by inspection—then the 'highest' centre of gravity can be obtained only by distributing all of the weights at just those two vertices. Reverting to expressions 4 (i) to 4 (iii), we have

$$S_i + S_j = \mathscr{S}',$$

$$P_i S_i + P_j S_j = \mathscr{P}'.$$

Whence
$$S_i = \frac{\mathscr{P}' - \mathscr{S}' P_j}{P_i - P_j} \quad \text{and} \quad S_j = \frac{\mathscr{S}' P_i - \mathscr{P}'}{P_i - P_j}$$

and
$$\mathscr{V} = V_i S_i + V_j S_j.$$

Similarly for the minimum.

(Alternatively, the appropriate values may be read off approximately from the graph and multiplied by $\mathscr{S}'$.)

4·3. Numerically, we have $\mathscr{S}' = £204{,}866$ (as before) and $\mathscr{P}' = 18{,}033s.$ per month. Using the graphical methods described above we arrived at the following optimum basic feasible distributions and their corresponding solutions.

Maximum

$$\left.\begin{aligned} S_{36} &= 120{,}048 \\ S_{49} &= 84{,}818 \\ \text{all other } S_x &= 0 \end{aligned}\right\} \quad \text{whence} \quad \mathscr{V} = V_{36} S_{36} + V_{49} S_{49} = 77{,}070.$$

Minimum

$$\left.\begin{array}{rl} & S_{28} = 73{,}191 \\ & S_{47} = 131{,}675 \\ \text{all other} & S_x = \phantom{131{,}67}0 \end{array}\right\} \quad \text{whence} \quad \mathscr{V} = V_{28}S_{28} + V_{47}S_{47} = 75{,}282.$$

The above range (75,282; 77,070) becomes $\pm 1 \cdot 17\%$ standardized. It is emphasized that subject to satisfying the two constraints 4 (i) and 4 (ii) and using the true valuation factors it is impossible to obtain an actual value outside these limits by any distribution whatsoever.

5·1. If we make use of the new business figures the range may be still further reduced. Thus the variables S_x in equation 4 (i) resulted from new business sums assured S_x^0 which are assumed to be available at each age.

Formally we have 35 constraints and a linear form

$$S_{17} + S_{18} + \ldots + S_{49} = \mathscr{S}', \qquad 5\text{ (i)}$$

$$P_{17}S_{17} + P_{18}S_{18} + \ldots + P_{49}S_{49} = \mathscr{P}', \qquad 5\text{ (ii)}$$

$$S_{17} \leqslant S_{17}^0, \qquad 5\text{ (iii)}$$

$$S_{18} \leqslant S_{18}^0, \qquad 5\text{ (iv)}$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$S_{49} \leqslant S_{49}^0. \qquad 5\text{ (xxxv)}$$

We require to optimize

$$V_{17}S_{17} + V_{18}S_{18} + \ldots + V_{49}S_{49} = \mathscr{V} \qquad 5\text{ (xxxvi)}$$

for non-negative values of the variables $S_{17} \ldots S_{49}$.

Construction II, which is described in the Appendix, was devised to provide a graphical solution to problems of this type.

It will be seen that the physical interpretation by distributions of (positive) weights at the 33 points (P_x, V_x) is still valid, but that the convex region of feasible solutions, i.e. possible centres of gravity, is less tangible. All we can say definitely about its position is that it lies within the convex polygon of Construction I. As before, however, the optimum values lie on the boundary, and Construction II determines the optimizing points on the appropriate edges.

The optimum distributions are shown to be obtained by placing as much 'weight' as the new business constraints allow at certain of the points, a lesser weight at exactly two of the points (i.e. part-filled points) and none at the rest.

Numerically, the effectiveness of the additional constraints was apparent although the total new business sum assured was £352,665 compared with only £204,866 still in force at the valuation date.

The resulting optimum basic feasible distributions and their corresponding solutions were

Maximum: $\mathscr{V} = 76{,}465$ (derived from ages 17–20, 34–43, 49; 44 and 48 part-filled).

Minimum: $\mathscr{V} = 75{,}685$ (derived from ages 21–33, 40–41, 44–48; 39 and 49 part-filled).

The range (75,685; 76,465) becomes $\pm \cdot 513\%$ standardized.

5·2. Reverting for a moment to the situation envisaged in § 5·1, the longer the duration, the wider, relatively, will be the feasible range, i.e. the larger the standardized feasible range since the restrictions imposed by the new business figures become less effective. Ultimately the range coincides with the two-point limits in § 4 and a position has been reached in which the inner convex region of § 5·1 has expanded to the convex polygon of § 4. The value of the new business restrictions may, however, be partially retained if one is prepared to make some assumption as to the likely maximum and minimum decrement rates experienced.

Numerically, the over-all persistence rate in our illustration was

$$\frac{\Sigma S_x}{\Sigma S_x^0}=\frac{204{,}866}{352{,}665}=\cdot 581,$$

and it might reasonably be assumed that $\cdot 2 < r_x < \cdot 8$ for all x, where $\frac{S_x}{S_x^0}=r_x$, the persistence rate at age x. The practical application of these rates consists of calculating $\cdot 2\,\Sigma S_x^0 V_x$ in full and applying the procedure used in Construction II after suitably modifying the right-hand side of equations 5 (i) and 5 (ii) to $(\mathscr{S}' - \cdot 2\,\Sigma S_x^0)$ and $(\mathscr{P}' - \cdot 2\,\Sigma S_x^0 P_x)$, respectively, and multiplying the right-hand side of equations 5 (iii) to 5 (xxxv) by $\cdot 6$. Under these conditions the optimum basic feasible distributions gave the following results:

$$\textit{Maximum}: \mathscr{V} = 76{,}306,$$

$$\textit{Minimum}: \mathscr{V} = 75{,}844.$$

The range (75,844; 76,306) becomes $\pm\cdot 304\,\%$ standardized.

6·1. In § 4·3 it was found that the tabulation of only the total sums assured and office premiums for each year of entry led to a narrow feasible range which was reduced in § 5 by knowledge of the new business distribution. In fact a narrow range will obtain for all years of assurance if V_x is approximately linear in P_x at all durations, i.e. if $V_x \simeq k_t P_x + \alpha_t$ where k_t, α_t are constants for fixed year of assurance t.

(In passing if k_t, α_t are found by the method of least squares an approximate valuation may be made taking the net liability as

$$\Sigma S_x V_x = k_t \Sigma S_x P_x + \alpha_t\,\Sigma S_x = k_t \mathscr{P}' + \alpha_t \mathscr{S}'.$$

This procedure is closely analogous to the methods and reasons underlying the fitting of a line of regression although the results will not necessarily be feasible.)

For all durations we may therefore write $V_x = k_t P_x + \alpha_t + \epsilon_x^t$ and for a particular t, $S_x V_x = k S_x P_x + \alpha S_x + S_x \epsilon_x$ and therefore

$$\Sigma S_x V_x = k \Sigma S_x P_x + \alpha \Sigma S_x + \Sigma S_x \epsilon_x \qquad 6\text{ (i)}$$

$$= k\mathscr{P}' + \alpha \mathscr{S}' + \Sigma S_x \epsilon_x. \qquad 6\text{ (ii)}$$

It is probable that for most offices ϵ_x^t will be small for all values of x and t, in which case the last term will be small, and it is clear that in so far as V_x can be expressed as $kP_x + \alpha$, $\Sigma S_x V_x$ will be constant (and the range zero) for all feasible solutions to equations 4 (i) and 4 (ii). Subject to these two constraints it follows from equations 6 (i) and 6 (ii) that, if for the purpose of evaluating $\Sigma S_x \epsilon_x$ only we make a reasonable (feasible) assumption as to the general shape of S_x, we can proceed without loss of accuracy direct to the left-hand side of equation 6 (i) by applying these same S_x to their respective V_x; no approximation is made to the valuation factors.

Consider again the numerical example; the new business figures are known at all ages and

$$S^0_{17}+S^0_{18}+\ldots+S^0_{49}=\mathscr{S}'_0,$$

$$S^0_{17}P_{17}+S^0_{18}P_{18}+\ldots+S^0_{49}P_{49}=\mathscr{P}'_0.$$

Since all S^0_x and $S^0_xP_x$ are known, xS^0_x and $xS^0_xP_x$ are also known. It is reasonable to suppose that over the ensuing 9 years the persistence rates r_x have varied with entry age, and if we assume that $r_x=a+bx$, then

$$S_x=(a+bx)\,S^0_x,$$

and since $\mathscr{S}'=\Sigma S_x$

$$\mathscr{S}'=a\Sigma S^0_x+b\Sigma x\,S^0_x.$$

Similarly $\mathscr{P}'=a\Sigma S^0_xP_x+b\Sigma x\,S^0_xP_x.$

Solving for a and b

$$\mathscr{V}=\Sigma S_xV_x=a\Sigma S^0_xV_x+b\Sigma x\,S^0_xV_x.$$

6·2. In our numerical example the values of a and b were ·22753 and ·008698 respectively, i.e. the assumed persistence rates varied from about ·375 at age 17 to ·654 at age 49. Applying these rates to S^0_x we obtained our hypothetical (feasible) distribution which was valued in full and gave an estimated value of 76,073 compared with the true value of 76,096.

The above satisfactory result is influenced by two independent factors:

(*a*) V_x is nearly linear in P_x,

(*b*) $r_x=S_x/S^0_x$ is nearly linear in x.

If either one of these conditions is exactly fulfilled it does not matter how wide of reality the other condition is, the answer will be exact. As regards (*a*) the general argument of this paragraph and the numerical results in § 5 led us to expect a narrow range in any event, and as regards (*b*), provided $0\leqslant a+bx\leqslant 1$ for all relevant x, reasonable decrement rates are employed to produce a reasonable and feasible distribution from which we obtained a close estimate within the range of feasible values. In practice the coefficient of correlation between S_x and $S^0_x(a+bx)$ was found to be ·997 at duration 9 and a similar block of business gave a coefficient of ·994 at duration 21.

7·1. To summarize the numerical results so far:

Feasible distribution	Minimum (£)	Maximum (£)	Standardized range (%)
One point (§ 3·2)	26,223	109,603	±61·4 (A)
Two point (§ 4·3)	75,282	77,070	± 1·17 (B)
Restricted by new business (§ 5·1)	75,685	76,465	± ·513 (C)
Restricted by k% new business (§ 5·2)	75,844	76,306	± ·304 (D)
Estimated by $(a+bx)$ (§ 6·2)	£76,073		
Actual net liability	£76,096		

Provided $0 \leqslant a+bx \leqslant 1$ for all relevant x the $(a+bx)$ procedure cannot give a value outside the limits of (C) and is unlikely to go outside the limits of (D). It may be thought that if advanced attained ages are involved a linear form for r_x is unlikely to fit owing to the mortality decrement, but in practice other sources of decrement are likely to preponderate, e.g. in § 6·2 the value found for b was positive.

The successive reductions in the standardized range resulting from additional information are marked and the advantages of tabulating a function which follows the general shape of the valuation factors are evident. In the example quoted the office premium satisfied this requirement since both V_x and A_x (the single premium) were found to be approximately linear in P_x at all durations. Since π_x (the net premium required for the fourth schedule of the 1909 Act) was also nearly linear in P_x no further tabulation appears to be necessary.

7·2. Since Industrial Assurance bonus cannot be surrendered apart from the policy the advantages of a year of issue classification for valuing bonus are obvious and the original numerical example was again employed, this time in order to estimate P.V.S.A., with the following results:

Feasible distribution	Minimum (£)	Maximum (£)	Standardized range (%)
One point	97,925	179,257	±29·3
Two point	147,517	160,652	± 4·26
Restricted by new business	153,722	159,313	± 1·79
Restricted by k% new business	154,979	158,119	± 1·00
Estimated by $(a+bx)$	£156,665		
Actual P.V.S.A.	£156,728		

Similarly close results were obtained for both the Net Liability and P.V.S.A. at other durations.

8·1. The following miscellaneous points arise in practice.

(*a*) The $(a+bx)$ method involves tabulating once and for all the new business sums assured S_x^0 and xS_x^0 at each age x together with the totals $\Sigma S_x^0 P_x$ and $\Sigma x S_x^0 P_x$. In practice $\Sigma S_x^0 V_x$ and $\Sigma x S_x^0 V_x$ may be calculated during the year; a and b will be determined as soon as the valuation data ($\mathscr{S}'$ and $\mathscr{P}'$) are available and the valuation reserve is at once $a\Sigma S_x^0 V_x + b\Sigma x S_x^0 V_x$. Similarly for A_x and π_x.

(*b*) In view of the relatively high decrement in the early years of assurance, it might be worth while retaining a full tabulation of data for the first two or three years, using the ultimate in force figures as data for the $(a+bx)$ technique.

(*c*) In the unlikely event of $a+bx<0$ or >1 at some ages (and we have found the risk of this happening to be more apparent than real) there are a number of modifications which the resourceful actuary can employ according to circumstances.

(*d*) Classification by year of issue avoids problems associated with change of rates. It also facilitates an exact calculation of the amount of bonus in force, and used in conjunction with the $(a+bx)$ method provides the bonus totals required for the fifth schedule of the 1909 Act.

(*e*) A further tabulation of totals (not a full tabulation) by year of maturity is required for fifth schedule purposes. This enables exact figures to be obtained in total for P.V.O.P. and, if a check on the $(a+bx)$ method is required, for P.V.S.A.

(*f*) If V_x (or A_x) is closely linear in P_x at the relevant ages some difficulty may be found in applying the technique indicated in Constructions I and II. In practice a more suitable diagram may be obtained by plotting the points $(1/V_x, P_x/V_x)$ rather than the points (P_x, V_x) and this is dealt with in Construction III in the Appendix.

8·2. Where it is intended to use an approximate method of valuation the information tabulated should be such that the resulting feasible range of values is as small as possible subject to the practical difficulty of tabulating more than a certain amount of information. This in itself is a minimizing problem, which may well become important with the advent of electronic computers. In the numerical example cited it was found that a tabulation of ΣS_x and $\Sigma S_x P_x$, used in conjunction with a knowledge of the new business figures, was sufficient, but in other cases a further tabulation may be required to reduce the feasible range to practical proportions. The advantages of tabulating a function (such as xS_x as suggested by Perks in *J.I.A.* 72, 396) which is independent of the valuation basis are obvious and it may be that the time has come for Ordinary offices to consider in what circumstances an alternative to the tabulation of valuation net premiums is possible.

APPLICATION TO ESTABLISHED METHODS

9. In the second part of this paper the techniques developed in the first part are applied to reinterpret some of the established methods of approximate valuation. In some instances these methods may have produced a 'good' result because no definition is given of a 'bad' result. It is suggested that these be interpreted as 'feasible value' and 'non-feasible value' since the actual value we are estimating

(*a*) can never lie outside the feasible range;

(*b*) could in fact be any of the values inside the feasible range.

Thus, in the situation of § 4·3, given only $\mathcal{S}'=204{,}866$, $\mathcal{P}'=18{,}033$, the actual value of the net liability cannot lie outside the range (75,282; 77,070). Provided we draw upon no other information, any value within these limits is as 'good' as any other since the public could have taken out its sums assured in such a way as to give an actual net liability of any value within the range. Provided all the available information is used in establishing the limits of the range of feasible values there will remain no criterion for judging any one value between the limits as 'better' than any other. A principle of indeterminacy applies and the size of the range will vary inversely with the amount of information available.

Given data which are fully tabulated and the factors for valuing them an exact total can be calculated. Once the data are grouped there is a loss of information and no exact value can be assigned. Instead, there exists a range of possible (feasible) values; this range is bounded, and with the aid of L.P. its maximum and minimum can be found with more or less work. It will be appreciated that with a large number of constraints and variables a computer would be necessary.

In general, it will be found that, apart from the maximum and minimum, to any particular value within the feasible range there corresponds an infinite number of feasible distributions. Some of these will be of the 'smooth' types that arise in practice whilst others will resemble a basic feasible distribution (i.e. $S_x = 0$ for many x); hence a basic feasible distribution cannot be rejected merely because it is unlikely-looking. Nevertheless, it is intuitively clear that if we can, in some way, find a 'smooth' distribution which approximates 'closely' to the actual distribution, we can expect it to give a value 'close' to the actual value within and in comparison with the size of the feasible range. In the $(a+bx)$ method we have in effect assumed that there is a probability distribution of actual values within the feasible range which can be built up on the lines of § 5·2. In passing, this suggests that an assumption of the form $r_x = (a+bx)\,r'_x$, where r'_x is a persistence rate derived from recent past experience of other tables, would be theoretically better.

It will be shown that some well-known approximate methods do in fact use unlikely-looking feasible distributions, and that in other cases it may be quite easy to find a basic feasible distribution and apply the true factors to it. Even in cases where the feasible range has not been found, if the use of a (basic) feasible distribution leads to a value which differs from another such value or the actual value by an unacceptably large amount, we may be sure that the range is large and more information will be needed to make any approximate method acceptable.

With these principles in mind we have selected the following four well-known methods for illustration:

(i) Average age (Austin's) method.
(ii) 'n-ages' method.
(iii) Perks's n-point method.
(iv) Z-method.

Austin's method

10. It may be asked why the average age at entry (Austin's method) which is frequently employed in the Industrial Branch for the valuation of endowment assurances, could not be used in the numerical example. The method was originally recommended and has given very satisfactory results for endowment assurances in which the grouping was primarily by original term and then by year of entry, each original term being treated virtually as a separate table. In the Industrial Branch this fitted in well with the practice of issuing policies for only a limited number of original terms, e.g. 15, 20 and 25 years. For an original term such as 15 years the reserve values at any duration t, ${}_tV_{x:\overline{15}|}$, will not vary greatly with $P_{x:\overline{15}|}$ so that if there exist k_t and α_t such that

$$ {}_tV_{x:\overline{15}|} = k_t\,.\,P_{x:\overline{15}|} + \alpha_t + {}_t\epsilon_{x:\overline{15}|} \qquad 10\,(i) $$

and ${}_t\epsilon_{x:\overline{15}|}$ is small for all x and t, then k_t will also be small. The advantage of this is that the curve of (P_x, V_x) will be roughly parallel to the abscissa, whereas the range is measured across the convex polygon parallel to the ordinate. Thus the result corresponding to (B) in § 7·1 for a block of 15-year endowment assurance business issued in the same year gave a standardized range of $\pm\,{\cdot}389\,\%$ compared with the very much larger $\pm\,1{\cdot}17\,\%$.

The essence of the method is to ascertain a mean age y, where $\mathscr{P}'/\mathscr{S}' = P_y$, by re-entering the prospectus and to value the whole business by $\mathscr{V} = V_y\,\mathscr{S}'$. In practice it is extremely unlikely that an exact age y will be obtained, but k_t and ${}_t\epsilon_{x:\overline{15}|}$ in 10 (i) may be so small that it will be sufficiently accurate to value the whole business by the nearest integral age to y. The resulting value may not be feasible, but this will be unimportant if the possible error is very small. Where, however, the feasible range is such that it is undesirable to risk going outside it two courses are open:

(A) Using the two integral ages w and $(w+1)$ on either side of y we have

$$S_w + S_{w+1} = \mathscr{S}', \tag{10 (ii)}$$

$$S_w P_w + S_{w+1} P_{w+1} = \mathscr{P}' = P_y \mathscr{S}'. \tag{10 (iii)}$$

These are solved for S_w and S_{w+1} giving

$$S_w = \frac{P_{w+1} - P_y}{P_{w+1} - P_w} \mathscr{S}' \quad \text{and} \quad S_{w+1} = \frac{P_y - P_w}{P_{w+1} - P_w} \mathscr{S}',$$

and we take as our estimated reserve value

$$S_w V_w + S_{w+1} V_{w+1} = \mathscr{V}. \tag{10 (iv)}$$

Putting all other $S_x = 0$ the distribution used is a basic feasible one and a feasible value will be produced.

(B) Assume that $\mathscr{V} = \mathscr{S}' V_y$ and find V_y by linear interpolation between V_w and V_{w+1}. This is the more practical approach, and if $y = w + \epsilon$

$$\mathscr{V} = \mathscr{S}' V_y = \mathscr{S}'[(1-\epsilon)V_w + \epsilon V_{w+1}]. \tag{10 (v)}$$

If y was found originally by interpolating between P_w and P_{w+1} it is clear that

$$1 - \epsilon = \frac{P_{w+1} - P_y}{P_{w+1} - P_w} \quad \text{and} \quad \epsilon = \frac{P_y - P_w}{P_{w+1} - P_w}.$$

Substituting in 10 (v) we get the same result as in 10 (iv). The closeness of the resulting $\mathscr{V}$ to the actual value depends on

(*a*) the feasible range (i.e. the 'extent' to which V_x is linear in P_x),

(*b*) the distance measured parallel to the ordinate between the true centre of gravity of the points (P_x, V_x) weighted with S_x and the join of the two points (P_w, V_w) and (P_{w+1}, V_{w+1}).

Two points emerge. Given only $\mathscr{P}'$ and $\mathscr{S}'$

(i) the method places a fortuitous emphasis on ages w and $w+1$;

(ii) any two ages, one on either side of y, could be used to provide a feasible value, which would be just as 'good'.

'*n-ages*' *method*

11. Assume that the total sum assured $\mathscr{S}'$ in force under a whole-life table is given, the total value of sums assured $\mathscr{A}'$ being valued by single-premium factors $A_x^{(1)}$ for attained age x. It is required to find the total value on a different basis with factors $A_x^{(2)}$.

Choose say four ages x_1, x_2, x_3, x_4 such that

$$\tfrac{1}{4}\mathscr{S}'(A_{x_1}^{(1)} + A_{x_2}^{(1)} + A_{x_3}^{(1)} + A_{x_4}^{(1)}) = \mathscr{A}'.$$

Formally, we have

$$\sum_x S_x = \mathscr{S}', \tag{11 (i)}$$

$$\sum_x S_x A_x^{(1)} = \mathscr{A}', \tag{11 (ii)}$$

required the value of

$$\sum_x S_x A_x^{(2)} = \mathscr{A}. \tag{11 (iii)}$$

The required value is taken as

$$\tfrac{1}{4}(A^{(2)}_{x_1}+A^{(2)}_{x_2}+A^{(2)}_{x_3}+A^{(2)}_{x_4})\mathscr{S}'.$$

The distribution used was

$$S_{x_1}=S_{x_2}=S_{x_3}=S_{x_4}=\tfrac{1}{4}\mathscr{S}',$$

all other $$S_x=0.$$

We note that the distribution satisfies equations 11 (i) and 11 (ii) and is thus feasible; hence a feasible value is produced and satisfactory results depend on $A^{(2)}_x$ being approximately linear in $A^{(1)}_x$.

Perks's n-point method

12. Assume that the following information is given concerning a particular distribution of sums assured:

$$\sum_x S_x=\mathscr{S}_0, \qquad 12\text{ (i)}$$

$$\sum_x xS_x=\mathscr{S}_1, \qquad 12\text{ (ii)}$$

$$\sum_x x^2S_x=\mathscr{S}_2, \qquad 12\text{ (iii)}$$

required the value of $$\sum_x V_xS_x=\mathscr{V}. \qquad 12\text{ (iv)}$$

Define $\bar{x}$, σ by $$\bar{x}\sum_x S_x=\sum_x xS_x, \quad \text{i.e.} \quad \bar{x}=\frac{\mathscr{S}_1}{\mathscr{S}_0},$$

$$(\bar{x}^2+\sigma^2)\sum_x S_x=\sum_x x^2S_x, \quad \text{i.e.} \quad \sigma=\frac{1}{\mathscr{S}_0}(\mathscr{S}_2\mathscr{S}_0-\mathscr{S}_1^2)^{\frac{1}{2}}.$$

The required value is taken as

$$\mathscr{V}=\tfrac{1}{2}(V_{\bar{x}-\sigma}+V_{\bar{x}+\sigma})\mathscr{S}_0. \qquad 12\text{ (v)}$$

Again as in § 10 it is extremely unlikely that exact integral ages $\bar{x}\pm\sigma$ will be obtained. Suppose $\bar{x}-\sigma=w+\epsilon$, $0<\epsilon<1$, w an integer. It can be demonstrated that a linear interpolation to obtain $V_{\bar{x}-\sigma}$ (or $V_{\bar{x}+\sigma}$) will satisfy 12 (i) and 12 (ii), but not 12 (iii), and thus the resulting underlying distribution is not feasible. A second-degree interpolation will satisfy all three constraints; thus if Lagrange is used

$$\frac{\mathscr{S}_0}{2}V_{\bar{x}-\sigma}=\frac{\mathscr{S}_0}{2}V_{w+\epsilon}=V_w\frac{(\epsilon-1)(\epsilon-2)}{4}\mathscr{S}_0+V_{w+1}\frac{\epsilon(2-\epsilon)}{2}\mathscr{S}_0+V_{w+2}\frac{\epsilon(\epsilon-1)}{4}\mathscr{S}_0,$$

and similarly for $V_{\bar{x}+\sigma}$. However, the implied distribution is still not feasible because one of the coefficients must be negative, although by obtaining $V_{\bar{x}-\sigma}$ from say V_a, V_b, V_c and $V_{\bar{x}+\sigma}$ from V_b, V_c, V_d, choosing $a<\bar{x}-\sigma<b<c<\bar{x}+\sigma<d$. four coefficients, each positive in total, may be obtained.

As suggested in §9, the simplest general line of attack is to find a basic feasible distribution, i.e. to solve the given n constraints in terms of n unknowns only (treating the rest as zero). The difficulty is to choose a set of n unknowns such that the solutions to the n equations will be positive. In §10 (the 'average age' method with two constraints) this difficulty was overcome by choosing the two ages one on either side of the average. In Perks's method (and in the Z-method) with three constraints, a basic feasible distribution may not be obvious and Construction IV in the Appendix indicates a method for finding such distributions. The conclusions are similar to those deduced for the average age method. In the latter, L.P. methods were used to determine the range, but with three constraints (or more), unless a computer is available, the sophisticated trial and error methods of L.P. become onerous. A series of basic feasible distributions can, however, be obtained and if the range derived from these is acceptably small, a mean value between the highest and lowest values so obtained will probably be acceptably close to the actual reserve.

Whilst it is accepted that Perks's method has given satisfactory results in practice, theoretically

(i) there seems to be no special virtue in placing emphasis on the ages $\bar{x} \pm \sigma$; any basic feasible distribution using only three ages will give a 'good' answer.

(ii) a three-dimensional model in which the points (x, x^2, V_x) are plotted shows that *ab initio* the feasible range will be small if these points are approximately coplanar since the range is measured parallel to the third axis across the convex region. Algebraically this may be stated that there exist α, β, γ such that

$$V_x \simeq \alpha + \beta x + \gamma x^2.$$

(iii) if a basic feasible distribution is valued the closeness of the result depends on the distance measured parallel to the third axis between the true centre of gravity of the points (x, x^2, V_x) weighted with S_x and the plane containing the three points chosen.

(iv) if V_x is not approximately quadratic in x then xS_x and x^2S_x are not good functions to tabulate, and in any event Perks's method would be unsuitable. This conclusion corresponds to the principle underlying the first part of this paper that if V_x is not approximately linear in P_x then the total sum assured and office premium alone are insufficient.

Z-method

13. In the 'Z-method', endowment assurances are grouped by unexpired term. For a particular unexpired term t, let S_M = sum assured in force which will mature at age M.

Given

$$\sum_M S_M = \mathscr{S}',$$

$$\sum_M S_M Z_M = \mathscr{Z}',$$

required to optimize $\sum_M S_M A_{M-t:\overline{t}|} = \mathscr{A}$ for non-negative S_M. Formally this is the same as the example in §4 as regards the optimizing solution. Current practice uses an average age, *vide* §10. This average age is also used to find the present value of the net premiums the amount of which is known. But since the amount of the net premiums is a further constraint the method fails to make use of all the information available; once the total net premiums in force is given the feasible range of $\mathscr{A}$ is affected.

If for a given unexpired term, t, $S_{x:M}$ = sum assured in force, age at entry x, age at maturity M, then formally

$$\sum_x \sum_M S_{x:M} = \mathscr{S}', \qquad 13\text{ (i)}$$

$$\sum_x \sum_M S_{x:M} Z_M = \mathscr{Z}', \qquad 13\text{ (ii)}$$

$$\sum_x \sum_M S_{x:M} \pi_{x:\overline{M-x}|} = \Pi', \qquad 13\text{ (iii)}$$

required to optimize

$$\sum_x \sum_M S_{x:M} A_{M-t:\overline{t}|} = \mathscr{A} \qquad 13\text{ (iv)}$$

and

$$\sum_x \sum_M S_{x:M} \pi_{x:\overline{M-x}|}\, \ddot{a}_{M-t:\overline{t}|} = \mathscr{B} \qquad 13\text{ (v)}$$

for non-negative $S_{x:M}$, where $\pi_{x:\overline{M-x}|}$ = net premium per £1 sum assured.* A further constraint may be provided by the office premium, $P_{x:\overline{M-x}|}$,

$$\sum_x \sum_M S_{x:M} P_{x:\overline{M-x}|} = \mathscr{P}'. \qquad 13\text{ (iii) }(a)$$

Using only constraints 13 (i) to 13 (iii) a basic feasible distribution may be found as in Construction IV and substituted in the expressions 13 (iv) and 13 (v).

In practice the Z-method normally gives satisfactory results since $A_{M-t:\overline{t}|}$ is approximately linear in Z_M (t constant) and the range is thus very small. In these circumstances it is not suggested that the additional work (although light) is worth while, but the fact remains that a basic feasible distribution obtained by the method outlined above will produce a feasible value which may well be closer to the actual value, since the inclusion of a third constraint cannot increase the range and may reduce it considerably.

SUMMARY AND CONCLUSIONS

14·1. Underlying the whole paper is a situation in which the valuation data were deemed to be permanently grouped. It was found that there exists a range of possible values each one of which could be the actual one; in a very real sense there is no 'true' value to which we can approximate. It is emphasized that this indeterminacy is inherent in the situation and is not due, for example, to small order errors in an approximating formula. However, no value outside this range is possible, and hence if the range is small enough the indeterminacy is not of practical importance. For this reason the techniques outlined above do not supersede existing methods, but rather run parallel to them and in the process not only shed new light on their action, but provide a means of judging the situation in which they are being applied.

Within the feasible range we may think of a probability distribution of values, but the situation is quite complex because there is not a one-to-one correspondence between the feasible distributions and the feasible values. Although a non-feasible distribution may produce a feasible value, only a feasible distribution will be certain of doing so, and then only if it is applied to the true valuation factors; once an approximation is made to these we lose sight of the feasible range. Thus it would appear that the following principles are of general application.

(1) The method of grouping and the amount of information tabulated should be judged by the size of the standardized feasible range to which they lead.

* But see remarks of N. Williams (p. 23) and C. W. Bennett (p. 35)—Eds. *J.I.A.*

(2) In order to ensure that a feasible value is obtained

(*a*) any approximate method should use or imply the use of a feasible distribution,

(*b*) no approximation should be made to the factors themselves.

The only justification for ignoring these points can be that the work involved is too great for the purpose in hand, e.g. when a rough approximate figure is required quickly, or when the absolute amount of the error involved is small. The above principles would appear to be of greatest importance in cases where it is not possible from time to time to check the working of the approximate method against the actual value.

14·2. In applying L.P. to valuation problems the coefficients of the variables in the L.F. may be considered as a set of 'valuation' factors, and the corresponding coefficients in the constraints as sets of 'information' factors. Many approximate methods of valuation depend for their success on a linear relationship between these two sets of factors. Thus Austin's method hinges on the relationship

$$V_x = \alpha(1) + \beta(P_x),$$

and the Z-method on $$A_{M-t:\overline{t}|} = \alpha(1) + \beta(Z_M).$$

In many cases of two constraints it has been found that the graph of the valuation factor treated as a function of the information factor is concave (or convex) throughout its age range. Where this is so, it follows that if the valuation data are in fact clustered around the mean age (as opposed to being evenly spread or concentrated at the ends of the age range), the standard approximate method of using an average age will give satisfactory results even when the functions are far from linearly related. If, as in § 13, a third constraint is introduced similar considerations apply in three dimensions.

14·3. The theory and techniques underlying L.P. are new and complex; this paper only touches the fringe of the subject and does not presume to be more than an introduction. The effect of the introduction of computers into life office work is difficult to forecast; a knowledge of L.P. may, in time, be required at clerical level; on the other hand, the subject may be by-passed as a result of the operation of the machines themselves. One point is clear, that even a computer will be unable economically to cope with either a vast array of constraints and variables or an excessive demand for storage-capacity, and the minimizing problem referred to in § 8·2 becomes paramount.

Generalizing, the problem reduces to the mathematical design of the information function or functions $f(x), g(x), \ldots$, to be tabulated, such that all the valuation functions may be approximately expressed as $\alpha + \beta f(x) + \gamma g(x) + \ldots$, both now and after any reasonable change in the valuation basis.

APPENDIX

Construction II

Given
$$S_{17}+S_{18}+\ldots+S_{49}=\mathscr{S}', \qquad 5\text{ (i)}$$
$$P_{17}S_{17}+P_{18}S_{18}+\ldots+P_{49}S_{49}=\mathscr{P}', \qquad 5\text{ (ii)}$$
$$S_{17}\leqslant S_{17}^{0}, \qquad 5\text{ (iii)}$$
$$S_{18}\leqslant S_{18}^{0}, \qquad 5\text{ (iv)}$$
$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$
$$S_{49}\leqslant S_{49}^{0}, \qquad 5\text{ (xxxv)}$$
we require to optimize
$$V_{17}S_{17}+V_{18}S_{18}+\ldots+V_{49}S_{49}=\mathscr{V} \qquad 5\text{ (xxxvi)}$$
for non-negative values of the variables $S_{17}\ldots S_{49}$.

Plot the points (P_x, V_x), $x=17, \ldots, 49$.

We place weights S_x at the points (P_x, V_x) in a specified order and call the point (P_x, V_x)

'filled' if $S_x=S_x^0$,

'part-filled' if $0<S_x<S_x^0$,

'empty' if $S_x=0$.

Take a line l such that all the points (P_x, V_x) lie to one side of it. Move l parallel to itself towards the points and 'fill' each point as it is crossed by l.

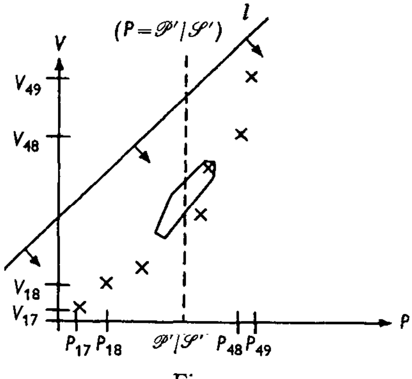

Fig. 2

Continue this process until the total $\mathscr{S}'$ is exhausted, i.e.
$$\sum_x S_x=\mathscr{S}'.$$

In general the result will give some points filled, one part-filled and the rest empty.

(If l happens to be parallel to a join of two points we may consider a small reorientation of l.)

Equations 5 (i) and 5 (iii) to 5 (xxxv) have been satisfied and the substitution of this distribution into the left-hand side of 5 (ii) and 5 (xxxvi) gives a pair of values ($\mathscr{P}$, $\mathscr{V}$) which may be represented in the above figure by the point ($\mathscr{P}/\mathscr{S}'$, $\mathscr{V}/\mathscr{S}'$), this point being the centroid of the distribution of weights. The set of all distributions which satisfy equations 5 (i) and 5 (iii) to 5 (xxxv) forms a convex region of centroids ($\mathscr{P}/\mathscr{S}'$, $\mathscr{V}/\mathscr{S}'$), and every distribution which is obtained by the 'l-construction' yields a vertex of this convex region, since the first point of any convex region to be reached by a line moving from a given direction must be a vertex. Two adjacent vertices may be recognized by the fact that a linear combination of the two corresponding distributions by λ, $(1-\lambda)$, where $0<\lambda<1$, will have only two part-filled points. For if there were more than two part-filled points it would be possible to redistribute the weights amongst the part-filled points used, so as to move the centroid of the system in any direction; this is not possible when the join of adjacent vertices, which is an edge of the bounding convex polygon, is involved, e.g. the distributions corresponding to two adjacent vertices might be

$$D_1 \equiv (0, 0, F, F, F, 0, PF, 0, \ldots, 0),$$

and

$$D_2 \equiv (0, 0, F, F, F, F, PF, 0, \ldots, 0),$$

and a linear combination (i.e. a point on the edge joining them) will be

$$\begin{aligned} D_\lambda &\equiv \lambda D_1 + (1-\lambda) D_2 \quad \text{for} \quad 0<\lambda<1 \\ &\equiv (0, 0, F, F, F, PF, PF, 0, \ldots, 0), \end{aligned}$$

where F stands for 'filled'

and PF stands for 'part-filled'.

To complete the construction we need to find the two adjacent vertices which lie one on either side of the line $P=\mathscr{P}'/\mathscr{S}'$, and hence find the point of intersection of the line $P=\mathscr{P}'/\mathscr{S}'$ with the edge joining these two vertices. The steps of the construction may be set out as follows; for definiteness we find the maximum value of $\mathscr{V}$.

(1) Take a line l_1 roughly parallel to the curve (P_x, V_x) such that all the points (P_x, V_x) lie below it.

(2) Move l_1 parallel to itself and fill each point as it is crossed until $\mathscr{S}'$ is exhausted. Note the resulting distribution D_1 of the S_x.

(3) Form $\sum_{D_1} S_x P_x$ and compare with $\mathscr{P}'$.

(4) If $\sum_{D_1} S_x P_x > \mathscr{P}'$ take a new line l_2 obtained by swinging l_1 slightly anticlockwise; if $\sum_{D_1} S_x P_x < \mathscr{P}'$ swing slightly clockwise.

(5) Repeat the above steps until two distributions $D_i D_j$ are obtained such that

(*a*) $\lambda D_i + (1-\lambda) D_j$ contains two 'part-filled' points, where $0<\lambda<1$ (by inspection);

(*b*) $\sum_{D_i} S_x P_x < \mathscr{P}' < \sum_{D_j} S_x P_x$ (by evaluation).

(6) Find λ such that $\lambda \sum_{D_i} S_x P_x + (1-\lambda) \sum_{D_j} S_x P_x = \mathscr{P}'$.

(7) Using the value of λ found in (6) form the distribution

$$D_\lambda \equiv \lambda D_i + (1-\lambda) D_j,$$

then $\sum_{D_\lambda} S_x P_x = \mathscr{P}'$ and D_λ is an optimum basic feasible distribution.

(8) Evaluate $\sum_{D_\lambda} S_x V_x$ which is the required maximum value of $\mathscr{V}$.

For a minimum the small modifications are obvious.

It can be demonstrated algebraically that the optimum basic feasible distributions will contain some filled points, precisely two part-filled points and the rest empty. As a check on the work, equations 5 (i) to 5 (xxxv) (with an additional variable in each of the constraints 5 (iii) to 5 (xxxv) to replace the inequality signs) were solved taking the non-zero variables in the maximum given by the graphical method as the basic variables (as in the Simplex Method of solving a problem in L.P. *vide* S. Vajda, ch. VI). Substitution in the expression for $\mathscr{V}$ gave negative coefficients to all the non-basic variables and a constant term equal to 76,465.

In passing, it will be noted that if we substitute $S_x = S_x^0 - T_x$ in 5 (i) to 5 (xxxvi) the l-line method with minor adjustments can be applied to T_x with a considerable saving of work at short durations. The diagram remains unaltered but the line

$$P = \frac{\Sigma P_x S_x^0 - \mathscr{P}'}{\Sigma S_x^0 - \mathscr{S}'}$$

takes the place of $P = \mathscr{P}'/\mathscr{S}'$.

Construction III

Consider the problem in §4 in which

$$S_{17} + S_{18} + \ldots + S_{49} = \mathscr{S}', \qquad 4\text{ (i)}$$

$$P_{17} S_{17} + P_{18} S_{18} + \ldots + P_{49} S_{49} = \mathscr{P}', \qquad 4\text{ (ii)}$$

and we wished to optimize

$$V_{17} S_{17} + V_{18} S_{18} + \ldots + V_{49} S_{49} = \mathscr{V}. \qquad 4\text{ (iii)}$$

Put $V_x S_x = y_x$ so that $S_x = y_x/V_x$ and dividing both the above constraints by the constant on the right-hand side we have

$$\frac{1}{V_{17}\mathscr{S}'} y_{17} + \frac{1}{V_{18}\mathscr{S}'} y_{18} + \ldots + \frac{1}{V_{49}\mathscr{S}'} y_{49} = 1, \qquad 4\text{ (i) }(a)$$

$$\frac{P_{17}}{V_{17}\mathscr{P}'} y_{17} + \frac{P_{18}}{V_{18}\mathscr{P}'} y_{18} + \ldots + \frac{P_{49}}{V_{49}\mathscr{P}'} y_{49} = 1, \qquad 4\text{ (ii) }(a)$$

required to optimize $\qquad y_{17} + y_{18} + \ldots + y_{49} = \mathscr{V}. \qquad$ 4 (iii) (*a*)

Plot the points $\qquad \left(\frac{1}{V_x \mathscr{S}'}, \frac{P_x}{V_x \mathscr{P}'}\right), \quad x = 17, \ldots, 49,$

and draw in the surrounding convex polygon. If $\mathscr{V}$ were unity the $y_{17} \ldots y_{49}$ would be a set of weights at the points $(1/V_x\mathscr{S}', P_x/V_x\mathscr{P}')$ giving a centroid at the point $(1, 1)$ by equations 4 (i) (*a*) and 4 (ii) (*a*). With $\mathscr{V} \neq 1$ we may consider the whole figure expanded (or contracted) about the origin by a factor $\mathscr{V}$ such that the point $(1, 1)$ still lies inside the convex region. The maximum and minimum factors $\mathscr{V}$ of expansion are obtained when the point $(1, 1)$ lies on an edge of the polygon.

Alternatively, we may go direct to the optimum basic feasible distributions by drawing in the line which passes through the origin and the point (1, 1) and finding its two points of intersection with the edges of the bounding convex polygon. The algebra is the same as before.

A simple alteration of the scale of the abscissa and ordinate enables us to plot points $(1/V_x, P_x/V_x)$, and use the line $(P/V)=(\mathscr{P}'/\mathscr{S}')(1/V)$ (cf. S. Vajda, ch. x). It can be shown algebraically that apart from certain exceptional points which need not concern us in practice the geometrical transformation from Construction I to Construction III

$$x'=\frac{1}{y},$$

$$y'=\frac{x}{y},$$

can also be applied to Construction II.

Thus (i) $(x, y)\to(x', y')$ uniquely, $y\neq 0$.

$$\text{(ii)}\quad \lambda(x_1, y_1)+[1-\lambda](x_2, y_2)\to\mu(x_1', y_1')+[1-\mu](x_2', y_2'),$$

if
$$\mu=\frac{\lambda y_1}{\lambda y_1+[1-\lambda]y_2}.$$

Hence, $0\leqslant\lambda\leqslant 1$ implies $0\leqslant\mu\leqslant 1$ if $y_1, y_2>0$, i.e. all the 'linearity' and 'convex' properties required are satisfied in practice.

Construction IV

If it is required to find a basic feasible distribution when three constraints have to be satisfied, the following graphical device is suggested.

Given
$$a_1w_1+a_2w_2+\ldots+a_nw_n=\mathscr{A}',$$
$$b_1w_1+b_2w_2+\ldots+b_nw_n=\mathscr{B}',$$
$$c_1w_1+c_2w_2+\ldots+c_nw_n=\mathscr{C}',$$
we require to optimize
$$d_1w_1+d_2w_2+\ldots+d_nw_n=\mathscr{D}$$
for non-negative values of the variables $w_1\ldots w_n$. If $a_i>0$, all i, put

$$\frac{a_iw_i}{\mathscr{A}'}=S_i,\quad\text{i.e.}\quad w_i=\frac{\mathscr{A}'}{a_i}S_i$$

and
$$\beta_i=\frac{b_i}{a_i},\quad\gamma_i=\frac{c_i}{a_i},\quad\delta_i=\frac{d_i}{a_i},$$

then we may rewrite the problem as follows:

Given
$$S_1+S_2+\ldots+S_n=1,$$
$$\beta_1S_1+\beta_2S_2+\ldots+\beta_nS_n=\left(\frac{\mathscr{B}'}{\mathscr{A}'}\right),$$
$$\gamma_1S_1+\gamma_2S_2+\ldots+\gamma_nS_n=\left(\frac{\mathscr{C}'}{\mathscr{A}'}\right),$$
and we require to optimize
$$\delta_1S_1+\delta_2S_2+\ldots+\delta_nS_n=\left(\frac{\mathscr{D}}{\mathscr{A}'}\right)$$

for non-negative values of the variables $S_1 \ldots S_n$. Plot the points (β_i, γ_i) in two-dimensional rectangular Cartesian co-ordinates. We are required to choose three of the points (β_i, γ_i) such that weights placed at them give a centre of gravity at the point $(\mathscr{B}'/\mathscr{A}', \mathscr{C}'/\mathscr{A}')$, i.e. any three points such that the triangle joining them contains the point $(\mathscr{B}'/\mathscr{A}', \mathscr{C}'/\mathscr{A}')$, will give a basic feasible distribution.

As an example consider the problem stated in § 13, i.e. given

$$\sum_x \sum_M S_{x:M} = \mathscr{S}', \tag{13 (i)}$$

$$\sum_x \sum_M S_{x:M} Z_M = \mathscr{Z}', \tag{13 (ii)}$$

$$\sum_x \sum_M S_{x:M} \pi_{x:\overline{M-x}|} = \Pi', \tag{13 (iii)}$$

for unexpired term t, we required to optimize

$$\sum_x \sum_M S_{x\ M} A_{M-t:\overline{t}|} = \mathscr{A}. \tag{13 (iv)}$$

Select three points
$$(Z_{M_1}, \pi_{x_1:\overline{M_1-x_1}|}),$$
$$(Z_{M_2}, \pi_{x_2:\overline{M_2-x_2}|}),$$
$$(Z_{M_3}, \pi_{x_3:\overline{M_3-x_3}|})$$

such that the centre of gravity $(\mathscr{Z}'/\mathscr{S}', \Pi'/\mathscr{S}')$ falls inside the triangle formed by the three points.

Solve the three equations

$$S_{x_1\ M_1} + S_{x_2:M_2} + S_{x_3:M_3} = \mathscr{S}',$$

$$S_{x_1:M_1} Z_{M_1} + S_{x_2:M_2} Z_{M_2} + S_{x_3:M_3} Z_{M_3} = \mathscr{Z}',$$

$$S_{x_1:M_1} \pi_{x_1:\overline{M_1-x_1}|} + S_{x_2:M_2} \pi_{x_2:\overline{M_2-x_2}|} + S_{x_3:M_3} \pi_{x_3:\overline{M_3-x_3}|} = \Pi',$$

for the three unknowns

$$S_{x_1:M_1}, \quad S_{x_2:M_2}, \quad S_{x_3:M_3}, \quad \text{and put all other } S_{x:M} = 0.$$

Then
$$S_{x_1:M_1} \cdot A_{M_1-t:\overline{t}|} + S_{x_2\ M_2} \cdot A_{M_2-t:\overline{t}|} + S_{x_3:M_3} \cdot A_{M_3-t:\overline{t}|}$$

is a feasible value.

ABSTRACT OF THE DISCUSSION

Mr S. Benjamin, in introducing the paper, said that as parents of the paper the authors claimed the right to name their own children; the hieroglyphics on the right-hand side of the equations were known as curly S, curly P and curly V. They apologized for the fact that Fig. 1 did not marry up with the text. They had stated in § 4·3 that they obtained a maximum from S_{36} and S_{49}; that was not apparent from the diagram. They had had the rather more general picture in mind when drawing it and there had been some confusion.

The general problem stated at the end of the paper was that of designing the best functions to be tabulated. The simplest case of all seemed to be that of an endowment policy issued for a single fixed term. They had considered a grouping by duration t, with only the sum assured and one constant tabulated according to the age x at entry. Using the method of least squares the constant per unit sum assured was obtained as a solution to the problem

$$\text{minimize} \sum_x \sum_t (V_{xt} - \alpha_t - \beta_t g_x)^2,$$

where g_x was the required constant, α_t, β_t constants depending on t only and V_{xt} was the net liability factor.

The problem with which they had been faced was a practical one. They did not envisage replacing the normal valuation calculations by an army of linear clerks stolidly moving parallel lines across invisible convex regions and shouting 'Eureka' whenever they hit a vertex.

Mr N. Williams, in opening the discussion, said that the paper dealt with a new application of linear programming which differed in its purpose from other applications of which he had heard. Most applications were made to find either the most profitable allocation of resources or the least costly way of achieving some given set of requirements. The purpose of the authors' application was to set limits within which the valuation liability had to lie, and thus to measure the effectiveness of an approximate method where that could not be measured otherwise because the true value could not be found exactly.

By using the original new business figures to set upper limits to the individual variables S_x, the authors had reduced the range of possible values to a fairly small percentage of the true, but unknown, value. The upper and lower values were based on two distributions of policies which, although theoretically possible, were inherently unlikely for two reasons: first, it was assumed that all policyholders entering at some ages were all living, and that, apart from two groups, those entering at other ages were all dead; and, secondly, the groups chosen would depend on the valuation basis, which was an estimate of expected future, not of past, experience.

To narrow the range further the authors had first applied a persistence rate to reduce the upper and lower bounds of the individual S_x, and then, by a more refined application of persistence rates, they arrived at a single value for the valuation of the group.

Up to and including § 5·1, the results depended on facts, but thereafter they depended on a combination of fact and more or less reasonable assumptions. Once assumptions were introduced the result could be made pretty well whatever was wanted, and it seemed to him that the assumption that S_x lay between $\cdot 2S_x^0$ and $\cdot 8S_x^0$ for all x was too crude. He would prefer to work out the expected S_x based on S_x^0 and past experience and then use limits of, say, expected S_x plus or minus three or four standard deviations.

He thought that the use of linear programming to test an approximate method could be useful provided that the range between the higher and lower feasible values was not too great. Although that range would undoubtedly include the true value, there was no reason to suppose that it would be mid-way; it might be worth comparing the most probable value with the upper and lower values in a number of cases to see whether it tended to be nearer the upper or the lower limit.

He agreed that it was desirable to use a computer if one was available. He had recently been using a fairly large computer for such problems, and matrices of the size under consideration would take about ten minutes each, or less, to solve. The main advantages were speed and, particularly, the accuracy of the answer. In working those problems out by hand it was very easy to go wrong. Moreover, it was very difficult to see that the calculation had gone wrong unless fairly elaborate checks were made, such as column sum or row sum, and then the calculations took longer.

He asked the authors to expand their explanation of the check on the Z method. He had always understood that it was not possible to optimize more than one thing at a time. How, therefore, did they arrive at the upper and lower values of the difference between the value of the sums assured and the value of the premiums?

The definition of a basic feasible distribution as a feasible distribution which contained $(n-k)$ zero elements and k non-zero elements occurred in §2·2, but there were also references in §5·1 to a lesser weight at exactly two of the points and on page 19 to precisely two part-filled points. In practice those were often true, but there was the possibility of degeneracy and it was better to define a basic feasible distribution as a feasible distribution which contained at least $(n-k)$ zero elements. In the same way there could be less than two part-filled points.

Another point concerned Construction II on page 17. When, as in that case, the matrix had a particular form, it was desirable to make use of that fact if the full set of calculations were to be reduced, and the authors had done that very neatly. He would, however, suggest an alternative method which might, perhaps, be simpler in application. It was as follows:

To find, for example, the maximum: join the two points which straddle $\mathscr{P}'/\mathscr{S}'$ and which would be chosen if there were no individual limits, and give one of them its upper limit. Mark that point as filled, and recalculate $\mathscr{P}'/\mathscr{S}'$, which would move to the left or right. Proceed as before until two points were less than or equal to their upper limits. Join those points and extend the line in both directions, then check that all points previously filled were above that line.

Mr S. S. Townsend pointed out that it was virtually impossible for a large industrial assurance office with millions of policies in force which was introducing an approximate method of valuation and thereby discarding its full valuation groupings to reconstitute its original groupings, without sorting millions of cards. The office therefore had to have a greater degree of confidence in its approximate method of valuation than, for example, a medium-sized ordinary office which could recreate its original data far more easily.

Although the industrial branch policy record card was usually referred to as a valuation card, that was a misnomer, since its purpose fundamentally was for movement and accounting.

No satisfactory method of approximate valuation for industrial policies maturing at age 65 had as yet been published and any of the orthodox approaches invariably broke down in one or more important particulars. Austin's method, which worked very well for policies for a fixed term of years, was unsatisfactory when applied to policies for a wide range of terms. Although the office would be faced with having 2000 individual valuation groupings eventually, the volume of business in force was likely to remain a fairly small proportion of the total business in force. The greater the need, therefore, to devise some method of reducing that vast array of groupings.

The authors, with the aid of linear programming, appeared largely to have solved the problem but they were careful to say, with regard to other classes of business, that the techniques which they suggested should not be regarded as superseding existing methods but could run parallel to them.

He had two comments to make on points of detail. In § 14·3 the authors concluded, 'both now and after any reasonable change in the valuation basis'. He felt that that was a possible over-simplification. If they considered an office which over the past generation had changed its valuation basis on six separate occasions and had in the process moved from O^{M} 4% to C.M.I. 1947–48 2%, he wondered whether the authors' assertion would apply so confidently not merely to each change of basis but to the series as a whole.

The second point was essentially practical and concerned Fifth Schedule requirements. The suggested grouping for the approximate method of valuation was by year of issue, which, of course, did not help in any way towards supplying the data by year of maturity for the Fifth Schedule. The authors dealt with that by saying that they would have a year-of-maturity classification of totals, which incidentally provided a check on the present values of sums assured. It did not, however, give the bonus additions by year of maturity. The authors anticipated that by saying that since the total bonus for each year of issue was known, it could be spread by applying the $(a+bx)$ method, so obtaining an assumed bonus at each age of entry which could be collated by year of maturity. The snag was whether those figures would be acceptable for Board of Trade purposes. If they were, it would be tempting to go a stage further and apply the same method for the sums assured and the office premiums and in that way dispense completely with a year-of-maturity classification. But since the sum assured and the office premium were the basic elements of the contract, he thought it was asking too much to expect assumed figures of that nature to be accepted. Bonus additions fell into a slightly different category and, moreover, the figures upon which assumptions had been made formed only a small proportion of the total bonuses, so that basis might perhaps be acceptable.

form of selected sample which would give a better result, although the extent of the improvement could not be measured? Perhaps that suspicion was groundless.

The authors said in §9 that 'Provided all the available information is used in establishing the limits of the range of feasible values there will remain no criterion for judging any one value between the limits as "better" than any other. A principle of indeterminacy applies.' In other words, they were treating any feasible value which arose as equally likely with any other feasible value. The possible range was between the minimum and the maximum value, and the best estimate would be halfway between those two, with a maximum error of half the range.

But was that the most reasonable solution of that indeterminacy problem? Surely it would be nearer the mark to say that any feasible distribution was equally likely and that, while the maximum or the minimum value would usually be given by one feasible distribution only, there would be a considerable number of such distributions which gave (or approximated closely to) any particular value in the middle of the range. That led to a frequency curve—he imagined in most cases a normal curve of error—and the best value would then be the mean of that curve (not necessarily the mean of the maximum and minimum) and the range for practical purposes would be two or three times the standard deviation, which he thought might well be less, in many cases, than the total possible range.

If the tabulated information was completely restricted to the total sum assured and total office premium, was it not a fact that their general knowledge of the business would tell them immediately that the business was not all concentrated at one or two particular ages but was spread, either reasonably uniformly or perhaps in a frequency distribution? If so, surely they could bring that general knowledge into their method of approximate valuation or use it for the purposes of assessing the value of such a method. In §9 the authors refer to a 'smooth' distribution which approximates 'closely' to the actual distribution, but it was not clear whether what they had had in mind was precisely the same point.

In §5·1, the procedure described appeared to involve a substantial tabulation and a considerable amount of calculation; when it had all been done he could not feel sure that a better result was obtained than could be got merely from a general knowledge of the run of the distribution.

In §12 there was a reference to the n-point method, but the authors' treatment of that would, he thought, be better headed '2-point method'. The n-point method included several other applications which would go beyond the 2-point form. Even assuming that the given data were limited to the three equations given, 12 (i), 12 (ii) and 12 (iii), a better value would usually be obtained by using three points rather than two, bringing in the principle of general knowledge of such distributions. Thus, if the third moment of the distribution were taken as zero and the fourth moment were assumed to be equal to 2·5 times the square of the second moment, the required value became

$$\mathscr{V}=\tfrac{1}{5}(V_{\bar{x}-d}+3V_{\bar{x}}+V_{\bar{x}+d}),$$

where $d^2=2{\cdot}5\,\mu_2=2{\cdot}5\sigma^2$.

That would usually give better results in practice than the 2-term formula.

He concluded with a query on the authors' criticism of the Z method. He could not see that the fact that the distribution was not feasible as regards the entry age would necessarily have any effect whatever on the feasibility of the value of the sum assured, which depended only on maturity age (or attained age derived from the maturity age by deducting a fixed term outstanding). The distribution by entry age might affect the value of the net premiums but not, surely, the value of the sum assured.

Dr S. Vajda (a visitor) said that whether by accident, by trial and error, or by systematic searching, the authors had found a new technique which, apparently, was useful for a purpose which they had in mind. They used their programming for their actuarial purposes, but another aspect of what they were doing might usefully be mentioned. Linear programming in its traditional application dealt with economic problems, where there were scarce commodities and where there was competition for them. The question was how to use those commodities in a way which raised something to a maximum or—and that was a separate problem—reduced something, such as cost, to a minimum.

The paper, however, did not deal with scarce commodities and, in fact, the authors had used a mathematical technique for a different aim; in order to get where they wanted they applied a rather interesting twist which he had never seen before in the literature of linear programming. That was the fact that they were not looking for a maximum only or a minimum only but for both at the same time.

When they had done that, the authors had also mentioned certain geometric representations. If he had understood the opener's remarks on how Construction II could be slightly altered, he hoped it would be agreed that even then it was worth mentioning what the authors said on page 19, 'It can be demonstrated algebraically that the optimum basic feasible distributions will contain some filled points, precisely two part-filled points and the rest empty.' That was useful, because it told them how many points and what type of points they ought to look for, whatever method of geometrical approach was used.

The authors had used linear programming for approximation. As they had said, many other methods of approximation were possible, more, indeed, than they had mentioned. But it seemed to him that they had added a new idea by saying in effect, 'We like this method because it leads us to an approximate value which is "possible"'. That was what they meant by 'feasible'. No doubt he would be contradicted about that; indeed, he had already been contradicted by Mr Jones, who was quite right to say that such an interpretation could not be made when there remained, eventually, only two points to be considered. However, the authors could be trusted to sort that matter out in their answer.

Now that linear programming had found its way into actuarial literature, he hoped that it might be used for other purposes, too. Two examples came to mind. One was the question of investment; after all, money was a scarce commodity. The second was a possible application of linear programming when a large office considered the locations for its mechanization department, for its records branch and for other places where it kept documents, taking into account the necessary transportation costs from various sources to various destinations. It would be very pleasant if further applications of the new technique were published soon.

Mr A. W. Joseph complimented the authors on a delightful paper. By considering a particular valuation problem, the authors were able to give the reader a good introduction to the methods of linear programming, including a simple solution of the linear programming problem with two constraints. Furthermore, Construction II of the Appendix was an ingenious solution of the problem where, in addition to two general constraints, there were other constraints of a very simple kind.

The valuation problem on which the paper was based was not important. In modern conditions, the class-book method of keeping records was replaced by a file of summary punched cards, each card containing the valuation data for each combination of year of entry and age at entry; punched cards with the valuation data for all 'ons' and 'offs' were machine sorted, merged with the summary cards and passed through tabulators to produce new summary cards. Thus, the clerical work which the authors sought to avoid was not material. But, of course, the value of the paper did not lie in its application to one particular problem, but in its salutary challenge to existing modes of thought on approximate methods of valuation.

Of the two items, data and valuation factors, which were multiplied together and summed in a valuation, the one which had always seemed the more amenable to treatment had been the valuation factor, because of its comparative regularity. Thus, the formulae associated with the names of Henry, Kenchington, King and Perks had been obtained by assuming that the valuation factor might be represented by a polynomial. Trachtenberg had differed a little from other authors by paying some attention to the valuation data. Until he (the speaker) had seen the authors' paper he had always thought that Trachtenberg was wrong in making approximations to irregular valuation data, but perhaps he had had the right idea after all. Certainly had he left the valuation factors severely alone, which in fact he had not done, he would have been a true precursor of the authors.

The paper was admirable in the attention which it paid to basic data. Different methods of approximate valuation were sometimes compared in such a way as to mask the fact that the basic data going into the formulae were not the same. For example, on pages 12 and 13 of the textbook *Approximate Valuation*, by Bizley and Lacey, an approximate valuation by Henry's method was given, producing a total value of sums assured of £543,590, which compared with a true value of £543,296. The data on which the approximate valuation was based consisted of sums assured in eight quinquennial groups. On page 29 the method of A. E. King was used to make an approximate valuation of what seemed to be the same data, and it produced the better approximation of £543,347. But for King's method the weighted mean age in decennial age groups was also given. In effect, the textbook had used a second summation in each decennial group. Thus the example of the textbook had not compared the accuracy of the two methods but only the amount of basic data used.

The only criticism he had to make of the structure of the paper was that at times it was difficult to disentangle linear programming from what was not linear programming. The method of §6 produced a spectacularly successful result, and because it produced a feasible value it might be regarded as an application of linear programming, but there were other ways of producing feasible values. For example, the averages of the minimum and maximum feasible values shown in the table in §7·1 were very good values indeed, being 76,176 for (B) and 76,075 for (C) and (D).

It was worth while paying some attention to the method of §6. That was a particular case of something much more general which had interesting implications and which had nothing to do with linear programming.

The problem was to evaluate $\Sigma S_x V_x$ over a range of values of x, where S_x was the sum assured and V_x the valuation factor at age x. He assumed that S_x might be represented approximately by the expression $af_1(x)+bf_2(x)+cf_3(x)$. (Three functions had been used, but, in fact, the method was quite general and would apply to n functions. In §6 the authors used two functions, S_x^0 and xS_x, to approximate to the persistence rate r_x.)

He also assumed that the constants a, b and c were obtained by means of three more functions, $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$. An equation in a, b and c was obtained by multiplying S_x and its representation in terms of $f_1(x)$, $f_2(x)$ and $f_3(x)$ by $\phi_1(x)$ and summing over the range. The same process was applied using $\phi_2(x)$ and $\phi_3(x)$ in place of $\phi_1(x)$. When all that had been done and the values of a, b and c as obtained from the three simultaneous equations were substituted in $\Sigma S_x V_x = a\Sigma f_1(x)V_x + b\Sigma f_2(x)V_x + c\Sigma f_3(x)V_x$, the result was the following quotient of two determinants:

$$\begin{vmatrix} \Sigma f_1(x)\,\phi_1(x) & \Sigma f_2(x)\,\phi_1(x) & \Sigma f_3(x)\,\phi_1(x) & \Sigma S_x\,\phi_1(x) \\ \Sigma f_1(x)\,\phi_2(x) & \Sigma f_2(x)\,\phi_2(x) & \Sigma f_3(x)\,\phi_2(x) & \Sigma S_x\,\phi_2(x) \\ \Sigma f_1(x)\,\phi_3(x) & \Sigma f_2(x)\,\phi_3(x) & \Sigma f_3(x)\,\phi_3(x) & \Sigma S_x\,\phi_3(x) \\ -\Sigma f_1(x)V_x & -\Sigma f_2(x)V_x & -\Sigma f_3(x)V_x & 0 \end{vmatrix}$$

$$\div \begin{vmatrix} \Sigma f_1(x)\,\phi_1(x) & \Sigma f_2(x)\,\phi_1(x) & \Sigma f_3(x)\,\phi_1(x) \\ \Sigma f_1(x)\,\phi_2(x) & \Sigma f_2(x)\,\phi_2(x) & \Sigma f_3(x)\,\phi_2(x) \\ \Sigma f_1(x)\,\phi_3(x) & \Sigma f_2(x)\,\phi_3(x) & \Sigma f_3(x)\,\phi_3(x) \end{vmatrix}.$$

The approximation given by the expression was obviously exact if, in fact, S_x was equal to and not just approximately equal to $a\,f_1(x) + b\,f_2(x) + c\,f_3(x)$. But it so happened that each of the two determinants was symmetrical with respect to the f functions and the ϕ functions. It therefore followed that the method also produced an exact result if the valuation factor V_x was exactly equal to

$$\alpha\phi_1(x) + \beta\phi_2(x) + \gamma\phi_3(x),$$

α, β, γ being constants which were determined by reference to functions $f_1(x)$, $f_2(x)$ and $f_3(x)$.

The application to the paper was that the functions $f_1(x)$, $f_2(x)$ and $f_3(x)$ were respectively S_x^0, xS_x^0 and zero and the functions $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ were respectively unity, P_x and zero. The authors' observation in §6·2 that their method yielded exact results if V_x were linear in P_x or if S_x/S_x^0 were linear in x was an example of the symmetrical property which he had mentioned.

Another example of the symmetrical property was given by the version of Henry's method outlined in *J.I.A.* 54, 313–7. In that case functions $f_1(x)$, $f_2(x)$ and $f_3(x)$ were respectively 1, x and x^2 and the same expressions were taken by functions $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$. Thus, an approximate method of valuation was obtained on the assumptions that the valuation factor V_x was approximately of the form $a+bx+cx^2$. When that approximation was exact the method gave exact results. But the method also gave exact results if the sum assured tabulated at age x was represented by $a+bx+cx^2$, no assumption whatever being made about valuation factors.

That gave him the opportunity to voice a small disagreement with §14·1. The authors said, 'In order to ensure that a feasible value is obtained (*a*) any approximate method should use or imply the use of a feasible distribution, (*b*) no approximation should be made to the factors themselves.' His quarrel was with (*b*). The Henry method, to which he had alluded, was developed by making an approximation to the valuation factors. Hence, the authors would imply that the Henry method could not be depended upon to produce feasible values. But that was not so, because the Henry method produced exactly the same result as if no graduation of the valuation factors had been made, but the sums assured had been graduated by the formula $a+bx+cx^2$; and such a graduation of sum assured would almost always lead to a feasible distribution.

Linear programming as applied to actuarial work was new, and time was needed to see whether the method would prove in the long run to be of real use. He had a distaste for graphical methods even though, as in the paper, the graphs were used only to find the place where a solution might be expected. He hoped that the authors and other members of the Institute would be encouraged to develop the technique of linear programming. Their work might prove valuable outside the profession.

Mr F. M. Redington said that the paper left him with the impression of, and was as attractive as, a landscape dappled with alternating sunlight and shadow.

He had seen with growing interest and rising hopes the light which was thrown on the subject by the rigorous analysis, in the first four sections, into feasible distributions. That phase of the paper culminated with the final words of §4·3, 'it is impossible to obtain an actual value outside these limits by any distribution whatsoever'. The authors might have added there the equally dramatic point that 'given the information in our possession it *is* possible to obtain *any* value within these limits'.

At that stage, however, while the method was admirable, the range of the results was too wide to be of practical value. A standardized range of over 1% was too much. It was worth pausing at that point to note that the smallness of even that range was attributable largely to the special valuation problem which the authors were considering. Had they taken whole-life policies they might have had a range of error of 20%–30%.

It was possible that a practical trick to give good results would be to interpolate between the maximum and minimum values in the same proportion as the true value for the preceding year lay between the then maximum and minimum. He would not pursue that because the authors had a better device, but not before the shadow of §5 had fallen over the scene.

In §5 the authors pursued their linear programming by adding the partial constraints that the sum assured at each age could not exceed the sum assured originally issued. That was an interesting examination, particularly the visual insight into what was happening given by Construction II. But the problem was beginning to slip through the fingers of the linear programming method. He had thought that the labour was outside the area of practical possibility; but the authors had told him that an intelligent girl could do it in an hour, and they had made the point that although Construction II was a little difficult it could largely be done in advance immediately the policies had been issued. Nevertheless, the feasible range was still not narrow enough.

The sunlight came back very brightly in §6. The assumption that the rates of decrement were linear in age led to extremely accurate and eminently practicable results. The one fault he had to find with the paper was that it failed to make the point explicitly that that powerful assumption was hardly 'linear programming'. The authors made two successful approximations *via* linear programming and threaded their way towards the centre of the maze; but then they became impatient and jumped over the hedge into the middle. He did not know whether they would agree with that comment. The solution the authors reached was not necessarily a 'feasible distribution'. They seemed to be worried if $(a+bx)$

should fall outside the range of 0–1, but he would not be unduly concerned; the method would be equally good, although it would not be a feasible distribution.

The authors had been exceedingly modest in presenting the results of their $(a+bx)$ method and could have tied their parcel a little tighter. In §8·1 they said that as soon as the totals $\mathcal{S}'$ and $\mathcal{P}'$ were available, a and b could be found and hence the reserve. It was simpler than that, and more effective, for there was no need to calculate a and b. Given the $(a+bx)$ assumption, the reserve at time t was equal to $\mathcal{S}'\alpha_t - \mathcal{P}'\beta_t$, where α_t and β_t had been calculated in advance, and, in fact, could be calculated on the first day of January of the year after issue.

The following was the formula: If S_0, P_0 and ${}_tV_0$ were the totals of the sums assured, office premiums and reserves at time t under the business originally issued, and S_0', P_0' and ${}_tV_0'$ were the first moments of those items with regard to age, then

$$\alpha_t = \frac{{}_tV_0P_0' - {}_tV_0'P_0}{S_0P_0' - S_0'P_0}, \quad \beta_t = \frac{{}_tV_0S_0' - {}_tV_0'S_0}{S_0P_0' - S_0'P_0}.$$

More work was, of course, involved but it was all done in advance. The amount left to the actual valuation was trivial and the error, it should be noted, was comparable with an error of 2 in the last place in the annuity tables. It was a very powerful method indeed.

The authors' examination of other established methods was interesting, but there was just a hint that they were judging them too closely from the special angle of linear programming. Their own highly successful $(a+bx)$ method, which must surely find a place in actuaries' established techniques, was not strictly linear programming, and he felt that the authors might, with a grateful wave to linear programming for bringing them so far, say 'au revoir' to it, but not 'adieu', and pursue their $(a+bx)$ method into wider fields.

An important point about the method was that it gave a distribution of the data which satisfied two extremely powerful conditions—that the sum assured was right and that the premiums were also right. That was more powerful than having the first or second moments, because the premiums were intimately connected with the problem being solved. While the use of the method of tracing policies back to issue was good, even more powerful results would be obtained by springing from the previous year as base. The method might have wider application than the rather simple case of industrial policies all tabulated by year of issue. He had not had an opportunity to test it, but he would like to test it on a block of ordinary endowment assurances tabulated according to year of maturity. The method would still give good results and the formula which he had suggested could basically be used.

Computers might alter the picture. A great deal of work in advance and at leisure might be no handicap when computers were available, and could be readily exchanged for a saving of work at the time of valuation.

Mr B. Benjamin welcomed the paper, first, because it was elegantly written and was a thoroughly competent piece of work making use of a technique normally considered outside the sphere of actuaries. He did not think that it mattered very much that the authors departed from the strict rules of the game of linear programming because the result was effective, and in a sense it was perhaps a very good thing that they were prepared to improvise what seemed to them to be the best techniques for the solution of their problems. They had marked out an area of precision in approximate methods which hitherto had been entirely pragmatic. They had also thrown valuable light on the more efficient organization of data.

The second reason for welcoming the paper was perhaps more important—at least, more important to him. When the formal treatment of statistics was given greater emphasis in the syllabus of the Institute examinations, it had been an act of faith because it had been thought, he believed rightly, that a new type of actuary would emerge with a wider horizon and a greater knowledge of the tremendous advances which had been made in statistics in recent years—an actuary who would be able to bring that knowledge advantageously to his ordinary work. He thought they were beginning to see results, and certainly the paper was a very good example. He felt sure that it was only one example and that many more actuaries would pick up the new techniques which were available. He believed that out of that transfusion of ideas the actuarial profession would begin to move forward more rapidly, that there would be more experiment and that their knowledge would be correspondingly richer.

Mr R. H. Daw was interested to experiment on the situation described in §4 in order to see whether the average rate of premium, or the duration since entry of a block of industrial branch business, affected the degree of indeterminacy in the net liability. For that purpose he had used a set of industrial office premiums for endowment assurances maturing at the age of 65 and the A1924–29, 3% reserves. (He had used that basis simply because the figures were readily available to him.) He did not think that the results of that investigation were of much interest in themselves, but he felt that they might be of some use because of the light they threw on the graphical methods described by the authors.

The first thing he found when using Construction I was that the points resulting from plotting V_x against P_x were rather close to a straight line. That should have been expected from the remarks in §6·1 that V_x might be nearly a linear function of P_x. The effect, however, was that the convex region was not nearly as easy to draw as he had expected from Fig. 1. That figure was, of course, for illustrative purposes and he was not criticizing it. It was, however, not easy to decide from his graph which were the appropriate ages to use in the equations for S_i and S_j in §4·2. While the choice of ages had some effect on the final result, it was not very large. He thought, however, that the fact that the points plotted were not very far from a straight line might be found more troublesome in Construction II, where it was necessary to decide when each point was crossed by a moving line. The moral of all that was that a large and very carefully drawn graph was required if reasonably accurate results were to be obtained. He had used a graph of foolscap size and that had certainly been much too small.

He had then tried Construction III on the same problem, using a graph of the same size. That method involved more work in determining the points to be plotted, but, at any rate in his example, it gave a wider convex region and one rather easier to draw. In fact, it was the better method.

A point to remember about the two constructions was that the graph of Construction I could be used for blocks of business having various average rates of premium (i.e. $\mathscr{P}/\mathscr{S}$) whereas Construction III required a separate graph for each average rate of premium.

Mr J. Stringer (a visitor) said that the Central Electricity Authority had a problem of deciding how to share out the coal which the National Coal Board supplied between the various power stations in such a way as to give minimum transport costs. That was one of the 'allocation of scarce resources' problems to which Dr Vajda had referred.

In one case they had found that the method of coal allocation already in use was giving an answer within 4% of the optimum found by calculation. They had thought 4% not too bad. But in that case, although 4% away from the minimum, they had been only 2% from the maximum. It was of value to explore the possibilities in the data at their command by the technique of linear programming.

It was natural to assume that an electronic computer would solve a fairly complicated problem more or less straight away, but they had been able to deal with problems of transportation by hand, using a few dodges similar in principle to the methods of the paper, although not graphical, in a time which made it doubtful whether it was worth paying to have them done by a computer. Those were fairly large problems, involving more equations than in the paper, and they were of a particular class which made solution by hand in that way possible, but in the general case also they had to think carefully before using a computer.

He had another method for solving a particular group of problems, and it might have an application where the same type of problem occurred frequently but with different numbers substituted. That was to use some sort of mechanical analogue. The construction which he had had in mind made use of a property that if a system of pulleys and weights and levers or some such mechanical system were connected up and knocked and tapped, then it would settle at a position of equilibrium with minimum potential energy. He could explain that only with a three-dimensional blackboard, but the property could be used to solve linear programming problems. If they could find the minimum for anything, they could postulate a related problem to which that gave the maximum.

A point which seemed to be peculiar to linear programming was that the more complicated a problem was made and the more restrictions added, the more quickly it could be solved. That seemed to be unique in mathematics.

Mr K. A. C. Wheeler said that the paper provided a powerful method of making an approximate valuation. A valuation in itself was an attempt to set up a figure which in certain economic circumstances was the reserve required by the office. The assets which matched that reserve consisted of investments of various types, some with known interest rates and some with unknown rates. But the whole integral of the valuation related to the valuation of the assets and the valuation of the liabilities. The answer produced was in terms of surplus.

There had been papers on matching, and it had occurred to him that the techniques of the paper might be applicable not only to valuation problems, but also to asset problems and surplus problems as an equally powerful method of arriving at approximate results.

Mr H. W. Haycocks, in closing the discussion, said that, as he had expected, the discussion had been directed to linear programming on the one hand and approximate valuation on the other hand, rather than to the application of linear programming to approximate valuation. The authors were perhaps themselves to blame somewhat for that. Before criticizing them in any way, he wished to say that he thought the authors had written a very interesting paper and had solved a practical problem in an ingenious and novel way.

He did not think that the authors had presented their results in the way in which they had arrived at them, a conclusion which Mr Townsend had confirmed. That was quite usual; normally results were arrived at first and then organized in a tidy manner. The authors, however, might have given the impression that they were setting up linear programming as a general and possibly best method of tackling approximate valuation. On the first page they commenced with some general principles about grouping and accuracy and showed that in certain cases linear programming gave satisfactory results. They then tried to judge traditional methods according to linear programming criteria. They had gone a long way in the right direction but he did not find their case completely convincing.

In practice, of course, the authors had not approached the problem in that general manner. They had taken a classification in use in their office and had wanted to cut out one of the variables—age at entry. They had asked themselves whether it was possible to use an approximate valuation method with only the resulting data. Presumably they had tried traditional methods and had found them wanting; indeed, there was a hint at that in the paragraph relating to Austin's method. But it would have been better had they started that way and shown to what extent traditional methods were wanting and why, for their problem, they had found it necessary to use a new method. The technique was new to them and as Dr Vajda had said, they were clever to notice so quickly that the mathematical form of the linear programming problem was the same as that underlying their problem.

As Mr Townsend had pointed out, the problem was not just a matter of approximate valuation. The grouping chosen must satisfy certain requirements, not only those of the Board of Trade returns but also the provision of information for internal purposes other than valuation, and it might also be undesirable to change existing methods too much. Traditional methods had been well tested for the purposes they served.

In the usual linear programming problem it was the actual distribution which would, for example, minimize cost or maximize profits that was required. If the authors had set themselves out to find a distribution of sums assured which in practice would maximize or minimize their liability, their problem would have been of the usual type, and the obtaining of a feasible distribution would have been a necessity. However, that was not so in the problem which the authors were trying to solve.

In dealing with the Z method they required a feasible distribution of sums assured which optimized the net liability, and the same distribution, he thought, would be taken for valuing the sums assured and valuing the net premiums. They would not optimize two things.

The great advantage of the authors' method was that it gave the range within which the actual value lay, and provided that was very narrow, the method was successful; if that were not so, however, the problem was more difficult to handle. In such cases they would have to rely on traditional methods, applying the kind of empirical tests that had been used with those methods.

Good results depended on the approximate linear relationship between V_x and P_x. In his 1933 paper (*J.I.A.* **64**, 264), Perks had noted that those functions were approximately of the same form, which implied a linear relation between them. Perks had adopted the form $an+b+c/n$ and had used it in order to find two terms at which to distribute the total sum assured. It was, therefore, likely that the use of the constraints ΣnS and $\Sigma S/n$ might give a narrower range than the use of ΣxS and Σx^2S.

In the discussion on Perks's 1946 paper (*J.I.A.* **72**, 377), both he and Redington had suggested that a generalization of Perks's method was to choose suitable properties of the actual distribution of S_x or S_n and find a simple distribution which possessed those properties. The simple distribution and the true factors were then used to make the valuation. The authors ended the paper with the statement that the problem was reduced to that of finding better information functions than the office premiums. The ideal was a function which was related linearly to ${}_tV_x$. In a way, the authors put the suggested generalization in a much more definite form.

The authors had pointed out that in practice they would have to be content with approximate linearity. He would have thought that net premiums would give better results than office premiums. He had tried a few experiments with net premiums and he had found the linear relation to be very nearly exact. In so far as the office tabulated net rather than office premiums, it should obtain even better results than those obtained by the authors.

He had thoroughly enjoyed trying to understand the paper; it had required much more than mere reading. Any actuary who did not find the idea fascinating and the neat method of solving an equation exciting must have lost some of the characteristics which first made him take up actuarial work. He thanked the authors for an extremely interesting paper, and for the same reasons as Mr B. Benjamin he was glad that it had been presented to the Institute. He hoped that some of the younger members would take up the ideas to see whether they could not apply them to organizational methods in life offices.

The Chairman (Mr M. E. Ogborn) said that the joint authors made an unusual team in that much of the paper was undoubtedly written while Mr Benjamin, who had just gained his Fellowship, was studying for the examinations and while Mr Bennett, who had since been appointed an examiner, was still a tutor. The authors deserved warm commendation for having produced a paper in such conditions.

Of necessity a sessional meeting could be devoted to the discussion of a completely new technique on comparatively few occasions; that evening was one of them. Any new method tended to attract its own critics. He almost blushed to mention the avuncular advice that it was better to travel hopefully than to arrive, but the paper was one of the occasions of which that was literally true. He thought that the authors did not set out to find any particular method of approximate valuation; they set out to explore the possibilities of linear programming. In that sense, the journey was the end itself; it was more important than the end reached.

It was appropriate that the Institute should discuss new methods, even if in the end they did not bear the fruit expected.

In the Board of Trade returns, actuaries had been accustomed to producing a great mass of figures which in the end came to one figure which was put in as the valuation liability. The activities of linear programming would be directed to defining an area within which the liability lay. That might be startling to the Board of Trade, but it was not necessarily wr ng; it was a problem to be considered. In certain circumstances it would be more appropriate to put in the maximum value obtained by the methods of linear programming, particularly if dealing with a relatively minor class. Apart from that, it must be remembered that, in addition to the valuation liability all sorts of other reserves were made, and the way in which the valuation liability had been calculated was considered when the additional reserves were made.

Were they to produce one value or an area? They were grateful to the authors for giving them the opportunity to discuss linear programming and hoped that their ideas would bear fruit in the years to come. In proposing a vote of thanks he especially added the thanks of the Research Committee, of which he was Chairman. Operational research was one of the subjects on the Committee's agenda; in stimulating interest in the subject the paper had done a good job, and the Committee were grateful to the authors.

Mr Bennett, in reply, thanked the meeting for the reception they had given to the paper. It was difficult to formulate verbal criticisms of a paper which was largely algebraic, and therefore the authors were grateful to those who had spoken, but by the same token it was equally difficult to reply verbally to such criticisms.

The opener had objected to the authors' two-point maximum and minimum and had given two reasons for his objection. The first was that the underlying distributions were inherently unlikely and the second was that the points chosen depended upon the valuation basis. The answer to the first objection lay in § 14·2; in considering the Z method, they had found in practice that a plot of Z against the single premium was in most cases convex to the abscissa throughout its length, and more often than not the valuation data were clustered around the mean age. In such circumstances the true distribution might give a value inherently likely and very close to the two-point minimum. So would a large number of inherently likely distributions. The actuary usually liked to be prepared for the worst, and the two-point upper and lower limit prepared him. In case the size of the range was sometimes frightening, it was worth mentioning that it was axiomatic that inherently likely distributions could not simultaneously yield values close to the maximum and close to the minimum. They might be close to one or the other, but he did not think they could be close to both.

As for the second objection, while he agreed that the valuation basis might be an estimate of the future, it was, nevertheless, a decision of the past, and it seemed to him that the objection was largely irrelevant. In the event of the valuation basis being changed, all they did was to recalculate the two-point maximum or minimum.

The opener's criticism of their check on the Z method, however, had been valid. He had pointed out that in § 13 they had sought to optimize two expressions, present value of sums assured and present value of net premiums, but if the net liability was under consideration it would be necessary to optimize the difference between those two expressions.

The methods outlined in the first part of the paper had come in for considerable bombardment, but some of the criticisms, although legitimate, had overlooked the fact that Part I was largely an illustration to show what could be done. Had it been a different block of business of a different class, they might have adopted different methods. The fact that the $(a+bx)$ procedure was used was neither here nor there. Had they had a block of whole life policies they might have finished with something quite different.

Furthermore, it was stated in § 14 that the techniques, at least in the first part of the paper, were not to supersede existing methods but to run parallel to them. If the paper had any merit it lay there. They had introduced a new tool, and although at the moment it might have only a limited application on its own—he personally felt that it might soon have very wide applications in conjunction with computers—nevertheless, it could be used in conjunction with existing methods and was a valuable addendum to them.

It had been objected that the r_x method was not linear programming. That was true, but if they had introduced their $(a+bx)$ method in the second paragraph of the paper and produced a good result, would there not have been criticism and scepticism? Without the rather tedious introduction and building up of the data, it was unlikely that anybody would have accepted the $(a+bx)$ method. It was not until the use of linear programming had shown that the range was exceedingly small that anybody could be moved to place any reliability in it.

Mr Joseph had criticized the remarks in § 14·1, where the authors had said that no approximation should be made to the factors themselves. The text, however, was merely a statement of fact that in order to be sure of a feasible value they had to have a feasible distribution and use the true factors.

He thanked the authors' office colleagues for their helpful criticisms and encouragement during the long period of gestation of their first-born. He also expressed gratitude to their departmental colleagues, who had helped, quite undismayed by the many abortive experiments and calculations inevitably associated with such a paper.