The Actuarial Profession
making financial sense of the future

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Capital aggregation and attribution

Background

- Capital is the amount of money, assessed on a realistic basis, which a firm requires to cover the risks that it is running or collecting as a going concern, such as market risk, credit risk, and operational risk
- Typical questions of senior management in respect of capital are:
  - What products, business units, or entities are capital efficient?
  - What actions can be taken to improve the capital efficiency of the organisation?
  - What is the impact of changes in business mix, organisational structure or strategy on capital?
  - What are the impacts of various market, environment and business effects on my capital profile?
- To be able to answer the above with any real confidence a firm must understand how capital is aggregated and attributed:
  - By capital aggregation, we refer to how the amount of capital across all the products sold by a firm is aggregated to give a single capital requirement
  - By capital attribution we refer to how the aggregate capital of a firm can be attributed to individual products, businesses, entities and risk classes

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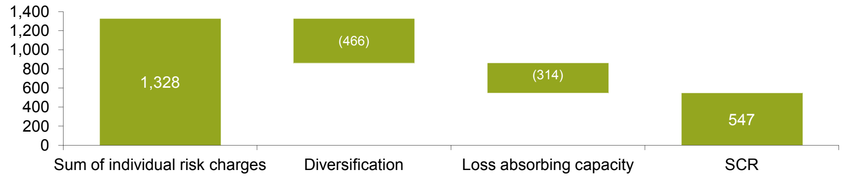
Why is aggregation important?

- The aggregation approach adopted by a firm will drive the diversification benefit
- Typically the diversification benefit for firms can be anywhere between 30-70% of the total capital requirement
- Diversification impacts on standard approach (in € billion) from the results of the QIS 5:
  - The diversification benefit between risks reduces the total capital requirement by €466 billion, i.e. approximately 35%
  - The final SCR of €547 billion was approximately 41% of the sum of individual risks
- Diversification therefore has a massive impact on the expected level of capital that a company holds



Source: [page 31, EIOPA Report on the fifth Quantitative Impact Study (QIS5) for Solvency II, EIOPA-TFQIS5-11/001, 14 March 2011, https://eiopa.europa.eu].

What are the requirements for aggregation techniques?

- Supervisory authorities must be satisfied that the system used for measuring diversification (i.e. the aggregation approach) is adequate
- What does adequate mean though from a regulatory and business view point?

EIOPA requirements

- Identifies the key variables driving dependencies
- Provides support for the existence of diversification effects
- Justifies the assumptions underlying the modelling of dependencies
- Takes into particular consideration extreme scenarios and tail dependence
- Tests the robustness of the system on a regular basis including sensitivity and stress testing
- Actively take diversification effects actively into account in business decisions

Business requirements

- Quick to run
- Understood by senior management
- Facilitate decision making
- Flexible so changes to the business model can be incorporated
- Able to be audited, controlled, managed so that management have confidence in the result
- Appropriately allows for the interactions of risks at a BU and Group level
- Appropriately reflects group structure and interactions between entities
- Adequately reflects risk profile of the company

Dependency

- A dependency structure describes the way that risk factors are linked together
- The most common and simplest form of dependency structure is linear correlation, usually measured by the Pearson's correlation
- There are many different types of dependency though that may be better captured by other correlation measures such as rank correlation or other techniques such as casual relationships etc
- Rank correlation measures the relationship between different rankings of the same set of items. Popular rank correlation statistics include
  - Spearman's ρ
  - Kendall's τ
- Rank correlations may be more appropriate when there is a non-linear relationship between risk factors
- Example of where this might occur are:
  - Tail correlation
  - Causal relationship, eg big earthquake and stock market crash, unidirectional nature of causality
  - Derivative non-linear price relationship to their underlying security

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Linear v Rank Correlation

- A Pearson correlation of 1 implies the relationship between X and Y can be described by a linear equation
- A Spearman correlation of 1 occurs when the two variables being compared are monotonically related, even if their relationship is not linear.
- Let (X_i, Y_i) and (X_j, Y_j) be a pair of (bivariate) observations. If $X_j - X_i$ and $Y_j - Y_i$ have the same sign, we shall say that the pair is *concordant. A* Kendall's Tau correlation of 1 is achieved if all $n(n-1)/2$ of possible pairs are concordant*

Pearson	1.00	Pearson	0.89	Pearson	0.51	Pearson	-0.28
Spearman	0.99	Spearman	1.00	Spearman	0.86	Spearman	-0.19
Kendall's Tau	0.95	Kendall's Tau	0.99	Kendall's Tau	0.73	Kendall's Tau	-0.13

**Refer: "Why Kendall Tau?" by G. E. NOETHER*

Importance of correlations

Base case

Correlation m/x	Market	Insurance	Credit	Operational	Market Vol	Hedge	Standalone Capital
Market	1	0.25	0.5	0.5	-0.75	-0.25	100
Insurance	0.25	1	0.25	0.25	0.25	0.25	35
Credit	0.5	0.25	1	0.25	-0.25	0.375	50
Operational	0.5	0.25	0.25	1	0.25	-0.25	20
Market Vol	-0.75	0.25	-0.25	0.25	1	-0.75	10
Hedge	-0.25	0.25	0.375	-0.25	-0.75	1	5

Standalone capital	220
Diversified capital	152

If we change some of these correlations there will be different impacts depending on which one we change...

Small change

Correlation m/x	Market	Insurance	Credit	Operational	Market Vol	Hedge	Standalone Capital
Market	1	0.25	0.5	0.5	-0.75	-0.25	100
Insurance	0.25	1	0.25	0.25	0.25	0.25	35
Credit	0.5	0.25	1	0	-0.25	0.375	50
Operational	0.5	0.25	0	1	0.25	-0.25	20
Market Vol	-0.75	0.25	-0.25	0.25	1	-0.75	10
Hedge	-0.25	0.25	0.375	-0.25	-0.75	1	5

Standalone capital	220
Diversified capital	151
% Change	1%

Here the change to the diversified capital is small - 1% as the altered correlation affects risk exposures that are relatively small

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Importance of correlations

Base case

Correlation m/x	Market	Insurance	Credit	Operational	Market Vol	Hedge	Standalone Capital
Market	1	0.25	0.5	0.5	-0.75	-0.25	100
Insurance	0.25	1	0.25	0.25	0.25	0.25	35
Credit	0.5	0.25	1	0.25	-0.25	0.375	50
Operational	0.5	0.25	0.25	1	0.25	-0.25	20
Market Vol	-0.75	0.25	-0.25	0.25	1	-0.75	10
Hedge	-0.25	0.25	0.375	-0.25	-0.75	1	5

Standalone capital	220
Diversified capital	152

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Small change

Correlation m/x	Market	Insurance	Credit	Operational	Market Vol	Hedge	Standalone Capital
Market	1	0.25	0.25	0.5	-0.75	-0.25	100
Insurance	0.25	1	0.25	0.25	0.25	0.25	35
Credit	0.25	0.25	1	0	-0.25	0.375	50
Operational	0.5	0.25	0	1	0.25	-0.25	20
Market Vol	-0.75	0.25	-0.25	0.25	1	-0.75	10
Hedge	-0.25	0.25	0.375	-0.25	-0.75	1	5

Standalone capital	220
Diversified capital	142
% Change	7%

Here the change in correlation is the same as before (reduction of 25%) but the affect the impact on the diversified capital is greater (7%)

7 times the impact of the previous case!

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Positive-semidefinite Matrices

- Positive-semidefinite (PSD) matrices are required in order to derive aggregated capital in a simulation based approach
- PSD matrices are required because their Cholesky decomposition or inverses are needed to generate the underlying correlated random scenarios
- Mathematically:
  - The n × n Hermitian matrix M is said to be positive-semidefinite (or sometimes nonnegative-definite) if $x^*Mx \geq 0$
  - for all $x \in \mathbb{C}^n$ (or, all $x \in \mathbb{R}^n$ for the real matrix), where x * is the conjugate transpose of x
  - An Hermitian matrix M is positive-semidefinite if and only if all of its eigenvalues are non-negative
  - Any positive semidefinite matrix M can be written as M = A*A (Cholesky decomposition)
- Correlation matrices generated by companies are unlikely to be PSD due to:
  - Elements of the correlation matrix may have been estimated separately and then combined together at a later stage
  - A mixture of data and expert judgement used to derive the correlation matrix
- A method is required to transform the correlation matrix of a company into a PSD matrix.
- In transforming the matrix though we need to understand how our correlations are affected!

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PSD Algorithms

- There are a number of different methods to derive a PSD matrix:
  - Trial and error
  - Simple decomposition *(aka zeriosing eignenvalues, brute force)*
  - Rebonato – Spectral decomposition *(aka weighted eigenvalue approach)*
  - Rebonato – Hypersphere decomposition
  - Higham method
  - Bending method *(including or excluding weights)*
- In the following slides we compare the attributes of a selection of the methods above
- We also focus on applying the PSD algorithm to the correlation matrix directly, rather than the covariance matrix given that typically insurance companies work directly with the correlation matrix.

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Simple Decomposition

Base case

Correlation m/x	Market	Insurance	Credit	Operational	Market Vol	Hedge	Eigen-values
Market	1	0.25	0.5	0.5	-0.75	-0.25	-0.40
Insurance	0.25	1	0.25	0.25	0.25	0.25	0.46
Credit	0.5	0.25	1	0.25	-0.25	0.375	0.61
Operational	0.5	0.25	0.25	1	0.25	-0.25	1.22
Market Vol	-0.75	0.25	-0.25	0.25	1	-0.75	1.82
Hedge	-0.25	0.25	0.375	-0.25	-0.75	1	2.29

Standalone capital	220
Diversified capital	152

Simple Decomposition

Correlation m/x	Market	Insurance	Credit	Operational	Market Vol	Hedge	Eigen-values
Market	1.12	0.20	0.47	0.47	-0.62	-0.14	0
Insurance	0.20	1.02	0.26	0.26	0.19	0.20	0.46
Credit	0.47	0.26	1.01	0.26	-0.29	0.35	0.61
Operational	0.47	0.26	0.26	1.01	0.21	-0.28	1.22
Market Vol	-0.62	0.19	-0.29	0.21	1.14	-0.64	1.82
Hedge	-0.14	0.20	0.35	-0.28	-0.64	1.09	2.29

Standalone capital	220
Diversified capital	155
% Change	2%

- Under this approach the eigenvalues of the matrix are forced to be greater than or equal to zero
- The matrix A has eigenvalue λ (lambda) with eigenvector x if the following equation holds:

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}.$$

- The correlation matrix is then derived using the transformed eigenvalues
- In our example a fundamental property of a correlation matrix has been destroyed using this naïve approach (diagonal entries do not equal 1)

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Rebonato Spectral Decomposition

Rebonato Method

Correlation m/x	Market	Insurance	Credit	Operational	Market Vol	Hedge	Eigen-values
Market	1.00	0.18	0.44	0.44	-0.55	-0.13	0.00
Insurance	0.18	1.00	0.26	0.26	0.18	0.19	0.45
Credit	0.44	0.26	1.00	0.26	-0.27	0.33	0.59
Operational	0.44	0.26	0.26	1.00	0.20	-0.27	1.15
Market Vol	-0.55	0.18	-0.27	0.20	1.00	-0.57	1.71
Hedge	-0.13	0.19	0.33	-0.27	-0.57	1.00	2.11

Standalone capital	220
Diversified capital	149
% Change	2%

Absolute difference to base correlation matrix

Correlation m/x	Market	Insurance	Credit	Operational	Market Vol	Hedge
Market	0.00	0.07	0.06	0.06	0.20	0.12
Insurance	0.07	0.00	0.01	0.01	0.07	0.06
Credit	0.06	0.01	0.00	0.01	0.02	0.04
Operational	0.06	0.01	0.01	0.00	0.05	0.02
Market Vol	0.20	0.07	0.02	0.05	0.00	0.18
Hedge	0.12	0.06	0.04	0.02	0.18	0.00

Max error	0.2
Standard Deviation	5%

- This approach is similar to the simple decomposition but weighted to ensure that the diagonal entries equal 1.
- The correlation matrix is then derived using the transformed eigenvalues
- This method **does not allow us to control which elements of the correlation matrix are altered**
- Our key corelations have all be shifted by greater than 5% leading to capital being understated by 2%

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$\tilde{A} = A + E$

Higham Method

- Highman provides 2 algorithms to derive a PSD matrix based on different weighting methodologies starting from a given non semi-definite matrix ..

Approach 1

R' = R + E such that R' is a correlation matrix and $\left|W^{\frac{1}{2}}EW^{\frac{1}{2}}\right|$ is minimum for some given PSD matrix W.

Approach 2

- R' = R + E such that R' is a correlation matrix and $|E^{\circ}H|$ is minimum for some given PSD matrix H.
- The values of H can be interpreted as weights which are predefined by the user prior to starting the required search

In the following slide we provide the results for the first approach

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Higham Method

Highman Method

Correlation m/x	Market	Insurance	Credit	Operational	Market Vol	Hedge	Eigen-values
Market	1.00	0.18	0.46	0.45	-0.56	-0.10	0.00
Insurance	0.18	1.00	0.27	0.27	0.17	0.19	0.41
Credit	0.46	0.27	1.00	0.26	-0.30	0.34	0.56
Operational	0.45	0.27	0.26	1.00	0.20	-0.29	1.13
Market Vol	-0.56	0.17	-0.30	0.20	1.00	-0.59	1.73
Hedge	-0.10	0.19	0.34	-0.29	-0.59	1.00	2.17

Standalone capital	220
Diversified capital	150
% Change	1%

Absolute difference to base correlation matrix

Correlation m/x	Market	Insurance	Credit	Operational	Market Vol	Hedge
Market	0.00	0.07	0.04	0.05	0.19	0.15
Insurance	0.07	0.00	0.02	0.02	0.08	0.06
Credit	0.04	0.02	0.00	0.01	0.05	0.04
Operational	0.05	0.02	0.01	0.00	0.05	0.04
Market Vol	0.19	0.08	0.05	0.05	0.00	0.16
Hedge	0.15	0.06	0.04	0.04	0.16	0.00

Max error	0.19
Standard Deviation	5%

- In this example we have used one version of the higham method that ensures the diagonal entries equal 1.
- The method is guarantees to compute the nearest correlation matrix
- In this example ,though the method gives very similar results to the Rebonato method

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Bending method - weighted

- This approach is based on a statistical method known as “bending”.
- Bending is an iterative process of updating the correlation matrix as follows:

1. Determine matrix of eigenvectors, U_n, and diagonal matrix of eigenvalues, D_n, of R. Hence, $R_n = U_n D_n U_n$, where n denotes iteration number;
2. Replace D_n with D^*_n, where d*i,i = 2ε, for $d_{i,i} < \varepsilon$ and $d^*_{i,i} = d_{i,i}$, otherwise. Set the value of ε to a small positive real number
3. Replace D^*_n with Δ_n, where $\delta_{i,i} = d^*_{i,i}$ (tr(D)/ tr(D*));
4. Replace D_n with Δ_n, where $\delta_{i,i} = \varepsilon$, for $d_{i,i} < \varepsilon$, and $\delta_{i,i} = d_{i,i}$, otherwise.
5. Calculate a new correlation matrix: $R^*_{n+1} = R_n – [R_n - U_n \Delta_n U_n']$ □ W (where □ is the Hadamard product)
6. Replace R^*_{n+1} with Rn+1, where $r_{i,j} = r^*_{i,j}/\text{sqrt}(r^*_{i,i}\, r^*_{j,j})$.
7. Repeat until R_{n+1} is positive definite

Refer: “A Simple Method for Weighted Bending of Genetic (Co)variance Matrices”; Journal of Dairy Science Vol. 86, No. 2, 2003; H.. Jorjani, L. Klei,† and U. Emanuelson,*

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Bending method - weighted

Bending method – Weighted

Correlation m/x	Market	Insurance	Credit	Operational	Market Vol	Hedge	Eigen-values
Market	1.00	0.25	0.50	0.50	-0.75	1.00	2.18
Insurance	0.25	1.00	0.26	0.26	0.17	0.25	1.72
Credit	0.50	0.26	1.00	0.26	-0.27	0.50	0.00
Operational	0.50	0.26	0.26	1.00	0.19	0.50	1.09
Market Vol	-0.75	0.17	-0.27	0.19	1.00	-0.75	0.57
Hedge	-0.12	0.19	0.32	-0.27	-0.55	-0.12	0.44

Standalone capital	220
Diversified capital	152
% Change	0%

Absolute difference to base correlation matrix

Correlation m/x	Market	Insurance	Credit	Operational	Market Vol	Hedge
Market	0.00	0.00	0.00	0.00	0.00	0.13
Insurance	0.00	0.00	0.01	0.01	0.08	0.06
Credit	0.00	0.01	0.00	0.01	0.02	0.05
Operational	0.00	0.01	0.01	0.00	0.06	0.02
Market Vol	0.00	0.08	0.02	0.06	0.00	0.20
Hedge	0.13	0.06	0.05	0.02	0.20	0.00

Max error	0.2
Standard Deviation	5%

- Weighted bending minimizes the changes to user selected correlations at the expense other correlations
- The choice of weights are key to the effectiveness of the methodology
- Many different weights can be chosen that focus on different aspects (eg reliability of correlation estimate, importance of correlation on capital etc)
- In this example we have used a weight based on the size of capital affected by the correlation

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PSD Algorithm Comparison

Algorithm \ Feature	Trial and Error	Simple Decomposition	Rebonato – Spectral Decomposition	Rebonato – Hypersphere Decomposition	Bending Method	Higham method
Accuracy (residual errors)	L	L	H	H	H	H
Preserves unity	Y	N	Y	Y	Y	Y
Convergence	N	N	Y	Y	N	Y
Calibration effort	H	L	M	H	L	M
User defined weighting	Y	N	N	Y	Y	Y*
Computation efficiency	L	H	H	M	L	M
Management understanding	H	M	M	L	L	L

* 2 different methods can be applied to weight the correlation matrix. It should be noted that weighting individual correlations is possible but its application is not straightforward

Legend:
H = High Y = Yes
M = Medium N = No
L = Low

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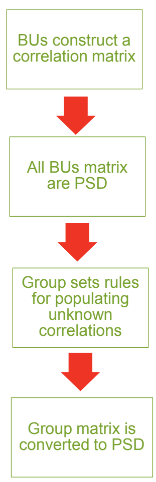
Correlation matrix construction

- Correlation matrices can become as big as large 1500 x 1500
- It is not feasible or practical to estimate the correlations of all possible combinations
- A pragmatic approach is therefore required to populate the remaining parts of the correlation matrix
- In the slides that follow we provide an overview of a possible process to construct the wider group correlation matrix.



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Correlation matrix construction (cont.)

BUs construct a correlation matrix
↓
All BUs matrix are PSD
↓
Group sets rules for populating unknown correlations
↓
Group matrix is converted to PSD

- BUs derive individual correlation matrices
- BUs convert matrices into PSD matrices using one of the algorithms described
- Supplying PSD matrices at this level generally means it is easier to derive the Group PSD and minimises the potential change in correlations

BU1	Risk 1	Risk 2
Risk 1	1.0	0.5
Risk 2	0.5	1.0

BU3	Risk 1	Risk 2
Risk 1	1.0	0.6
Risk 2	0.6	1.0

BU2	Risk 1	Risk 2
Risk 1	1.0	0.5
Risk 2	0.5	1.0

BU4	Risk 1	Risk 2
Risk 1	1.0	0.25
Risk 2	0.25	1.0

GROUP	BU1 Risk 1	BU1 Risk 2	BU2 Risk 1	BU2 Risk 2	BU3 Risk 1	BU3 Risk 2	BU4 Risk 1	BU4 Risk 2
BU1 Risk 1	1.0	0.5	ρ(1,1)	ρ(1,2)	ρ(1,1)	ρ(1,2)	ρ(1,1)	ρ(1,2)
BU1 Risk 2	0.5	1.0	ρ(2,1)	ρ(2,2)	ρ(2,1)	ρ(2,2)	ρ(2,1)	ρ(2,2)
BU2 Risk 1	ρ(1,1)	ρ(1,2)	1.0	0.5	ρ(1,1)	ρ(1,2)	ρ(1,1)	ρ(1,2)
BU2 Risk 2	ρ(2,1)	ρ(2,2)	0.5	1.0	ρ(2,1)	ρ(2,2)	ρ(2,1)	ρ(2,2)
BU3 Risk 1	ρ(1,1)	ρ(1,2)	ρ(1,1)	ρ(1,2)	1.0	0.6	ρ(1,1)	ρ(1,2)
BU3 Risk 2	ρ(2,1)	ρ(2,2)	ρ(2,1)	ρ(2,2)	0.6	1.0	ρ(2,1)	ρ(2,2)
BU4 Risk 1	ρ(1,1)	ρ(1,2)	ρ(1,1)	ρ(1,2)	ρ(1,1)	ρ(1,2)	1.0	0.25
BU4 Risk 2	ρ(2,1)	ρ(2,2)	ρ(2,1)	ρ(2,2)	ρ(2,1)	ρ(2,2)	0.25	1.0

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Correlation matrix construction (cont.)

BUs construct a correlation matrix
↓
All BUs matrix are PSD
↓
Group sets rules for populating unknown correlations
↓
Group matrix is converted to PSD

- Large parts of the correlation matrix still remain undefined
- A rules based approach is typically used to determine correlations between the same risks across business units (eg same region/ same risk = 100%, different region / same risk = 50%.
- But there may still be about the remaining correlations? For example, how do we derive a correlation between UK/equity and France/property?

		UK		Europe	
		Equity	Property	Equity	Property
UK	Equity	1.0	0.5	0.8	r=?
	Property	0.5	1.0	0.5	0.4
Europe	Equity	0.8	0.5	1.0	0.5
	Property	r=?	0.4	0.5	1.0

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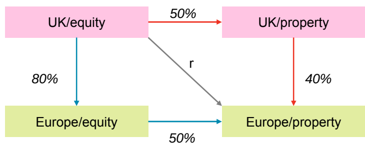
Correlation matrix construction (cont.)

- Remaining correlations can be calculated by any number of different approaches.
- The most common methods are:
  - Max Rule
  - Average Rule

Example

- The correlation between UK equity and Europe Property can be derived by taking the red or the blue route:
- Under the Max Rule, we have:
  r = Max(50% * 40%, 80% * 50%) = 40%
- Under the Average Rule we have:
  r = Ave(50%, 50%) * Ave(80%, 40%) = 30%



- The final step in the process is to convert the group matrix to a PSD matrix

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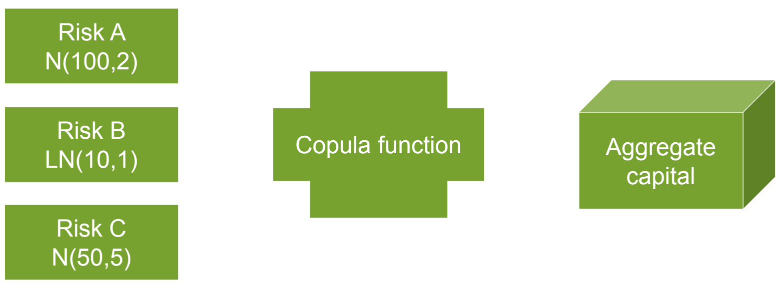
Copulas

- A Copula allows us to aggregate individually defined loss functions of the risks that are relevant to a firm into a coherent set



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Copulas

- There are different types of copulas and the choice of copulas reflect the prior beliefs of a firm
- There are copulas that allow for a greater proportion of extreme events at the tail, that is fat tailed, such as the T- copula compared to Gaussian Copula
- There are copulas that are symmetric, that is the proportion of adverse scenarios are the same as favourable scenarios eg Gaussian and the student – T copulas, as opposed to Gumbell or Frank copulas
- Elliptical(Gaussian and the student – T copulas) copulas are commonly used in aggregating capital in insurance firms. The main reason for this is that:
  - It is not straightforward to implement Archimedean copulas such as Gumbell or Frank for a large number risks, use of the vine copulas approach
  - Elliptical copulas are an improvement on the correlation matrix approach that most senior management are familiar with

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Copulas

Correlations

	Equity	Property	Lapses	Mortality
Equity	1	0.5	-0.2	0
Property	0.5	1	0.1	0
Lapses	-0.2	0.1	1	0
Mortality	0	0	0	1

Risks

Risk	Capital Req (£m)	Base Case	Case 1
Equity	100	N(180,38.82)	LN(4.53,0.28)
Property	10	N(6.14,3.88)	LN(2.38,0.25)
Lapses	100	N(65,38.82)	N(65,38.82)
Mortality	10	N(16.99,3.88)	N(16.99,3.88)

Results

Approach	Base Case	Case 1
Correlation matrix	131.91	131.91
Gaussian Copula	132.24	124.02
Student T (50 dof) Copula	133.03	124.56
Student T (10 dof) Copula	139.12	130.11
Student T (5 dof) Copula	145.79	136.97
Student T (1 dof) Copula	174.06	166.11

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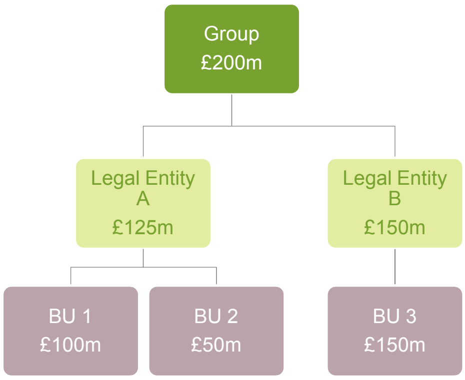
Attribution of Capital (to products)



- The diagram on the left, shows the diversified capital at different hierarchies of an insurance group
- By attributing capital, we are trying to answers questions like:
  - How much of the £200m group diversified capital is due to BU1?
  - How much of the £125m legal entity capital is due to BU2?
  - How much would the group diversified capital increase by, if the amount of business in BU3 increases 15%?
  - What would be the impact of buying another legal entity (C), should legal entity C be merged with legal entity A or if should be a separate legal entity?
  - What products in BU1 contributes most to the diversified capital of BU1?

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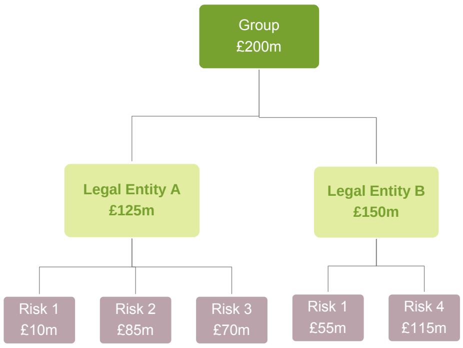
Attribution of Capital (to risks)



- Attribution of capital to risks, we are interested in the following questions:
  - how much of the £200m group diversified capital is particular risk (a risk family such as insurance risk or US equity risk)?
  - What are the most material risks for legal entities and BUs?
  - What would be the impact of undertaking an equity hedge on the group diversified capital?

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Attribution of Capital (to risks)

- Most of the capital used to attribute capital to risks is based on Taylor Series expansion
- Let assume a very simple case in which a firm is exposed to a single risk factor x, then the capital of the firm is a function of x, f(x)
- If we assume that f(x) is differentiable, then Taylor Series tells us that
  $f(x_0 + \Delta x) = f(x_0) + \Delta x f'(x_0) + \frac{1}{2!}(\Delta x)^2 f''(x_0) + \cdots$
- If we assume that higher differentials are not material, then change in capital due to a very small change in risk x is $\Delta x f'(x_0)$
- If we express $\Delta x = k(x_0)$ where k is a very small number, then the change in capital can be expressed as $\frac{f(x_0 + \Delta x) - f(x_0)}{k}_{x_0}$
- If the firm is exposed to more than one risk factor, x, y, z for example, then the capital attributed to each of these risks can be shown to be:
  $\frac{f(x_0 + \Delta x) - f(x_0)}{k}_{x_0}$ $\frac{f(y_0 + \Delta y) - f(y_0)}{k}_{y_0}$ $\frac{f(z_0 + \Delta z) - f(z_0)}{k}_{z_0}$
- This is the so-called **risk marginal attribution approach**. The risk value that is perturbed in the risk marginal approach is the 1 in 200 scenario (could be smoothed or unsmoothed 1 in 200 scenario)

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Attribution of Capital

- An alternative approach is to assume that a given proportion of the diversified group is due a particular risk
- The proportions are determined mainly by expert judgement (that is guessing)
- The use of assumed proportions can also be used to attribute capital to products. This approach assumes that x% of the group diversified capital is due to BU1 or legal entity A
- Other approaches that can be used to attributed capital to products/BU are:
  - Use of group 1 in 200 scenario: the group 1 in 200 scenario is used to calculate the capital required for the products or BU that make up the group
  - Use of weights (if a smoothing approach is used)
- We set out a simple case study to explain these approaches

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Case Study

Step 1

Scenario	Product A	Product B	Group
1	-20	-222	-242
2	-9	56	47
3	43	-258	-215
4	45	-141	-96
5	-6	-136	-142
6	-200	41	-159
7	-100	91	-9
8	-9	-17	-26
9	88	-191	-103
10	34	45	79
11	29	87	116
12	-88	-36	-124
13	-45	-78	-123
14	-767	-174	-941
15	34	125	159
16	-234	-1111	-1345
17	234	151	385
18	209	158	367
19	-101	-125	-226
20	-52	41	-11

Step 1

- The change in NAV at each scenario is calculated for each product
- The sum of the change in NAV is the group change in NAV

Case Study

Step 2

Scenario	Product A	Product B	Group	Weights
16	-234	-1111	-1345	0
14	-767	-174	-941	0.3
1	-20	-222	-242	0.4
19	-101	-125	-226	0.3
3	43	-258	-215	0
6	-200	41	-159	0
5	-6	-136	-142	0
12	-88	-36	-124	0
13	-45	-78	-123	0
9	88	-191	-103	0
4	45	-141	-96	0
8	-9	-17	-26	0
20	-52	41	-11	0
7	-100	91	-9	0
2	-9	56	47	0
10	34	45	79	0
11	29	87	116	0
15	34	125	159	0
18	209	158	367	0
17	234	151	385	0

Step 2

- The data in step 1 is sorted by the change in NAV of group
- The weights are calculated
- The attributed capital is the sum product of the weights and change in NAV

	Product A	Product B	Group
Standalone capital	-340.4	-446.9	-446.9
Attributed capital	-268.4	-178.5	-446.9

Attribution: Issues/Decisions

- Attribution to products first and to risks or vice versa
- Attribution to primary or secondary risk factors
- Stability of attributed results when different sets of scenarios are used:
  - Through more scenarios at it (subject to time and computing resources restrictions)
  - Implement variance reduction techniques
  - Implement smoothing
- Allowing for fungibility restrictions

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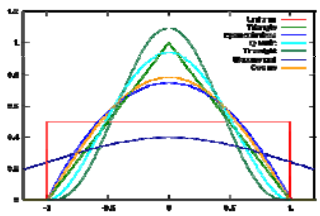
Smoothing

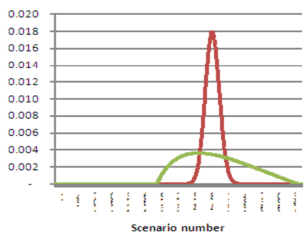
PGT Note: Graphs are current pasted in as pictures. If they require formatting please supply data or requests graph retracing.

- Smoothing is simply using a weighted average instead of a single number.
- Smoothing can be applied to capital or to scenarios.
- Kernel weights are the most common type of weights used in smoothing. A kernel is any function that satisfies the following conditions

$$K(-u) = K(u) \quad \text{and} \quad \int_{-\infty}^{\infty} K(u)\,du = 1$$

- The first graph on the right shows some of the different shapes that the kernel weights can assume depending on how the kernel function is defined.
- Most academic literature on Kernel smoothing state that the Epanechnikov Kernel is the optimum kernel function
- Harrell Davies weights is another type of weights that can be used in smoothing. The second graph comes HD and kernel smoothing. The HD is red and it smoothes over a relatively smaller window compared to Kernel smoothing
- The key question about smoothing is the number of scenarios to smooth over. HD does not allow you to choose this, whilst Kernel smoothing does.





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Validation of Internal Model numbers

- Run 1 in 200 scenario on actuarial models (Prophet/MoSes) and compare stressed value of assets and liabilities to those given by internal model – this tests how good the formula fitting is
- Derive 1 in 200 uni-variate capital from internal model and compare that with that from standard formula
- Run shredded 1 in 200 scenario – the sum of the capital requirements and the SCR is the so-called non-linearity
- Sensitivities run and assess if the movement in capital is intuitive:
  - Change copula, or degrees of freedom of copula
  - Change marginal distribution
  - Change correlations

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Variance Reduction Techniques

- Quasi Monte Carlo eg SOBOL
  - insert graph of psuedo random number and SOBOL
  - difference between convergent and stable results
- Importance sampling
  - What is importance sampling
  - Why it is not easy to implement this
- Why other variance reduction techniques are not appropriate
  - Antithetics
  - Control variate
  - Stratified sampling

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Questions or comments?

Expressions of individual views by members of The Actuarial Profession and its staff are encouraged

The views expressed in this presentation are those of the presenter

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