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THE INSTITUTE OF ACTUARIES

APPROXIMATE INTEGRATION

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FOR the first eighty years of our Institute's existence, approximate integration of actuarial functions was confined to ascertaining the area of a plane surface bounded by three straight lines and one curved; although the function to be integrated is usually a product, its components were ignored, and the product itself treated as the integrand.

2. This is the more curious since so long ago as November 1868, in his Presidential address to the Actuarial Society of Edinburgh (*J.I.A.* XV, 257), Edward Sang pointed out that 'the integral or primitive of a product is a series composed of the products of the derivatives of the one by the primitives of the other factor', that 'the primitives and derivatives of v^n are known' and that accordingly we can integrate the product $u_{x+n}v^n$ when the derivatives of u_{x+n} terminate—an invitation in the clearest terms to abandon the product-integrand, to treat u_{x+n} as a polynomial, and to make use of the reduction formula

$$\int u_{x+t}v^t\,dt=\frac{u_{x+t}v^t}{\log_e v}-\int\left(\frac{du_{x+t}}{dt}\frac{v^t}{\log_e v}\right)dt.$$

3. In 1927 Prof. J. F. Steffensen wrote in the *Skandinavisk Actuarietidskrift* upon the sum or integral of the product of two functions, and in 1933 G. J. Lidstone wrote upon product finite integration by parts (*J.I.A.* LXIV, 160). From 1918 increasing attention had been paid in this country to the use of moments in approximate valuation, led by Henry, and there naturally followed the application of such methods to the approximate calculation of isolated values. This brief statement may assist in allaying the surprise of the student of the future when he is told to look for our own first systematic contribution to the evaluation of the product-integral in a paper the title of which commences *On a modification of the net premium method of valuation*... (W. Perks, 1933, *J.I.A.* LXIV, 286).

4. Perks was followed by H. G. Jones (p. 318) and by A. W. Joseph (p. 329 and *J.I.A.* LXV, 277), and in the following year in a more elementary field, and by a geometrical instead of an analytical approach, by the author of *The Curve of Deaths* (1934, *J.I.A.* LXVI, 17). There the matter rested (if that be the right word to apply to what was evidently a considerable display of industry) until 1945, when Perks presented *Two-variable developments of the n-ages method* (*J.I.A.* LXXII, 377), followed by R. E. Beard in 1947 with *Some notes on approximate product-integration* (*J.I.A.* LXXIII, 356), and in 1948 with *Some experiments in the use of the Incomplete Gamma Function for the approximate calculation of actuarial functions* (*Proc. Cent. Assembly Inst. Actuaries*, II, 89).

5. In the discussion on Beard's 1947 paper, F. M. Redington neatly summarized the position (p. 407) by saying that into the theoretical problem of integrating the $f\phi$ product

f and ϕ entered symmetrically, although in the practical problem the two factors might be introduced in very different shapes, and might have to be treated separately.

However it was certainly important to choose suitable mathematical adjectives to describe both f and ϕ. The adjectives used in most of Perks's work and in the present paper could briefly be described as 'moments' for ϕ and 'points' for f—one or two or three or four specimen points....The use of moments for both f and ϕ had been examined in their different ways by Henry and Joseph.

Perks (p. 411) visualized the possibility of product-integration 'by substituting a few rectangular distributions in the form of a histogram', which system 'might be called "n-slab" formulae to distinguish them from the n-point formulae'.

6. In the course of the same discussion Dr L. J. Comrie said that 'on first reading the paper he had encountered a difficulty at the very start....Why could not $f(x)$ and $\phi(x)$ be multiplied together and the ordinary methods of quadrature then applied?' (p. 406). C. D. Rich (p. 413) threw light upon this question by pointing out that $\bar{A}$, for instance 'was the integral of a continuous function which could be tackled directly, or which could—on the lines adopted by the author—be split into two component factors. It seemed that, perhaps because of the nature of the function, the latter course was probably advantageous.'

7. It may be convenient to deal a little more fully with this question at once. It will be recalled that the symbol $(d)_x$ was tentatively used in 1934 for the ordinate of the Curve of Deaths, but one felt then, and still feels, that a Greek letter is to be preferred for this 'instantaneous d', by analogy with μ and δ, and we will here follow R. D. Clarke in *A bio-actuarial approach to forecasting rates of mortality* (1948, *Proc. Cent. Assembly Inst. Actuaries*, II, 12) in using ϕ_x for this ordinate. Also d_{a-b} was used for $(l_{x+a} - l_{x+b})$, i.e. the deaths between ages $(x+a)$ and $(x+b)$, but a more convenient notation is required. Regarding l_x as 'all the deaths after age x', it is hoped that $l_{x:\overline{n}|} = l_x - l_{x+n}$ for 'all the deaths after age x within n years' will be readily recognizable. Then, in deciding whether, for instance,

$$l_x \bar{A}_x = \int_0^h \phi_{x+t} v^t dt + \int_h^{2h} \phi_{x+t} v^t dt + \ldots,$$

should be treated as a series of areas, or as a series of solids in each of which the face-area is $l_{x:\overline{n}|}$ and the third dimension is the ordinate of the die-away curve, v^n, we shall get some assistance from the block shown by solid lines in diagram 1, all the six surfaces of which are planes, regarded as a first approximation to

$$\int_a^{b=a+h} \phi_{x+t} v^t dt.$$

8. The broken lines show the cross-section of the block at the mid-point of h, and the rectangular prism formed by extending this cross-section over the whole length of h. The dotted lines delineate a pyramid; the dimensions of the rectangular base (shaded) of the pyramid are manifestly $(\phi_a - \phi_b)$ and $\frac{1}{2}(v_a - v_b)$,* and its height $\frac{1}{2}h$. A moment's inspection will show that, if this pyramid is cut away, the remainder of the original block can be converted into the rectangular prism by turning wedge A through 180° on axis A, and wedge B through 180° on axis B, the new position of the wedges being indicated by dot-dash lines. From this it follows that the volume of the whole original block is

$$V = h\{\tfrac{1}{2}(\phi_a + \phi_b) \cdot \tfrac{1}{2}(v_a + v_b) + \tfrac{1}{3} \cdot \tfrac{1}{2}(\phi_a - \phi_b) \cdot \tfrac{1}{2}(v_a - v_b)\}.$$

* Suffixes have been used for v in place of indices, because of the assumption of a straight line.

Clearly this formula is equally applicable if ϕ slopes in the opposite direction, for then the turning of the two wedges will leave the rectangular prism deficient by the same pyramid, and the term $(\phi_a - \phi_b)$ rightly becomes negative. The formula reduces to

$$\frac{h}{6}\{\phi_a v_a + (\phi_a + \phi_b)(v_a + v_b) + \phi_b v_b\},$$

at which we could have arrived very much more simply, since it is a well-known rule of mensuration. Indeed, since for each function the mid-value is the mean of the terminal values, the mensuration corresponds precisely to Simpson's Rule, although it preceded it by about 3500 years, having been known since *circa* 1850 B.C. However, there was a reason which will immediately appear for expressing V by the formula above, and a reason which will emerge later for arriving at that formula geometrically.

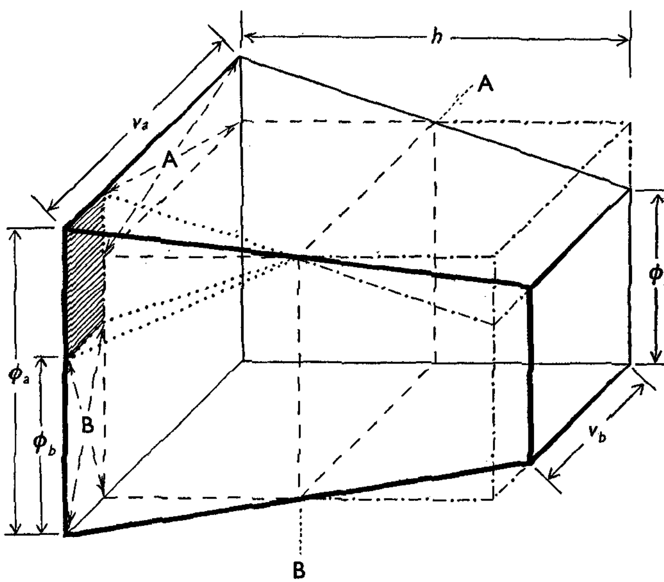

Diagram 1.

9. A quadrator who has not seen the block, and who is not told that ϕ and v are each of the first degree, is given only the three figures $f(a)$, $f(b)$, and h, where $f(a)$ is $\phi_a v_a$ and $f(b)$ is $\phi_b v_b$. What can he do with these three figures further than to treat $f(n)$ as a function of the first degree? His best estimate can be no better than

$$Q = \frac{h}{2}\{f(a) + f(b)\},$$

which reduces to

$$\begin{aligned} Q &= \frac{h}{4}(\phi_a + \phi_b)(v_a + v_b) + \frac{h}{4}(\phi_a - \phi_b)(v_a - v_b) \\ &= V + \frac{h}{6}(\phi_a - \phi_b)(v_a - v_b) \\ &= V + e \quad \text{(say).} \end{aligned}$$

10. A geometer is supplied with two figures only, namely, $l_{a:\overline{h}|}\left(=\frac{h}{2}(\phi_a+\phi_b)\right)$ and the mean value of $v\ (=\frac{1}{2}(v_a+v_b))$. He knows that he will not get the correct volume of the block by multiplying the area of the face by the average thickness, unless the height be constant; but as he does not know whether ϕ is, in fact, constant, or whether it is increasing or decreasing, it is the best he can do. He thus reaches

$$\begin{aligned} G &= \frac{h}{4}(\phi_a+\phi_b)(v_a+v_b) \\ &= V - \frac{h}{12}(\phi_a-\phi_b)(v_a-v_b) \\ &= V - \tfrac{1}{2}e. \end{aligned}$$

His error is precisely half that of the quadrator.

11. In practice ϕ_x will usually not be a straight line. Whether it be concave, or convex, the quadrator will arrive at the same value as previously; the geometer, on the other hand, will arrive at a value differing from that for a straight line, for he uses the true value of the area $\int_a^b \phi_t\,dt$ in his calculations. Consider the case where ϕ_x is convex; above the plane of the straight line ϕ_a to ϕ_b is a segment-shaped slab which the geometer's method allows for by multiplying the area of the segment by the mean value of v. There can be little error in so doing, since the ordinate of the segment is zero where the value of v_t differs most from the mean value, and the 'weight' of the slab is in the centre, where the mean value is approximately correct. Furthermore, although ϕ_x and v^n are not the simple functions we have assumed as a first approximation, the same principles apply in a more complex way to the more complex functions.

12. Reverting to the block with six plane surfaces, we can improve upon the results of both the quadrator and the geometer. Since, as we have seen, the true value lies one-third of the way between the two, we shall arrive at the exact volume of the block if we add Q to twice G, and divide by 3. We will leave the exciting possibilities of this artifice for later exploration.

13. It is the purpose of this paper to consider the practicability of obtaining monetary values, $\bar{A}_x$ for instance, from mortality functions, either ϕ_x or l_x, available only for every nth age, either by means of existing formulae, or by modifications of existing formulae, incidentally adapting to equally spaced datum points the simple cubature formula suggested in 1934 (*J.I.A.* LXVI, 28):

$$\bar{A} = 1{\cdot}05 - \frac{l_0+l_a+l_b+\dots}{10 l_0}.$$

This formula depended for its rough accuracy partly upon the feature illustrated in § 10 *supra*, partly upon the deliberate use (as a compensating error) of ·95 as the mean value of v^n from $v^0=1$ to $v^a=\cdot 9$, etc., but also to a considerable extent upon the equal decrements of v^n which resulted from the use of unequally spaced datum points (the simplification of the arithmetic being an incidental result of, and not the motive for, those equal decrements). However, before considering the adaptation of this formula to equally spaced datum points, before returning to the artifice of § 12, or to 1868 and Edward Sang and the use of integration by parts, it is appropriate to see whether we need anything more than our existing equipment.

HISTORICAL SURVEY

'The manner in which engineers, physicists and actuaries so readily use parabolic curves is open to considerable criticism.'—KARL PEARSON (1902, *Biometrika*, II, 19).

14. Formulae of quadrature which use 'selected values of the function to be integrated' have been in actuarial use only since 1883. From a date considerably earlier than the foundation of our Institute in 1848, the method used was to divide the area into vertical strips, assumed rectangular, to sum them and to apply a correction for curvilinearity in the form of the leading differences (Lubbock), until in 1863 W. S. B. Woolhouse applied the Euler-Maclaurin expansion to life contingencies (*J.I.A.* XI, 61), replacing leading differences by leading differential coefficients. That paper, interesting historically and otherwise, is also entitled to our admiration for the skill with which Woolhouse dealt with the integral calculus without terrifying his contemporaries with integral notation. It is not difficult to read once one has become accustomed to A being used as the symbol for a_x, and from it those who have forgotten may be reminded that the original form of the familiar formula for the value of an annuity payable m times a year was to obtain a_x from values of D_{x+n}/D_x taken at intervals of n years, e.g.

$$a_x \doteq \frac{7}{D_x}\{D_{x+7}+D_{x+14}+D_{x+21}+\dots\}+3-4(\mu+\delta).$$

Such formulae were applicable to more complicated functions than those involving only one life, and apparently they were immediately adopted by the profession in 1863, in place of Lubbock.

15. Twenty years elapsed. In a paper presented to the Institute in January 1883 (*J.I.A.* XXIV, 97) G. F. Hardy said, of the Euler-Maclaurin formulae: 'So far as the accuracy of these formulas is concerned, they scarcely admit of any improvement; the calculation of the differential coefficients...is occasionally, however, attended with some inconvenience'; and he added that 'a similar objection lies against Lubbock's formula in consequence of the manipulation of the successive differences involved in its use'. To avoid these difficulties Hardy drew attention to the so-called Simpson's Rule (published by Simpson in 1743, but given earlier by Cavalieri in 1639—in geometric form, be it noted!—and by James Gregory in 1668), to the 'three-eighths' rule, and to Weddle's rule. Incidentally Hardy demonstrated that Simpson's Rule usually gives better results than the 'three-eighths' rule, and showed why.

16. Hardy then turned his genius upon

some important formulas of integration first introduced by Gauss. This mathematician has shown that, by properly selecting the ordinates, we can with n values effect the exact integration of a function of the $(2n-1)$th degree.

The Gauss formulae, it will be remembered, use ordinates which are not only not equidistant, but which are not for integral arguments; but Hardy skilfully selected those for which integral datum points could be substituted for the irrational with good approximation, and produced four formulae of which the most elaborate was

$$\int_0^{22h} = 11h\{\cdot 23u_h+\cdot 48u_{5h}+\cdot 58u_{11h}+\cdot 48u_{17h}+\cdot 23u_{21h}\}, \qquad \text{(H 1)}$$

but in none of which the datum points were equidistant.

17. Finally, Hardy set out into what appears to have been a new field, and used Jacobi's method for the development of Gauss-type formulae 'to find what ordinates should be selected to obtain the best results when the initial and final values of the function are already known' (and for that reason are to be included in the formulae); for the cases where the function is of the 7th, 9th and 5th degree respectively, he obtained three formulae (A), (B) and (C), using three, four, and two ordinates respectively, in addition to the initial and final values. He gave worked examples of formulae (B) and (C), as also of the four preceding formulae, but not, rather curiously, of formula (A):

$$\int_0^{6h} = h\{\cdot 28\,(u_0+u_{6h}) + 1\cdot 62\,(u_h+u_{5h}) + 2\cdot 2u_{3h}\} \qquad \text{(H 37)}$$

It is known to many as 'formula (37)', having borne that number in the chapter which Hardy later wrote for King's Text-book, Part II.

18. In February 1887 George King presented a paper (*J.I.A.* XXVI, 276) dealing exclusively with (H 37) and modifications of it, and in particular made the valuable suggestion that a very powerful formula could be derived from (H 37) by summing in sections

$$\begin{aligned}\int_0^{\omega} = h\{\cdot 28u_0 &+ \cdot 56\,(u_{6h}+u_{12h}+u_{18h}+\ldots)\\ &+ 1\cdot 62\,(u_h+u_{5h}+u_{7h}+\ldots)\\ &+ 2\cdot 2\,(u_{3h}+u_{9h}+u_{15h}+\ldots)\},\end{aligned} \qquad \text{(HK 38)}$$

which Hardy later reproduced in the Text-book, Part II, as formula (38). Most of the Gauss-type formulae are unsuited to our present purpose, namely, to use datum points at intervals of n years, but one is not prepared to agree entirely with H. and B. S. Jeffreys when they say, in their monumental *Methods of Mathematical Physics* (1946, Cambridge University Press), at p. 264, that all such formulae as those due to Gauss 'are best regarded as museum pieces'. There are exceptions.

19. Formula (H 37), for example. Hardy achieved integral datum points by the not noticeably close assumption that $\cdot 1727$ and $\cdot 8273$ can respectively be replaced by $\frac{1}{6}$ and $\frac{5}{6}$. It is true that the formula uses unequally spaced datum points, but only to the extent of eliminating u_{2h} and u_{4h}. It is correct to fifth differences, and as such it keeps reappearing in actuarial and quasi-actuarial literature; it has become the habit so to develop it for the student, and as Jones pointed out recently (*J.I.A.* LXXIII, 409) everyone appears to have forgotten that it was developed by Hardy to be approximately true as far as seventh differences. Its disadvantage is that it requires the interval over which we are integrating to be divisible by 6, but this disadvantage does not arise when we use it in its replicated form (HK 38), with h constant ($=5$, say), truncated where the data peter out.

20. By truncating at $7h$, King arrived at the classic formula 39 (a), and his paper was devoted to numerical tests of that formula. For the first three demonstrations he chose h so that $(x+7h)$ was just under ω; but then, pointing out that u_{7h} was contributing nothing to his results, he thereafter chose h so as to eliminate u_{7h} (and it is believed the profession has done so ever since) in which form 39 (a) differs from (H 37), applied with the same value for h, only

in using ·56 instead of ·28 as the coefficient of u_{6h}. Both Wm. Sutton and T. G. Ackland asked why ·56 was better than ·28, but the question remained unanswered.

21. Theoretically it is easy to see that, with $(x+6h)$ just under ω, (H 37) gives, for instance, $\bar{A}^{1}_{x:\overline{6h}|}$, and that 39 ($a$), with its double dose of u_{6h}, provides for what happens after age $(x+6h)$, and gives $\bar{A}_x$. The trouble with the theory lies in its practical application. Of the six examples worked in the current edition of *Life Contingencies*, three would have given precisely the same result had (H 37) been used instead of 39 (a) (pp. 262, 313, and 344), and in two cases (H 37) would have given better results (pp. 295 and 335)! Incidentally, it has been said of both (H 37) and 39 (a), but admittedly not by either Hardy or King, that they are recommended by the simplicity of their coefficients. Since the formulae were originally demonstrated, and still are to this day, worked by logarithms, the only advantage of the simple coefficients would seem to be that they are easily remembered.

22. Woolhouse was back at the Institute in the following year with an elaborate paper *On integration by means of selected values of the function* (1888, *J.I.A.* XXVII, 122) in which he expounded a technique for deriving Gauss-Hardy type formulae (i.e. Gauss formulae approximately correct to the $(2n-1)$th difference, but with integral datum points) in wholesale quantity. Fifteen formulae were set out (and many more implied) of which ten use 'boundary' ordinates, and with two of them Woolhouse was so delighted that he called them 'Nugget No. 1' and 'Nugget No. 2' respectively. The eminent astronomer was, perhaps, possessed of a sense of humour which some of his more ponderous writings did not always reveal.

23. No. 5 of these fifteen formulae is, as Woolhouse himself pointed out, merely a reappearance of (H 37) in a new guise, his coefficients being set out in fractional, instead of in decimal form; but Woolhouse did not point out, and presumably did not realize, that his No. 9 is approximately the same as (H 1). No. 10, the so-called Nugget No. 1, becomes, with coefficients in decimal form,

$$\int_0^{10h} = h\{\cdot 28093\,(u_0+u_{10h}) + 1\cdot 61899\,(u_h+u_{9h}) + 2\cdot 18490\,(u_{3h}+u_{7h}) + 1\cdot 83036 u_{5h}\}. \qquad \text{(W 10)}$$

The resemblance of the first two coefficients to those of (H 37) will be noticed. This formula is so good, Woolhouse said, that it may be truncated beyond u_{6h}; and he calculated $\bar{a}_{40}$ at H^{M} $3\frac{1}{2}$ % by putting $h=9$, reaching 16·5978 as against the true value 16·5989. It is a little unfortunate that if we use the whole formula by putting $h=6$, we arrive at a value which is not so good, namely, 16·5970. However, we will include (W 10) in the tests to be made in the next section of this paper; it is the only one of the fifteen formulae, apart from (H 37), of course, that is of use to us here, the others having datum points at uneven intervals.

24. The appearance of Henry's *Calculus and Probability* in 1922, prompted Chas. H. Wickens to contribute (1923, *J.I.A.* LIV, 209) a note in which he made the interesting point that there are an unlimited number of formulae using five equidistant ordinates, including the boundary ordinates, which will exactly integrate a function of not more than the third degree, and that such formulae can be scheduled systematically by writing down any multiple whatever of (1, 0, 4, 0, 1), attaching the appropriate factor, and repeatedly subtracting (1, −4, 6, −4, 1) = 0. He arrived at Simpson (in two different forms) and also

four other simple formulae 'not so well known', of one of which he said that probably for most purposes it would 'give as good results as any':

$$\tfrac{4}{9}(1, 2, 3, 2, 1), \qquad \text{(Wi. 1)}$$

but unfortunately he gave no worked examples.

25. Repeating the process with seven equidistant ordinates, exactly to reproduce a function of not more than the fifth degree, he found that, in order to have positive coefficients throughout, the coefficients of the boundary ordinates must lie between ·28 and ·39, both inclusive; and commencing with

$$(\cdot 28,\ 1\cdot 62,\ 0,\ 2\cdot 2,\ 0,\ 1\cdot 62,\ \cdot 28)$$

[(H 37) again!], by repeated addition of

$$(\cdot 01,\ -\cdot 06,\ \cdot 15,\ -\cdot 20,\ \cdot 15,\ -\cdot 06,\ \cdot 01),$$

he reached Weddle's rule in two steps, and tabulated twelve formulae in all, of which he selected the sixth (which has a strong resemblance to Simpson's Rule applied three times):

$$\int_0^6 = \cdot 3\,(1\cdot 1,\ 4\cdot 4,\ 2\cdot 5,\ 4,\ 2\cdot 5,\ 4\cdot 4,\ 1\cdot 1), \qquad \text{(Wi. 2)}$$

and suggested that it has some advantages over both Hardy's formula and Weddle's rule; but again, alas, no worked examples.

26. In 1924 *The Calculus of Observations* by Whittaker and Robinson was published, and as its Preface acknowledges assistance from Lidstone it is not surprising that in Chapter VII entitled *Numerical Integration and Summation* we meet many old friends, commencing with the Euler-Maclaurin expansion, and Gregory's and Lubbock's formulae. Passing on to formulae which involve only values of the function to be integrated, the authors start from Gregory's formula to point out (p. 152) that Weddle can be reached at once from (H 37) by adding the zero quantity $\frac{1}{50}\Delta^6 f_a$, but they do not mention Wickens, nor that a whole series can be obtained by the repeated addition of $\frac{1}{100}\Delta^6 f_a$—or (for that matter) ten times as many by the repeated addition of $\frac{1}{1000}\Delta^6 f_a$, and so on, *ad infinitum*. They do not mention that Hardy's formula was designed to be approximately correct to seventh differences; and later they give Woolhouse's 'Nugget No. 1' and 'Nugget No. 2' with the same omission. Shovelton's formula is given:

$$\int_0^{10} = \tfrac{5}{126}(8, 35, 15, 35, 15, 36, 15, 35, 15, 35, 8), \qquad \text{(Sh.)}$$

and the Newton-Cotes series are dealt with systematically, including Simpson's and the 'three-eighths' rules. The chapter concludes with Tchebychef's formulae using equal coefficients for unequally spaced datum points, upon which Elderton in 1924 contributed a note (*J.S.S.* II, pt. 2, 140), and with Gauss's formulae (which should have preceded any references to (H 37) and Nuggets 1 and 2, instead of following them) giving the four formulae set out by Hardy in *J.I.A.* XXIV, 97. With regard to the Cotes formulae, M. T. L. Bizley drew attention in 1946 to one which integrates between limits 0 and $5n$, and so can be employed when other formulae fail, i.e. when the number of years in the period of integration is an odd multiple of 5 and not also a multiple of 3 (*J.S.S.* VI, pt. 2, 90), the Editor having apparently overlooked that this formula was given in 1922 in A. E. King's essential paper (*T.F.A.* IX, 218).

TESTS OF EXISTING FORMULAE

'The need of the moment...is arithmetic and arithmetic and arithmetic.'*—*J.I.A.* LIV, 206.

27. For testing existing quadrature formulae we will assume that the following is the whole of the information available of a particular, but for the moment unidentified, mortality table. (Later we shall use the l_x column alone to test a suggested cubature formula):

x	$\frac{1}{10} l_x$	ϕ_x	$\bar{A}_x$
20	1000	39	·2917
25		42	
30	958	43	·3587
35		45	
40	913	52	·4484
45		68	
50	843	94	·5577
55		133	
60	708	183	·6742
65		240	
70	470	286	
75		298	
80	182	254	
85		161	
90	21	66	
95		14	
100	—	1	

28. If it is intended to put a replicated quadrature formula to any extensive use with h constant, $=5$, it is convenient to rewrite it in the form

$$l\bar{A} = a\phi_0 + b\phi_5 + c\phi_{10} + d\phi_{15} + \dots,$$

where

$$a = c_0 \times 5,$$

$$b = c_5 \times 5 \times v^5,$$

$$c = c_{10} \times 5 \times v^{10},$$

etc.

The values of $a, b, c, \dots$ at 3 % are set out in the following schedule for (HK 38), (W 10), Simpson's Rule, (Wi. 2), and (Sh.). A test was made of (Wi. 1) also, but the results did not encourage the tabulation of the coefficients. With $h=5$, it replicates every twenty years, as compared with the ten years of Simpson, with which—on the basis of this one test—it cannot compete. Indeed, the arithmetic for this paper, only a small part of which is exhibited, has proved how powerful Simpson's Rule is, in actuarial use, and how right Rich has been persistently to remind us of it.

29. The values of ϕ_0, ϕ_5, ϕ_{10}, etc., are written in sequence one under another on a strip of paper, which is laid upon the schedule with the 'age at entry' in

* I will not insult the reader's memory by naming the author.

line with $t=0$. The multiplications are accumulated upon the arithmometer without being recorded, and the total, divided by l_x, gives $\bar{A}_x$ at once. The

t	HK 38	W 10	Simp.	Wi. 2	Sh.
0	1·4	1·405	1·667	1·650	1·587
5	6·987	6·983	5·750	5·693	5·990
10	0	0	2·480	2·790	2·215
15	7·060	7·012	4·279	3·851	4·457
20	0	0	1·846	2·076	1·648
25	3·869	4·371	3·184	3·152	3·412
30	1·154	0	1·373	1·360	1·226
35	2·879	3·882	2·369	2·346	2·468
40	0	0	1·022	1·150	·912
45	2·909	2·141	1·763	1·587	1·836
50	0	·641	·760	·855	·724
55	1·594	1·593	1·312	1·299	1·366
60	·475	0	·566	·560	·505
65	1·186	1·599	·976	·966	1·017
70	0	0	·421	·474	·376
75	1·198	·997	·726	·654	·778
80	0	0	·313	·352	·280
85	·657	·886	·540	·535	·563
90	·196	0	·233	·231	·208
95	·489	·488	·402	·398	·419

results, with the errors 'Calc. – True' immediately beneath, are shown in the following schedule:

3 %	$\bar{A}_{20}$	$\bar{A}_{30}$	$\bar{A}_{40}$	$\bar{A}_{50}$	$\bar{A}_{60}$	Mean error
True	·2917	·3587	·4484	·5577	·6742	
HK 38	·2922	·3594	·4485	·5582	·6746	
Error	5	7	1	5	4	4·4
W 10	·2920	·3595	·4486	·5578	·6752	
Error	3	8	2	1	10	4·8
Simp.	·2920	·3591	·4483	·5578	·6745	
Error	3	4	−1	1	3	2·4
Wi. 2	·2920	·3590	·4485	·5578	·6741	
Error	3	3	1	1	−1	1·8
Sh.	·2920	·3592	·4482	·5580	·6743	
Error	3	5	−2	3	1	2·8

In comparing the results it may be borne in mind that to obtain the five $\bar{A}$ values, (HK 38) has involved us in only 43 multiplications, and (W 10) in only 39, whereas the other three have each required 65.

30. As applied to this particular purpose, with these particular figures, it does not appear that (W 10) is a nugget of such dazzling lustre as to put (HK 38) off the gold standard. (Wi. 2) brings out, in this case, the lowest mean error, but it is probably true to say that this test shows no significant difference between any of the three formulae which use every fifth value of ϕ_x. It should be noted that, since it will presently be proposed to calculate $\bar{A}_x$ from decennial values of $l_{x:\overline{10}|}$ with radix $l_{20}=1{,}000$, it was thought not unfair to limit the scale of the ϕ_x figures to radix $l_{20}=10{,}000$; but such small-scale figures naturally cannot guarantee the fourth place of $\bar{A}$, whatever the formula employed.

CUBATURE

'When I was at school the mean distance from the earth to the sun was stated as 95,142,357 miles. I wonder why furlongs and inches were not mentioned.'—PROFESSOR JOHN PERRY, in a lecture on *Practical Mathematics*, 1899.

31. We return to solidity, and to the intention to adapt the 1934 formula to equal age-groups. That formula was evolved for the sole purpose of evaluating $\bar{A}_x$, and made use *inter alia* of error-offsetting between the years immediately following the age at entry, and the later years of life. The formula about to be discussed, on the contrary, is intended to be approximately correct for each section of $2n$ years, and it is not put forward for one purpose only, but is intended for the integration of any product function

$$\int_0^{2n} v^t u_{x+t}\, dt,$$

where we can readily obtain, and more especially if we already have available,

$$\int_0^{n} u_{x+t}\, dt = U_{\overline{n}|} \quad \text{(say)},$$

a simple case of the general formula being that for obtaining $\bar{A}^{1}_{x:\overline{2n}|}$ directly from $l_{x:\overline{n}|}$ and $l_{x+n:\overline{n}|}$.

32. Write

$$l_x \bar{A}_x \equiv l_x \bar{A}^{1}_{x:\overline{2n}|} + v^{2n} l_{x+2n} \bar{A}^{1}_{x+2n:\overline{2n}|} + \ldots, \tag{a}$$

and, by analogy with the formula for V in § 8, in which $h.\frac{1}{2}(\phi_a+\phi_b)$ is the equivalent of $l_{a:\overline{h}|}$, write

$$\begin{aligned} l_x \bar{A}^{1}_{x:\overline{2n}|} &\doteq l_{x:\overline{n}|}\cdot\tfrac{1}{2}(1+v^n) + l_{x+n:\overline{n}|}\cdot\tfrac{1}{2}(v^n+v^{2n}) \\ &\qquad + \tfrac{1}{3}\{l_{x:\overline{n}|}\cdot\tfrac{1}{2}(v^n-v^{2n}) - l_{x+n:\overline{n}|}\cdot\tfrac{1}{2}(1-v^n)\} \\ &= \tfrac{1}{2}(l_{x:\overline{n}|} + v^n l_{x+n:\overline{n}|})(1+v^n) + \tfrac{1}{6}(v^n l_{x:\overline{n}|} - l_{x+n:\overline{n}|})(1-v^n). \end{aligned} \tag{b}$$

This formula is put forward as a piece of geometrical mensuration in three dimensions, but in case the analogy is troublesome an alternative method of deriving it is shown in the Appendix. A formula which is closely similar is reached by integration by parts in a later section of this paper, the difference being slight and probably not significant. Furthermore, as noted in the Appendix, formula (*b*) as applied to the evaluation of $\bar{A}$ is closely similar to a known process, the difference here being not quite so slight, and perhaps significant. Formula (*b*) may be (but is not here) reproduced by the two-to-one error artifice of § 12 if we treat both ϕ and v as of the first degree, or by simple algebra if we treat either ϕ or v as of the first degree and the other as of the second degree. However, because it does in fact make use of the true values of $l_{x:\overline{n}|}$ and $l_{x+n:\overline{n}|}$, it is very much more accurate than any of these three modes of reproducing it would suggest. It is in every sense an approximate formula, it is not correct even if ϕ is constant, but it is very nearly correct in all circumstances likely to arise in actuarial practice.

33. Formula (*b*) transforms to

$$l_x \bar{A}^{1}_{x:\overline{2n}|} \doteq \{\tfrac{1}{2}(1+v^n)+\tfrac{1}{6}v^n(1-v^n)\}\, l_{x:\overline{n}|}+\{\tfrac{1}{2}v^n(1+v^n)-\tfrac{1}{6}(1-v^n)\}\, l_{x+n:\overline{n}|}$$

$$=f_0\, l_{x:\overline{n}|}+f_n\, l_{x+n:\overline{n}|},$$

whence

$$l_x \bar{A}_x \doteq f_0\, l_{x:\overline{n}|}+f_n\, l_{x+n:\overline{n}|}+v^{2n} f_0\, l_{x+2n:\overline{n}|}+v^{2n} f_n\, l_{x+3n:\overline{n}|}+\ldots, \qquad (c)$$

where $\quad f_0=(\tfrac{1}{2}+\tfrac{2}{3}v^n-\tfrac{1}{6}v^{2n})$

and $\quad f_n=(-\tfrac{1}{6}+\tfrac{2}{3}v^n+\tfrac{1}{2}v^{2n})$.

34. Substituting (l_x-l_{x+n}) for $l_{x:\overline{n}|}$, etc., in formula (*c*), we have

$$l\bar{A} \doteq f_0 l_0-\{(f_0-f_n)\, l_n+(f_n-v^{2n}f_0)\, l_{2n}+v^{2n}(f_0-f_n)\, l_{3n}+\ldots\},$$

or, for tests upon existing mortality tables for which D_x is tabulated,

$$\bar{A} \doteq F-\frac{1}{D_0}\{GD_n+HD_{2n}+GD_{3n}+HD_{4n}+\ldots\},$$

where $\quad F=f_0=(\tfrac{1}{2}+\tfrac{2}{3}v^n-\tfrac{1}{6}v^{2n})$,

$$G=\frac{f_0-f_n}{v^n}=\tfrac{2}{3}\{(1+i)^n-v^n\},$$

$$H=\frac{f_n-v^{2n}f_0}{v^{2n}}=\tfrac{1}{6}\{-(1+i)^{2n}+4(1+i)^n-4v^n+v^{2n}\},$$

i.e.

$$\bar{A} \doteq F-\frac{GN_n^{(2n)}+HN_{2n}^{(2n)}}{D_0}, \qquad (1)$$

where $\quad N_n^{(2n)}=D_n+D_{3n}+D_{5n}+\ldots,$

$\quad N_{2n}^{(2n)}=D_{2n}+D_{4n}+D_{6n}+\ldots.$

35. We will presently put $n=10$ and compare the results obtained by formula (1) with those summarized in § 29, but first we will put $n=8$. At 3 % *formula (1) reduces to*

$$\bar{A} \doteq \cdot 92241-\frac{\cdot 3182\, N_8^{(16)}+\cdot 1547\, N_{16}^{(16)}}{D_0}, \qquad (1a)$$

which is far too powerful a formula with which sensibly to compare the results of the quadrature formulae. Instead, in the schedule on p. 171, $\bar{A}_x$ is calculated for A 1924–29 Ultimate, from tabulated D_x cut down by one figure, for every eighth age from 12 to 60, to five places of decimals, with a slightly greater accuracy, in the aggregate, than is obtained by multiplying the tabulated A_x values by $(1+i/2)$.

36. If values of $\bar{A}$ are required at high ages to obtain terminal figures for endowment assurances and temporary annuities, they can be calculated very accurately (if l_x or D_x is available from age 68 onwards at every fourth age) by the formula

$$\bar{A}=\cdot 96076-\frac{\cdot 1580\, N_4^{(8)}+\cdot 0785\, N_8^{(8)}}{D_0}. \qquad (1b)$$

The calculations made by this formula, having been at first compared with tabular $A_x \times (1+i/2)$, over 82 % of the apparent discrepancies turned out to be due to the use of this formula, unreliable at old ages. In other words, the error in $\bar{A}_x$, calculated to five places of decimals by formula (1 *b*) for ages 68

A 1924–29 ult. 3 %

x	D_x Tabular cut by one figure		$N^{(16)}_{x+8}$	$\cdot 3182\, N^{(16)}_{x+8}$ $+ \cdot 1547\, N^{(16)}_{x+16}$	$\bar{A}_x$ Calc.	$\bar{A}_x$ True*	Deficiency True − Calc.
12	699,123		1,148,544	365,467 130,349 495,816	·21321	·21322	1
20		542,745	842,590	268,112 93,717 361,829	·25574	·25574	0
28	420,459		605,799	192,765 65,304 258,069	·30863	·30869	6
36		325,219	422,131	134,322 43,406 177,728	·37592	·37596	4
44	249,047		280,580	89,281 26,776 116,057	·45641	·45645	4
52		186,720	173,084	55,075 14,520 69,595	·54969	·54976	7
60	132,849		93,860	29,866 6,224 36,090	·65075	·65077	2
68		83,298					
76	39,117						
84		10,533					
92	1,118						
100		29					

* True values are $\bar{A}_x = \delta/d \,.\, A_x + \delta/12 \,.\, \mu_x$.

to 84 inclusive, is little more than one-fifth of the error involved in taking tabular A_x and multiplying by $(1+i/2)$ (see p. 172).

37. This accuracy is vastly greater than we need, and may even be greater than the accuracy of the tabulated data can justify. We are therefore encouraged to abandon $n=8$, and substitute $n=10$, although a wide range of calculations

indicates that the mean error with $n=10$ is about five times as great as with $n=8$, providing we are content to calculate $\bar{A}$ to four places of decimals instead of five. If, in addition, we reduce the scale of our data we shall have a recording error which may exceed $\pm\cdot00005$; with radix $l_{20}=1{,}000$, and with $n=10$, we must be prepared for errors up to $\pm\cdot0002$.

x	$\bar{A}_x$ 3 %			A $(1+i/2)$	Error
	Calc.	True*	Calc. − True		
68	·74888	·74888	0	·74888	0
72	·79257	·79255	2	·79249	6
76	·83064	·83059	5	·83046	13
80	·86300	·86294	6	·86272	22
84	·88952	·88949	3	·88915	34
			16		75
		Mean error	3·2		15

* True values are $\bar{A}_x=\delta/d\,.\,A_x+\delta/12\,.\,\mu_x$.

38. The following table contains all the data now supposed available of the specific but hitherto unidentified mortality table (it is, in fact, the AM[(5)] Table) used for the quadrature formulae tests:

x	$l_{x:\overline{10}\rceil}$	x	$l_{x:\overline{10}\rceil}$
20	42	60	238
30	45	70	288
40	70	80	161
50	135	90	21

Formula (1) now becomes

$$\bar{A} \doteqdot \cdot90378-\frac{\cdot3999\,N_{10}^{(20)}+\cdot1911\,N_{20}^{(20)}}{D_0}, \tag{1c}$$

but, since it is assumed that D_x values are not now available, it will be more convenient to return to formula (*c*) rewritten as

$$l\bar{A}=f_0\,l_{0:\overline{10}\rceil}+f_{10}\,l_{10:\overline{10}\rceil}+v^{20}(l\bar{A})_{20}, \tag{2}$$

where, at 3 % $\quad f_0=\cdot90378$ and $f_{10}=\cdot60623$,

so that we can use the continuous method of the following schedule (cutting f_0 and f_{10} to three figures):

(1) x	(2) l_x	(3) $l_{x:\overline{10}\rceil}$	(4) $l_{x:\overline{10}\rceil}$ × ·904	(5) $l_{x+10:\overline{10}\rceil}$ × ·606	(6) $(l\bar{A})_{x+20} \times v^{20}$ (= ·55368)	(7) $(l\bar{A})_x$ = (4)+(5)+(6)	(8) $\bar{A}_x$ = (7) ÷ (2)	(9) $\bar{A}_x$ True	(10) = (8) − (
90	21	21	18·98	—	—	18·98	—	—	—
80	182	161	145·54	12·73	—	158·27	—	—	—
70	470	288	260·35	97·57	10·51	368·43	—	—	—
60	708	238	215·15	174·53	87·63	477·31	·6742	·6742	0
50	843	135	122·04	144·23	203·99	470·26	·5578	·5577	1
40	913	70	63·28	81·81	264·28	409·37	·4484	·4484	0
30	958	45	40·59	42·56	260·37	343·52	·3586	·3587	−1
20	1000	42	37·97	27·27	226·66	291·90	·2919	·2917	2

39. Now we have a mean error of ·00008, as against quadrature mean errors in the tests summarized in § 29 varying from ·00018 (Wi. 2) to ·00048 (W 10); but it should be said at once that the close agreement in § 38 must be partly fortuitous with so low a scale as $l_{20} = 1{,}000$, according to the argument of § 37. On the basis of comparative mean errors the case for a new formula is not made out conclusively, but its support is also argued from

(i) the use of decennial figures only, whereas quinquennial figures were used for the quadrature formulae,

(ii) the use of l_x only, to the scale of $l_{20} = 1{,}000$, instead of having also ϕ_x to the scale of $l_{20} = 10{,}000$, and

(iii) the greater arithmetical simplicity.

In particular, if $U_{\overline{n}|} = \int_0^n u_{x+t}\, dt$ is tabulated, but u_{x+t} itself is not, we cannot use a quadrature formula to evaluate $\int v^t u_{x+t}\, dt$ directly until we have first found u_x, if we can.

40. There is an obvious transformation of formula (*c*) if we want only one or two isolated values, or if for some reason we want our $(l\bar{A})_x$ values displayed in their twenty-year component parts:

$$l\bar{A} \doteq f_0\, l_{0:\overline{10}|} + f_{10}\, l_{10:\overline{10}|} + f_{20}\, l_{20:\overline{10}|} + f_{30}\, l_{30:\overline{10}|} + \dots, \tag{3}$$

where $f_{20} = v^{20} f_0$, $f_{30} = v^{20} f_{10}$, $f_{40} = v^{20} f_{20}$, etc. At 3 % we have

$$\begin{aligned} f_0 &= \cdot 90378, & f_{50} &= \cdot 18585, \\ f_{10} &= \cdot 60623, & f_{60} &= \cdot 15341, \\ f_{20} &= \cdot 50040, & f_{70} &= \cdot 10290, \\ f_{30} &= \cdot 33566, & f_{80} &= \cdot 08494, \\ f_{40} &= \cdot 27706, & f_{90} &= \cdot 05697, \end{aligned}$$

these factors being cut to four or to three figures when we are working with a radix of 1,000.

41. For studying curves of death in arithmetical form with equally spaced datum points as advocated by A. W. Evans in 1934 (*J.I.A.* LXVI, 56), we need not be limited to a constant radix at only one age; we can show our mortality table in composite form with a constant radix at every decennial age, and yet, since we require l_x at decennial ages only, keep it compact if we are satisfied with the degree of accuracy in our monetary values which can be obtained with a radix of 1,000—still more compact, of course, if it is conceded that the essence of a mortality table is preserved when the radix is reduced to 100. For example, the following schedule shows the English Life Table No. 10 (Males), by means of nine curves of death in arithmetical form. These $l_{x:\overline{10}|}$ figures, and others which follow, have been obtained by scaling down the l_x column and differencing. Accumulation of recording error is thereby eliminated at the expense of such inconsistencies as 7 deaths for $l_{40:\overline{10}|}$ with the radix at age 30, as against 8 when the radix is at age 20.

English Life Table No. 10 (Males)—'100-deaths'

f	x									
·904	0	11	—	—	—	—	—	—	—	—
·606	10	2	2	—	—	—	—	—	—	—
·500	20	3	3	3	—	—	—	—	—	—
·336	30	3	4	4	4	—	—	—	—	—
·277	40	6	7	8	7	8	—	—	—	—
·186	50	11	13	13	14	13	15	—	—	—
·153	60	21	22	22	24	25	27	32	—	—
·103	70	27	31	31	32	34	36	43	63	
·085	80	14	16	17	17	18	20	22	33	90
·057	90	2	2	2	2	2	2	3	4	10

42. From a mortality table arranged in this form, using formula (3) with the coefficients cut to three figures, we can obtain approximate values of $\bar{A}_{10x}$ with a likely error of not more than $\pm$·001 at the ages mostly required for assurance purposes, which appears to be within the variation of the current retail selling price. In applying formula (3) there is no division to be performed for l_{10x} is universally 100. One merely writes the coefficients on a strip of paper, places it upon the schedule against the appropriate $l_{x:\overline{10}|}$ column, and accumulates the products on the arithmometer. The results for the English Life Table No. 10 (Males) are:

Age	10	20	30	40	50	60	70	80
Calc. $\bar{A}$ 3 % =	·222	·282	·352	·444	·552	·670	·789	·874
True =	·222	·282	·352	·444	·552	·670	·787	·875
Calc. − True	0	0	0	0	0	0	2	−1

43. The somewhat remarkable results at ages 60, 70 and 80 are not so fortuitous as might be supposed. For example, from the A 1924–29 (ultimate)—'100-deaths':

60	28	—	—	
70	42	58	—	we have $\bar{A}_{60}$ = ·651, true ·651; $\bar{A}_{70}$ = ·773, true ·771; $\bar{A}_{80}$ = ·865, true ·863
80	26	37	87	
90	4	5	13	

and from the $O^{[NM]}$ ultimate:

60	36	—	—	
70	41	64	—	we have $\bar{A}_{60}$ = ·686, true ·687; $\bar{A}_{70}$ = ·794, true ·790; $\bar{A}_{80}$ = ·877, true ·876
80	21	33	91	
90	2	3	9	

44. English Life Table No. 8 (Males)—'100-deaths':

x										Calc. $\bar{A}_x$	True	Calc. − True
0	19	—	—	—	—	—	—	—	—	—	—	—
10	2	2	—	—	—	—	—	—	—	·242	·242	0
20	3	4	4	—	—	—	—	—	—	·307	·306	1
30	4	6	6	6	—	—	—	—	—	·382	·381	1
40	8	9	9	10	10	—	—	—	—	·470	·471	−1
50	12	15	15	16	17	19	—	—	—	·575	·575	0
60	19	23	24	24	26	29	36	—	—	·684	·684	0
70	21	26	27	28	30	33	41	64	—	·792	·788	4
80	11	13	13	14	15	17	20	32	89	·871	·871	—
90	1	2	2	2	2	2	3	4	11	—	—	—

45. H^M (Makeham graduation)—'100-deaths':

x									Calc. $\bar{A}_x$	True	Calc. − True
10	4	—	—	—	—	—	—	—	·272	·272	0
20	6	7	—	—	—	—	—	—	·333	·333	0
30	8	7	8	—	—	—	—	—	·396	·397	−1
40	9	10	11	12	—	—	—	—	·478	·478	0
50	14	15	15	16	19	—	—	—	·575	·575	0
60	21	21	24	26	29	35	—	—	·682	·683	−1
70	24	25	26	29	33	41	63	—	·790	·789	1
80	13	14	15	15	17	22	34	91	·877	·876	1
90	1	1	1	2	2	2	3	9	—	—	—

46. For select tables it is, of course, essential to have a curve of deaths for each of the decennial ages at entry:

A 1924–29 (select)—'100-deaths'

x									Calc. $\bar{A}_{[x]}$	True	Calc. − True
10	2	—	—	—	—	—	—	—	·203	·203	0
20	2	2	—	—	—	—	—	—	·254	·255	−1
30	3	3	3	—	—	—	—	—	·324	·323	1
40	5	5	5	5	—	—	—	—	·414	·413	1
50	10	10	10	11	11	—	—	—	·524	·523	1
60	21	22	23	23	25	27	—	—	·647	·646	1
70	33	34	34	35	37	42	56	—	·766	·762	4
80	21	21	22	23	23	27	38	85	·859	·848	11
90	3	3	3	3	4	4	6	15	—	—	—

It will be noticed that for one select table, at least, the formula tends towards values of $\bar{A}_{[x]}$ higher than the true values, particularly at the older ages. This was to be expected from the nature of the formula.

47. The following is the whole of the working for the values of joint-life annuities for two, three and four lives, aged 10, by H^M 3 %:

	100-lives				100-deaths		
10	100	100	100	100	8	11	15
20	96	92	89	85	11	16	19
30	90	81	73	66	14	18	21
40	82	67	55	45	14	16	17
50	73	53	39	28	18	18	16
60	59	35	21	12	21	16	10
70	38	14	5	2	12	5	2
80	14	2	—	—	2	—	—
90	1	—	—	—	—	—	—

Calculated $\bar{A}$	= ·3655	·4275	·4788
Converted to A	= ·3601	·4212	·4717
Converted to a	= 20·97	18·87	17·14
Tabular a	= 20·98	18·80	17·10

The corresponding figures for age 20 are:

Calculated a	= 18·55	16·14	14·50
Tabular a	= 18·62	16·29	14·55

These results are interesting, having regard to the small scale of the figures. It will be appreciated that while we have used equal ages in order to be able to compare our results with tabular values, we could as easily have used different ages, and indeed different mortality tables.

RATIONALE OF THE CUBATURE FORMULAE

48. We return to 1868, and Edward Sang. When u and w are both rational functions of t we have

$$\int uw\,dt = u\int w\,dt - \int\left(\frac{du}{dt}\times\int w\,dt\right)dt,$$

whence, by reduction,

$$\int v^t t^n\,dt = -v^t\left\{\frac{n!}{n!}\frac{t^n}{\delta}+\frac{n!}{(n-1)!}\frac{t^{n-1}}{\delta^2}+\ldots+\frac{n!}{1!}\frac{t}{\delta^n}+\frac{n!}{0!}\frac{t^0}{\delta^{n+1}}\right\}$$

49. Now suppose that from x to $(x+20)$, $\phi_{x+t}=u_{x+t-10}$ is a polynomial of not more than the fourth degree, and write:

$$u_{x+t}=a+\frac{t}{10}b+\left(\frac{t}{10}\right)^2c+\left(\frac{t}{10}\right)^3d+\left(\frac{t}{10}\right)^4e,$$

so that

$$\begin{aligned}\phi_x &= u_{x-10}=a-b+c-d+e,\\ \phi_{x+10}&=u_x \quad =a,\\ \phi_{x+20}&=u_{x+10}=a+b+c+d+e.\end{aligned}$$

Then

$$\begin{aligned} l_x\bar{A}^1_{x:\overline{20}|} = \mathrm{I} &= \int_{-10}^{10} u_{x+t}v^{10+t}\,dt\\ &=\int_{-10}^{10} v^{10+t}\left(a+\frac{t}{10}b+\left(\frac{t}{10}\right)^2c+\left(\frac{t}{10}\right)^3d+\left(\frac{t}{10}\right)^4e\right)dt\\ &=\Bigg[-v^{10+t}\left\{\frac{a}{\delta}+\frac{b}{10}\left(\frac{t}{\delta}+\frac{1}{\delta^2}\right)+\frac{c}{100}\left(\frac{t^2}{\delta}+\frac{2t}{\delta^2}+\frac{2}{\delta^3}\right)\right.\\ &\quad\left.+\frac{d}{1{,}000}\left(\frac{t^3}{\delta}+\frac{3t^2}{\delta^2}+\frac{6t}{\delta^3}+\frac{6}{\delta^4}\right)+\frac{e}{10{,}000}\left(\frac{t^4}{\delta}+\frac{4t^3}{\delta^2}+\frac{12t^2}{\delta^3}+\frac{24t}{\delta^4}+\frac{24}{\delta^5}\right)\right\}\Bigg]_{-10}^{10},\end{aligned}$$

or, for convenience, writing p for $\frac{1}{10\delta}$:

$$\begin{aligned}\mathrm{I}=10[&-v^{20}\{ap+(p+p^2)b+(p+2p^2+2p^3)c+(p+3p^2+6p^3+6p^4)d\\ &\qquad+(p+4p^2+12p^3+24p^4+24p^5)e\}\\ &+\{ap-(p-p^2)b+(p-2p^2+2p^3)c-(p-3p^2+6p^3-6p^4)d\\ &\qquad+(p-4p^2+12p^3-24p^4+24p^5)e\}].\end{aligned}$$

At 3 % $\quad \mathrm{I}=15{\cdot}0995a-1{\cdot}4792b+5{\cdot}0912c-{\cdot}8905d+3{\cdot}0488e. \qquad$ (i)

50. Now

$$\begin{aligned} l_{x:\overline{10}|}&=\int_{-10}^{0}u_{x+t}\,dt\\ &=\left[t\left\{a+\frac{b}{2}\left(\frac{t}{10}\right)+\frac{c}{3}\left(\frac{t}{10}\right)^2+\frac{d}{4}\left(\frac{t}{10}\right)^3+\frac{e}{5}\left(\frac{t}{10}\right)^4\right\}\right]_{-10}^{0}\\ &=10\left\{a-\frac{b}{2}+\frac{c}{3}-\frac{d}{4}+\frac{e}{5}\right\},\end{aligned}$$

and similarly $\quad l_{x+10:\overline{10}|}=10\left\{a+\frac{b}{2}+\frac{c}{3}+\frac{d}{4}+\frac{e}{5}\right\},$

so that, by formula (2) at 3 %,

$$l_x \bar{A}^1_{x:\overline{20}|} = \cdot 90378 l_{x:\overline{10}|} + \cdot 60623 l_{x+10:\overline{10}|}$$

$$= 15\cdot 1001a - 1\cdot 4878b + 5\cdot 0334c - \cdot 7439d + 3\cdot 0200e, \qquad \text{(ii)}$$

a result which supports the remarks at the end of § 32.

51. Had one started with the approach of §§ 48 and 49, instead of with a geometrical approach, being nevertheless seised of the idea of expressing $\bar{A}^1_{x:\overline{2n}|}$ in terms of $l_{x:\overline{n}|}$ and $l_{x+n:\overline{n}|}$, one might have been bold enough to ignore the coefficients of c, d and e, in (i) above, obtaining by solving the two equations of the coefficients of a and b:

$$f'_0 = (1\tfrac{1}{2} + \tfrac{1}{2}v^{20})p - (1 - v^{20})p^2 = \cdot 902895 \text{ at } 3\ \%,$$

$$f'_{10} = (1 - v^{20})p^2 - (\tfrac{1}{2} + 1\tfrac{1}{2}v^{20})p = \cdot 607055 \text{ at } 3\ \%,$$

and instead of (ii) we should have had

$$15\cdot 0995a - 1\cdot 4792b + 5\cdot 0332c - \cdot 7396d + 3\cdot 0199e. \qquad \text{(iii)}$$

This hardly differs from (ii) for a, b, c and e, but where (ii) fits (i) badly, namely, the coefficient of d, (iii) is worse. However, the differences between f_0 and f_{10}, and f'_0 and f'_{10}, are probably not significant. Over any period of 20 years it depends upon the slope of the curve of deaths which of these two sets brings out the larger value for $\bar{A}^1_{x:\overline{20}|}$, but over the whole future lifetime of (x) we may expect ·9038/·6062, or ·904/·606 to bring out larger values than will ·9029/·6071 or ·903/·607. Whichever set of factors we adopt, we have the answer to the question when is it better not to deal with the product of u_{x+t}, a mortality function, and v^t, as the integrand for an ordinary quadrature integration—when u_{x+t} more closely resembles a polynomial than does $u_{x+t}v^t$. Some light is thrown on this by supposing that in § 27 we had given the quadrator only the ϕ_x figures, and not the l_x figures, which accordingly, since he needed them

x	ϕ_x	$\frac{6}{10} l_{x:\overline{10}|}$	Σ	$\frac{1}{10} l_x$
20	39	250	6001	1000
25	42			
30	43	275	5751	958
35	45			
40	52	418	5476	913
45	68			
50	94	809	5058	843
55	133			
60	183	1429	4249	708
65	240			
70	286	1732		
75	298			
80	254	964		
85	161			
90	66	123		
95	14			
100	1	1		

for his denominators, he calculated by Simpson's Rule. He would, in fact, have reached precisely the l_x figures which were supplied to him. Let it be said at once that the AM(5) is by no means the only mortality table in which, for the main ages, ϕ_x over 10-year periods can be assumed, with great accuracy, as being of not more than the third degree.

52. In order to show that the essence of a mortality table can be contained in a schedule of small compass, emphasis has been placed in the later part of this paper upon simplicity rather than upon accuracy. By way of contrast, the schedule below sets out calculations of $\bar{A}_x$ at quinquennial ages from 10 to 65 by the O^M table at 3 %, to five places of decimals, with $n=5$, by the continuous method of § 38, with a mean error for the twelve values of ·00001 precisely. Similar tests with the H^M and English Life Table No. 10 each gave a mean error of ·00002.

x	l_x (tabular)	$l_{x:\overline{5}\rceil}$	$l_{x:\overline{5}\rceil}$ × ·95106	$l_{x+5:\overline{5}\rceil}$ × ·78045	$(l\bar{A})_{x+10} \times v^{10}$ (=·74409)	$(l\bar{A})_x$	$\bar{A}_x$	Calc. − True
100	7	7	6·7	—	—	6·7	—	—
95	186	179	170·2	5·5	—	175·7	—	—
90	1,596	1,410	1,341·0	139·7	5·0	1,485·7	—	—
85	6,359	4,763	4,529·9	1,100·4	130·7	5,761·0	—	—
80	15,530	9,171	8,722·2	3,717·3	1,105·5	13,545·0	—	—
75	27,752	12,222	11,623·9	7,157·5	4,286·7	23,068·1	—	—
70	40,615	12,863	12,233·5	9,538·7	10,078·7	31,850·9	—	—
65	52,307	11,692	11,119·8	10,038·9	17,164·7	38,323·4	·73266	+3
60	62,073	9,766	9,288·1	9,125·0	23,699·9	42,113·0	·67844	+1
55	69,919	7,846	7,462·0	7,621·9	28,516·1	43,600·0	·62358	+1
50	76,185	6,266	5,959·3	6,123·4	31,335·9	43,418·6	·56991	−1
45	81,262	5,077	4,828·5	4,890·3	32,442·3	42,161·1	·51883	+2
40	85,467	4,205	3,999·2	3,962·3	32,307·3	40,268·8	·47116	0
35	88,995	3,528	3,355·3	3,281·8	31,371·7	38,008·8	·42709	0
30	91,942	2,947	2,802·8	2,753·4	29,963·6	35,519·8	·38633	0
25	94,387	2,445	2,325·3	2,300·0	28,282·0	32,907·3	·34864	0
20	96,453	2,066	1,964·9	1,908·2	26,429·9	30,303·0	·31417	−1
15	98,284	1,831	1,741·4	1,612·4	24,486·0	27,839·8	·28326	0
10	100,000	1,716	1,632·0	1,429·0	22,548·2	25,609·2	·25609	−3
						Total of 12 errors		12
						Mean error ·00001		

APPENDIX

Since this paper was in draft form, attempts have been made to improve it at a number of points as a result of suggestions from Perks and others, to whom I am greatly indebted. In particular, the original Appendix reproduced formula (*b*) of § 32 by the two-to-one error artifice of § 12, and the § 33 transformation of that formula by algebra in two different ways, but all three of these demonstrations involved 'intuitive' substitutions such as do not call for any great ingenuity when one knows the expression to be reproduced, but which ordinary people like the author do not hit upon when they are attempting to derive a formula, as distinct from reproducing one. Thus these demonstrations were clumsy, and I am accordingly grateful to Perks for a chance but very pregnant remark which has stimulated a neater method of reproduction, which, by your leave, I substitute for the original Appendix.

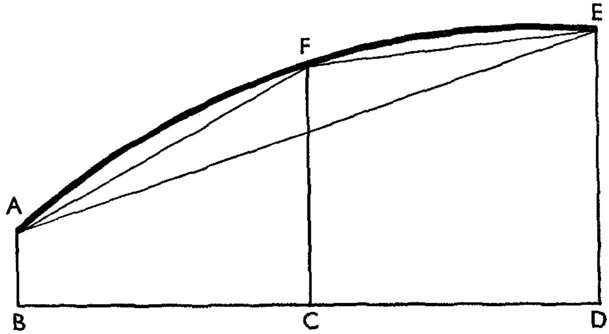

Diagram 2.

Consider the area I under the arc AFE. With BC = CD, assume that the area of segment AE is four times the area of the two segments AF and FE together. This is manifestly equivalent to assuming that in using the area of the quadrilateral ABDE as an approximation to I the error involved, say $4e$, is four times the error, e, in using the sum of the areas of the quadrilaterals ABCF and FCDE. Then

$$\mathrm{I} = \mathrm{ABDE} + 4e,$$

and alternatively

$$4\mathrm{I} = 4\mathrm{ABCF} + 4\mathrm{FCDE} + 4e.$$

By subtraction

$$3\mathrm{I} = 4(\mathrm{ABCF} + \mathrm{FCDE}) - \mathrm{ABDE},$$

which reduces at once to Simpson's Rule.

Now apply the same error assumption to actuarial cubature; for instance, assume

$$l_x\bar{A}^{1}_{x:\overline{20|}} = l_{x:\overline{20|}}\cdot\tfrac{1}{2}(1+v^{20}) + 4e, \tag{i}$$

and alternatively assume

$$4l_x\bar{A}^{1}_{x:\overline{20|}} = 4l_{x:\overline{10|}}\cdot\tfrac{1}{2}(1+v^{10}) + 4l_{x+10:\overline{10|}}\cdot\tfrac{1}{2}(v^{10}+v^{20}) + 4e. \tag{ii}$$

Subtract (i) from (ii) and divide by 3, and formula (*b*) of § 32 follows immediately.

This stresses the relationship, already implicit in the 'analogy' of § 32, between formula (*b*) and Simpson's Rule, but I am indebted to Joseph for having pointed out the nature of this relationship, namely that as applied to $\bar{A}$ the cubature formula is very near indeed to the evaluation of $\bar{A}$ by conversion of $\bar{a}$ obtained by Simpson replicated, a process which for obvious reasons produces better values of $\bar{A}$ than are obtained by direct quadrature; see, for example, the remarks of Rich in 1934 (*J.I.A.* LXVI, 46).

ABSTRACT OF THE DISCUSSION

Mr William Phillips, in introducing the paper, said that for the past several weeks he had been in a state of some excitement upon discovering that three hundred years ago a professor of mathematics at Bologna had shown how to correct a bad integration by means of a worse. If the area beneath a curve were divided into an even number of vertical strips of equal width, from which the primitive process imposingly named the 'trapezoidal rule' produced results insufficiently accurate, he thought the reaction of the average man would be to demand a lot more ordinates; not so Cavalieri, who said, in effect, 'Let us discard the even-numbered ordinates. By doubling the width of each strip we shall get a worse value than the one which is not good enough, we shall get one, in fact, *which is just four times as bad.*' Then by blending the two, Cavalieri produced Simpson's Rule, seventy years before Simpson was born!

Had he, the speaker, known of that aspect of the Rule a year or so previously, he might have been deprived of some amusing work, for an application of the same principle to cubature would have produced formula (*b*) of § 32, without invoking the aid of solid geometry. The calculated value of $\bar{A}_{36}$ in the schedule on p. 171 had, in effect, been obtained from a bad first attempt of

$$\frac{1}{l_x}\{l_{x:\overline{8}|}\cdot\tfrac{1}{2}(1+v^8)+l_{x+8:\overline{8}|}\cdot\tfrac{1}{2}(v^8+v^{16})+\ldots\}=\cdot 37898,$$

and a second, and worse, attempt of

$$\frac{1}{l_x}\{l_{x:\overline{16}|}\cdot\tfrac{1}{2}(1+v^{16})+l_{x+16:\overline{16}|}\cdot\tfrac{1}{2}(v^{16}+v^{32})+\ldots\}=\cdot 38817,$$

by treating the difference between the two values as being three times the error of the first. The final error was ·00004, reached by blending errors of the order of ·00303 and ·01221, a result which he found somewhat astonishing.

He thought the underlying assumption, true in fact for a curve of the second degree, that the area of a segment is proportional to the cube of the base-line which subtends it, might have further applications. As a simple example, the geometry leads to the general formula

$$\int_0^n u_x\,dx=\frac{n}{2}(u_0+u_n)+\frac{1}{12}\frac{n^3}{a(n-a)}\{(u_a+u_{n-a})-(u_0+u_n)\},$$

which is in fact true to the third degree because it is symmetrical. The expression reduces to Simpson's Rule upon putting $a=n/2$ and with other values of a produces a range of formulae in terms of four ordinates, including the 'three-eighths' rule in the special case of equidistant ordinates, with $a=2n/3$.

Mr L. V. Martin, in opening the discussion, said that he thought no one could accuse the author of being dull. The author had talked to the Institute of many things, such as backward time and light rays and, on one famous occasion, the carpenter's shop in the life office basement. The new paper which was the subject of discussion that evening was a real *tour de force*, being both of great interest and of undoubted originality, and yet containing a mathematical theory so simple that any competent Part I student could follow it. Such a combination of virtues in a paper was rare, and there was cause for gratitude that the author's legal light had not entirely dimmed his actuarial light.

In the section on the tests of the existing formulae the author had used a very useful artifice in the application of the various rules. The tabulation of the coefficients in the form $5c_0$, $5v^5c_5$, etc., made possible a neat method of carrying out the calculations on a machine. Once the coefficients had been calculated, they could be used again and again; $\bar{a}_x$ and $\bar{A}_x$ could be calculated from the values of l_x or ϕ_x alone, and the D_x values were not required.

The values to four decimal places which the author had obtained should not be taken as a reliable guide to the relative merits of the various formulae. In making the calculations, the final step had been to divide by a value of l_x which was tabulated to only three significant figures. There was quite a possibility, therefore, of a large error, perhaps of 3 or 4 in the fourth place. It was possible that, if l_x were taken to one more figure (and, of course, the author's ϕ_x figures were taken from a table with a radix of 10,000), comparison of the values of $\bar{A}_x$ given by the various formulae might bring out rather different results.

The main section of the paper dealt with the suggested cubature formula. Anyone who had looked through the results would agree that by the use of surprisingly few figures the author had obtained amazingly accurate results. In the section in which the author dealt with '100 deaths' and obtained values of $\bar{A}_x$ to three decimal places it might be wondered whether a similar criticism should be made of the accuracy of the last decimal place. In effect, however, the author was weighting the various values of $l_{x:\overline{10}|}$ with a value of v^n at a particular age n. Since the l_x column was exactly the sum of the $l_{x:\overline{10}|}$ column, the errors due to rounding-off had a comparatively small effect on $\bar{A}_x$. For example, if the deaths in one 10-year age-group were given as 8, and in the next 10-year age-group were given as 7, and there should be 7·5 deaths in each, the error, even if it was in the first two age-groups, would not be greater than about $\cdot 5v^6(1-v^{10})$ in the value of $l_x\bar{A}_x$, resulting in a comparatively small error in the third decimal place of $\bar{A}_x$.

In the table on p. 171, he thought that the author had been a little luckier than he deserved. The author had taken G and H, the two coefficients in formula (1) of § 34, to four decimal places only; the use of four-figure coefficients might well produce an error of as much as 1 in the fourth decimal place, whereas the results were given to five decimal places. The values to five decimal places of G and H were ·31824 and ·15465+, so that the errors tended to counterbalance; had they been in the same direction there might have been a substantial error in the fifth place.

While admiring the results which had been obtained, he wished that the author had adopted, not the formula which he had in fact adopted, but the alternative which was given in § 51. As the author said, the formula which he used had been obtained by analogy with his geometrical example. That was an approach which might be expected to appeal to him; but surely the point was that the author was expressly considering integrals of the form $\int v^t u_{x+t}\,dt$—and it was known that v^t was not a straight line. An algebraic approach would show that the formula used by the author was accurate if v^t was of the first degree in t, and u_{x+t} of the second. It was not strictly accurate if $v^t u_{x+t}$ was of the fourth or higher degree in t, though the fourth difference error was small.

There was an alternative approach. Were it supposed that $l_{20}\bar{A}^1_{20:\overline{10}|}$ was required from given values of l_{20} and l_{30} it could only be assumed that the deaths were evenly spread, and the result would be $l_{20:\overline{10}|}\times\dfrac{1-v^{10}}{10\delta}$. If that method were extended, values of $l_x\bar{A}_x$ would be obtained by summing $v^x l_{x:\overline{10}|}\times\dfrac{1-v^{10}}{10\delta}$, i.e. by assuming a series of histograms. The results given by such a method would be too high at the young ages and too low at the old ages.

The values of $\bar{A}_x$ obtained by that method from the l_x data of § 27 at ages 20, 30, etc., were ·2931, ·3601, ·4504, ·5590 and ·6726. Despite the crudity of method, the results were within $\frac{1}{2}$% of the true values. That was surely a good starting point, and one from which the formula developed by the author in § 51, could be reached. In effect, the values $l_{x:\overline{10}|}$ were grouped in pairs and ϕ_x was assumed to be of the form $a+bx$ over a range of 20 years.

The formula of § 51 could be expressed in two other ways. First of all, it could be expressed as

$$l_x\bar{A}^1_{x\,\overline{20}|}=\frac{1-v^{20}}{20\delta}\,l_{x:\overline{20}|}+C(l_{x:\overline{10}|}-l_{x+10:\overline{10}|}),$$

that was the histogram method applied over the whole 20 years with an adjustment of $(l_{x\,\overline{10}|} - l_{x+10:\,\overline{10}|})$, multiplied by a constant C, where C was ·14792. Alternatively, the formula could be expressed in terms of two separate histograms and written

$$l_x \bar{A}^1_{x\,\overline{20}|} = \frac{1-v^{10}}{10\delta}(l_{x:\,\overline{10}|} + v^{10} l_{x+10:\,\overline{10}|}) + K(l_{x:\,\overline{10}|} - l_{x+10:\,\overline{10}|}),$$

where K was ·03714—about one-quarter of the value given for C.

The fact that the formula could be obtained as an adjustment of the original estimate made him feel that it would have been preferable had the author adopted that formula instead of the one derived by geometrical analogy, although, as the author said, for the calculation of $\bar{A}_x$ the difference between the two formulae was small. Taking § 35 as an example, the author gave the coefficients as ·92241, ·3182 and ·1547. Those obtained by applying Simpson's Rule for $\bar{a}_x$ and then converting were ·92118, ·3153 and ·1576—figures which confirmed the remark made in the paper that the suggested formula was very close to an inversion of Simpson's Rule. The formula in § 51 was even closer to the inversion of Simpson's Rule, giving coefficients of ·92193, ·3170 and ·1559. A test on the table on p. 171 showed, as the author said, that there was no significant difference in the results whichever one of the three sets of coefficients was used.

The author, of course, had obtained his formula by grouping values of $l_{x:\,\overline{10}|}$ in pairs. There was no particular reason why that should have been done, and it suggested that a series of other formulae could be obtained by different groupings. For example, by analogy with central differences, a good value of $l_x \bar{A}^1_{x:\,\overline{10}|}$ might be expected to be obtained from $l_{x-10:\,\overline{10}|}$, $l_{x:\,\overline{10}|}$ and $l_{x+10:\,\overline{10}|}$.

In actuarial work there were three main occasions when approximate integration formulae might be needed. One was when it was desired to know the monetary effect of a change of basis on say premiums or reserves; the second was for approximate valuation, and the third for evaluating actuarial functions such as $\bar{A}^1_{xy}$. For the first, he did not think that any existing methods would help to obtain $\bar{A}_x$ from q_x without obtaining l_x first (except perhaps Beard's differential analyser). For the second, R. E. Beard in 1947 (*J.I.A.* LXXIII, 356) dealt with the problem in some considerable detail, and in the discussion Perks made some suggestions about the possible use of an 'n-slab method'. The method of the paper, though somewhat akin to the 'n-slab method', was hardly appropriate, because it was specifically confined to integrals of the form $\int v^t u_{x+t}\,dt$, so that he thought that, in its present form at least, the formula was restricted to the calculation of actuarial functions.

There again, he feared, its use was limited by the fact that there were so few actuarial functions for which it was possible to obtain $\int u_{x+t}\,dt$. It was possible to obtain $\bar{A}_x$ in that way, and also $\bar{A}_{xy}$, but there were comparatively few other functions for which $\int u_{x+t}\,dt$ was available. Contingent assurances could not be calculated in that way, nor reversionary annuities; in fact, the method could not be used for most of the awkward functions for which Simpson and Hardy-King formulae were used. That might seem to be a pessimistic estimate of the chances that the paper would be put to practical use, but it did not mean that the paper was not of very great interest. Had the author led his readers into a cul-de-sac? He hoped not, and perhaps someone that evening would show how the formula could be profitably developed.

Mr G. V. Bayley referred to pp. 176 and 177 of the paper. The formal solution of the problem which the author had set himself was given there on familiar lines. He (the speaker) agreed with the opener that a pure mathematician who was concerned with product integrations in general, rather than simply $\bar{A}_x$, would have derived formula (iii) rather than formula (ii), and would then, probably, have expressed his solution in terms of the true integral plus an error. If the error were expressed in terms of u_x and its differences, it became much easier to appreciate the power of one particular formula as compared with another. Thus, solution (ii) on p. 177 was equal to the true integral

$$+\cdot 0006u_x - \cdot 0086\Delta u_x - \cdot 0246\Delta^2 u_x + \cdot 0505\Delta^3 u_x + \cdot 0387\Delta^4 u_x \ldots,$$

and solution (iii) was equal to the true integral

$$-\cdot 0290\Delta^2 u_x + \cdot 0542\Delta^3 u_x + \cdot 0428\Delta^4 u_x \ldots.$$

There was no need to stop there. Instead of supposing u_x to be of the first degree it could be assumed to be of the second degree, which would lead to a formula involving three terms instead of two, and so on; a whole family of formulae could be derived with obvious advantage. If u_x were assumed to be of the second degree over periods of 18 years, the formula corresponding to (1*a*) in § 35 would be

$$\cdot 93767 - \frac{\cdot 21320\ \mathrm{N}_6^{(18)} + \cdot 18456\ \mathrm{N}_{12}^{(18)} + \cdot 13430\ \mathrm{N}_{18}^{(18)}}{\mathrm{D}_0}$$

That gave results as accurate as those of the author for whole-life assurances, and better results for temporary assurances.

The author's 16-year temporary assurance errors were not shown on p. 171. They were small at the younger ages, but at age 60 amounted to as much as 48, compared with the 'whole-life' error of 2 shown in the last column. He thought that the accuracy of some of the 'whole-life' integrations depended on compensating errors. At age 68 the error had been kept rather 'under the counter'. It was -22, and from there onwards he thought that the compensation disappeared. If the 3-term formula which he suggested had been used, the 18-year temporary assurance error at age 60 would have been 34—compared with 48—and the '15-year' error 11. He mentioned that because the suggested formula might be more dangerous than appeared from the 'whole-life' results.

The author, at the suggestion of Mr Joseph, had explained that the calculation of $\bar{\mathrm{A}}$ by formula (1*a*) in § 35 was similar to the calculation of $\bar{a}$ by Simpson's Rule replicated, $\bar{a}$ then being converted to $\bar{\mathrm{A}}$, a process which might be called the inversion of Simpson's Rule. The procedure led to a formula which was similar to (1*a*) and had been given by the opener. In comparing the formula with the author's, he, also, had found it necessary to use five significant figures rather than four in the numerator of that expression; attention could then be confined solely to the power of the formulae. The results were similar to those obtained by the author's formula as modified. The inversion of Simpson's Rule led to errors which were only slightly larger up to age 60, and at age 68 the error was -2 instead of -22.

He had emphasized the closeness of those formulae because the inverted approach need not be confined to Simpson; other quadrature formulae could be inverted also. The 'three-eighths' rule could be inverted to give a 2-term formula, with results admittedly not quite so good as the inversion of Simpson's Rule. The close similarity of (1*a*) to the inversion of Simpson's Rule did suggest, however, that § 36 displayed the results of an approximate integration of $\bar{a}$ rather than $\bar{\mathrm{A}}$; that was, he thought, the reason why the results were better than those given earlier in the paper for the calculation of $\bar{\mathrm{A}}$ by direct quadrature.

He asked the author to reconsider the notation in § 7 for the deaths between ages x and $x+n$. He submitted that, by analogy with $\mathrm{A}_{x:\overline{n}|}$, $l_{x:\overline{n}|}$ defined the number of people who died between ages x and $x+n$ plus the number who survived age $x+n$, and therefore became indistinguishable from l_x. He thought that $l^{1}_{x:\overline{n}|}$ would be consistent with the international notation, or possibly $d_{x:\overline{n}|}$, which had been suggested by Mr Ogborn.

Mr R. E. Beard, in a contribution which was read to the meeting, wrote that the subject of approximate integration was essentially practical, and thus it was not surprising that from time to time valuable suggestions for the development of new formulae arose from essentially practical considerations. The author, in a characteristic manner, had discovered a new formula, the analysis of which gave a valuable insight into the nature of product-integration formulae, and led to useful ideas for further development.

In any particular problem the type of formula to be selected for use depended on the nature of the available facts. If $\int_0^{2n} u(t)\, w(t)\, dt$ was to be computed from the three values $u(0)$, $u(n)$, $u(2n)$ and the first two moments of $w(t)$, then a convenient formula was available in equation (71) of the writer's paper on Product Integration (*J.I.A.* LXXIII, 371). In the particular case where $w(t)=v^t$ the solutions had been worked out in the same paper for various values of i and n (Table 7). The author's approach in § 51 used precisely the same facts, and his formula (iii), which he did not use, could be obtained directly from Table 7 of the paper referred to, in the following way:

$$l_x\, \bar{A}^1_{x:\overline{20}|} \doteqdot l_x - \delta \bar{a}_{\overline{20}|}\, (\cdot 21757 l_x + \cdot 66282 l_{x+10} + \cdot 11961 l_{x+20}) - v^{20} l_{x+20}$$

$$= \cdot 90289 l_{x\,\overline{10}|} + \cdot 60706 l_{x+10\,\overline{10}|}\,.$$

The formula was rigorously true if l_{x+t} was of the second or lower degree in t and would appear to be the proper basic approximation when $w(t)=v^t$.

If, however, the moments of $w(t)$ were not available, but only the values $\int_0^n w(t)\, dt$ and $\int_n^{2n} w(t)\, dt$—which were the conditions assumed by the author—a formula could be derived by assuming that $w(t)$ could be replaced by a straight line passing through the points $(n/2, \int_0^n w(t)\, dt/n)$ and $(3n/2, \int_n^{2n} w(t)\, dt/n)$. Denoting $\int_0^n w(t)\, dt$ by A_1 and $\int_n^{2n} w(t)\, dt$ by A_2, the formula resulting from the calculation of the first two moments of the straight line would be

$$\int_0^{2n} u(t)\, w(t)\, dt \doteqdot \frac{1}{6}\{(3A_1 - A_2)\, u_0 + 4(A_1 + A_2)\, u_n + (3A_2 - A_1)\, u_{2n}\}.$$

Putting $w(t)=v^t$, formula (*b*) of § 32—which was the basis of the author's paper—followed after a slight rearrangement.

The author's formula (*b*) involved an unnecessary assumption, since the moments of v^t were known; yet it was an extraordinarily good approximation, and would, in fact, be better than the formula in § 51 in those freak cases where the deviation of $u(t)$ from a second degree function just counteracted the error in substituting the approximate moments of v^t.

In view of the closeness of the approximate results it was of interest to compare the moments of v^t with those of the substituted straight line. The former were tabulated in Table 5 of the writer's paper and the latter were

$$m_1 = n(1+5v^n)/3(1+v^n) \quad \text{and} \quad m_2 = 8n^2 v^n/3(1+v^n).$$

Comparative values were shown in the following table:

i	Range	Subst. m_1	True	Subst. m_2	True
·02	10	4·835	4·835	31·685	31·695
	20	9·342	9·342	120·175	120·340
	30	13·526	13·523	255·768	256·579
	40	17·394	17·387	429·078	431·555
·06	10	4·518	4·517	28·512	28·598
	20	8·111	8·100	95·550	96·801
	30	10·888	10·837	176·651	182·244
	40	13·005	12·854	253·535	268·892

Clearly for low values of ni the true and approximate formulae gave identical results and confirmed the author's comments in his § 51. Alternatively, $w(t)$ might be expanded in powers of t when it would be found that m_1 was exact for expressions of the second degree and m_2 for expressions of the first degree.

The analysis could be concluded by a reference to the error term. Since the author's formula was a close approximation to the more accurate formula (91) of the writer's paper, the error terms tabulated in Table 7 might be used. For $n=10$ the error term (true − calculated) in calculating $\bar{A}^1_{x\,\overline{20}|}$ would be $-\dfrac{39{\cdot}1\times\delta\bar{a}_{\overline{20}|}}{3!}\dfrac{1}{l_x}\left\{\dfrac{d^3 l_{x+t}}{dt^3}\right\}_{t=\zeta}$, where $0<\zeta<20$ or approximately $2{\cdot}8\dfrac{\Delta^2\phi(\zeta)}{l_x}$. For $n=8$ the numerical factor was about ·8 as compared with 2·8 so that a considerable, if not a fivefold, increase in the error by increasing n from 8 to 10 was confirmed. Owing to the lack of smoothness in the A 1924–29 table it was difficult to obtain reasonable values of the error term from $\Delta^2\phi$, although it would be found that they were of the right order of magnitude.

In conclusion, he congratulated the author on his new approach to the problem of quadrature, which opened up a new set of formulae based on different conditions from those previously used, and which clearly merited analytical investigation. There were, of course, other ways of looking at the new formula, but in his opinion the most convenient for analytical development was the systematic development of the substitution principle.

Mr A. W. Joseph called attention to a paper by Prof. Steffensen entitled *On certain formulas of mechanical quadrature* (*Skand. Akt.* 1945, p. 1), where G. F. Hardy's process, in *J.I.A.* XXIV, 97, of substituting near integral arguments for the irrational arguments of Gaussian formulae was carried a stage further. Prof. Steffensen seemed to have been unaware of G. F. Hardy's earlier work.

Interesting though the author's historical survey was, he felt sure that the author would want and expect the paper to be judged by the new formula in § 32 and its transformation in §§ 33 and 34. The formula in § 32 was quite remarkable and of a new type. It expressed the integral of $\phi_x v_x$ in terms of 'slabs' of ϕ_x and points of v_x. By 'slabs' was meant the integral of ϕ_x, i.e. $l_{x\overline{n}|}$. Actuarial literature was studded with pleasing and remarkable formulae, but a formula needed to be something more than that if it was to become a valued and useful tool. There was an obvious disadvantage in a 'slab' formula, because the 'slab' itself had to be evaluated. Whenever that could be done easily, the area of the 'slab' was the difference between two values of the integrated function; in other words, it was a points relation of the integrated function. Thus whenever it was practicable to use the author's formula (which introduced the slabs) the formula turned out to be a 'points' formula.

The transformation in §§ 33 and 34 brought that out clearly, because formula (1) in § 34 was only a simple relation for $\bar{A}_x$ in terms of D_x, and as had been said many times already, when converted into a formula for $\bar{a}_x$ it took on a familiar form—almost, but not quite, Simpson. The use of Simpson's Rule replicated for the calculation of $\bar{a}_x$ would enable calculations to be made on parallel lines to those in §§ 38–47. He had done the whole work quite easily, and, in order to compare the results with those of the author, he had converted $\bar{a}_x$ to $\bar{A}_x$. At many ages the results were identical with the author's, and where they differed they were slightly less accurate. Was a slight gain in accuracy enough to win allegiance from the well-tried and—in the words of *1066 and all that*—'memorable' Simpson formula?

What was the real problem with which they were faced? It was to compute the integral from 0 to infinity of the product of a number of functions. In that product v^t usually appeared, and also ${}_tp_x$ or ${}_tp_x\mu_{x+t}$; frequently there were other actuarial functions or less restricted functions of t such as 't' itself. It would be very desirable for each such function to be typified by one or two constants which could be manipulated in a simple way to produce the desired integral. Almost all attempts so far had been by way of a polynomial approach. The present paper followed the same lines.

Personally, he felt sure that the solution was waiting round the corner, but it would be a process as transcendental and revolutionary as the process by which logarithms changed multiplication into addition.

Mr W. Perks welcomed the paper, because, as had already been suggested, it opened the door to an entirely new set of product-integration formulae.

Before coming to that, however, he desired to say something about the author's mensurator and quadrator. He did not think that the author had been quite fair to the quadrator, because he had given more knowledge to the mensurator than to the quadrator. If the quadrator had the same knowledge as the mensurator—i.e. if he knew the values of each of ϕ_a, ϕ_b, v_a and v_b—he would surely fit a straight line to ϕ and another straight line to v, and then calculate an approximate value of ϕv at the midpoint between a and b. He would thus have three values of the product. As the product of two straight lines was a parabola, he would automatically apply Simpson's Rule and get the same answer as the mensurator. At any rate, that was a much easier way of getting the result than by trying to unravel the diagram on p. 161.

The analogy by which the author had reached formula (b) of § 32, which was the main formula of the paper, had given him a great deal of trouble, but he had produced the formula by three other processes, which he thought would illuminate the author's methods.

First, a straight line could be fitted to the curve of deaths from age (x) to age $(x+2n)$ using the deaths from (x) to $(x+n)$ and from $(x+n)$ to $(x+2n)$. One nth of these numbers represented ordinates at $(x+n/2)$ and $(x+3n/2)$ and one nth of the difference represented the slope of the straight line. From that straight line, the ordinates at (x), $(x+n)$ and $(x+2n)$ could be calculated and the results multiplied by 1, v^n and v^{2n} respectively; Simpson's Rule applied to the products to obtain the integral from (x) to $(x+2n)$ would produce the author's formula.

The second process was to fit a straight line to the curve of deaths, exactly as in the first method; the first and second moments of the straight line could then be used in Beard's formula (71) to obtain a 3-point solution, which produced the author's formula in the form of coefficients of 1, v^n and v^{2n}, i.e. $\frac{1}{2}l_{x\,\overline{n}|}-\frac{1}{6}l_{x+n\,\overline{n}|}$, $\frac{2}{3}(l_{x\,\overline{n}|}+l_{x+n\,\overline{n}|})$ and $\frac{1}{2}l_{x+n\,\overline{n}|}-\frac{1}{6}l_{x\,\overline{n}|}$. That method had a certain similarity to the method described by Mr Beard—but he had, in effect, fitted a straight line to v^t.

The third process was to fit a parabola to v^t, using 1, v^n and v^{2n}, and then to compute the area and first moment of the curve as a distribution in n. The distribution could then be replaced by two slabs on bases (0 to n) and (n to $2n$) with the same area and first moment as the parabola. The heights of the slabs represented the author's f_0 and f_n.

Those various ways of reaching the author's formula showed that it was a remarkable formula indeed. It represented an ordinary approximate integration formula, an n-point formula with moments of one factor and a slab formula with moments of the other factor. There were obvious ways of developing other formulae of the same family. For example, three slabs could be used instead of two, or four points instead of three and so on. Thus there seemed to be an entirely new series of possible formulae.

If in the third method outlined the true area and first moment of v^t were taken instead of the area and first moment of the fitted parabola, the following expressions for f_0 and f_n were obtained:

$$f_0=\frac{3+v^{2n}}{2n\delta}-\frac{1-v^{2n}}{n^2\delta^2} \quad \text{and} \quad f_1=\frac{1-v^{2n}}{n^2\delta^2}-\frac{1+3v^{2n}}{2n\delta}.$$

The process led to the formula in § 51, but by the opposite route from that used by Mr Beard.

The symmetry seemed to him at first sight to be remarkable, but when it was realized that the data assumed in the author's formula were really the values of each of l_{x+t} and v^t at $t=0$, n and $2n$, the symmetry was not so surprising. The fitting of a straight line to the curve of deaths was really the same as fitting a parabola to l_x, and so the formula assumed that l_x and v^t were both of the second degree. That explained the symmetry, but it did not explain why the application of Simpson's Rule to the product of two second degree functions (i.e. to a fourth degree function) should give an exact answer very close to Simpson's Rule applied directly to $v^t\, l_{x+t}=f(t)$ which implied the assumption that $f(t)$ was a curve of the third degree.

The analogous formula corresponding to the three-eighths rule could be easily obtained by fitting a parabola to three slabs of deaths, obtaining the ordinates of that parabola at 0, n, $2n$ and $3n$, multiplying by 1, v^n, v^{2n} and v^{3n} respectively, and then applying the three-eighths rule to the products. He had done that and worked out a few examples, and there was, as would be expected, no improvement over the author's results. It was clear that there was a similar formula corresponding to Weddle's Rule.

Mr F. M. Redington said that he might be able to help Mr Perks in the difficulty which Mr Perks mentioned towards the end of his remarks. The basic formula in § 32 of the paper was, as one or two speakers had implied, accurate to the third degree—i.e., if the functions ϕ and v were expressed as polynomials the formula was correct as far as the third degree in the product. But it was something more than that: if the second function, v, was a constant, the formula was exact whatever the other function, ϕ, and that would be true for the constant part of v, even if as a whole v were not constant.

That led him to a consideration which might be of help in answering some of the criticisms made in the discussion. The formula was accurate to the third degree and, as he had explained, a little more. But that statement was insufficient to explain the marked success of the author's formula, which exceeded what might be expected from a formula theoretically accurate to a much higher degree. It was profitable to ask why it was so accurate.

A peculiar feature of many of the functions with which actuaries had to deal, and particularly mortality functions, was that if differences were taken of, say, l_x or q_x, it would be found that the successive differences diminished rapidly down to the third, but then started to rise. The rise in the values of the differences was due partly to the roughness caused by limiting the number of decimal places taken. If, for example, the last column of a set of logarithm tables were differenced, the differences began to mount very quickly. In addition to that roughness, the waves in the curve, whether intrinsic or due to graduation, themselves gave rise to differences which were liable to increase rapidly after the third.

That was the explanation of the disconcerting fact that a formula accurate to the fifth or sixth degree might produce results which were less accurate than those of a formula accurate only to the third or fourth degree. There was a good deal of chance in the problem beyond the third or fourth degree, and elaborate mathematics might be out of place.

In that connexion he would take up again the comments of his which were quoted in the paper. A function such as v^t was perfectly smooth, and, for that reason, it was safe to typify the curve by a number of points on it. The fact that two or three points would not exactly identify v^t was immaterial; the error was small and regular. On the other hand, it was unsafe to rely upon a few points when dealing with a mortality function. It was possible to describe a house by a certain number of point descriptions—length, breadth and so on—but it was not possible to describe a face with a number of points like that; to describe a face it was necessary to take some integrating adjective and to say that it was kindly or miserable, for instance. That was the secret of the author's success. For the v^t curve, which was mathematical and smooth, the author had used points, but for the mortality curve, which was complex and 'organic', the author had used integrals.

Mr H. A. R. Barnett remarked that in the discussion most of the speakers had been concerned largely with the theory of the author's formula, but had not considered the extent to which it might be applied. He did not know how it would work out in practice, but he would like the author to think about it. It could be applied to A and to $\bar{a}$, and it would be interesting to know whether it could also be applied to certain functions required in pension fund valuations, provided the progression of the salary scale was not too irregular. He did not see why it should not be so applied, and, if so, it would certainly be very valuable for making an approximate valuation of a pension fund. That was a practical question to which some thought might be given.

Mr C. D. Rich regarded it a privilege to be allowed to close the discussion on a paper by Mr Phillips, for the papers previously submitted by him to the Institute had proved him to be an actuary of original ideas. Moreover, his ideas, besides being original, had usually proved to be of real importance. Thus the main idea underlying the author's 1936 paper on *Binary Calculation* had since turned out to be invaluable in the construction of electronic calculators.

Certain previous speakers, including the opener and Mr Bayley, had commented on a question of arithmetic which arose more than once out of the paper, namely, that the result of a calculation could not be expected to be correct to more figures than were used in the working. As an example, the values of $\bar{A}_x$ in the tables on p. 171 and near the top of p. 172 were given to five places of decimals, although the formulae by which they were calculated, i.e. (1*a*) and (1*b*) of §§ 35 and 36, contained coefficients taken to only four places. It seemed that the extent of any correspondence or divergence in the fifth place between the true and calculated values must be largely fortuitous.

Many people would probably regard the paper as giving confirmation of the excellence as an approximate integration formula of the repeated Simpson's Rule

$$\tfrac{1}{3}(1, 4, 2, 4, 2, \ldots, 4, 1).$$

That was borne out not only by the schedule in § 29, where the repeated Simpson's Rule was among the three formulae giving the best results out of the five formulae tested, but also by the author's own cubature formula itself, for that formula—as was mentioned in the Appendix to the paper, and had been referred to by other speakers—was almost identical with the result of applying the repeated Simpson's Rule to determine $\bar{a}_x$ and thence deducing $\bar{A}_x$ by the conversion formula $\bar{A}_x = 1 - \delta\bar{a}_x$. The 'pattern' of the repeated Simpson's Rule formula (in which, apart from the end terms, the even coefficients were double the odd coefficients) could in fact be seen in the examples of the author's formula given in the paper, e.g. in (1*a*) of § 35, where the coefficient ·3182 was not very different from twice the coefficient ·1547.

Indeed, if the coefficients of the author's formula (as given in (1) of § 34) and of the formula using the repeated Simpson's Rule were both expanded in ascending powers of δ, the force of interest, the difference between the formulae, ignoring fourth and higher powers of δ, was

$$\tfrac{1}{9}\delta^3\, n^3(D_0 - 2D_n + 2D_{2n} - 2D_{3n} + \ldots)/D_0.$$

The difference was easily seen to be a small quantity, for δ^3 itself was small and the series—which represented the repeated (1, −2, 1) pattern, just as the repeated Simpson's Rule represented the repeated (1, 4, 1) pattern—had a sum whose value to first differences was zero.

He himself had made some comparisons between the values of $\bar{A}_x$ calculated by the author's formula and by the repeated Simpson's Rule formula (using the same intervals in each case) and had obtained results which, in his opinion, showed that for accuracy there was little to choose between the two formulae:

	Total of errors in fifth place of decimals	
	Author's formula	Repeated Simpson's Rule
Table on p. 171 (7 values)	30	37
First table on p. 172 (5 values)	21	10
Table on p. 178 (12 values)	12	13

The differences between the figures he gave for the errors by the author's formula in the tables on pp. 171 and 172 and those given in the paper were explained by the fact that he had recalculated to six places of decimals the coefficients in the author's formulae (1*a*) and (1*b*) of §§ 35 and 36, their values together with those for the corresponding repeated Simpson's Rule being:

(1*a*) Author's formula: $\cdot 922412-(\cdot 318241 N^{(16)}_{8}+\cdot 154651 N^{(16)}_{16})/D_0$,

Repeated Simpson's Rule: $\cdot 921177-(\cdot 315294 N^{(16)}_{8}+\cdot 157647 N^{(16)}_{16})/D_0$,

(1*b*) Author's formula: $\cdot 960756-(\cdot 158015 N^{(8)}_{4}+\cdot 078454 N^{(8)}_{8})/D_0$,

Repeated Simpson's Rule: $\cdot 960588-(\cdot 157647 N^{(8)}_{4}+\cdot 078823 N^{(8)}_{8})/D_0$.

It was interesting to consider why the repeated Simpson's Rule formula—correct only to third differences—gave such good results that it was often more successful than formulae theoretically correct to a higher order of differences. Mr Redington's remarks had touched on that point. The answer probably was that the repeated Simpson's Rule fitted a number of third degree curves over short intervals; the error in each was small because the interval was small, and the total error over n intervals might be less than that involved in fitting a single curve of higher degree over the whole range. Mathematically, if Simpson's Rule were applied once over the interval 0 to $2h$, the error would be proportionate to the fourth difference multiplied by h^5; in applying the repeated Simpson's rule over the range 0 to $2nh$ the maximum error was thus proportionate to nh^5. If, on the other hand, a single curve of, say, the seventh degree were fitted over the range, the error would be proportionate to the eighth difference multiplied by h^9. The significance of the comparison lay in the ratio between h^5 and h^9, and it was clear that, though over a short range the higher degree formula might be more accurate, over a sufficiently long range the reverse would apply.

He did not remember ever having seen it mentioned that any approximate integration formula of the kind considered on pp. 163–168—i.e. formulae in terms of values of the function at integral points—could, if it extended over an even number of intervals, and if it were correct to at least third differences, be expressed as sums or differences of applications of Simpson's Rule. A simple example was Wicken's formula, denoted in § 24 as (Wi. 1), which could be written in the form

$$\tfrac{4}{9}(1, 2, 3, 2, 1)=\tfrac{2}{9}(1, 0, 4, 0, 1)+\tfrac{2}{9}(1, 4, 2, 4, 1),$$

where $\frac{2}{3}$ (1, 0, 4, 0, 1) represented Simpson's Rule applied once over the whole range and $\frac{1}{3}$ (1, 4, 2, 4, 1) represented Simpson's Rule applied twice, i.e. once over each half of the range. The proof of the general proposition, which was not difficult, involved repeated use of the pattern (1, −4, 6, −4, 1) which itself could be written as

$$2(1, 0, 4, 0, 1)-(1, 4, 2, 4, 1).$$

Since all such formulae were capable of being represented by combinations of Simpson's Rule, he thought it probable that the most successful formulae were those which came nearest to the simple repeated Simpson's Rule. That was confirmed by the schedule in § 29, where (Wi. 2) and (Sh.) gave better results than (HK 38) and (W 10), it being observed from the formulae set out in §§ 18–26 that the coefficients of the two former came nearer the pattern (1, 4, 2, 4, 2, ..., 4, 1) than those of the two latter.

In the paper the author had given three different demonstrations of his cubature formula. They should be called demonstrations rather than proofs, for none of them gave any measure of the size of the error involved in the formula, and an approximate equality could hardly be said to be proved unless some limit to the degree of approximation were established. Although, therefore, he regarded parts of the paper as plausible rather than convincing there were two good reasons why he was satisfied about the merits of the new formula. One was the fact that it was almost identical with a recognized formula—the repeated Simpson's Rule—under a new guise, and the other was that, the author having previously shown himself to be a good judge of approximate integration formulae, his recommendation could be accepted with confidence.

The President (Mr F. A. A. Menzler, C.B.E.), in proposing a vote of thanks to the author, said that he had achieved a number of things that evening. First of all, he had succeeded in writing in a characteristic and lively manner about a subject which was not inherently colourful. He had brought together a great deal of material in a systematic fashion and thereby provided a focus for an interesting debate. He had rendered the President a personal disservice by bringing about an acute attack of nostalgia. The very sight of the fraction 1/720 and the reference to formula 39(*a*) made him think back 35–40 years. He at once had recourse to Freeman's *Actuarial Mathematics*, where there was a chapter on approximate integration which he read and, to his surprise, understood. He then remembered that that did not exist when he took the examinations, and he tried to recall what he did read. His mind went back to King's Text-book, Part II, 1902 edition, and he remembered very well looking at the chapter on what King called 'Summation'. At the time he had found it very heavy going, so he had asked his tutor what he would recommend for the systematic study of the subject. He was told that 'Nobody has written anything since 1860, the date of Boole's *A Treatise on the Calculus of Finite Differences*'—a revised edition of which was published in 1872. In the preface Boole said that he had paid particular attention to 'the connexion of the methods of finite differences with those of the differential calculus'. There was a chapter in Boole's book on 'Approximate Summation of Series', which was well worth re-reading.

When he was taking the examinations, the modern idea of the progressive approach had not been invented. There was a formula which did what was wanted, and that was largely the end of it. Nobody bothered too much about the systematic underlying theory so long as the right answer was obtained. They had progressed a long way since then, and students were much better looked after.

He remembered, when re-reading those ancient text-books, that formulae of the kind under discussion had to be used when tabulated values of functions involving two or more lives were not available; but the author had drawn all his examples from the single-life function $\bar{A}_x$. In real life commutation columns would be run off and the complete table of $\bar{A}_x$ would be ready-made for future use, without any approximation at all. He asked the author whether, if his formula were applied to complicated functions, the results obtained would be just as good.

Mr Phillips, in reply, agreed with Mr Bayley that formulae (1) and (3) were partially compensatory as between one integration interval and another, though not to nearly so great an extent as was the 1934 formula.

Many speakers had referred to the opportunities for further exploration; that pleased him, because he occasionally came across such statements as that of E. T. Bell in *Development of Mathematics* (1940): 'Of actuarial science in general it may be said that it has been so thoroughly explored that little remains to attract a professional mathematician'—a view with which he was in profound disagreement.

As to those opportunities, there was nothing he could say at short notice, except that—corresponding to the assumption that the error of the trapezoidal rule was increased four-fold by doubling the datum interval—when Simpson's Rule was applied upon a curve of the fifth degree, first with an interval of $2h$, and then in succession on each interval of h, the respective errors were in the ratio of 16 to 1. From that intuitive guess, since verified with no great difficulty, blending reproduced the formula (true to the fifth degree)

$$\int_0^4 u_x dx = \tfrac{2}{45}(7u_0 + 32u_1 + 12u_2 + 32u_3 + 7u_4),$$

a formula which might have been more often remembered had it not become an obsession to integrate over an interval of $5h$ years. He had been tempted to try the effect of modifying that formula to cubature, in the way Simpson's Rule modified to formula (*b*), and a number of speakers had also given attention to the possibilities of using more than two slabs and three points; but he thought that Mr Rich had given the right answer to anyone who might attempt too wide a spread: it was better to assume a curve of the third degree over a succession of short intervals, than one of a higher degree over a long span. Whether or not they had reached a cul-de-sac, as the opener had suggested, they had apparently come to a door with the name 'Simpson' upon it.

The criticism of Mr Rich and other speakers that it was unpardonable to attempt to test two things at the same time—a new algebraical expression and the effect of condensing the arithmetic—was unanswerable, at least in the case of the schedule on p. 171, for there had been no striving for brevity at that stage. However, he had in fact known that he was reducing one coefficient by ·000041 and increasing the other by ·000049, so that his luck had been limited to finding that chance opportunity for saving the time of a computer who had no arithmometer. He had, in fact, had another, indeed a double, piece of luck, which could not be expected to recur at a different rate of interest, namely, that when ·90378 and ·60623 were cut to four figures one went up by 2 and the other down by 3, or when cut to three figures one went up by 22 and the other down by 23.

With no special reference to the opener's criticism of the comparison of the five classic formulae, where at least it could be said that all five had been subjected to the same strain, he felt that actuaries were sometimes a little over-anxious to attain an accuracy which might prove illusory. He would offer a comment in the form of a dream in which he was being shown round a life office. He came to a clerk who was sorting cards, and was told 'These are a hundred lives insured by our Taunton broker last year.' When asked 'What will happen to your Taunton broker if all the hundred die this year or next?' the clerk smiled in a superior way and murmured something about the law of averages. On the way back a little later he found the same clerk adding up figures, to three places of decimals, in a large book, a hundred lines to the page. As the clerk started on the third place of decimals, he asked him 'Why don't you save time by assuming the total is 450?' The clerk replied haughtily 'They might all be 8's or 9's.' The moral of that story was that it could be seen at a glance that the figures were *not* all 8's or 9's, whereas there was nothing to show that those hundred lives might not all die that year or the next. If actuaries were willing to trust human beings to accord with the law of averages, even in small groups, why could they not trust figures to do so?

Mr C. D. Rich subsequently wrote as follows:

In the course of my remarks in the discussion, I drew attention to the fact that any formula of approximate integration which (1) is in terms of values of the integrand for integral values of the variable, (2) extends over an even number of intervals, (3) is correct to third differences, can be expressed as sums or differences of applications of Simpson's Rule. This can be demonstrated by an example, and I give below details of the working by which Hardy's (37) formula can be expressed in the manner referred to.

The formula (which is given in § 17 of the paper) can be written as (·28, 1·62, 0, 2·2, 0, 1·62, ·28). In the working the coefficients have been multiplied by 150 to avoid decimals and fractions. The notation used is explained as follows:

$(\mathrm{H})_0^6$ denotes Hardy's (37) formula for $\int_0^6 u_x dx$,

$(\mathrm{S})_0^2$ denotes Simpson's Rule for $\int_0^2 u_x dx$, i.e. $\frac{1}{3}(u_0+4u_1+u_2)$,

$(\mathrm{RS})_0^6$ denotes the repeated Simpson's Rule for $\int_0^6 u_x dx$, i.e.

$$(\mathrm{S})_0^2+(\mathrm{S})_2^4+(\mathrm{S})_4^6, \qquad \text{(i)}$$

which may be written as $\frac{1}{3}$ (1, 4, 2, 4, 2, 4, 1),

$$\Delta^4 u_0 = u_0 - 4u_1 + 6u_2 - 4u_3 + u_4 = 3(S)_0^4 - 3(RS)_0^4. \quad \text{(ii)}$$

We commence by writing

$$(H)_0^6 = \tfrac{1}{150}(42 + 243E \qquad + 330E^3 \qquad + 243E^5 + 42E^6)\, u_0$$

and $$(RS)_0^6 = \tfrac{1}{150}(50 + 200E + 100E^2 + 200E^3 + 100E^4 + 200E^5 + 50E^6)\, u_0,$$

where E has its usual meaning $1 + \Delta$.

Subtracting:

$$(H)_0^6 - (RS)_0^6 = \tfrac{1}{150}(-8 + 43E - 100E^2 + 130E^3 - 100E^4 + 43E^5 - 8E^6)\, u_0. \quad \text{(iii)}$$

Since $(H)_0^6$ and $(RS)_0^6$ are both approximations to the integral correct to the third (or higher) degree, the difference between them must vanish up to and including this order of differences and consists therefore of fourth and higher differences of u. Hence $\Delta^4 = (E-1)^4$ must be a factor of the expression in brackets on the right-hand side of (iii). Dividing the expression by

$$(E-1)^4 = 1 - 4E + 6E^2 - 4E^3 + E^4,$$

the quotient is found to be

$$-8 + 11E - 8E^2.$$

Hence $(H)_0^6 - (RS)_0^6 = \tfrac{1}{150}(-8 + 11E - 8E^2)\, \Delta^4 u_0$

$$= \tfrac{1}{150}(-8\Delta^4 u_0 + 11\Delta^4 u_1 - 8\Delta^4 u_2).$$

Substituting from (i) and (ii), we obtain

$$(H)_0^6 = \tfrac{1}{50}\{58\,(S)_0^2 - 11(S)_1^3 + 66(S)_2^4 - 11(S)_3^5 + 58(S)_4^6 - 8(S)_0^4 + 11(S)_1^5 - 8(S)_2^6\}.$$

If at the beginning we had deducted $(S)_0^6 = (1 + 4E^3 + E\ )\, u_0$ instead of $(RS)_0^6$, we would have arrived at a different expression for $(H)_0^6$ in terms of applications of Simpson's Rule. There are, in general, many alternative ways of expressing a given formula in the required manner.

It is easily seen from the above that the difference between two approximate integration formulae, each of which is correct to the nth degree, can be expressed in terms of $(n+1)$th differences of the integrand. This leads to a generalization of the remarks of Wickens in *J.I.A.* LIV, 209.

Mr M. E. Ogborn has sent the following note:

Quadrature formulae which use selected values of the function to be integrated derive from the work of Newton whose name is not usually associated with the formulae. It is an interesting question how Simpson's name became attached to the three-ordinate formula. The author, and also Whittaker and Robinson in *The Calculus of Observations*, p. 156, refer to Simpson's *Mathematical Dissertations* published in 1743, and by the kindness of the author I have been able to borrow a copy from the London Mathematical Society. But on p. (vii) of the preface to that work Simpson states:

The ninth (part) relates to mechanic Quadratures, or the Method of approximating the Areas of Curves, by Means of equidistant Ordinates. This Method was originally an Invention of Sir Isaac Newton's, since prosecuted by Mr. De Moivre, Mr. Stirling, and others: However, as I here assume nothing to myself, but a Liberty of putting the Matter in such a Light, as I judge will be most plain and satisfactory to the Reader, I see no reason why I may not be allowed the same Privilege as Others.

Simpson gives two propositions for finding the areas of curves by approximation; Proposition I deals with the simple case of three ordinates, and the repeated application of the process; Proposition II deals with the general form for any number of ordinates.

A year earlier, in 1742, in *The Doctrine of Annuities and Reversions*, Simpson had quoted the formulae for 3, 4, 5, 6 and 7 ordinates which, he stated, were 'by a known method for approximating the areas of curves by means of equidistant ordinates'.

Simpson, it is clear, did not claim that the three-ordinate formula, 'Simpson's Rule', was original. The method went back to the *Methodus differentialis* of Newton, written before October 1676, though the trapezoidal rule and the three-ordinate formula were known earlier.

Proposition VI of the *Methodus Differentialis*, as translated by D. C. Fraser in *J.I.A.* LI, 100, states:

> To find the approximate area of any curve a number of whose ordinates can be ascertained.
>
> Let a parabolic curve be drawn through the extremities of the ordinates by means of the preceding propositions. This will form the boundary of a figure whose area can always be ascertained, and its area will be approximately equal to the area required.

Newton's proposition is general in form and applies to ordinates at unequal as well as at equal intervals. The general form of the coefficients for equidistant ordinates, the Newton-Cotes formula, was given by Newton in the *Principia* (see *The Calculus of Observations*, p. 154).

Newton develops the four-ordinate formula, the 'three-eighths' rule, as an example of Proposition VI—perhaps because the three-ordinate formula had already been given by James Gregory a few years earlier.

Mr William Phillips writes:

A number of my friendly critics have expressed a preference for the f_0' and f_n' of § 51; originally I had no great preference either way, but now that I am aware of the relationship between Simpson's Rule and its modification, I have a personal preference for the f_0 and f_n of § 33, because I now know the rationale of their emergence, whereas f_0' and f_n' are reached by arbitrarily ignoring the coefficients of c, d and e in formula (i) of § 49.

In speaking of the relationship between the formulae demonstrated and Simpson's Rule, I am referring only to their common use of the four-to-one error assumption, and not to the closeness of the results to those obtained by calculating $\bar{a}_x$ by Simpson and converting to A_x. As Mr Rich said, other formulae can be used for the first half of this double process; for example, the formula of the paper. Choosing for a test English Life Table No. 8—Males, because all the data required, and the answers, are readily available in *Life Contingencies*, I have calculated $\bar{a}_{10x}$ from values of l_{10x} by Simpson replicated, and then from values of T_{10x} by the formula

$$(l\bar{a})_x = f_0 T_{x:\overline{10}|} + f_{10} T_{x+10:\overline{10}|} + v^{20}(l\bar{a})_{x+20}, \qquad \text{(X)}$$

with the following results:

Age	(S) Calc. − True	(X) Calc. − True
10	·004	·001
20	·006	·003
30	−·006	·004
40	·009	·004
50	−·012	·007
60	·022	·005
70	−·061	·006
80	·083	−·032
90	·911	−·059
100	1·825	−·065

On converting we have the following values for $\bar{A}_x$

Age	(S)	(X)	True	Calc. – True (S)	(X)
10	·2424	·2423	·2424	0	–1
20	·3060	·3061	·3062	–2	–1
30	·3811	·3808	·3809	2	–1
40	·4712	·4713	·4715	–3	–2
50	·5751	·5746	·5748	3	–2
60	·6835	·6840	·6842	–7	–2
70	·7899	·7881	·7883	16	–2
80	·8685	·8719	·8709	–24	10
90	·8950	·9237	·9220	–230	17
100	·9033	·9596	·9554	–521	42

However, this process of conversion does not lend itself very readily to the evaluation of each 20 years separately, as will be appreciated from the somewhat chaotic form of 'Simpson converted':

$$l\bar{A} = \cdot 90147 l_{0:\overline{10}|} + \cdot 60821 l_{10:\overline{10}|} + v^{20}(\cdot 90144 l_{20:\overline{10}|} + \cdot 60817 l_{30:\overline{10}|})$$
$$+ v^{40}(\cdot 90136 l_{40:\overline{10}|} + \cdot 60810 l_{50:\overline{10}|}) + \dots,$$

a point which does not emerge when it is put in the form of the formulae of § 35 which Mr Martin quoted.

I feel that a wide vista has been opened up by Mr Joseph's suggestion that future progress in approximate integration for actuarial purposes lies in a direction away from polynomials. True though it be that any continuous curve can be reproduced with any required degree of accuracy by a polynomial of the nth degree, if n is sufficiently large, this might prove to be a cumbersome method of judging the efficacy of a formula which was being used to integrate a function which turned out to be of a precise mathematical form other than a polynomial, and the whole proceeding would appear faintly ridiculous if it transpired that the formula integrated that particular function with mathematical precision. Perhaps I am guilty of having started a wild goose chase in mentioning polynomials in connexion with formula (b) and its variations, for I cannot feel that either Mr Perks or Mr Redington are quite happy in their minds about the attempts to measure the formulae with this particular yard-stick. On the other hand it is possible to postulate a number of mathematical non-polynomial forms for ϕ_x for which formula (b) brings out exact results.

I am much intrigued by Mr Rich's ingenious analysis of those formulae which use an even number of intervals into what one may call, perhaps, 'molecules' of Simpson, a logical development of the interesting Note in *J.I.A.* LIV, 209. If the process Mr Wickens there suggested is employed in a more elaborate form, commencing with

$$\tfrac{1}{90}(60, 0, 240, 0, 60),$$

subtracting from it continuously

$$\tfrac{1}{90}(1, -4, 6, -4, 1),$$

a vast number of formulae are obtained, all correct to the third degree, and eventually

$$\tfrac{2}{45}(7, 32, 12, 32, 7),$$

the formula mentioned in the reply to the discussion as being true to the *fifth* degree. Perhaps the analysis of Mr Rich will reveal some simple method of identifying the 'best' formulae in a schedule constructed by Mr Wickens's continuous method.

If from the general formula given in my opening remarks u_0 and u_n are eliminated by putting

$$\frac{n}{2} = \frac{1}{12}\frac{n^3}{a(n-1a)}, \quad \text{and} \quad a = 1,$$

we arrive at the Gauss formula, true to the third degree

$$\int_0^{3+\sqrt{3}} u_x\,dx = \frac{3+\sqrt{3}}{2}(u_1 + u_{2+\sqrt{3}}),$$

which Hardy gave in 1883 (*J.I.A.* XXIV, 104) in its approximate form

$$\int_0^{19} u_x dx \doteqdot 9\cdot 5\ (u_4 + u_{15}).$$

This is also a Tchebychef formula, and as such was duly noted by Elderton (*J.S.S.* II, pt. 2, 140).

Hardy tested it in its replicated form

$$\int_0^{\alpha} u_x dx \doteqdot 9\cdot 5\ (u_4 + u_{15} + u_{23} + u_{34} + \ldots),$$

and further recent tests suggest that it should not be forgotten when a quick rough approximation is required; certainly it would seem that simplicity can go no farther when the function to be integrated is available at all integral ages.

With the greatest respect for the President's contention that 'in real life' values of $\bar{A}_x$ would be obtained by constructing the commutation columns, it happened a short while ago that I wanted to compare the values of 3% $\bar{A}_x$ for the English Life Tables. As we say in another place, 'I have been informed (by those most likely to know) and verily believe' that these values are not available for English Life Tables Nos. 4, 5 and 7—Males, and I accordingly calculated them for quinquennial ages from the l_x columns by formula (3) of § 40, at a cost of about twenty minutes for each table. I seem to remember that there is a short method for obtaining, from a body of data, the values of ${}_5p_{5x}$; perhaps one day we shall be satisfied to obtain from the data the values of ${}_{10}p_{10x}$.

I shall not be lured into an argument by the picture Mr Perks has drawn of the quadrator who turned out to be a mensurator after all; it is not a picture of the quadrator I had in mind who, far from looking at u_x and w_x separately, or at all, takes the logarithm of each and adds them together.

Geometry may yet prove of actuarial value, if only for its powers of suggestion, and I am glad to find that I am not the only remaining disciple of Thomas Young, though 'for a hundred years and more, his was a voice crying in the wilderness that few appear to have heard'; that I am not alone in disputing that geometry ceased to be of any practical value in the middle of the eighteenth century. Perhaps I may be permitted to quote the concluding words of the preface which Thomas Simpson wrote in his *Miſcellaneous Tracts on ſome curious, and very intereſting ſubjects*, published in 1757:

And it appears clear to me, that, it is by a diligent cultivation of the Modern Analyſis, *that* Foreign Mathematicians *have, of late, been able to puſh their* Reſearches *farther, in many particulars, than* Sir Iſaac Newton *and his* Followers *here, have done: tho' it muſt be allowed, on the other hand, that the ſame* Neatneſs, *and* Accuracy of Demonſtration, *is not every-where to be found in thoſe* Authors; *owing in ſome meaſure, perhaps, to too great diſregard for the* Geometry of the Antients.