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A BASIC CURVE OF DEATHS

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THE two-fold purpose of this paper is:

(i) To hypothesize the existence of a Basic Curve of Deaths, a curve to which it is suggested all curves of deaths are, as it were, striving to attain—the 'idea' of a curve of deaths, almost in a Platonic sense, a curve applicable not only to humanity, but equally to all forms of life.

(ii) To seek a representation of that curve by a function containing as few parameters as possible, on the principle that while there are other functions which fit the observations more exactly, we here prefer a simple function which fits them well enough.

As customarily visualized, the curve of deaths is a plane surface the area of which represents the radix l_0, and the ordinate at x represents $\mu_x l_x$, for which as before we shall here use the symbol ϕ_x. Then, writing $l_{x:\overline{n}|}$ for $(l_x - l_{x+n})$, any two ordinates ϕ_x and ϕ_{x+n} enclose an area which represents $l_{x:\overline{n}|}$.
The distribution of deaths over time can equally be represented by a solid block such as that shown in axonometric projection in Fig. 1, its volume representing l_0, the areas of any two cross-sections at x and $(x+n)$ representing ϕ_x and ϕ_{x+n} respectively, and the volume enclosed by them representing $l_{x:\overline{n}|}$.

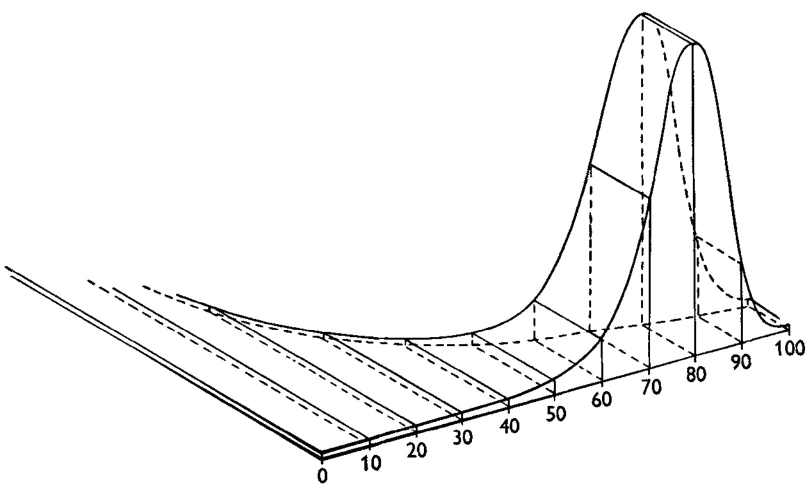

Fig. 1.

2. In order to discuss the measurement of lifetime we must know what we mean by 'life' and what we mean by 'time', two fields in which anyone is free to range widely, since, despite extensive exploration, the limitations of both are still discerned but dimly, and indeed appear now to be further away than

they seemed to be thirty or forty years ago. Until recently sidereal time was regarded as flowing uniformly, without anyone explaining, or indeed realizing the need to explain, with what it was supposed to be uniform. Here 'time' will be thought of as a three-dimensional continuum, analogous to three-dimensional space.

3. By the Cartesian convention (for it is, after all, only a convention) any movement in space can be resolved into a combination of movements in three mutually independent directions. Here 'life' will be considered as though it possessed the capacity to move simultaneously in three mutually independent directions, say x, y and z, in a three-dimensional time continuum. It is suggested:

(i) that the x-direction represent sidereal time; that so much of the living thing as is composed of matter is, like any other matter, travelling through time in this direction at a speed which is common to all matter, whether organic or inorganic;

(ii) that the y-direction represent deterioration, or 'ageing', in which direction 'life' is thought of as travelling through time at a speed which, by comparison with that in the x-direction, is an accelerating speed;

(iii) that the z-direction represent development, in which direction 'life' is thought of as travelling at first very rapidly and constantly decelerating.

It should be made clear that we are now departing from the common perception that during the first n years a living thing develops, and thereafter begins to age. The present analogy, on the contrary, supposes the two processes to run concurrently throughout the lifetime.

4. It is a commonplace that to a child a year appears longer than it does to an adult; it is here suggested that it appears longer because, in the hypothesized z-direction, it *is* longer. *Per contra* it is common observation that a man 'ages' more between 70 and 80 than he does between 60 and 70; it is here suggested that this is analogous to an accelerating speed in the y-direction.

5. It is realized that this is not the place to discuss either biology or cosmology, but after some hesitation two appendices have been included epitomizing the biological and cosmological background against which the author was led, by considerations outlined in the first of those appendices, to make tests of the formula

$$l_{x:\overline{n}|} \propto \int_{x_{,}^{3}}^{(x_{,}+n)^{3}} \frac{1}{x_{,}^{3}} e^{-a(x_{,}^{3}-c^{3})^{2}} dx_{,}^{3}$$

(where $x_{,} = x + \cdot 75$ and a and c are parameters) representing the solid block shown in Fig. 2 with a base of diminishing width and a normal curve elevation.

Notwithstanding that the normal curve extends indefinitely in both directions, it has been used, as so often in so many other connexions, as an arithmetically convenient approximation.

6. It is more convenient to reconstruct the block of Fig. 2 so that the cross-sections are equidistant. This is equivalent to changing the variable in the integration of § 5 from $x_{,}^{3}$ to $x_{,}$, and has the effect of changing the depth of the block from $1/x_{,}^{3}$ to $1/x_{,}$. Writing $b = ac^{6}$ and $j = x_{,}/c$, we have

$$l_{x:\overline{n}|} \propto \int_{x}^{x+n} \{j \,.\, e^{b(j^{3}-1)^{2}}\}^{-1} \, dx \quad \text{(Fig. 1)}.$$

7. Calculations by this formula have been made by the following process, for selected values of b and c:

(i) Tentatively putting $\phi_c = 10{,}000$, merely for arithmetical convenience, calculate ϕ_x for every fifth age from the formula

$$\phi_x = \phi_c\{j \cdot e^{b(j^3-1)^2}\}^{-1}.$$

(ii) Calculate $\cdot 6 l_{x:\overline{10}|}$ for every fifth age by Simpson's Rule.
(iii) Sum to obtain $\cdot 6 l_x$ at every fifth age.
(iv) Prepare columns of $10\phi_x$ and $\cdot 6 l_x$ with radix $l_{20} = 1000$.
(v) Calculate $\bar{A}_x 3\%$ by the formula

$$(l\bar{A})_x = \cdot 9038 l_{x:\overline{10}|} + \cdot 6062 l_{x+10:\overline{10}|} + v^{20}(l\bar{A})_{x+20}$$

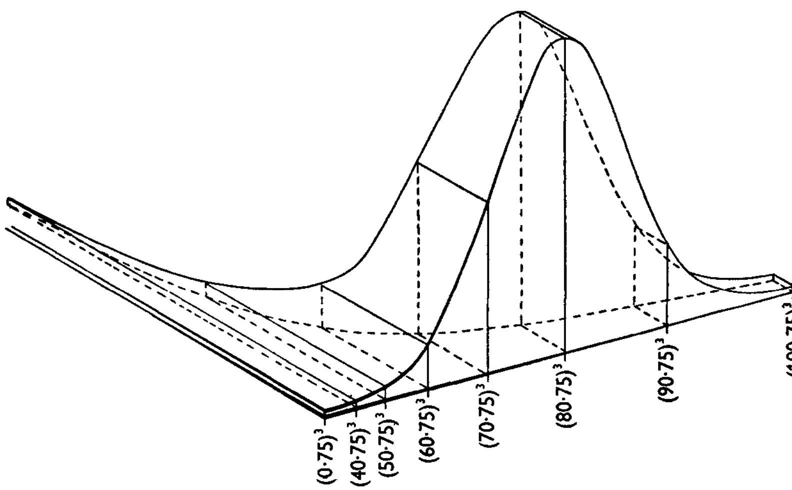

Fig. 2.

(*J.I.A.* 77, 159, at 172). In the light of the discussion upon that paper the author's tenacity in adhering to this formula might occasion comment were it not explained that all the calculations for the present paper were completed before the 1951 paper was presented, that paper being in fact composed of material which outgrew the original Appendix to an early draft of this paper.

8. By taking $c = 78\frac{1}{4}$ and $b = 4\cdot 00$, a curve of deaths is obtained by the procedure of § 7 which bears a considerable resemblance to the ϕ_x curve of the A 1924–29 (ultimate) Table, as will be seen from Fig. 3 and Table 1. It may well be objected that the author is behaving illogically when, in suggesting a formula for a basic curve of deaths, he emphasizes that this is only to be regarded as a Platonic 'idea', and then proceeds to make a comparison between the synthetic curve and an experience table, particularly one which is very far indeed from being a generation mortality table. This entirely valid criticism the author is unable to refute. Continuing his illogical course he set out to calculate ϕ_{5x}, l_{10x} and $\bar{A}_{10x}$ at 3% for all values of c from 72 to 79 by quarter-unit intervals, and all values of b from 2 to 4 by one-tenth intervals, with the intention of seeing which of them most closely corresponded to various

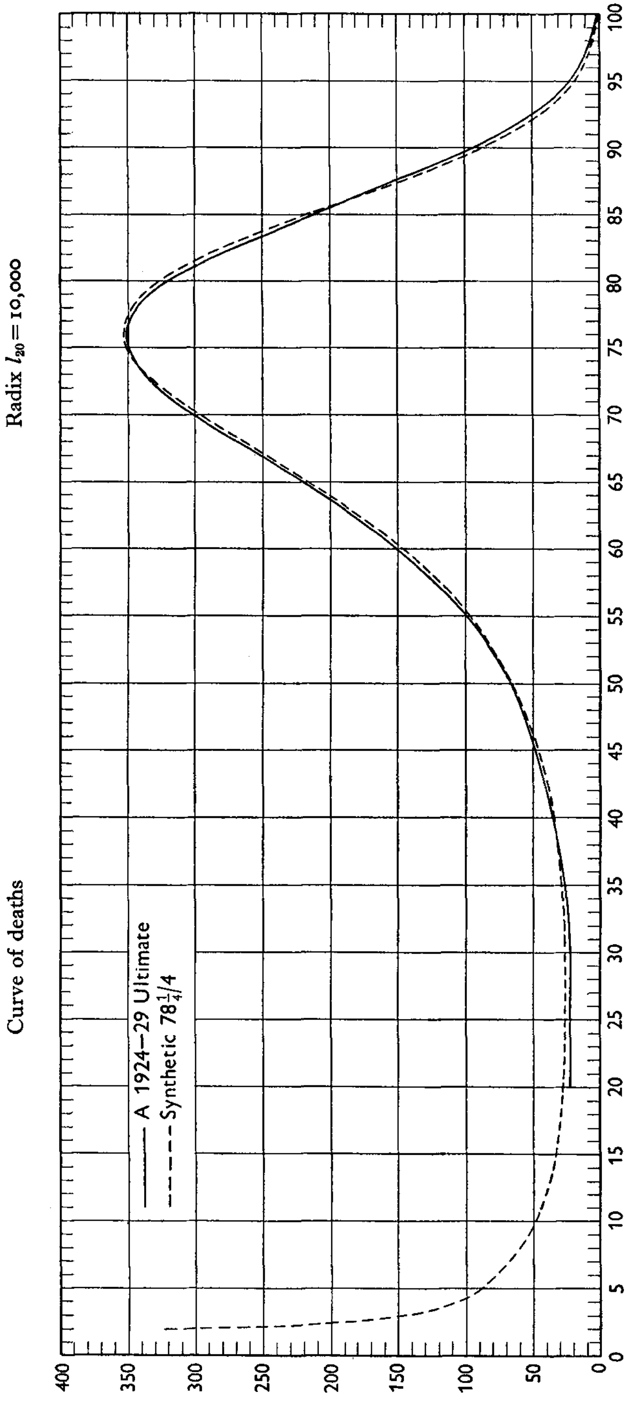
Curve of deaths
Radix l_{20} = 10,000
A 1924–29 Ultimate
Synthetic 78¼/4
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95 100
0 50 100 150 200 250 300 350 400

Fig. 3.

experience tables. He soon discovered that, of this potentially rectangular field, only a diagonal curved strip was required; that is to say, there was a marked tendency for b and c not to vary independently.

Table 1. A 1924–29 (ultimate) compared with $78\frac{1}{4}/4.00$

x	l_x		ϕ_x		$1000\,\bar{A}_{x3}$ %		
	Tabular	Synthetic	Tabular	Synthetic	Tabular	Synthetic	Deficiency
20	1000	1000	23	28	256	258	−2
25	—	—	23	26	287	289	−2
30	977	974	23	26	324	325	−1
35	—	—	27	29	367	367	0
40	949	944	36	35	415	413	2
45	—	—	48	46	467	466	1
50	900	897	66	66	525	525	0
55	—	—	98	97	587	587	0
60	799	797	151	146	651	651	0
65	—	—	218	216	713	713	0
70	578	578	300	297	771	775	−4
75	—	—	350	352	822	831	−9
80	241	240	320	328	863	870	−7
85	—	—	211	216	895	892	3
90	32	27	93	87	—	—	—
95	—	—	23	18	—	—	—
100	—	—	3	2	—	—	—

Table 2. H^m aggregate compared with $75\frac{1}{4}/2{\cdot}5$

x	l_x		ϕ_x		$1000\,\bar{A}_{x3}$ %		
	Tabular	Synthetic	Tabular	Synthetic	Tabular	Synthetic	Deficiency
20	1000	1000	55	84	333	338	−5
30	934	924	72	71	397	395	2
40	857	850	85	80	478	476	2
50	758	755	117	114	575	576	−1
60	613	611	179	181	683	684	−1
70	395	390	251	253	789	789	0
80	146	144	217	207	—	—	—
90	13	17	48	51	—	—	—
100	0	0	0	1	—	—	—

9. Table 2 compares the H^m aggregate with synthetic $75\frac{1}{4}/2{\cdot}5$, and Table 3 compares the O^m aggregate with $75\frac{3}{4}/2{\cdot}55$. By the deliberate selection of synthetic curves which best represent the values of $\bar{A}_x$ for ages 30 and over, we reach, not only in these three examples but for every table for which tests have been made, a curve of deaths which is not sufficiently low at the adolescent ages. To a limited extent this is due to the experience tables not being single-generation tables, and so showing mortality at the older ages which belongs to a more ancient generation, and is accordingly out of gear, so to speak, with the low mortality of those who, besides being younger, have the advantage of being born later.

10. The comparison of E.L.T. No. 10 (Males) and $76\frac{5}{8}/3\frac{1}{3}$ by quinquennial ages may be of interest (Table 4). However, from the author's point of view, it is of more interest that the synthetic formula appears to have about the same degree of relationship to a number of standard tables, than that it should fit closely any one standard table.

Table 3. O^m aggregate compared with $75\frac{3}{4}/2{\cdot}55$

x	l_x		$10\phi_x$		$1000\,\bar{A}_x 3\,\%$		
	Tabular	Synthetic	Tabular	Synthetic	Tabular	Synthetic	Deficiency
20	1000	1000	40	81	314	332	−18
30	953	927	56	69	386	390	−4
40	886	855	80	78	471	470	1
50	790	764	116	110	570	570	0
60	644	623	182	177	678	678	0
70	421	406	259	253	784	784	0
80	161	155	230	216	—	—	—
90	17	20	57	57	—	—	—
100	0	0	1	2	—	—	—

Table 4. E.L.T. No. 10 (Males) compared with $76\frac{5}{8}/3\frac{1}{3}$

x	l_x		$10\phi_x$		$1000\,\bar{A}_x 3\,\%$		
	Tabular	Synthetic	Tabular	Synthetic	Tabular	Synthetic	Deficiency
20	1000	1000	31	47	282	290	−8
25	984	977	33	43	314	320	−6
30	968	957	33	42	352	356	−4
35	950	934	39	46	396	397	−1
40	928	911	51	53	444	443	1
45	898	880	69	66	496	495	1
50	857	843	94	88	552	552	0
55	803	790	126	122	610	611	−1
60	729	718	171	170	670	670	0
65	629	618	232	231	730	730	0
70	497	487	295	291	786	787	−1
75	340	333	324	315	835	837	−2
80	186	184	278	270	875	873	2
85	73	74	166	164	—	—	—
90	18	19	60	62	—	—	—
95	3	3	12	12	—	—	—
100	0	0	1	1	—	—	—

11. It has already been remarked that b and c do not vary with complete independence. The following results are not so crude as was expected. Put

$$b = 4{\cdot}648 \times 10^{-19} \times c^{10}$$

The empirical coefficient has been chosen to reproduce $b = 4{\cdot}00$ when $c = 78\frac{1}{4}$, thus leaving undisturbed the curve which approximately represents the A 1924–29 (ultimate) Table. Table 5 compares the ϕ_x of English Life Tables (Males) Nos. 4–10 inclusive with those of the synthetic curve for various stated values of c and Table 6 makes corresponding comparisons for $\bar{A}_x 3\,\%$.

Table 5. Comparisons of ϕ_x for English Life Tables (Males) Nos. 4–10 inclusive, and A1924–29 (ultimate) with ϕ_x by one-parameter synthetic curves with c as stated

x	E.L.T. No. 4 ($c=73\frac{1}{2}$)		E.L.T. No. 5 ($c=73\frac{1}{4}$)		E.L.T. No. 6 ($c=73\frac{1}{2}$)		E.L.T. No. 7 ($c=74\frac{3}{4}$)		E.L.T. No. 8 ($c=75\frac{1}{4}$)		E.L.T. No. 9 ($c=76\frac{1}{2}$)		E.L.T. No. 10 ($c=76\frac{3}{4}$)		A1924–29 (ultimate) ($c=78\frac{1}{4}$)	
	Tab.	Syn.	Tab.	Syn.	Tab.	Syn.	Tab.	Syn.	Tab.	Syn.	Tab.	Syn.	Tab.	Syn.	Tab.	Syn.
20	60	108	47	113	45	108	37	83	34	74	34	52	31	48	23	28
25	74	94	60	98	53	94	44	74	39	66	39	47	33	44	23	26
30	86	89	77	93	63	89	53	71	46	63	41	46	33	43	23	26
35	98	90	90	93	80	90	66	73	57	66	51	49	39	46	27	29
40	113	97	105	100	100	97	81	80	73	74	61	57	51	54	36	35
45	126	110	122	112	118	110	101	94	92	88	76	70	69	67	48	46
50	141	130	141	132	141	130	127	116	117	110	95	92	94	89	66	66
55	165	157	166	159	169	157	161	147	153	142	133	126	126	122	98	97
60	188	189	197	189	204	189	200	184	195	183	174	172	171	169	151	146
65	211	221	227	219	231	221	233	228	236	230	234	229	232	229	218	216
70	228	238	241	233	250	238	261	259	270	267	278	285	295	288	300	297
75	220	223	229	217	234	223	264	276	279	270	305	305	324	312	350	352
80	171	170	179	164	189	170	206	203	228	218	260	260	278	269	320	328
85	101	95	101	92	107	95	124	118	134	128	158	159	166	166	211	216
90	38	35	34	34	33	35	49	44	53	48	61	61	60	64	93	87
95	8	8	6	7	5	8	9	9	11	10	14	13	12	13	23	18
100	1	1	—	1	—	1	1	1	2	1	2	1	1	1	3	2

Table 6. Comparisons of $1000\bar{A}_x 3\%$ by English Life Tables (Males) Nos. 4–10 inclusive, and A 1924–29 (ultimate), with $1000\bar{A}_x 3\%$ by one-parameter synthetic curves with c as stated

x	E.L.T. No. 4 ($c=73\frac{1}{2}$)		E.L.T. No. 5 ($c=73\frac{1}{4}$)		E.L.T. No. 6 ($c=73\frac{1}{2}$)		E.L.T. No. 7 ($c=74\frac{3}{4}$)		E.L.T. No. 8 ($c=75\frac{1}{4}$)		E.L.T. No. 9 ($c=76\frac{1}{2}$)		E.L.T. No. 10 ($c=76\frac{3}{4}$)		A 1924–29 (ultimate) ($c=78\frac{1}{4}$)	
	Tab.	Syn.	Tab.	Syn.	Tab.	Syn.	Tab.	Syn.	Tab.	Syn.	Tab.	Syn.	Tab.	Syn.	Tab.	Syn.
20	360	367	347	372	338	367	318	339	306	326	293	297	282	291	256	258
25	393	391	385	396	374	391	354	365	341	354	326	326	314	321	287	289
30	430	423	423	427	415	423	394	398	381	388	363	361	352	356	324	325
35	469	459	463	463	457	459	437	436	425	427	405	402	396	397	367	367
40	509	501	506	505	502	501	482	480	471	471	451	448	444	443	415	413
45	552	547	553	550	549	547	532	529	522	521	501	499	496	494	467	466
50	599	598	600	600	598	598	584	581	575	574	556	555	552	551	525	525
55	648	649	651	651	650	649	637	635	630	630	614	613	610	610	587	587
60	696	701	701	703	701	701	689	690	684	685	673	672	670	669	651	651
65	747	751	752	753	751	751	742	745	738	742	730	730	730	728	713	713
70	796	800	801	801	799	800	795	796	788	794	783	787	786	786	771	775
75	840	841	845	842	842	841	838	839	834	838	831	836	835	835	822	831
80	872	873	877	874	883	873	872	874	871	873	870	872	875	872	863	870
85	890	890	897	890	913	890	893	893	900	894	900	892	905	892	895	892

The so-called 'tabular' values of $\bar{A}_x$ for English Life Tables (Males) Nos. 4, 5 and 7, not being in fact tabulated anywhere known to the author, have been calculated by him by the formula of § 7 (v) from '1000 deaths'.

APPENDIX I

TIME

What, then, is time—if nobody asks me, I know; but if I try to explain to one who asks me, I do not know. AUGUSTINE.

12. Alexander (1920) was of the opinion that the most characteristic feature of thought of the last few decades was the discovery of time. Anyone who still clings to the belief that the idea of time is something simple may be referred with some confidence to Cleugh (1937) for an excellent and representative collection in summary form of the views of the great thinkers of all eras, from Augustine to Whitehead. Probably few will deny that they have often visualized time as analogous to space of one dimension, and I believe that Sullivan (1933) was right in thinking that the general conception has not been merely that of a line, but essentially of a line that is straight; but the very conception of a straight line in space, which is the analogy being employed, implies one other dimension at least by which its straightness can be determined. In an isolated unidimensional space it would be manifestly impossible to distinguish between straightness and curvature; neither the one nor the other would have any meaning. Those who have been accustomed to visualizing time as a straight line have, it would seem, tacitly made the assumption that time is analogous to travel in a straight line through a two- or three-dimensional continuum.

13. Since Karl Pearson (1892) there have been almost as many kinds of 'time' as there have been writers on the subject; but for the moment we shall concern ourselves only with what we will call 'biological' time. According to a study by Shapley, the normal rate of progression of ants is a function of temperature. For every rise of 10° C. the ants go about twice as fast, with such close agreement to this 'law' that their rate of locomotion may be employed as a thermometer, giving the temperature to within 1° C. On a warm day an ant does more, thinks more, lives more; has more Bergsonian *durée* (Huxley, 1923). Prof. Child, of the University of Chicago, was able immensely to prolong the lives of certain primitive flatworms by retarding their development. It begins to appear that the 'developing' and the 'ageing' processes have some sort of a reciprocal relationship.

14. Hertwig divided a batch of frog eggs into four portions, and kept them at temperatures of 11·5°, 15°, 20° and 24° C. respectively. After three days, the first had completed their primary ground plan, but were still simple spheres; the last were tadpoles ready to hatch; and the other two were intermediate. In this connexion Huxley (1926) raises the problem, which in another form has been propounded by Bergson, of the relation of time to true being:

> What is the true age of these four batches of embryos? They have all been alive the same length of days, hours, and minutes. But each has accomplished a different portion of its essential cycle of being, each has penetrated a different distance along the road which leads to old age and death. Judged by outer standards, they are of the same age; judged by standards relative only to themselves, they are of different ages.

15. Paul Janet, who seems to have been the first, at any rate in modern times, to point out that to the child a year appears longer than it does to an adult, thought this might be explained by the fact that a year represents a much

longer time in proportion to the past lifetime. This view has been strenuously disputed, but it is here suggested that a year appears longer to a child than to an adult because it *is* longer, and to the precise extent of Janet's explanation. This is tantamount to suggesting that biological time is logarithmic, that an instant at age x is twice as long as an instant at age $2x$, since

$$d \log t/dt = 1/t.$$

16. There would have been much more hesitation in suggesting that biological time is logarithmic, if logarithmic time had not already appeared in physics. It is true that it finds no place in the Einstein-de Sitter universe, which avoids it only at the expense of introducing a curvature into space. For the theory of E. A. Milne, whose mathematics are such that when he comes to 'a little straight-forward hyperbolic trigonometry' he leaves out the working, one may rely with confidence upon Martin Johnson (1945), with due regard to the Preface which Milne himself contributed. Milne's theory postulates a logarithmic time, with origin at $-\infty$ instead of at 0, and it immediately attracts notice by virtue of the fact that it provides a stable universe, without the necessity for any cosmical constant, and in the sequel supersedes the Doppler effect as the explanation of the 'shift to the red', thus negativing that expansion of the universe which it has already rendered unnecessary.

LIFE

I am well aware of the danger of analogies and mine may be no better than most.
Per Vaisey, J. [1950] 2 All E.R., 1034.

17. If we fix our attention upon a newly made brick, and an acorn which has just germinated, we shall presently be conscious that as the brick and the sprouting oak tree journey side by side through time, they do so; in essentially different fashion. Outside influences aside, the brick will go steadily downhill to eventual disintegration; it will age, and will eventually decay. But this decay will not be at a uniform rate, measured by sidereal time; the brick will age slowly at first, and then more and more rapidly. To express its existence mathematically it might, indeed, be found helpful to plot its course through time visualized as a continuum of two dimensions.

18. The oak tree shares with the brick this characteristic of an ageing process, at first so small as to be imperceptible, and later increasing with an ever-increasing rapidity; but simultaneously a great deal more is happening to the oak tree than is happening to the brick—it is growing and developing from a comparatively simple organism into one vastly complex, one which in an astonishingly short ration of sidereal time is passing uphill, through an equally astonishingly long chain of 'events', in which it inevitably follows the lead of its forerunners of all the ages, so that, for example, its first two leaves will be so primitive as to be unrecognizable as oak leaves. It is as though it were passing, and at first at a tremendous speed, along a groove* made for it by its succession of ancestors, a groove of life along which any saplings that germinate from its own acorns will as inevitably follow it.

19. Walter Shepherd (1939) suggests that for living creatures such as ourselves to try to appraise life is rather like a man trying to gauge his weight by tugging at his bootlaces. It has only been with considerable temerity that one

* In the sense developed in § 32.

has ventured to suggest that to express life mathematically it may be helpful to plot its course through time visualized as a three-dimensional continuum.

20. In place of the analogy of a lifetime growing in length only it is here suggested that it should be regarded as growing in volume, say xyz, retaining x with its current meaning of 'number of years lived' subject only to measuring it from approximately the moment of conception instead of from the moment of birth. On the one hand we have the enormous 'distance' (say z) travelled through time, from the origin $-\infty$, during early development, with rapid deceleration, and reflected in our conceptual time by which the older we become the less 'length' each succeeding year appears to have. On the other hand, we have at first the small 'distance' (say y) travelled in the direction of deterioration, with acceleration as we become older.

For simplicity I have taken the yz curve as being an equilateral hyperbola, i.e. $y = -1/z$, so that the volume (taken positively) $x_{,}yz = x_{,}$, i.e. it grows at the same rate as $x_{,}$ grows. Then when $z = -\infty$, $y = 0$, and the analogy is of a volume one dimension of which is at first indefinitely large and the other two dimensions are zero, the first of these three dimensions decreasing as the other two increase.

21. To change the analogy, some may find it preferable to think of 'life' as a capacity to move through a three-dimensional time continuum. Then $x_{,}, y, z$ is to be thought of as a specification of the locus of a point whose motion may be resolved into three mutually exclusive motions, as suggested in § 3. We may say, perhaps not too fancifully, that the point $x_{,}$, y, z moves in a curve such that it traces out equal volumes in equal (sidereal) times.

22. If the curve of this movement is constructed in space and projected orthogonally on to each of the $zx_{,}$, $x_{,}y$, yz planes, we have three plane curves of which the second has here been taken to represent the combined 'ageing' process. From § 5 it will be seen that y has been taken as proportionate to $x_{,}^{2}$, so that $x_{,}y = kx_{,}^{3}$; I might have reflected, though in fact I did not, that age may well be as three-dimensional as our weight, and that if we were applying a normal curve to the distribution of the weights of, say, apples we should find it more profitable to use the cube of the diameter, rather than the diameter, as our class interval. The suggestion that $x_{,}y = kx_{,}^{3}$ is not really so startling as it might at first appear; if the 'ageing' process combined with the normal passing of time is, in fact, proportional to the cube of the age, this means no more than that a man 'ages' 33% more between 70 and 80 than he does between 60 and 70, twice as much between 60 and 70 as between 40 and 50; that a child 'ages' nearly three and a half times as much between $x = 1{\cdot}25$ and $x = 2{\cdot}25$ as it does between $x = {\cdot}25$ and $x = 1{\cdot}25$; and that a man $39\frac{1}{4}$ years old 'ages' in twelve months 700 times as much as does a three-month-old child in the same period.

23. It does not seem to be necessary to say more about that 'travel' in the y-direction which it is suggested causes 'ageing' to accelerate in terms of x, and the more so since the author is far from being the first to suggest some such change of the time scale (in particular see Du Nouy, 1936); but the somewhat startling picture of the motion in the z-direction, with its origin at $-\infty$, may excuse the brief biological excursion of Appendix II.

LEMMA I

In Fig. 4, *ABCD* is a rectangular area marked out on the frozen surface of a lake, divided into rectangular strips by parallel lines *a*, *b*, *c*, ... at equal intervals. The arrows at *A* and *B* respectively represent the direction towards the winning-post *C*, of ice-yachts *A* and *B*.

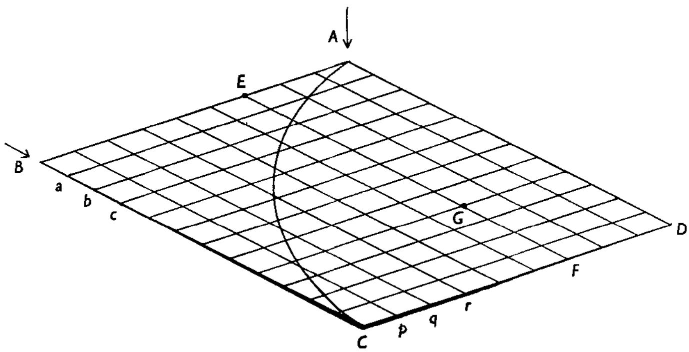

Fig. 4.

It is manifest that an ice-yacht, travelling directly before the wind, cannot move faster than the wind. Ignoring friction, it will, in fact, travel at the precise speed of the wind.

It is a commonplace than an ice-yacht with the wind abaft its beam travels faster than the wind. Indeed, with the wind blowing in the direction of arrow *B*, yacht *B* and yacht *A* will (friction ignored as before) arrive at *C* at the same moment.

If instead of moving in a straight line from *A* to *C*, yacht *A* follows a curved line such as *AC*, the result will be precisely the same. Thus, suppose that at the commencement of the race yacht *A* does not know where the winning-post is, and accordingly steers as nearly as may be towards yacht *B*, thereafter curving its course more and more in the direction of *C* as the position of *C* becomes more and more definite; then, as before, both yachts will successively cross each of the lines *a*, *b*, *c*, ... simultaneously, and will arrive at *C* together.

Let the rectangle be further divided into strips of equal width by the parallel lines *p*, *q*, *r*, ... Suppose, now, that we are on yacht *A* with no chronometer. It is not possible for us to calculate our speed, but we can *assume*, if we wish, that the speed of *B* is uniform, and accordingly *observe* that of the two components of our speed, that of our movement parallel to *BC* is uniform. Suppose, further, that we have means of satisfying ourselves that the intervals between lines *a*, *b*, *c*, ... are constant, and that the intervals between *p*, *q*, *r*, ... are constant, but that we have no means by which we can compare the length of the first of these with the length of the other. Then, because we have assumed that our speed in the direction parallel to *BC* is uniform, we can measure the variation of our speed in the direction parallel to *AB*, and satisfy ourselves that it is decelerating. We can go further, and relate our movement in one direction with movement in the other, by plotting our course amongst the small rectangles, provided we incorporate into our ratios an unknown and, to us, unknowable constant, namely, the ratio of one dimension of each small rectangle to the other dimension. We can now talk about speed south, say, in units travelled south per unit of movement east, or (what is the reciprocal of the first) speed east in units travelled east per unit of movement south.

LEMMA II

Suppose the rectangular field $ABCD$ of Fig. 4 such that if a man moves uniformly along any segment $[EF]$ parallel to BC he will encounter particles at points between E and F the distribution of which can be approximately represented by a normal curve upon EF, symmetrical about G, a point on $[EF]$. Let the ordinate of this curve at any distance l from E be $n(l)$.

If now he moves instead along the curved line AC in such a way that the orthogonal projection on AD of his position moves uniformly from A to D, then the number of particles he will encounter will vary from point to point in proportion to the tangent of the angle which the curve makes at each point with AD, say $\tan\theta$. Thus the particles he will 'collect' will be represented by the solid block normal curve of Fig. 2, in which the ordinates of elevation are $n(l)$ and the ordinates of the plan are $\tan\theta = df(l)/dl$, where $f(l)$ is the function represented by the curve AC.

24. To apply Lemmata I and II by analogy to the measurement of lifetime, for the 'man' of Lemma II substitute George King's '100,000 children all born at the same moment', for 'encounter particles' read 'encounter lethal particles each a one-man dose', for '$n(l)$' substitute '$n(x,y)$' and for 'the curve AC' substititute 'the curve $zx_{,}$'. Then by taking $z = -1/y$ and $y = kx_{,}^{2}$, so that $dz/dx_{,} = (2/k)(1/x_{,}^{3})$, the distribution of deaths over (sidereal) time is represented by the solid block of Fig. 2 and the accompanying formula of § 5.

If instead of the transformation made in § 6 we change the variable of integration of § 5 from $x_{,}^{3}$ to $\log x_{,}$, we get rid of the factor $1/x_{,}^{3}$ in the integrand and arrive at a solid of constant thickness, which may just as well be represented by the plane curve of Fig. 5.

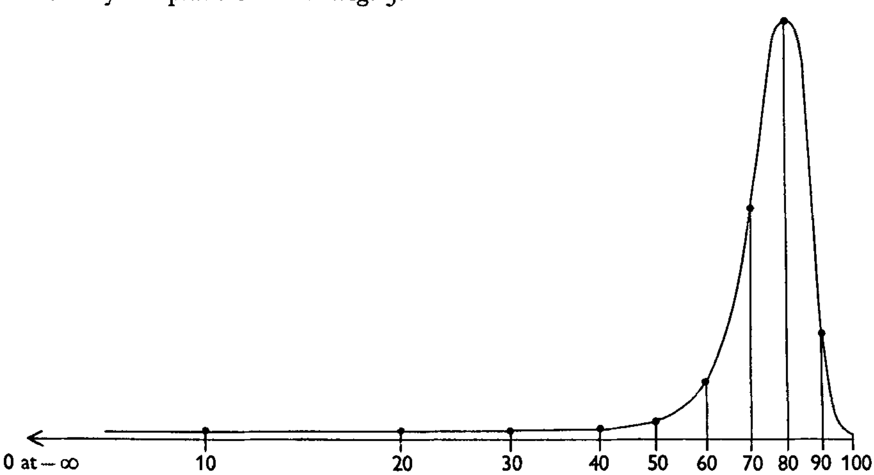

Fig. 5.

A 'CURVE OF LENGTHS'

25. I have already suggested it is not necessarily an objection to the normal curve as an approximation that it extends to infinite values of the argument in both directions; but as it does not account for the large quantity of nuts, apples, pea-pods, *et alia*, which perish almost immediately after they have 'set' (not to mention that 'the cod fish lays a million eggs'), I am disposed to doubt the validity of its direct application to the measurement of living things.

26. Suppose a statistical research worker completely strips a bean row, and groups by length all those beans which exceed m inches, using class boundaries which are in G.P. instead of the customary A.P. With the intention that the rectangles of the histogram his assistant is to construct shall be on bases respectively proportionate to the class intervals, he calculates 'weighted' heights of the rectangles by dividing the class frequencies, the physical 'counts' of the classes, by amounts proportionate to the respective class intervals; that is to say, he multiplies the 'counts' respectively by c/m, c/ma, c/ma^2, ...

Now suppose that by mistake his assistant uses, instead of the 'weighted' heights, heights proportionate to the physical 'counts', and finds that the resulting histogram, which of course is systematically erroneous, can be very closely represented by a normal curve; if these suppositions were repeatedly realized in practice the curve of distribution of bean lengths would be such that its ordinate at l is proportionate to $n(l)/l$, writing $n(l)$ to represent the ordinate of a normal curve at l.

27. The shape of a normal curve is determined by two parameters, as also, therefore, is the 'curve of lengths', to coin a phrase for the curve just imagined. When a histogram for the range from m to ω appears to be capable of being represented by almost the complete area of a normal curve, the ordinate at m is nearly, and can legitimately be assumed to be, zero. If $n(m) \doteqdot 0$, $c.n(m)/m$ is not significantly larger, while over the range where $n(l)$ is large, the value of $1/l$ does not vary greatly. If the observed span is, say, from 4 in. to 6 in., with the mode at 5 in., the difference between 99% of the area of a normal curve over this span and the corresponding curve of lengths $5n(l)/l$ would hardly be detected were the curves drawn as separate diagrams instead of being superimposed, and even the difference between 90% of the area of a normal curve over this span and the corresponding curve of lengths is small.

It is believed that investigators have often found, on fitting a normal curve to a physical count of animal or vegetable dimensions, that there has been a tendency for that curve somewhat to understate the experience for values of the variate less than that at which the mode occurs; and I have previously suggested that when this negative skewness is not observable the explanation may lie in the method of selection; instead of using a complete stripping of the pea plants, for example, the pods measured are those someone else has thought worth the trouble of picking, thereby reducing the 'count' of the shortest class, and negativing the small excess required to support the hypothesis.

But while the curve of lengths does not differ materially from a normal curve over most of its span, it 'kicks up' as the variate approaches zero, as does the curve of deaths.

28. It will be seen that the curve of lengths can be represented equally well by a solid such as that of Fig. 2 (ignoring the cross-sections), but with a varying depth $1/l$. It will further be seen that the curve of lengths can be repre-

sented in the plane in the form shown in Fig. 5 by erecting normal curve ordinates for values of the variate in A.P. at distances proportionate to the difference between the logarithms of the variates, since this curve equally transforms to such a solid as that of Fig. 2 but with a varying depth $1/l$, as can be seen by a simple change of the variable of integration:

$$\int \{n(l)/l\}\, dl = \int n(l)\, d(\log l).$$

APPENDIX II

29. It is difficult to grasp just how stupendous the 'speed' of growth and development is initially, and how rapidly it diminishes. In connexion with the material of the next paragraph which is taken from Huxley (1926) he has this pregnant phrase: 'Thus the rate of change during development on the whole becomes slower and slower.' If we measure growth, not by total bulk but, as we logically should, by the percentage of the previous total added in a given time, Minot's estimate that under 2 % of the potentiality of growth resident in the human ovum is still present at birth seems to Huxley to be, if anything, above the mark.

30. The process by which every living thing 'climbs up its own genealogical tree' is well illustrated by the life history of the frog. The bewildering complication of processes by which the single fertilized egg-cell generates the rudiments of all the future organs of the body is the affair of a day. In a week the embryo becomes enough of a self-sufficing organism to hatch as a larva. In a month it has become a complete tadpole, which is in effect a fish, breathing by gills, swimming by means of its tail-fins, with special sense organs for perceiving long-frequency vibrations in water which, like any true fish, it carries along its flank. In three or four months the sprouting of limbs from its sides, the alteration of body shape and colour, the disappearance of tail and gills, and the remodelling of lengthy coils of intestine into a simple loop, adapted to a flesh instead of a vegetable diet, have metamorphosed it into a frog. Thereafter there will be little change of function, and scarcely any change of form. Huxley (1926) has said:

> The problem of development is to understand how it is possible that this simple motionless sphere can within a few days give rise to a tadpole, a creature capable of swimming, seeing, hearing, smelling, feeling, feeding, growing....And, for all its littleness and insignificance, possessing an organization of no mean complexity, with heart, muscles, brain, sense organs, stomach, digestive glands, kidneys, ductless secretions, protective skin and skeleton.

31. One can more readily understand how the development of organic chemistry might have held out possibility of a belief that a purely chemical explanation of life would be forthcoming, if it had been possible to continue to hold the view of a hundred and fifty years ago that every acorn contains a miniature oak tree folded up inside; but modern descriptive embryology shows that development is truly epigenetic, and no mere unfolding of a preformed miniature. We now know that the acorn contains nothing but a bud and a store of food. As Shepherd (1939) has said, it is not a question of the 'ingredients' of the oak being in the acorn. Unless some elusive biotic force is postulated the story of life would appear to be one in which the impossible continually happens. What forces are at work, besides those concerned with

the absorption of the food by the bud, *disposing* the young plant to become an oak tree, true to pattern?—and, more, itself in the fulness of time to produce more acorns?

32. The phrase 'ontogeny repeats phylogeny' seems to have been used originally in the sense that before becoming an infant the embryo passes through a succession of adult phases. Thus Garstang (1929) remarked that to some zoologists

> as the chain of adult ancestors is drawn out, at each new evolutionary advance the former adult is succeeded by a new one, and slips back into the ontogeny as a developmental stage

and he gave three reasons for not accepting this theory. It is not surprising that de Beer (1930, 1940) should write of 'the outworn theory of recapitulation'. Writing in 1927, a biologist was able to say:

> Sixty million years ago our ancestors were mammals not unlike lemurs, 300 million years ago amphibians somewhat resembling newts or mud-puppies, and 500 million years ago very primitive fish.

The reproductive cell of an organism is derived solely from the reproductive cell of its parent (Sullivan, 1933). I do not believe that man is descended from (adult) mammals 'not unlike lemurs', but that man develops from an embryo which descended from the embryo of such an animal, which in turn descended, not from (adult) mud-puppies, but from those amphibians' embryos. As Garstang (1929) pointed out, there is evidence to the contrary that new steps in evolution are first manifested in adult life. It seems that the phylogeny which is repeated by ontogeny is the phylogeny of the embryo; that what we should picture is not a following in atavistic footsteps, but a 'curve of pursuit', a curve to which there are at every point tangential blind alleys leading to completed more primitive adulthood. For the rest, it is noteworthy that Dawes (1950) has said:

> There was not long to wait...for proof that an embryo which proceeds from a simple to a more complex type of organization does not necessarily follow in the footsteps of its ancestors. First came a realization of the fact that ontogeny, *being of very limited duration*, could not possibly provide more than the broadest indication of racial history.

The italics are my own.

33. The story unfolded in Keith's *Human Embryology* (1948) has been described as 'more astounding than any detective story ever written'. This story commences at the moment when the unicellular ovum first performs that supreme miracle of life of dividing itself into two cells, a trick which the primitive protista may have taken several million years to learn, and which the modern protista have not learned yet. A further stage is the refission of the cells, and their differentiation into three layers, a rough ground-plan which the human shares with every form of animal life above the level of the jelly-fish. Then comes the development of the organs, and the successive fish, amphibious, and reptile stages of phylogenetic rehearsal, until the mammalian adult stage is reached.

To-day our blood plasma has much the same composition as sea water diluted to about what was the composition of sea water at that remote period when our marine ancestors first began to develop impermeable gills. Prior to that time every marine organism was permeable to sea water, as marine invertebrates are to this day, whose plasma differs from our own in being nearly of the same composition as is sea water now (Haldane, 1927).

34. After mitotic division has taken place for several days, the human embryo assumes a plate-like form, with what appear to be fine lines from side to side. These are the 'segments' such as are visible on every earthworm, and which persist in our spinal column and ribs. At about the thirtieth day the embryo looks very like any other mammalian embryo of the same size. Instead of limbs it has tiny paddle-like organs not unlike the flippers of a turtle, it has a freely projecting tail, and a pair of mammary ridges extending up and down its front. The neck band has four gill-clefts, and the embryonic neck circulatory system at this time also resembles that of a fish. The human foetus does not need gills; the clefts disappear (Dorsey, 1927, summarizing an earlier edition of Keith, 1948). It is even possible to distinguish in the human heart successively the fish-heart with single ventricle, auricle and venous sinus; there follows the stage of the amphibian heart in which the single ventricle has been divided into two sections; later a median wall is formed representing the stage which is permanently present in the reptiles. Finally, by a complete separation of both ventricles and other modifications, the type of the mammalian heart is reached. At a later stage the human embryo, like that of the anthropoid ape, possesses thirteen pairs of ribs; with the ape they persist, but man has but twelve. The *os centrale* of the *carpus* which is present in reptiles and amphibians is absent in man; Rosenberger discovered that a rudimentary *os centrale* exists at an early stage in the human embryo (Thesing, 1911, *Lectures on Biology*).

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ABSTRACT OF THE DISCUSSION

Mr William Phillips, in introducing his paper, said that his was by no means the first suggestion of a variation of the time scale for the purpose in view. He was referring not only to Perks's classic paper of 1931 (*J.I.A.* **63,** 12), the spirited paper of Rich in 1939 (*J.I.A.* **70,** 314) and Barnett's interesting contribution to the discussion on Ogborn's paper of 1953 (*J.I.A.* **79,** 170) about which no doubt more would be heard. As long ago as 1870 Opperman had found that for mortality prior to age 20 'more progress could be made by taking the square root of time, instead of time itself, as the independent variable'. The idea of commencing the measurement of human mortality from 9 months before birth appeared to have originated in Karl Pearson's lecture at Leeds in 1895, so that it was by no means a novelty.

Obviously the conception of three-dimensional time might lead to at least as many 'theories of mortality' as had the conception of time of one dimension. The formula with which the paper commenced was not put forward as a necessary deduction from the conception of time of three dimensions; all that he claimed was that that formula could not have been arrived at by him from any other conception.

Mr J. Hamilton-Jones, in opening the discussion, recalled that the author's previous paper on the curve of deaths (*J.I.A.* **66,** 17) had included an extensive analysis of the general properties of the curve, but his numerical examples had been based on values of l_x given in standard tables. In the paper to be discussed he attempted to introduce an intermediate stage—the fitting of the curve of deaths by a mathematical curve—followed by a development of the numerical values of assurance functions. The matters included in the paper, however, were not entirely, or even mainly, concerned with curve-fitting, because the author introduced philosophical ideas of a stimulating although perhaps controversial nature as a background to the general mathematical curve.

The author introduced his philosophical ideas in §§2, 3 and 4 and in the Appendices, while the curve-fitting occupied the rest of the paper. That arrangement made the paper somewhat difficult to follow, and he himself had found it essential, on first reading it, to leave §§ 5–11 until the end.

In §§ 3 and 4 it was suggested that the progress of living organisms should be measured not according to age alone but according to age, amount of deterioration experienced since conception, and amount of development, so that a three-dimensional space was used to map the course of a lifetime, the progress of any group of individuals supposed to undergo a more or less identical development being represented by a twisting line in space.

It was, of course, extremely difficult for the ordinary person to think abstractly in three dimensions. All the visual information which reached a human being was represented by a two-dimensional picture on the retina, and the third dimension was supplied partly by experience and partly by a conscious use of the imagination. They were therefore led to question whether the three-dimensional pattern was essential. It was difficult to believe that development and deterioration could usefully be separated, since no criterion could be introduced which distinguished them except some arbitrary idea such as the author's scheme in § 20, which assigned to development a measure which was the reciprocal of deterioration but was negative. Moreover, the deterioration of a living organism was compared in § 18 with that of inert matter, but it was surely necessary to take into account that an inert object such as a brick was composed of particles which remained *in situ* so long as the brick existed as a brick. An animal, on the other hand, was composed at any time of living cells all of which were destined to die or to be removed within a comparatively short fraction of the animal's lifetime. What persisted was not the physical matter but an elusive quality which might be described as the capacity for organizing cells. Despite that criticism, it was clear that the author had found a helpful clue to biological ageing when he focused attention on the cube of the age as a measure, though it seemed questionable whether anything was gained by adding 9 months to the ordinary age.

One of the results of the author's three-dimensional scheme, not commented on in the paper, was that development obeyed the inverse square law, which was attractive because of its wide application to other natural phenomena. For all that, they were in the realms of conjecture. The only possible evidence for saying that a year appeared longer to a child than it did to an adult was a statement by someone who had experience of both childhood and adulthood and who was comparing his memory of recent events with his memory of distant events. His standards were arbitrary, personal ones; in the existing state of knowledge it was a long step to pass from such ideas to a mathematical expression for biological age upon which all could agree. On the fascinating excursions into embryology in Appendix II few of those present would be qualified to speak; but, having studied the subject in very broad outline some years ago, he ventured to congratulate the author on having assembled in so short a space the salient features of the subject, and in particular on his interpretation in § 32 of the phrase 'ontogeny repeats phylogeny', or in other words that the development of the individual repeats the history of its race.

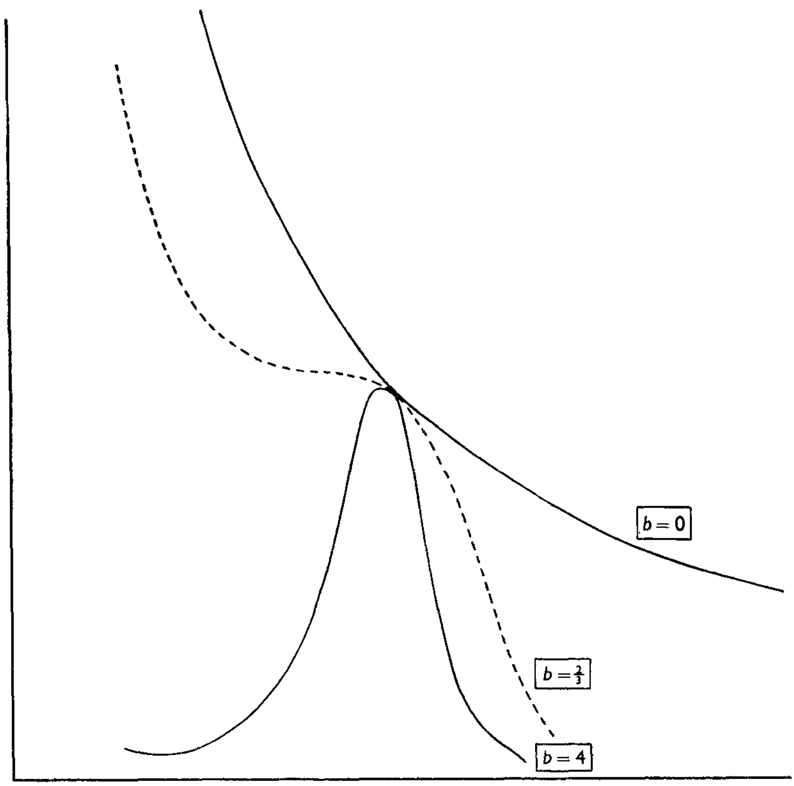
Lemmata I and II seemed to him to be the crux of the author's approach. It would not have escaped notice that, of the three directions in which life was considered to be moving, one, as he had already suggested, could be suppressed. In a somewhat similar way to that of the author, a normal curve related in a subtle way to the curve of deaths would be substituted in the third dimension. The three-dimensional pattern of biological development and decay, instead of being thought of as a twisting line in space, then became the solid block normal curve of Fig. 2, which was so much kinder to the imagination. The deaths among a group of mortals were represented by the successive slices of the solid cut out by the sketched rectangles in Fig. 2.

It would be realized that there were various ways of determining the volume of any part of the block by integration. Those ways could be represented by changing the variable in the integral of § 5, but after a few experiments the speaker had been unable to arrive at an alternative form which lent itself to mathematical analysis. It would, of course, be an immense advantage if the synthetic ϕ_x were an integrable function, but the integral of the author's function, if he might so christen the expression for ϕ_x in § 7, was divergent. The area from 0 to ∞ was infinite, and so far as he knew mathematicians were unable to deal with finite portions of that curve. He had not himself been able to break down the integral into component parts the values of which had been tabulated.

The fitting of the curve of deaths had been discussed by other writers, notably by Beard in his paper on the use of the incomplete gamma function (*Proc. Cent. Assembly Inst. Actuaries*, **2**, 89), where he dealt amply with the advantages to be gained by using a mathematical representation of ϕ_x, the integral of which could be expressed in terms of tabulated functions.

He, the speaker, had therefore been led to an analysis of that curve itself, and there he had first attempted to assign some meaning to the parameter c. Fortunately, that presented little difficulty. If $j_{\text{max.}}$ and $j_{\text{min.}}$ denoted the real values of j, other than 0 and ∞, for which ϕ was a maximum and minimum respectively, then $j^3_{\text{max.}} + j^3_{\text{min.}} = 1$. If the ages $(x_{\prime})_{\text{max.}}$ and $(x_{\prime})_{\text{min.}}$ were known (the dash denoting the addition of 9 months to the ordinary age at which it was desired to maximize and minimize the synthetic ϕ_x), it was possible at once to determine c. In practice, that meant that c was a year or two beyond the age at which ϕ_x was a maximum.

The parameter b clearly represented the dispersion of the curve. It might be derived from $j^3_{\text{max.}}\, j^3_{\text{min.}} = 1/(6b)$.



Perhaps the most interesting property of the curve was that the gradient at the point $j = 1$ was independent of b. That meant that once c had been fixed, all the possible curves that could be drawn would touch each other at a fixed point. The diagram above showed the position. The unit of measurement in both the horizontal and the vertical directions was the quantity corresponding to ϕ_c in § 7 of the paper. The family of curves (of which three were shown on the diagram) was derived by assigning different values to b. As b increased, the ordinates of the curve corresponding to a given value of j diminished, so that each curve lay entirely below the curve derived from a smaller value of b. The rectangular hyperbola (the continuous curve at the top) represented the limiting curve $b=0$. The curve $b=\frac{3}{8}$ represented the limiting curve which could possibly be of interest to them, because it was the first curve for which the maxima and minima had non-trivial real values, and their coincidence was at the point $j^3=\frac{1}{2}$. The lowest curve represented $b=4$, and that was the curve which the author had used to fit the A1924–29 Table.

The remarkable fact about that family of curves was how vigorously the tail of the curve to the left of the common meeting point swung round as the value of b was changed. In other words, the fitting of the younger ages would be the problem in finding a suitable synthetic curve. It would be noticed from Table 5 of the paper that there was a considerable distortion at the younger ages for smaller values of c.

In a table such as the Oriental 1925–35 experience, which was a modern table, the maximum ϕ_x occurred at about age 67, and it was necessary therefore to take a lower value of c than any of the author's values. If $c=70$ was tried to obtain a reasonable ratio between ϕ_{84} and ϕ_{70}, b was 1·4, and with that value of b, ϕ_{35}/ϕ_{70} was ·68 against a true value of ·17. In other words, there was a stage at which the tail presented insuperable difficulties.

It had therefore occurred to him to experiment with the introduction of more parameters. The search for a modification which would allow fitting by the method of moments seemed promising, and the obvious curve to try was one of the early English Life Tables. He had therefore tried $\phi_j=\frac{1}{p(j)}e^{-b(j^3-1)^2}$, where $p(j)$ was a polynomial in j of the third degree. That operation was heavy, but he had taken his work to the stage where he felt safe in saying that the suggested curve did not alter the pattern very much, because the coefficients of j^3 and j^2 in the polynomial $p(j)$ turned out to be small.

After that and other trials, which he did not propose to mention, he felt that the author had not passed over any obvious method of extending the usefulness of his curve.

To sum up, the author's paper represented a thoughful attempt to fit the curve of deaths, which had met with remarkable success in the case of the A1924–29 Table. The function fitted was not an observable quantity, so that before applying the author's method to fit a curve of deaths it would be necessary to make a preliminary graduation. Owing to the form of the author's curve, there was no obvious way of testing fidelity to data, but the outstanding feature of the paper was that it directed their thoughts to the problem in its most general form. They would never get to the truth, but they could approach it by stages. The author had taken them one of those stages, and the speaker had thoroughly enjoyed the journey. A fitting symbol of the journey was the author's ice yacht, which could operate only in an invigorating climate where it had plenty of room.

Mr M. E. Ogborn wanted to pass over the philosophy behind the paper and to deal more with the mathematics. Before passing on from the philosophy, however, he asked the author on what principle would he say that some well-defined feature of the curve of deaths could be accepted and another feature rejected. In the curve of deaths in Fig. 3, the minimum in early life was shown somewhere about 25–30, but in fact the speaker believed that the minimum was about the age of 10. That was what had occasioned him so much trouble in preparing his own paper, so that he was painfully conscious of the fact that the minimum was at that age. It was possible, he supposed, to say that that feature should be swept away.

If the formula in § 5 of the paper were regarded as a frequency curve in a variable that might be called z $(=x^3)$, it would be found that the curve had some similarity to a frequency curve, and the way to discover its properties, as with frequency curves generally, was to take the differential coefficient of the ordinate. It would be remembered that Pearson, in his system of curves, took the ratio of a straight line to a quadratic, i.e. $\frac{1}{y}\frac{dy}{dx}=\frac{a+x}{b_0+b_1x+b_2x^2}$. If the ordinate were called ϕ_z, and the equivalent of $\frac{1}{y}\frac{dy}{dx}$ were formed, the expression reduced to a quadratic divided by z, namely, $\frac{1}{\phi_z}\frac{d\phi_z}{dz}=-\frac{1-2ac^3z+2az^2}{z}$, that was to say a ratio of polynomials such as he had been investigating in the previous year, and it linked up with Beard's work on the Type III frequency curve, which was derived from a ratio of two straight lines. It had to be remembered, of course, that there was the change of variable, and that Beard and the author had both been working on the curve of deaths, whereas the speaker had been working on the l_x curve.

In any frequency curve, the range of the curve was fixed by the denominator and, since that was simply z, or x^3, the range was from $z=0$ to ∞, as the author remarked. The roots of the numerator fixed the maximum and minimum of the curve. The quadratic in the numerator was first negative, giving a decreasing curve; after the first root it became positive, and after the second root it was negative again. Taking the roots of the numerator and calling the ages at which the maximum and minimum occurred m and n, then, as the opener had said, $m^3+n^3=c^3$. In fact, the larger of the two varied very little from c, and the maximum was somewhere near the value of c. The difference between the two cubes was given by $m^3-n^3=c^3\sqrt{(1-2/b)}$. As b increased $1-2/b$ increased, so that the larger the value of b the larger the disparity; b was essentially a measure of the disparity in the ages of the maximum and minimum, not actually of ϕ_x, but of the altered curve in ϕ_z. If b was less than 2* there was a negative sign in the square root, and there was no real root; i.e. there were no maximum and minimum. If b was greater than 2 there were a maximum and minimum; their disparity depended on the size of b, and it was noticeable that quite a small value of b in excess of 2 could give a reasonable curve.

He did not claim that the author's curve was one of the curves which he had been discussing. It was different, because in his paper he had been dealing with l_x, and he had a numerator which was a quadratic without real roots. The purpose of that had been to give a curve which continuously decreased or increased. The author had shown that the same technique could apply to the curve of deaths, though in that case there was need for a quadratic which had two real roots, in order to give the maximum and minimum. The speaker had felt convinced, when writing his own paper, that that would not work, and that was one reason why he did not adopt that approach. The author had shown another of his innate prejudices to be wrong.

The author said nothing about the fitting of the curve to data, and in a sense, from his approach, it would be wrong to try to fit it; but the speaker thought

* Mr Hamilton-Jones finds the critical value of b to be $\frac{8}{3}$ by working on the variable j or x_t/c instead of z or x^3. Eds.

that it might be useful to consider how the curve should be fitted. The moments of ϕ_z, say m_n, were linked by the recurrence equation

$$m_{n+2} - c^3 m_{n+1} - \frac{n}{2a} m_n = 0 \quad \text{for} \quad n > 0.$$

In the special case where $n = 0$, $m_2 - c^3 m_1 = \frac{1}{2a} e^{-ac^6}$. Although that sounded an easy fit, he was sorry to say that in practice it did not seem so easy, because the variable was $z = x_{,}^3$, and the first and second moments of z involved the fifth and the eighth moments of $x_{,}$, so that the method might not be practicable.

His only other comment related to the choice of an origin of $-\cdot 75$; the opener had also referred to that. To his mind it could not be claimed to have any real significance. Those who had worked on frequency curves would readily appreciate the need for the curve to start at some time before birth, in order to give a real value at birth; but it seemed preferable to leave the origin to be fixed by the data. Perhaps that was because he approached the matter from the point of view of fitting the statistics, whereas the author approached the problem more as a philosophic one of what was the proper curve to choose.

Mr A. W. Joseph remarked that the author's new curve brought for the first time the principles of the Welfare State into mortality. In contrast to Gompertz or Makeham, it was a curve that extended from the cradle, or even earlier, to the grave. The normal curve resembled the curve of deaths from about age 60 onwards, but at the early ages the two curves separated. It had been a bright idea to make the normal curve kick up at the early ages by dividing it by the age measured from conception. Even then the curve did not represent the curve of deaths satisfactorily, but a change of scale from $x_{,}$ to $x_{,}^3$ made a great improvement. It might even be possible to obtain a closer fit to the curve of deaths by dividing the normal curve by a power of $x_{,}$ other than the first, or by changing the scale from $x_{,}$ to a power of $x_{,}$ other than the third.

The above development of the author's formula was much simpler to understand than Appendices I and II, but, if he criticized those Appendices, it should be borne in mind that in fact he had not thought along the lines which he had indicated and had not discovered the formula, whereas the author did think along the lines of his Appendices and had discovered the formula.

The author had felt impelled to explain the two features of the formula to which the speaker had referred, namely, why he divided by $x_{,}$ and why he altered the scale from $x_{,}$ to $x_{,}^3$. The first explanation he found intelligible and helpful; the suggestion that so far as life processes were concerned time was logarithmic fitted in well with the biological data advanced by the author—that in the early stages of life the embryo appeared to run through past history at an enormous but ever-decreasing pace, and that with age the life processes slowed up, so that the older the organism the faster time appeared to pass. In order to justify the alteration of scale from $x_{,}$ to $x_{,}^3$, the author considered that it was helpful to regard life as moving through a three-dimensional time continuum. That was an idea which the speaker found difficult to grasp; it was not easy to see its value. The author used three co-ordinates, x, y and z; y and z were tightly linked together by the relation $yz = 1$. The author did not at first directly link his z co-ordinate, which represented development starting at time $-\infty$, with his x co-ordinate, which represented sidereal time starting at an arbitrary finite origin, but Appendix II suggested that the two motions took place simultaneously

—if that had any meaning whatever in that context—and that the relationship between them was in the nature of $z=\log x_{,}$, which made z equal to $-\infty$ when $x_{,}=0$. The so-called three-dimensional motion really depended on one variable, x.

He failed to see the essential difference in marching in time along a straight line or a curved line. A one-to-one correspondence could be constructed between each point of the curved line and each point of the straight line, and so time could be regarded as moving along the straight line. In other words, he saw no real use being made of the additional dimensions, and they might just as well be omitted. It was as easy to say 'Let the ageing process vary as $x_{,}^{3}$' as to say 'Let the ageing process vary as $x_{,}y$ and let y vary as $x_{,}^{2}$'. If y varied as $x_{,}^{2}$, then z varied as $1/x_{,}^{2}$, which knocked on the head the impression, gained from the study of Appendix II, that $z=-\infty$ when $x_{,}=0$. He found it all very confusing.

If at each point of the line of time being traversed a bundle of lines was imagined to emerge, each one of those being a possible path, and likewise at each point of each possible path fresh possible lines emerged, and so on, that would be a conception of time which needed more than one dimension. He thought that that was J. W. Dunne's conception. He was not championing it, but he mentioned it to contrast a conception of time needing more than one dimension with the author's conception, which in his opinion gained nothing by introducing more than one dimension. He was a great believer in knocking down scaffolding, however entertaining and thought-provoking.

The mathematics of the new curve had been little explored in the paper, and there was a field there for further research. It would be a pity if attention were diverted from the mathematics of the curve itself because of impatience with the manner in which the curve had been introduced. He was glad to find from the discussion that his fears were unwarranted.

The area of the 'curve of lengths' which the author discussed in §§ 25–28, if taken back to $l=0$, was infinite. If he had divided $n(l)$ by $l^{\frac{1}{2}}$ or $l^{\frac{3}{4}}$, or any power of l less than unity, the area would have been finite, but the integral of a non-zero quantity divided by l introduced log 0, i.e. $-\infty$.

Mr H. A. R. Barnett noted that the author, instead of restricting himself entirely to death, considered life, thereby giving a new approach to an old problem. He spoke of the facts of life rather than of death, and for that reason he had operated on the function ϕ. As previous speakers had pointed out, his formula did not appear readily capable of being transformed into a function of, say, μ. The trouble, as the speaker saw it, was that $\log \phi_x = \log \mu_x - \int \mu_x \, dx +$ a constant, and there were not many formulae which could be found for μ_x which were readily both integrable and whatever the right word might be for finding an expression for its logarithm.

He had carried out somewhat similar research, to which he had referred in the discussion on Ogborn's paper. He liked the author's conception of an ideal to which the curve of deaths was striving. He himself had tried to devise an ideal to which the force of mortality might be striving, and in certain respects his conclusions were similar to, if not identical with, those of the author. His conclusion, so far as he had gone—and he was thinking as he spoke of rate of mortality or force of mortality, not of the curve of deaths—was that the ideal might consist of a harmonic curve decreasing throughout life—the author might tell him that he was there 'monkeying' with the age scale, but he did not think that

that mattered—plus a geometrical or exponential curve increasing throughout life. There was a similarity between his conclusions and those of the author, in the first place because there were two forces (one increasing and one decreasing) operating simultaneously, and secondly because he found himself in line with the author and others in finding that the harmonic curve started approximately $\frac{3}{4}$ year before birth. He had in mind a remark made by Perks in the discussion on Ogborn's paper the previous year, about certain things which happened just after birth; that remark was a red herring, because to his mind what happened at birth or just after was merely a temporary disturbance from the ideal.

He did not think that it was necessarily a praise of the author's formula to demonstrate that it reproduced very closely the tabular values shown in the A 1924–29 Tables. The author had rather cunningly avoided possible criticism by comparing tabular values; it was essential in the fitting of his formula to have a preliminary graduation. A further small point was that he was only comparing the male curve of deaths; had he attempted to investigate the female curve of deaths the result might have been different.

Mr F. M. Redington was afraid that on the subject of laws of mortality he found himself among the sceptics, not temperamentally, but reluctantly and in the face of facts. He did not find his own understanding of the subject advanced much further without pursuing the questions of heredity and environment and their interrelation, on the lines suggested by R. D. Clarke (*Proc. Cent. Assembly Inst. Actuaries*, **2**, 12), and when he did that he obtained a different picture. It was that picture which made him sceptical about laws of mortality. Taking heredity first, he thought it was quite certain that genetic constitution must affect longevity, just as it affected height, colour, intelligence, musical capacity and so on. The oak lived longer than the pine, and the elephant lived longer than the rabbit. Men lived longer than mice, and probably Tom was predisposed to live longer than Harry. All living things had a genetic constitution which predisposed them towards some ω, and that ω could not fail to be different for different people, if only slightly. Within a given species that ω must be closely grouped but dispersed to some extent in the inevitable frequency curve which was found with everything —height, weight and so on. That must be so for man, and the familiar hump-backed frequency curve of deaths was primarily a consequence of the varied genetic constitution of the lives examined.

If for a moment a hypothetical population was considered consisting entirely of persons as similar to each other as were identical twins, he was not sure there would be the familiar hump-backed curve of deaths; it would be conceivable that their curve of deaths might rise to a maximum at their common ω. The mere examination of that line of approach threw doubt on any law of mortality that was based on the hump-backed frequency curve of deaths. He thought that that hump-backed curve of deaths was fundamentally statistical and not a law of mortality.

So much for heredity. He would add in parenthesis that he was making a sharp distinction between heredity and environment, a distinction which might not exist and which, even if it did exist, was perhaps impracticable, but he was using it to emphasize the main line of his argument.

Turning to environmental factors, he thought that the case for a law of mortality was even less admissible. When mortality was examined it was customary to omit the mortality from war, influenza epidemics and so on, because that was regarded as exceptional. But in what sense was it exceptional?

He suggested that mortality from those causes was exceptional only because it was exceptionally exceptional, and that all mortality, apart from the real decline to complete old age, was exceptional in some degree. There were environmental bumps in the mortality curve owing to war, influenza epidemics, motor cycles, cancer and so on, and there were hollows in the curve owing to penicillin, the sulphonamides, etc.

The conclusion could only be, therefore, that environmental mortality was a clumsy package of very varied articles. He believed that mortality up to age 60 at least was just such a clumsy package of miscellaneous articles, and that the search for any law other than a rough graduation was vain.

However, he did not believe that investigation was fruitless, and there was one fascinating aspect of the subject to which the author had drawn attention. The author used two main parameters (apart from the formula, which was a sort of parameter itself), and, as he examined different experiences, found that those two parameters did not vary independently. The fundamental reason for that, the speaker was convinced, was that all the experiences which were examined were bound to have almost the same underlying genetic constitution. Genetic changes were slow and took thousands, perhaps tens of thousands, of years to be appreciable; in any event they were quite negligible over the period of 100 years with which they were then concerned, compared with the vastly bigger environmental changes in mortality which were quite measurable every year, as was well known.

He could picture the many British mortality experiences of the past 100 years as all being examples of a common optimum hereditary curve of deaths located in very old age, between 80 and 110, distorted and accelerated in different degrees and places by environmental factors which were all pushing the curve leftwards. He felt, therefore, that the author's efforts were strictly graduations rather than any real descriptions of 'life forces', but they were powerful and interesting graduations, and he had to thank the author for making him turn out some thoughts which had been put away for too long.

Mr R. E. Beard said that a good deal of activity had been devoted during the past few years to applying a formula in which the parameters were made to vary with time to describe some function of mortality. The author had presented another variation on that theme, and in addition to a liberal ration of arithmetic he had drawn on his extensive reading for ideas to support his formula. It was possible to criticize, and various speakers had criticized, many points in the paper, but to do that tended to detract from the real value which lay behind it. Personally, he did not think that the general reasoning of the Appendix led to a sound synthesis of the formula, nor did he regard the arithmetic as supporting the formula which the author had put forward; nevertheless, he regarded the paper as extremely valuable and suggestive of ideas for further research.

A point which emerged clearly from the arithmetical results of applying the author's formula, was that the curve had only the approximate shape of the curve of deaths at the younger ages. On careful examination it was clear that the minimum value was quite different from any examples which occurred in practice. Mr Ogborn had already commented on that point, and the speaker also had found the problem of the minimum at the younger ages a difficult one when trying to find an expression to cover the entire age range of the mortality table. He thought that to use assurance factors as comparative indices was dangerous, because they were specially weighted functions. From a scientific point of view they covered up essential variations, although from a practical point of view they might well be adequate as functions to be used in reaching an opinion about the suitability of a mortality table for a specific purpose. Finally, in connexion with the one-parameter formula it might well be that the limited success was due to the various parameters being highly correlated with time.

It was of considerable interest to make a comparison with other attempts to express mortality over the past 100 years by a single formula. There was the author's attempt, there were Starke's recent calculations (*J.I.A.* 78, 171), and there were the calculations by the incomplete gamma function which the speaker had published (*J.I.A.* 78, 341). In the author's formula, c roughly measured the position of the mode of the curve of deaths, and a convenient characteristic to consider was the movement of the mode of the curve of deaths over the period. The figures in Table 5 of the paper suggested that the mode had advanced over the past 60 years by about 3 years of age, or about ·05 year of age for each year of time. That was the figure which he had found when using the incomplete gamma function for annuitants' mortality over the same period, and the same value was implicit in Starke's graduations. Those three separate calculations showed that over the past century the mode of the curve of deaths had shifted at about the rate of a year of age for every 20 years of time, and whatever the opponents of laws of mortality might say, that was a feature of the period. He thought that that simple comparison brought out one of the advantages of the curve of deaths for describing mortality experiences.

In statistical practice four parameters were usually needed to give a good description of any distribution, one each for position, spread, skewness and kurtosis. In bi-modal distributions these would not necessarily be the best ones, and in point of fact they might be dangerous, since they would not necessarily give unique comparisons. As Mr Ogborn had mentioned earlier, the slope relation of the curve of deaths was a very convenient way of looking at the mathematical properties of the curve of deaths. He had found that to be the best way when dealing with the gamma function, and it also provided a link with the observed data. The form of that slope curve for a curve of deaths going from 0 to ω was an inverted U, which cut the x axis at the minimum in the younger ages, and at the mode of the curve of deaths at the older ages.

In the author's formula, that expression was a sextic expression in x' divided by x'. The zeros of the numerator could easily be found, and there were two real zeros over the practical range of values of b. The maximum was between 96 and 99% of c, which confirmed the figures quoted by earlier speakers, and showed the mode to be in a reasonable position. The minimum value was between 35 and 45% of c, which was between ages 25 and 35. Mortality tables over the past 100 years exhibited the minimum between ages 8 and 15, which showed how far the curves put forward that evening were deviating from the actual shape of observed curves of deaths.

There was one further point he wished to make, which linked up with Mr Ogborn's remarks. The formula proposed by the author was a case of the confluent hypergeometric function with a transformed variable. That was one of the higher transcendental functions and explained why the integrals were complicated. It was defined largely by its numerical values, but over quite an area the curves had the same shape as the curve of deaths throughout the whole range of life, and the problem was to find a suitable fitting process. Whether one of those curves would be found to fit the whole range of life remained to be seen, but at least there was hope, and the fact that the author had found one such curve which had not quite fitted did not imply that one which would fit did not exist.

It was significant also that that function was a distribution function in certain conditions, although it was, of course, bi-modal in the useful case. In recent years it had entered into various statistical applications and, furthermore, it was a specialized solution of a very general equation which entered into theoretical physics. All those things seemed pointers to the fact that somewhere in that region at least there might be a linkage between the mortality table for the whole range of life and statistics and mathematics. As to the fact that it had a physical background, fundamentally all life had a physical background. It might be complicated to apply, but personally he would go on trying, and he was much encouraged in the quest by the author's efforts.

Mr R. H. Daw remarked that he had recently been studying the recorded mortality from degenerative heart disease in the United Kingdom, and in the course of that work* he had divided the male death-rate by the female rate for the same age. He had done that for a number of calendar years. The resulting table of ratios showed various trends, and made him wonder whether it was altogether appropriate to compare male and female death-rates for the same age, or whether it might not be better to use rates for females a few years older than the males. He argued that, since females had lighter mortality than males and he was dealing with degenerative disease, it might be that females degenerated at a slower rate, or started doing so at a later age, than males. He felt, therefore, that in comparing male and female death-rates it might be more appropriate to use age recorded on some scale of degeneration or ageing, rather than a scale of years lived, and to compare male and female rates for equal degeneration ages.

He had therefore been interested to find, on the second page of the paper, that the author seemed about to use a scale based on the ageing process; however, on getting to the Appendix he found that the scale was to be the cube of the age in years, which was apparently to apply to both males and females. On getting over that disappointment, he had wondered how the fitted formulae for the male English Life Tables would compare with those for females, but he had then found that the author only quoted results for the male tables. Also in fitting his formula to the English Life Tables the author reduced the two constants b and c to one by using the relation given in § 11 of the paper. Therefore the only measure of the difference between the sexes that would be available was in the values of the constants c.

It would be interesting to know whether the author thought that his formula gave any answer to the speaker's problem. For instance, did he consider that the difference between the values of c for males and females (if they were also calculated for the female English Life Tables) would give some clue to the age differences which should be used when comparing male and female mortality rates, presuming, of course, that that was a legitimate method of making the comparison?

Mr J. Womersley (a visitor), said that he felt, as a mere outsider whose knowledge of assurance was confined entirely to glancing at premium renewals, somewhat out of place after the detailed technical comments which had been

* See *J.I.A.*, **80**, 69.

made on the paper. It was interesting, however, to someone outside the profession to put together in his head the different things which had been said that evening and see what sort of picture resulted.

The first point that struck him on reading the paper was what the author said in § 1, where he referred to 'the "idea" of a curve of deaths, almost in a Platonic sense'. If any answer came out at all from the postulates on which he had based his paper, it would obviously apply only to Bernard Shaw's *Back to Methuselah* —to the ideal environment. There was development and there was ageing, but it was important not to forget environment. In other words, what had become of the old Darwinian struggle for existence, and where was the parameter for that?

It was interesting, when Mr Beard spoke of the confluent hypergeometrical function, to remember that that was not only a generalized form of many of the functions used every day in mathematical physics, mechanics and engineering, but something that was also used in connexion with the formulae concerned with complicated inspection problems, particularly in the field of sequential analysis discovered by Edwin Mold and Bernard. Those schemes of sequential analysis could all be converted into gambling games; the solutions of them were all allied to gambling games in Laplace's *Théorie Analytique*. They therefore ended with a nice picture of the author discovering a function for the curve of deaths which showed them all dicing with death.

Mr L. G. K. Starke, in closing the discussion, said that both the paper and the discussion had left him with some sense of bewilderment. He was not, perhaps, altogether alone in that. He felt a little inclined to borrow the quotation with which the author prefaced Appendix I: 'if nobody asks me, I know; but if I try to explain to one who asks me, I do not know.' That was something with which he could not agree more. At the same time, he had found the paper extraordinarily interesting. He had no doubt that he would feel the same about the discussion when he was able to read it in print but, at the moment, he rather shrank from the idea of attempting any extempore running commentary on it.

He thought that not only those who had taken part in the discussion but anyone who had studied the paper would agree that the result of a first reading had been to give rise to some slight degree of mental dyspepsia. There were passages which, in the absence of the wide reading and intellectual agility of the author, had frankly to be taken as read, but fortunately they were interlarded with passages which they could all follow and try to get their teeth into. He had had no difficulty, for example, in focusing his attention, as the author recommended in § 17, on the essential differences between the newly-made brick and the newly-germinated acorn. In the last sentence of § 16, however, he had been faced with some difficulty and he was tempted to paraphrase the quotation from Augustine and say 'I know what I think they are trying to say, but have they said it?'

Reading § 12 of the same Appendix, he had flattered himself for quite a long time into thinking that if he were in the middle of a train he would not need any system of co-ordinate axes to enable him to decide that if he could see the front or the back coaches the train must be going round a curve; but in the end he had been content to give it up and to say that the author was probably right.

It was interesting to note that the author had succeeded in presenting a mathematical paper without invoking χ^2, minimum χ^2 or weighted minimum χ^2. The paper purported to be about death, but although the Appendices at any rate had some illuminating things to say about birth, or what happened before birth, it was difficult to find any reference to death in the paper, and much less to what, in a three-dimensional time continuum, might be conjectured to happen after death. It would be interesting to know what the author's view was about the popular supposition that at the instant of death—at any rate of sudden, violent death—the whole of past life was relived; that had led him (the speaker) to wonder whether the idea underlying the paper could not be developed in a way which would lead to the base of the three-dimensional structure becoming a closed circuit. Again, the author had introduced his readers to a number of words which were outside the scope of a normal vocabulary. Finally, the paper was surely almost unique in that all the underlying ideas were tucked away in Appendices, while the paper itself was concerned almost entirely with arithmetical illustrations of a particular formula. He (the speaker) hesitated to add to the author's collection of zoological specimens, but that last characteristic tempted him to wonder which was the hen and which was the egg.

Some of those phenomena were undoubtedly due to the fact that, as was known from previous experience, the author's mind worked in a refreshingly original way; but a further explanation probably lay in the fact that, as the author himself had said, the paper was the result of a major surgical operation; a section dealing with approximate integration had been removed and presented as a separate paper a year or two earlier. In one respect the result of that dichotomy seemed to be very unfortunate, because it had left behind considerable remnants of the foreign body which was supposed to have been excised. Abridged methods of calculating $\bar{A}_x$ had been a primary theme of the approximate integration paper, and if the paper under discussion had been concerned in the slightest degree with the graduation of mortality rates, he would have welcomed the retention of some of the references to $\bar{A}_x$ to demonstrate something which he himself had tried to point out on one occasion—namely that the tests of a graduation by the customary statistical methods were unnecessarily meticulous from the practical point of view of the effect of graduation on the monetary values; but, the subject of the paper being what it was, he suggested that it was entirely irrelevant to clutter up the three-dimensional time continuum with what seemed to be a fourth kind of time (it might perhaps be called moneylender's time), the relation of which to sidereal time was geometric. To come across the tables of the money value of 1 payable on death in the general context of the paper was as if a reader of a treatise on relativity suddenly came across a recipe for Christmas pudding.

He was entirely with the author in his desire to find a simple formula to describe mortality generally rather than a complicated formula that would reproduce almost exactly a particular mortality experience. He knew that the latter had an almost irresistible attraction for some people, but it was surely from the former that most would be learnt. He did not think that he was being fussy, however, in suggesting that it was a great pity that the formula which had been arrived at by such an ingenious process of general reasoning should have been confined in its application entirely to adult mortality, as had been pointed out already by two or three previous speakers, because the essential differences between childhood, maturity and old age dominated the whole thesis of the paper—the device of adding 9 months to the age to bring the development of the embryo into the picture was an example of that. But in all the illustrative calculations the whole of the downward sweep of the curve of deaths during childhood was completely ignored. It was true that the general resemblance was there, to the extent that the curve produced by the formula curled up as the age decreased, but if any or all of Tables 1–5 were carried back from age 20, the numerical resemblances between the tabular and the synthetic values would soon disappear. That had already been commented on, and it was noticeable that in all cases the synthetic values had their minimum at about age 30, whereas the minimum ordinate of the curve of deaths was—and he thought had always been—somewhere between ages 10 and 15.

He (the speaker) had attempted a very rough calculation to find how the parameter values would be affected if it were made a condition that the minimum should lie between those ages. He found—as others appeared to have found—that it did not seem to make much difference to the value of c, but the value of b, instead of being of the order of 3, as in the author's examples, might have to be something like 20. Alternatively, if an attempt was made to keep b within reasonable limits the formula had to be varied by using for the denominator not $x_{\prime}$ but $x_{\prime}^{n}$, where n was a very small fraction. That had the effect of bringing the curve which formed the base of the solid into more of an L-shape, instead of a J the wrong way round.

To alter the formula in that way would involve reconsidering the basic principles from which it had been developed, at any rate so far as the developing, as distinct from the ageing, process was concerned. He had really come in the end to what he thought was the same conclusion as that which the opener had reached, that from about age 25 or 30 as good a fit as, if not a better one than, the author's could be obtained by leaving out the $x_{\prime}$ in the denominator altogether and coming down to two dimensions. He did not suggest for a moment that the idea of providing for the development factor by working on a three-dimensional basis could not be made to work out—he thought that it could; but, even if it were found to be unworkable, the author would be entitled to high marks for showing that the curve of adult deaths could be made to resemble very closely the normal curve if μl were plotted against $x_{\prime}^{3}$ instead of against $x_{\prime}$. Even that he would regard as a very substantial achievement, providing much food for thought and further research, although it had to be remembered that there was a considerable number of curves which, over a wide range of values, were almost indistinguishable from the curve of error. In that connexion he would mention Prof. Kendall's monograph on the so-called cube law for parliamentary election results.

He was interested in the author's reference to Du Nouy, because it was Du Nouy who, on the basis of extensive observations carried out in military hospitals, claimed to have discovered an observational law connecting age with the healing rate of a wound of a given superficial area. He (the speaker) had often wondered whether that could not be shown to have some bearing on their own discussions on the matter of ageing.

The author had admitted that he was far from being the first to suggest that the time scale varied with age. Looking back to childhood, did not the great contrast then appear to be between term time and holiday time? In particular, used not the weeks before Christmas to seem intolerably long drawn-out? Further, the dictionary meaning of 'pastime' was a way of passing the time, and if time was much longer in youth than at other ages it would be supposed that pastimes would be the monopoly of the young; but that, of course, was not so. While he would agree that a feeling that time was passing quickly was to be found among the middle-aged and the elderly, he had met old people who—strange as it might seem—complained that their time passed very slowly.

However, the author had certainly some of the poets on his side in the matter; he was so catholic in his reading that he was probably already familiar with certain lines which had been written more than a hundred years ago, but in case he had not read them he would like to offer them to him as a small token of the interest, and the great admiration, with which he had read the paper. The lines were taken from Thomas Campbell's *The River of Life*:

The more we live, more brief appear
  Our life's succeeding stages;
A day to childhood seems a year,
  And years like passing ages.

Heaven gives our years of fading strength
  Indemnifying fleetness;
And those of youth, a seeming length
  Proportion'd to their sweetness.

The President (Mr W. F. Gardner, C.B.E.), in proposing a vote of thanks to the author, remarked that it was nearly 20 years since Mr Phillips had submitted a paper with a title similar to the one discussed that evening, and on that occasion the then President had proposed a very warm vote of thanks to the author for his efforts; they were no less grateful to him for his further studies on the subject. Whatever the differences of view which had been expressed between speakers and the author, and as between one speaker and another, he was sure that they were all in agreement that they owed a debt of gratitude to the author for putting such an interesting and provocative paper before them. He deserved a very warm vote of thanks for his work for the Institute.

Mr William Phillips, in reply, said that from W. L. Sumner's *Progress in Science* (1946) he had culled the following: 'The so-called laws of science are only generalizations which are useful to give the scientist a summary of what has gone before and a stone to stand on until he can jump to the next.' The subject so far as it had been developed had been presented to the Institute in order that better minds than his own could get to work upon it. He hoped they would work on ϕ and not on μ, that they would seek for one curve and not for a combination of two or more, and that they would not be side-tracked by thoughts of epidemics and cortisone and the disjunction which occurred at the moment of birth. He also hoped that they would have good luck.

At so late an hour it would not be possible to reply to all the speakers fully, but he would like to express at once his deep appreciation of a very thoughtful discussion. He wished to thank the President for his kind remarks, and the members for the generous way in which they had received them.

Mr Phillips writes:

At first I found it rather terrifying that so much highly skilled consideration was given, in particular by Hamilton-Jones and Ogborn, to a rather complete analysis of the curve of § 6, and felt a little guilty at the thought that it was being treated a great deal more seriously than it deserved. On second thoughts it occurs to me that these analyses may be just what is wanted as a point of departure by someone who will take anything of value from the philosophy of the paper and give it a better mathematical shape. I am glad that Hamilton-Jones experimented with more parameters, and still more glad that he found no obvious signs of hope in doing so, for I have an instinctive feeling that whatever else in the paper is wrong, at least the figure 2 is correct—the number of parameters!

Beard has said bluntly that in its present form the curve will not do, and Beard is right. The author confesses to some warm paternal regard for his philosophy, but none for the precise shape of the formula in § 6. Barnett was another speaker who flatters that formula by taking it more seriously than the author does. I am not satisfied that it is correct to say that in the fitting of that formula it is essential to have a preliminary graduation, but as I hope that someone will improve upon that formula I am prepared to wait a reasonable time for the improvement before considering how to fit a curve to experience data.

By my procedure I may have misled Beard and some others into speaking a little too positively about the impossibility of using the curve to approximate to the curve of deaths at the younger ages. As stated in § 9, synthetic curves were deliberately selected that deal best with ages 30 and over, but I believe that by abandoning that period one might find curves which fit for ages 5–30, if only one knew what is the shape of the curve of deaths between those ages. Several speakers spoke rather positively about the curve of deaths having a minimum at about age 10, but presumably this is in mortality tables which are not generation tables after they have been graduated to remove the 'hump' which the data for most non-generation mortality tables indicate. As a geometer I find it a little difficult to speak positively about where precisely the minimum value of ϕ occurs in the case of tables where the curve of ϕ is nearly flat over a period of 20 years or more.

It is obvious that where the ϕ curve is flat a small difference in the value of the ordinate has a considerable effect upon the area (which is what matters), and where the curve is steep a considerable difference in the value of the ordinate has a comparatively small effect upon the area. Some speakers criticized the $\bar{A}$ comparisons, but for every table the paper gives also ϕ comparisons, and for all except those dealt with by the procedure of § 11 gives also l comparisons. In my view the essence of a mortality table is to be found not in the ordinates of the curve of death but in the shape of the histogram of that curve. In 1952 I satisfied myself that that histogram is, except for the first ten years of life, sufficiently represented with the class intervals ten years apart, that for example all the A 1924–29 (ultimate) figures which occupy six heavy volumes could be sufficiently reproduced from age 20 to the end of life, should the need ever arise, from the following schedule of '1000 deaths':

x	$l_{x:\overline{10}\rvert}$
20	23
30	28
40	49
50	101
60	221
70	337
80	209
90	31
100	1

I am asked what I consider would be a proper test of a representation of an experience curve by a synthetic one, and reply 'its ability to produce something resembling this schedule sufficiently for all practical purposes', but of course extending over the whole range of life. $\bar{A}_x$ is certainly not a proper criterion, being open to Beard's criticism of it theoretically, and to the practical objection that to the actuary $\bar{A}_{x:\overline{n}\rvert}$ is as important as $\bar{A}_x$, or even more important.

Long before the formula, or even the philosophy, of the paper had crystallized, I had reflected that we are not all conspicuously familiar with the shape of the curve of deaths, and decided not to rely upon comparisons of histogram figures such as in the schedule just given. I thought it would be complained that it was difficult to adjudge the practical effect of the divergences between the experience and the synthetic histograms, and decided that $\bar{A}$ would be an interesting function of which to attempt the reproduction. Therefore I made the 'tools' for use in this paper, and tested their efficacy in an appendix that presently became the paper of 1952, so that when they were used in this paper everyone might know how far they could be trusted. This should be sufficient to show how wrong Starke is when, the tools having been used here for the very purpose for which they were made, he speaks about 'a major surgical operation' upon this paper which has 'left behind considerable remnants of the foreign body', namely, the calculation from synthetic histograms of comparative $\bar{A}$ values.

Synthetic curves had been selected as representative of each of the English Life Tables Nos. 4 to 10 (Males) some time before the procedure of § 11 was designed, and these, on the whole, naturally approximate more closely to the tabular values than do the synthetic curves exhibited in Tables 5 and 6. Synthetic curves were also selected for some other life tables not yet mentioned, including the Japanese Post Office Table. For this the ϕ curve has two minima (namely, at about ages 12 and 31) as all non-generation mortality tables have unless the intervening 'hump' is removed by graduation. The following are some synthetic curves not reported in the paper:

Table	From about age 25 approximately represented by
Japanese Post Office Table	71/2·20
$O^{[NM]}$ ultimate	75/2·30
$AM^{(5)}$	76¼/3·25
E.L.T. (Males), No. 6	73½/2·15
E.L.T. (Males), No. 8	75½/2·80
E.L.T. (Males), No. 9	76½/3·10

Because the formula for b in § 11 is not more than an empirical expedient, tentatively used only between the limits $73 < c < 78\frac{1}{2}$ as a rough approximation to a rational formula until (if ever) such a rational formula can be found, the calculated values of b have not yet been set out. Now I give those values so that it may be seen how far the parameters may be modified without losing recognizable resemblance between the synthetic curves and the tabular values, and that certain tables, for example the H^m aggregate (75¼/2·5), the O^m aggregate (75¾/2·55), the $O^{[NM]}$ ultimate, and conspicuously the Japanese Post Office Table, clearly do not belong in the same 'family' as that of the E.L. Tables (Males). Anyone who attaches any importance to the synthetic curves may care to consider why the A1924–29 (ultimate) should find itself comfortable in the 'family' to which the following schedule relates, while other British life-office tables do not. Redington has told us why the $AM^{(5)}$ finds itself uncomfortable there, and why the Japanese table is quite out of line.

73¼	2·07	75	2·62	76¾	3·30
73½	2·14	75¼	2·71	77	3·41
73¾	2·21	75½	2·80	77¼	3·52
74	2·29	75¾	2·89	77½	3·63
74¼	2·37	76	2·99	77¾	3·75
74½	2·45	76¼	3·09	78	3·87
74¾	2·53	76½	3·19	78¼	4·00

Hamilton-Jones referred to the fact that the volume of the solid curve of deaths was indefinitely large if one completed it back to age $x_{\prime}=0$. While the paper was still in draft form Beard raised the cogent objection that, even though presumably no one intended to trouble themselves with any curve of deaths at a younger age than $x_{\prime}=\cdot 75$, anyone with a mathematical intuition must feel a revulsion for a solid block which has an infinite volume! Even without any mathematical intuition I feel that revulsion, but I have allayed it by reflecting that I am using the normal curve only as a convenient approximation to some curve that someone else is to put in its place, and by piously hoping that curve will provide both a better fit and a finite volume.

It is a more serious matter that the curve of lengths outlined in §§ 25–28, and illustrated in Fig. 5 (ignoring the ordinates), should have an infinite area, and what Joseph has remarked upon this may pave the way for someone to hypothesize a better suggestion. I have no data to offer in support of the particular distortion which is implied by §26, and it may well be that $\sqrt{l}$ or some other function of l will be found, as a result of experiment, to be the denominator which is required in substitution for l; but if a denominator emerges experimentally that does, in fact, give an infinite area in taking the curve back to 0, I shall take refuge in the knowledge that no fertilization occurs in a body of zero magnitude. A few weeks before his premature death in 1938 my father, W. J. Phillips, the best amateur mathematician I ever met, and no mean horticulturist, had undertaken to sow a row of peas and make a physical count, in classes with class boundaries in geometrical progression, of every pea which 'set'.

A number of speakers complained of complexities in the paper, and I can only conclude that I have been guilty of a lack of clarity in the matter of the 'curve of life' through three-dimensional time, and the shape of the 'curve of deaths' that might result therefrom. If I have not made it crystal clear that these are two different things I am much at fault. I can at short notice think of no better remedy for this than to refer to the analogy contained in paragraph 5 of my paper 'The curve of deaths' (*J.I.A.* **66**, 17), where I considered the distribution of the points of impact of N shells fired from one gun and from n guns of different calibre. I have referred to this not for the usual reason, but in order to repudiate it, notwithstanding that it seems to be roughly the view point of Redington. I would now say that what we want in the analogy is a set of guns of continuously graded calibre, with an orderly system for continuously varying the charge on the one hand, and the retarding force on the other, and that the resulting distribution would then be represented not by a more complex curve, but by a very simple one.

But the curve of flight is a conception quite distinct from that of the curve of distribution of the points of flight cessation. May I recapitulate? I have visualized the curve of life as a movement through a three-dimensional continuum, which by the convention of Cartesian co-ordinates may be resolved into three independent movements, I have talked of the varying speeds of these movements, but I have expressed algebraically not the speeds but the resulting positions in the continuum, i.e. $x_{\prime}$, y, z at x years after birth. Then the Lemmata show with what I claim is geometrical precision, capable I hope of being expressed equally precisely in other mathematical form, what might, *upon certain suppositions*, be the distribution of the points measured along the x-axis of the deaths of N people, originally all alive at the same age, who pursued the life-line of which the locus is $x_{\prime}$, y, z.

I know that men and women are very different, but I shall not be persuaded by Daw that they are so different that if, indeed, it should be found that the cube of sidereal time elapsed since conception comes into the proper measure of the degeneration age of males, the same cubing would not be appropriate also to females. In fact I do not take the degeneration age of the male to be x^3, but x^3/c, which is quite a different matter. By concentrating one's attention on j, which has the further advantage that it is independent of any arbitrary unit of time such as that taken by the earth to circumnavigate the sun, one would not feel, perhaps, that the word 'only' can be very forceful when Daw speaks of 'the only measure of the difference [of ageing] between the sexes' being c.

Both Barnett and Daw made the point that nothing is said in the paper about female mortality. The point is a good one, although I am of the opinion that, life office records perhaps apart, the inaccuracies which are inherent in male experience tables are small by comparison with the inaccuracies in female data. I have made no calculations for values of c over 79, and apparently for E.L.T. No. 10 (Females) we need a value for $c \geqslant 79{\cdot}5$. I have therefore looked at E.L.T. No. 8 (Females) for which it seems that the 'best' value of c lies not far from 78, and E.L.T. No. 9 (Females) for which a somewhat remarkable fit is obtained by 79/3·5. The following schedule compares the histograms of these tables for $l_{20} = 1000$ with the histograms for the 78/3·2 and 79/3·5 curves respectively:

y	$l_y : \overline{10}\vert$			
	Tabular E.L.T. No. 8 (F.)	78/3·2	Tabular E.L.T. No. 9 (F.)	79/3·5
20	33	47	34	38
30	49	48	42	39
40	75	67	60	58
50	125	116	105	104
60	208	211	196	203
70	289	290	304	303
80	187	188	219	215
90	33	32	39	39
100	1	1	1	1

Though doubtless a slightly better fit could be found for Table No. 8 by testing other values of c in the neighbourhood of 78, it is suggested that this schedule suffices to show that it was right to assume (as we all assumed) that the E.L. Tables (Females) do not belong to the 'family' of curves which roughly represent the E.L. Tables (Males).

I think there must be some essential difference of outlook which is at present separating Joseph and myself. Whenever I see a heavy body falling in a vacuum and observe that at time t, measured from the moment it commenced to fall, the distance it has fallen is proportionate to t^2, I define its speed as one with acceleration. If I know that in some other sense its speed is uniform (as, for example, in 'keeping in step' with a drifting balloon from which it has fallen), I conclude that to track its course I shall require at least two dimensions. Similarly, if the human beings I see around me are ageing at a speed such that at time t they have reached on the degeneration scale a distance proportionate to t^2, while they are manifestly 'keeping in step' with ordinary sidereal time in many other respects, I can only track their course by using an independent dimension for that movement which gives them room for accelerated ageing. I then find that on the plane the area swept out is proportionate to t^3.

I conclude with some quotations which do not seem to be capable of repudiation:

Dr Farr (1859, *J.I.A.* 9, 121):

The Life Table...may be called a biometer, for it gives the exact measure of the duration of life. It represents a generation of men passing through time.

A. R. Davidson (1924, *T.F.A.* **10**, 35):

One is forced to consider whether the standard process of preparing mortality tables ...is really a sound or useful one....It seems to be clear that the expectation of life becomes a function of the year of birth as well as of the age.

(1925, *T.F.A.* **10**, 284):

The search for a suitable curve to graduate mortality tables is not likely to be successful so long as it is prosecuted on data which are an amalgamation of rates not forming a sequence representing the actual trend of human life.

(1925, *T.F.A.* **10**, 331):

It appears that the series before us is a sequence of points on a number of separate series of rates of mortality, each representing consecutive human life.

It may be that a mathematical expression will satisfactorily represent the rate of mortality for *consecutive* human life, but if one proposes to use a mathematical formula for graduating this series, one would require to employ a larger number of constants than would normally be necessary in a series representing *consecutive* life.

I would emphasize the dates of these quotations, for it appears to be thought by some that we owe the origin of the idea of 'consecutive human life', the reversion to the 'generation' life table of 1859 and earlier, to two papers presented in Edinburgh and London respectively on 31 January 1927.

H. Hosking Tayler (1950, *J.I.A.* **76**, 134):

The records...provided material from which they could learn what had been the shapes of the curves representing the experiences of successive generations over considerable stretches of the generation histories. If that data were collected and examined, they would begin to know what a real generation curve of deaths looked like and, more important, whether one curve looked like another.

So far I am in whole-hearted agreement with Hosking Tayler, and I differ from him only where he speculated upon the possibility that

most of the difference between one curve and another consisted in the location of the mean;

for in accordance with my philosophy, as the mean moves towards an older age, so also to a lesser extent does the upper limiting age. I do not think that any formula for ϕ will be acceptable that does not preserve this feature, so very evident in all the experience curves at present available for examination. Incidentally we have here another advantage to place beside the observation of the shifting mean which Beard stresses as 'one of the advantages of looking at the curve of deaths for describing mortality experience'.

Beard has, of course, done more than anyone towards finding a shape for the curve of deaths (on the basis that finding a curve for μ is an essentially different thing), but he has so far not had available to him, any more than I have had, those generation tables of consecutive human life which, I so agree with Davidson, we must await before we can hope to express in the simplest terms the mathematics which, if not a necessary consequence of, at least do not quarrel with, our philosophical ideas.