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BRIDGING THE GAP BETWEEN FINANCIAL ECONOMISTS AND ACTUARIES: A NEW ASSET-LIABILITY MANAGEMENT MODEL

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Bridging the Gap Between Financial Economists and Actuaries: A New Asset-Liability Management Model

SYNOPSIS

This paper describes a new approach to asset allocation modelling. The traditional risk/reward framework of financial economics is used as a starting point, but the definitions of "risk" and "reward" are refined so that they have specific regard to the liabilities of the particular investing institution in question. Reference is made to the recent work that has been done in the field of stochastic solvency testing by the actuarial profession and, indeed, this paper represents a fusion of the ideas of financial economics (on the assets side) and actuarial theory (on the liabilities side).

The first part of this paper is a brief historical overview of the development of modern day asset/liability management theory and practice, and it is interesting to see how actuaries and financial economists have been tackling the same issues but from different angles.

The second part of this paper sets out some basic principles underlying the model, and uses an extreme hypothetical example to illustrate how the traditional risk/reward approach could be misleading if used without reference to the liability profile of the financial organisation under consideration.

The third part of the paper then develops the model. For illustrative purposes, a hypothetical (and highly simplified) UK life office is examined, and it is assumed that the only investments available to the office are UK government bonds and domestic equities. Using historical data for these asset classes, stochastic simulation is used to create a large number of future investment scenarios. The ability of the institution to meet its liabilities under each scenario is examined, from which a "risk" characteristic, defined simply in terms of solvency rather than asset volatility, is assigned to the particular asset mix under consideration. A risk/reward profile is then created for all the possible asset mixes, from which investment policy decisions can then be made or reviewed.

I An Historical Overview

Both financial economists and actuaries have been involved in the development of quantitative asset allocation techniques for many years. Indeed, it is ironic that, given the different approaches to the subject taken by the two professions, their origins of inspiration can be traced back to the same year (although not the same source).

The "theory of immunisation" put forward by Redington (1952) has had a profound influence on the way actuaries approach the valuation of insurance companies and their assessment of solvency. For insurance operations, as a practical asset/liability management model, to date it has had very little competition: the idea of equating the mean term (or duration) of assets with the mean term of liabilities has been used by many insurance companies worldwide. But immunisation does have its limitations. In particular, it has little relevance to interest-sensitive products, such as universal life, and performance-linked products, such as unit-linked life. Moreover, immunisation immunises against profits as well as against loss.

More recently, the notion of "convexity" (which is the same as duration but with second derivatives replacing first derivatives) have given immunization new life. Tilley (1988) comments at the 23rd International Congress of Actuaries held in Helsinki, that "a whole investment advisory business has grown up in the United States around immunisation concepts."

Redington's ideas may today be viewed as the "classical" actuarial approach to asset/liability management. The success of the Redington "model" as an accepted asset/liability management tool lies in its relative simplicity and the ease with which the calculations necessary to test immunisation can be made. As Buff (1989) commented at a Society of Actuaries' meeting in Vancouver, "if you can't compute it, you can't compute it", which means to say that it is not possible to make use of theoretical advances unless it is feasible to execute the calculations called for by these advances.

Actuarial research into the area of asset/liability modelling was conspicuously muted for many years after Redington's work, but the actuarial profession seems to have recently found a new interest in the subject matter. Some of the most interesting work has been done in the UK where some pioneering stochastic investment modelling has been carried out by Wilkie (1986). However, the bias of the UK work is in the area of solvency testing -see Hardy (1993) for an excellent example of this - while it is in the US that actuaries have begun to apply some of the new ideas concerning stochastic modelling to practical asset-liability management.

Although the primary catalyst behind new actuarial approaches to asset/liability management is undoubtedly the advancement of computer power, and hence the ability to use stochastic models in practice, there is an important side-catalyst: this being the growing interest taken by the actuarial profession in financial economics and the potential application of asset/liability management techniques to the solving of actuarial problems.

The same year that Redington published his ideas on immunisation, one of the most important papers of modern financial economics was also published, this being the pioneering work of Markowitz (1952) introducing the idea of asset allocation modelling within a risk/reward trade-off framework.

Markowitz noted that a reduction in risk, measured by the standard deviation of return on assets, could be achieved by diversification (into assets whose returns were correlated), without any reduction in return. Markowitz also introduced the idea of an "efficient frontier", which is a curve joining the risk/reward combinations of asset mixes which give the highest reward for any given level of risk. The original ideas were subsequently updated by Markowitz (1987).

At the time, the financial world was not ready for the concept of an efficient frontier, or rather, to return to Buff's truism, computer power had not yet got to the stage where Markowitz's ideas could be put into practice. Indeed, a practical adaption of these ideas had to wait over a decade, when Sharpe (1963) introduced the "diagonal model" which suggested that the future price of a security depended on its "alpha", the market return through its "beta", and a random error term, the values being based on simple linear regression on historical data. This was the birth of the now widely used "Capital Asset Pricing Model."

Sharpe (1970) was again influential in the book, "Portfolio Theory and Capital Markets", in which it is suggested that mean and variance alone "may suppress too much reality", and a different utility curve may be needed to compare portfolios of different riskiness. This delves into the area that has most concerned financial economists over the past two decades when looking at asset allocation: how to incorporate "utility" into their calculations within the context of a risk/reward framework.

Modern day approaches to the efficient frontier idea have attempted to experiment with constraints which can be used to narrow down acceptable portfolio mixes lying on the efficient frontier, and also attempt to be "dynamic" in that the acceptable portfolio mixes change and reflect the particular market conditions present at any particular time. A good example of such a model is that developed by Leibowitz, Kogelman, Bader and Dravid (1994). Looking at a one year time horizon, the model takes on a dynamic form by up-dating the asset allocation strategy whenever interest rates move. The model does not just look at portfolios on the "efficient frontier" but introduces the further constraint that portfolios must give no more than a specified probability of generating one-year returns that fall below a certain level. This is incorporated by the introduction of "shortfall line", such that all portfolios above the line of constraint meet the maximum probability criterion. If interest rates now fall, with the equity risk premium, equity and bond volatilities, and equity/bond correlations all remaining constant, the entire return/risk curve will correspondingly shift down, decreasing the expected returns of all potential portfolio mixes. With the shortfall line unchanged, the market conditions make all portfolios riskier in shortfall terms, and fewer portfolios will now fall above the shortfall line. This will require a revision of the equity/bond mixes that were previously deemed acceptable.

This approach is an interesting refinement of the original Markowitz risk/reward idea, but, like all its predecessors, the work has one major flaw as an asset allocation model: the model says nothing at all about the liabilities that underlie the assets. In effect, the models are nothing more than "asset/asset" models rather than true asset/liability models.

Late in the 1980s, Arthur and Randall (1989) expressed that they had yet "to see a valid allocation system" which concerned itself with the overriding principle of "investing to meet liabilities." Here, as we approach the mid-1990s, such a system has still not made itself evident, at least in the academic literature.

Perhaps progress on developing a true asset-liability system has been slow because many practitioners remain unaware that there is a fundamental flaw in the traditional approach of financial economics. In a presentation to the Institute of Actuaries, Arthur (1993) quotes Harrington (1987) that "all investors have identical time horizons", a sentiment Arthur has little sympathy for. In the same presentation, Arthur also quotes Brealey & Myers (1991) who go as far as to say that "risk is the same for all investors."

This introductory section has provided a brief overview as to how actuaries and financial economists have arrived at where asset allocation theory and practice is today. In the remainder of this paper, a new approach to asset allocation will be discussed, an approach which makes some attempt to really tackle the problem of investing to meet liabilities, in so much as the model that is put forward recognizes that liabilities play as much a part of the asset allocation decision as the assets.

II What is Return?

Although it is usually the case that the meaning of "return on assets" in financial modelling can be taken for granted - being based on market value changes after allowing for positive and negative cash flows - this is not the case in the context of an asset/liability model.

This extra consideration arises because of the need to ensure that the definitions of "return" and "risk" are consistent.

In this paper, risk is viewed as having something to do with the ability of the financial institution to demonstrate, from time to time, that it is in a financially stable situation. This requires making an assessment of the "solvency" of the institution by comparing the "actual" value of assets with the value of assets "required" to enable the future liabilities to be met by the institution.

For a UK life office, a "solvency valuation" is required by regulation, and asset values in such a solvency test generally require carrying assets at market values subject to certain limitations designed to dilute the valuation of the assets with a default risk greater than that of government stock. In any case, in order to be consistent with the risk/solvency assessment, return must be defined in terms of return on the "actuarial value" of assets as carried in the solvency valuation.

This paper considers primarily asset allocation decisions made in the context of a UK life office which may only invest in UK government stock or domestic equities. On the prescribed solvency valuation approach, this would mean that returns should be assessed on the basis of market values but with allowance made for the permissibility restrictions. However, for purposes of this paper, and to avoid developing an overly complicated hypothetical model office, the actuarial value has been defined as the market value without restriction. In any case, the reserving method and basis for valuing the liabilities should strictly also correspond precisely to the statutory model, but again, to avoid undue complication, the reserves and, therefore, the value of assets "required" from year-to-year, are based purely on the assumptions used in originally setting the premium rates.

III Underlying Principles

In order to examine the rationale behind our asset/liability model, in this section an extremely simple "model" life office is introduced: a more realistic (albeit still highly artificial) model office is used to test the ideas later in this paper.

In this model life office, the only liability is a policy which has been written today and pays £1,000 in two years' time. The actuary has calculated a single premium based on his or her estimate of the future expected returns on investments made by the office; this is calculated as $1,000/(1 + i)^2$, where i is the annual rate of interest assumed.

In estimating the rate of interest the actuary will look at the actual (or, in this case, the anticipated) investment policy. The actuary will also recognize that investment policy can alter (often dramatically) over time, and will allow for potential fluctuations in returns by using a deliberately conservative rate of interest assumption.

For the model office concerned, it is assumed that there are only two types of investments available, and the only possible annual rates of return on these investments, which in any year are equally probable and independent, are as shown in Table I.

Table I

Possible rate of return scenarios for hypothetical investments (A) and (B)

This table shows the possible annual rates of return on two hypothetical asset classes, investment (A) and investment (B). The possible rates of return are equally probable and independent.

Investment (A)		Investment (B)	
Annual Rate of Interest	Probability	Annual Rate of Interest	Probability
6%	0.5	2%	0.5
8%	0.5	12%	0.5

For the two year horizon for which the actuary of the model office needs to estimate an annual rate of return (ie the value of i in calculating the single premium), the possible internal rates of return are easily calculated, and are as shown in table II.

Table II

Two year projected internal rate of return scenarios for hypothetical investments (A) and (B)

This table shows the possible two year projected internal rate of return scenarios for two hypothetical asset classes, investment (A) and investment (B), using the basic data of Table I.

Investment (A): possible internal rate of return scenarios			
Scenario	Return in Year 1 (i_1)	Return in Year 2 (i_2)	Effective Internal Rate of Return ($=\sqrt{(1+i_1)(1+i_2)}-1$)
1	6%	6%	6%
2	6%	8%	6.9953%
3	8%	6%	6.9953%
4	8%	8%	8%

Investment (B): possible internal rate of return scenarios			
Scenario	Return in Year 1 (i_1)	Return in Year 2 (i_2)	Effective Internal Rate of Return ($=\sqrt{(1+i_1)(1+i_2)}-1$)
1	2%	2%	2%
2	2%	12%	6.8831%
3	12%	2%	6.8831%
4	12%	12%	12%

As the scenarios detailed in table II are equally probable and independent, then it is readily determined that investment (A) has a mean internal rate of return of 6.9977% with standard deviation of 0.7071% and investment (B) has a mean internal rate of return of 6.9416% with a standard deviation of 3.536%.

According to the conventional theories of financial economics there is no question whatsoever as to what investment should be held by the office: not only does investment (A) have the highest expected internal rate of return but it also has the lowest risk (since according to conventional theory "risk" is defined as the standard deviation of the returns).

However, assume that the actuary of the model office has anticipated in pricing the single premium policy that the future rate of return on the fund will be 9% per annum (perhaps anticipating favourable investment conditions over the next two years, or competition in market rates may have forced the actuary to adopt a rate of interest at the top end of his or her expectations).

This produces a single premium of $1{,}000/(1.09)^2 = 841.68$.

If the office were then to hold the "less risky" investment (A) it is clear that at the end of the two year policy term the office would be "insolvent" (ie it would have insufficient funds available to meet the £1,000 liability at the end of the term), simply because the yield could never get to the required 9% per annum. Table III details the simple calculations involved in arriving at this conclusion.

Table III

Possible performance of model office over a two year projection period where assets are held exclusively in hypothetical investment (A)

This table shows the possible performance of the model office over a period where assets are held exclusively in hypothetical investment (A). It is clear that at the end of the projection period the office would be insolvent under all possible performance scenarios.

Investment (A): Possible fund performance scenarios					
Scenario	Probability	Annual rate of return in Year 1 (i_1)	Annual rate of return in Year 2 (i_2)	Target fund at end of term	Actual fund at end of term ($=841.68 \times (1+i_1) \times (1+i_2)$)
1	0.25	6%	6%	£1,000	£945.71 (Insolvent)
2	0.25	6%	8%	£1,000	£963.56 (Insolvent)
3	0.25	8%	6%	£1,000	£963.56 (Insolvent)
4	0.25	8%	8%	£1,000	£981.74 (Insolvent)

On the other hand, if the office were to hold the "more risky" investment (B), although there is still a high likelihood of insolvency, the office at least has a 25% chance of being solvent at the end of the policy term - see table IV.

Table IV

Possible performance of model office over a two year projection period where assets are held exclusively in hypothetical investment (B)

This table shows the possible performance of the model office over a two year projection period where assets are held exclusively in investment (B). Although the office is insolvent under three of the possible outcomes, it does achieve solvency in one of the scenarios, from which it can be deduced that by holding investment (B) the office has a 25% probability of being solvent at the end of the projection period.

Investment (B): Possible fund performance scenarios					
Scenario	Probability	Annual rate of return in Year 1 (i_1)	Annual rate of return in Year 2 (i_2)	Target fund at end of term	Actual fund at end of term (=841.68 x $(1+i_1)$ x $(1+i_2)$)
1	0.25	2%	2%	£1,000	£ 875.68 (Insolvent)
2	0.25	2%	12%	£1,000	£ 961.54 (Insolvent)
3	0.25	12%	2%	£1,000	£ 961.54 (Insolvent)
4	0.25	12%	12%	£1,000	£1,055.80 (Solvent)

On this basis, there is strong justification that it is the "probability of insolvency" rather than "asset return volatility" which is the true measure of asset riskiness; thus, in this example, investment (B) is less risky than investment (A) and would be the preferred choice of investment if the objective of investment policy were solely to minimize risk. This fits in neatly with the intuitive knowledge that investment (A) is totally inappropriate as it could never yield enough to make the accumulated premium enough to meet the promise of £1,000 at the end of the policy term.

It is instructive to examine the performance of the office where investments are diversified between the two asset classes rather than being confined to investment exclusively in a single asset class. Table V summarizes the results on the 9% interest rate assumption.

Table V

Expected internal rate of return/probability of insolvency profiles for various asset allocation strategies involving hypothetical investment (A) and investment (B) where liabilities have been priced using a 9% rate of interest

This table shows the expected internal rates of return and corresponding probabilities of insolvency for various asset allocation strategies involving hypothetical investment (A) and investment (B) where liabilities have been priced using a 9% rate of interest. It is clear that in order to minimise the probability of insolvency it is not necessary to go exclusively into the low yielding and highly volatile investment (B), and that by allocating the fund 50% in each asset class a higher yield is achieved without increasing risk.

Investment (A)	Investment (B)	Expected (Mean) Internal Rate of Return	Risk (Probability of Insolvency)
100%	0%	6.9977%	1
90%	10%	6.9975%	1
80%	20%	6.9962%	1
70%	30%	6.9936%	0.9375
60%	40%	6.9898%	0.9375
50%	50%	6.9848%	0.75
40%	60%	6.9786%	0.75
30%	70%	6.9712%	0.75
20%	80%	6.9625%	0.75
10%	90%	6.9526%	0.75
0%	100%	6.9416%	0.75

The figures of table V lead to the sensible conclusion that in order to minimise risk it is not necessary to go 100% into the low yielding and highly volatile investment (B): indeed, by holding 50% in investment (A) and 50% in investment (B) a satisfactory expected yield can be achieved with minimum risk (see figure 1. Note the upward sloping nature of the resulting risk/reward trade-off curve).

The analysis so far has looked at the case of a relatively high rate of interest assumed in calculating premium rates. It is instructive to look at the more typical case where the rate of interest assumption is conservative and there is a relatively low probability of it not being achieved.

The results using a rate of interest assumption of 6% per annum in pricing the liabilities are shown in Table VI.

Table VI

Expected internal rate of return/probability of insolvency profiles for various asset allocation strategies involving hypothetical investment (A) and investment (B) where liabilities have been priced using a 6% rate of interest

This table shows the expected internal rates of return and corresponding probabilities of insolvency for various asset allocation strategies involving hypothetical assets investment (A) and investment (B) where liabilities have been priced using a 6% rate of interest. It is clear that whatever mix of assets are held there is a very little chance of being insolvent, although because investment (B) could yield less than 6% per annum in an extreme downturn in the market, a very different profile to that shown in table V becomes evident.

Investment (A)	Investment (B)	Expected (Mean) Internal Rate of Return	Risk (Probability of Insolvency)
100%	0%	6.9977%	0
90%	10%	6.9975%	0.0625
80%	20%	6.9962%	0.1875
70%	30%	6.9936%	0.1875
60%	40%	6.9898%	0.25
50%	50%	6.9848%	0.25
40%	60%	6.9786%	0.25
30%	70%	6.9712%	0.25
20%	80%	6.9625%	0.25
10%	90%	6.9526%	0.25
0%	100%	6.9416%	0.25

In this case, whatever mix of assets are held there is very little chance of being insolvent, although the fact that investment (B) could fall below 6% per annum does mean that in an extreme downturn in investment markets the office could run into difficulties; hence the downward sloping curve in contrast to the upward sloping curve produced by using a relatively high rate of interest (again see figure 1).

The low interest rate "run" of this simple model office gives a result which is very similar to the prediction of conventional financial economics (ie investment (A) in preference to investment (B) because returns are less volatile). This is not surprising because the nature of the liabilities has effectively become relatively unimportant in choosing the correct portfolio as there should be little difficulty in achieving the required rate of return whatever asset mix is chosen. In this respect, the traditional ideas of financial economics can be seen to be a special case of the more general asset allocation system using a true asset/liability model.

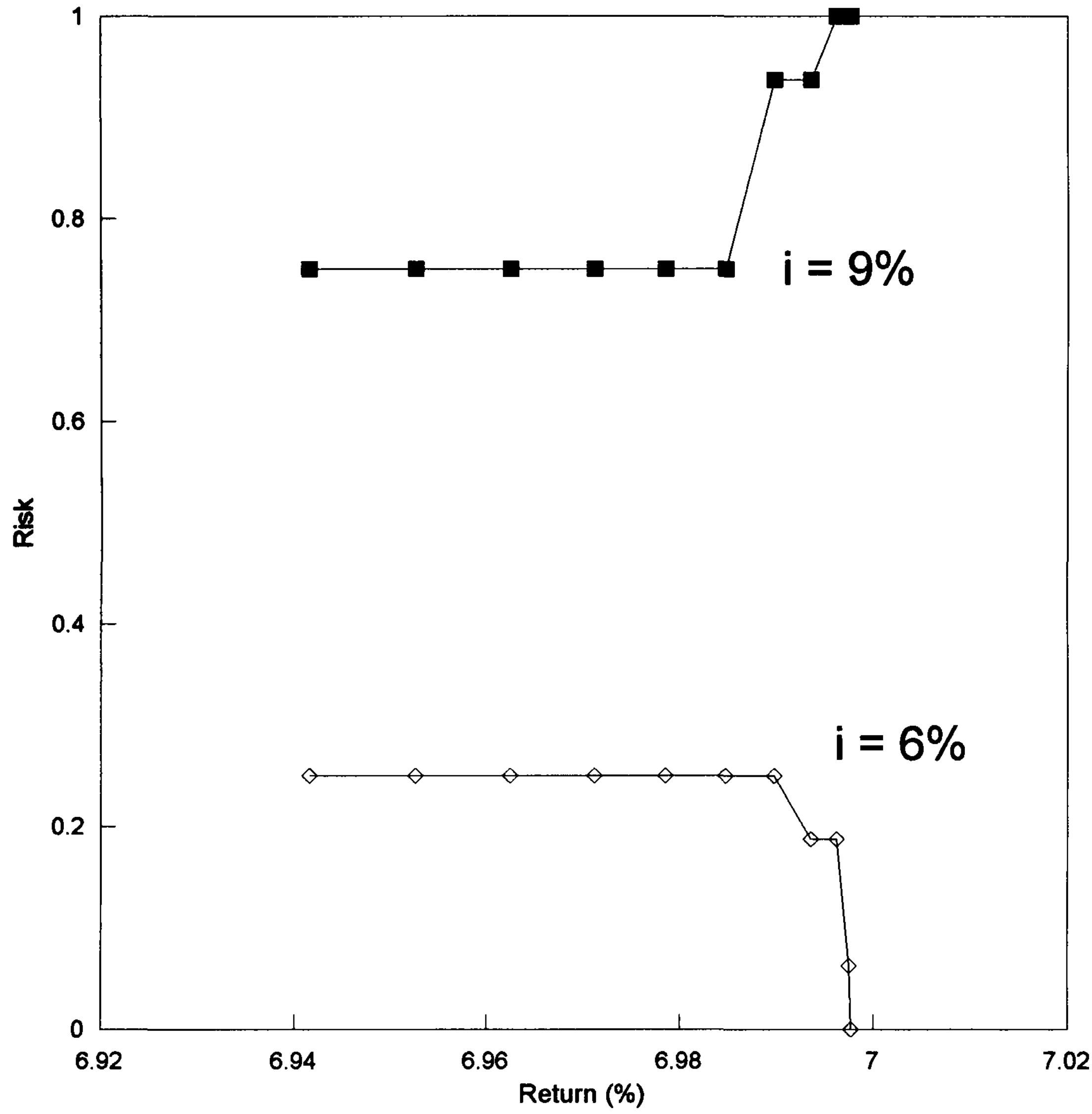

Figure 1. Risk/reward profiles for a simple fund. The risk/reward profile looks very different depending upon the liability structure under consideration. For liabilities priced using a relatively high rate of interest, an upward sloping curve is evident, and a fund which is heavily invested in investment (A) could be heading for financial difficulties. Alternatively, for liabilities priced using a relatively low rate of interest, a downward sloping curve is evident, so that a minimum risk position can be achieved by maximising the holdings in investment (A).

IV Model Framework

To transform the basic ideas illustrated in the idealised financial world of the previous section into a model which may have practical applications, there are four important steps that need to be taken:

1 An assessment has to be made as to the probability distribution of the returns on assets available to the financial institution.

2 A cash flow projection must be made of the future liability outgo of the financial institution as accurately as possible.

3 Using the information about the probability distribution of asset returns, a large number of possible investment scenarios need to be derived and the performance of the fund in meeting the liabilities under each scenario needs to be examined.

4 A large number of runs will enable an assessment to be made as to how a particular mix of the various asset classes will succeed in meeting the liabilities. This assessment can then form the basis for the construction of a "risk/reward" profile from which possible optimal asset mixes can be considered for investment policy.

The model will immediately be recognised as being a simulation process, where the simulated variable is the return on assets. Simulation is necessary because a mathematical solution to the model is far too complicated (compared to the adequate approximation provided for by a simple simulation technique), not because the return on assets is a complicated variable but because the other variable making up the model - the "risk" variable - is extremely complicated (the issue of how "risk" should be defined is central to the model and will be discussed in detail later in this paper). Thus the simulation process starts by generating random observations for the random variable with a known distribution (or at least a distribution for which a reasonable assessment can be made) which can then be used to calculate random observations for the complicated random variable. From these observations it is then possible to make inferences about the distribution of the complicated variable.

For the purpose of this paper the financial institution being assessed is taken to be a life office which issues a large number of level annual premium 30 year endowment policies on male lives aged 50 at entry, and these policies all incept today. The only decrement is mortality, and this is assumed to be according to the A67/70 2 year select and ultimate table, both in setting premiums and in projecting future cash flows. All expenses and commissions are assumed to be zero.

The model tracks forward for each of the years for which the whole life contracts are expected to be in force, and computes, for each year, the following:

Fund at end of year t = F_t

$$= (F_{t-1} + P_t)(1+i) - C_t(1+i)^{1/2}$$

where P_t = premium received (at start of year t) based on actuarially calculated net premium at rate i

i = rate of interest assumption corresponding to the calculation of the net premium

C_t = claims in year t (assumed to occur half way through the year on average)

Thus, F_t represents the "target fund" (at the end of any year t) to which the office should strive, being based on an investment return equal to that assumed in the premium basis. If, in practice, the actual fund falls persistently below this target fund then the office would be heading towards financial difficulties. It is therefore appropriate to examine the success of any particular investment policy in generating a fund size which is consistently at least as great as the target fund.

V The probability distribution of asset classes available to the institution

The most difficult element of the construction of the model is the determination of the probability distribution of the available asset classes. To avoid going into a vast amount of analysis, this paper considers primarily UK government bonds and domestic equities; this is a reasonable starting point for any discussion of the basic asset allocation decision process for a UK financial institution.

Using the figures for total returns given in the appendices to a recent paper presented to the Institute of Actuaries by P D Jones (1993), which go as far back as 1923, and using various statistical tests for randomness, there is inconclusive evidence that equity and gilt returns are either random or non-random (section A of the appendix shows the basic raw data underlying all the calculations presented in this paper). In the absence of contrary evidence, and further because the whole issue as to whether markets do move in a random fashion remains highly contentious, for the purposes of demonstrating the model it has been deemed reasonable to assume that both equity and gilt returns move randomly rather than non-randomly. In any case, it would not be a big step to re-run the model using a stochastic approach which correlated successive returns in some fashion, or indeed recognised a relationship between equity and gilt returns.

On the assumption of random returns, in order to test the success of a fund in meeting its liabilities using any particular mix of equity and gilts, the approach taken is to derive a large number of potential individual investment scenarios by creating a set of random rates of return for each year for which the projection is made, where these random rates of return are based on cumulative probability distributions constructed from the historical data. Section B of the appendix gives these cumulative distributions for equity and gilts constructed from the full historical data available, while section D of the appendix shows how a single investment scenario is constructed from a sample set of random numbers using a Monte Carlo sampling method. The projection period extends to the year in which all policies are expected to have matured - in this case, 30 years.

VI Application of the complete simulation process

The simulation process must then be built into the liability cash flow framework of the life-office, ie the "target fund" needs to be compared with the "simulated fund" in each year of projection, as derived under each simulated investment scenario.

This simulated fund is found by:

Simulated fund at end of year t = N_t

$$= (N_{t-1} + P_t)(1+s_t) - C_t(1+s_t)^{1/2}$$

where P_t and C_t are as previously defined

and s_t = the simulated annual rate of return in year t

Thus, for example, consider year 1 under the simulated rates of return given in Section D of the appendix, and assume that a mix of 50% in equities and 50% in gilts is being considered. Then the simulated annual rate of return in year 1 is given by:

$$0.5 \times 0.4574 + 0.5 \times -0.0524$$

$$= 20.25\%$$

The level net annual premium for a 30 year endowment policy covering a male aged 50, sum assured of £1,000, using the A67/70 2 year select and ultimate tables, and assuming a rate of interest of 6%, is £22.01. This produces a "target fund" at the end of year 1 of:

$$F_1 = (0 + 22.01) \times 1.06 - 2.86 \times 1.06^{1/2}$$

where 2.86 is the expected claims cost for the year

$$= £20.39$$

This compares with the simulated fund of:

$$N_1 = (0 + 22.01) \times (1 + 0.2025) - 2.86 \times (1+0.2025)^{1/2}$$

$$= £23.33$$

Thus, in this case, the simulated fund is in excess of the target fund - a good start for the office.

The progress of the target and simulated funds is then tracked through for the full expected future term of the business in force. This is then repeated for various simulated equity and gilt returns - the model has been run using 50,000 simulations - using all possible combinations of equities and gilts in steps of 1%, and using liability profiles based on actuarial interest rate assumptions of 0%, 2%, 4%, 6%, and 8%. (50,000 simulations have been used to ensure an extremely smooth curve could be drawn between any set of risk/reward points at a particular interest rate assumption. See appendix E for a mathematical justification for the number of simulations used.)

For each simulated investment scenario, the internal rate of return for each mix of equities and gilts is calculated as:

$$r = \left[\prod_{t=1}^{n} (1 + s_t) \right]^{1/n} - 1$$

where s_t is as previously defined and n = projection period (in years)

This return is then averaged over the 50,000 simulated scenarios to derive an expected rate of return on the fund for any particular mix of equities and gilts. This expected rate remains the same regardless of the liability profile under consideration.

The next step is then to determine how "risk" should be specified within the framework of the cash flow projections for any particular liability profile. An extremely "aggressive" definition would be to express risk as the "probability of insolvency" where this is taken to be equal to 1 if, in any year of the projection, the simulated fund falls less than the target fund. However, given the volatility of equity and gilt returns, and given that no contingency margin has been built into our premium rating process, it would not be unexpected that in any particular year the simulated fund fell below the target: thus, a more flexible definition of risk should be adopted.

Clearly, this is a subjective decision and settling on an appropriate definition is really a decision for senior management. For the purposes of this paper, risk has been defined as the probability of the simulated fund being less than the target fund for 3 consecutive years during the full projection period. The 3 year period is chosen on the premise that if the fund has gone this amount of time in an unbalanced financial position, then this would suggest it may have run into long-term financial problems.

Thus risk for any simulated investment scenario and a particular mix of equities and gilts is defined as:

$R = 1$ if $N_t < F_t$ and $N_{t-1} < F_{t-1}$ and $N_{t-2} < F_{t-2}$

for any t where $3 \leq t \leq n$

Otherwise, $R = 0$

The risk for any particular mix of equities and gilts is then the sum of all values of R over the 50,000 simulations, divided by 50,000 to give an average "probability of insolvency."

It is interesting to note at this stage that a new concept is beginning to appear in the asset/liability management literature in the US which is concerned with the consideration of solvency in connection with asset/liability management, this being the idea of asset/liability surplus management (ALSM). ALSM refers to asset/liability management which focus on the NAIC risk-based capital standards. These standards require certain minimum surplus amounts to be maintained in respect of various classifications of risk - an ALSM model might then assess how well the required minimum surplus levels are likely to hold up using the potential investment strategies up for consideration. Hepokoski (1994) gives an excellent introduction to ALSM as an extension of asset/liability management.

VII "Pseudo optimisation"

On the surface, the model looks like an exercise in optimisation but in actuality it is a little more than that - the approach taken might be labelled "pseudo" optimisation.

The objective of the model is to find an "optimal" allocation between two (or more) asset classes, with "optimal" defined in its most general sense, that is to maximise the return and minimise the probability of insolvency. More generally:

Find the value of each y_i such that we can:

(1) Maximize the return on $\sum_{i=1}^{t} y_i A_i$ where $\sum y_i A_i = Z$

$\sum y_i = 1$, $0 \leq y_i \leq 1$ and Z is a fixed monetary amount and $A_i = Z$ for all i (with no restriction placed on the value of A_i that may be taken)

and

(2) Minimise the probability of insolvency for the mix $\sum y_i A_i$

(1) is a linear programming problem, while (2) is a non-linear programming problem. Expressed in this way, the optimisation problem is complicated and might not even be mathematically solvable. However, a computer algorithm can be used to solve the problem, hence the "pseudo optimisation" approach.

The approach taken is to determine a large number of interest rate scenarios and to then run each potential asset mix through each scenario. For any one scenario, the rate of return and the corresponding probability of insolvency for each asset mix is computed and the results stored. The process is then repeated for another scenario until all potential scenarios have been tested. The flow chart below (figure 2) illustrates the algorithm. The final step of the algorithm is to average all rates of return and insolvency probabilities for each asset mix and to then plot these results on a graph.

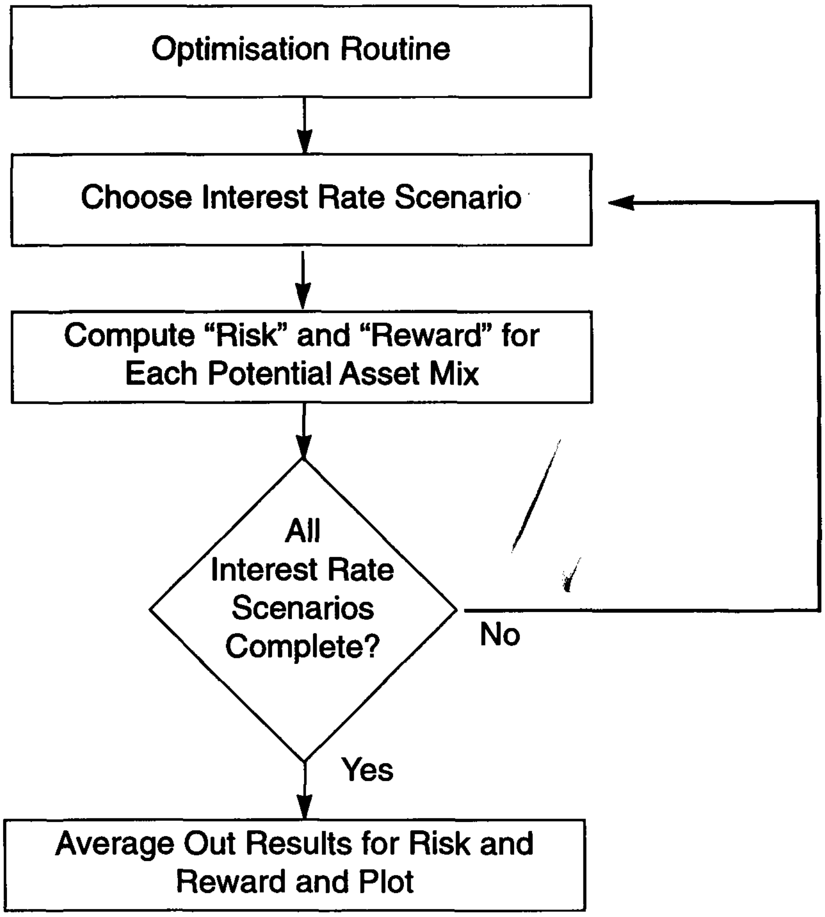

Figure 2. Flow chart of the model algorithm. A large number of interest rate scenarios are determined and then each potential asset mix is tested against each scenario. For any one scenario, the rate of return and the corresponding probability of insolvency is computed and the results stored. The process is repeated until all potential scenarios have been tested. The final step of the algorithm is to average all rates of return and insolvency probabilities for each asset mix and to then plot these results on a graph.

VIII Results of the equity/gilt model

The results of the model for each liability profile under consideration (ie for each rate of interest assumption) and using various combinations of equities and gilts are given in section F of the appendix. These results are also summarised in figure 3. Each risk/reward point is plotted and a curve drawn through these points to create a risk/reward profile for each rate of interest assumption used.

All points on any line which lie to the left of the minimum risk point can be ignored, since it is possible to achieve simultaneously a higher return and a lower risk by altering the mix of equities and gilts.

One of the interesting features of the chart is that the point at which the minimum level of risk is achieved is heavily dependent on the liability structure under consideration. At a rate of interest of 0% the minimum risk is achieved where 38% of the fund is held in equities and 62% of the fund is held in gilts. However, this minimum risk point shifts more and more towards a heavier weighting in equities as the rate of interest rises - reflective of the points made in the simple example given in the earlier parts of this paper - and at an 8% rate of interest the minimum risk point is not achieved until virtually 100% is held in equities.

The curves based on the low rate of interest assumptions look rather like traditional "efficient frontiers." This is not surprising - at relatively low rates of interest the nature of the liabilities becomes relatively unimportant so that the model reverts to the conventional "asset/asset" model. But at relatively high rates of interest the concept of an "efficient frontier" collapses, and at a rate of interest of 8% there are only one or two "efficient" points (where 100%, or close to 100%, is held in equities).

In practice, the "efficient frontier" in itself may be of limited use, because a life office may be required to hold certain asset categories. For example, there may be an investment policy constraint within the office that at least 50% of the portfolio must be held in gilts. Moreover, in many countries there are legal restrictions on the extent to which certain categories of asset may be held by life offices. Thus, this paper does not concentrate unduly on an analysis of the "efficient" combinations of the various asset classes, which in reality may be little more than an idealistic, theoretical concept.

The final part of the exercise is then to determine what is an acceptable level of risk; having decided on this it is possible to derive a uniquely defined optimal asset mix. For example, for the fund which has used a rate of interest assumption of 2% in its pricing assumptions, it may be appropriate to go 100% into equities (and therefore go for the maximum possible return) if a "probability of insolvency" level of around 25% were deemed acceptable.

Clearly, the setting of an acceptable level of risk is a largely subjective decision and, in practice, the usefulness of a model such as this would be in assessing the relative "riskiness" of various portfolio mixes rather than hoping to make any sense out of the absolute values generated for the risk and reward of any particular investment policy in isolation. Indeed, the absolute values for the probability of insolvency in the model look extremely high across the board, the result of the relatively large probability of a market crash in any one investment scenario - see Hardy for similar findings when using a stochastic model.

Although it should be stressed that a highly artificial liability profile is being considered in the illustration used here, the above results will come as no surprise to fund managers who are aware that a well diversified portfolio of equities should, in the long-term, considerably out-perform a well diversified gilt portfolio. Ironically, in this instance, it is where the rate of interest assumption used in pricing the liabilities is relatively low, hence giving more freedom in investment policy, that opportunities arise to invest in gilts - contrary to the textbook principle that the more freedom an office has to choose its investments the more scope it should have for investments in equities.

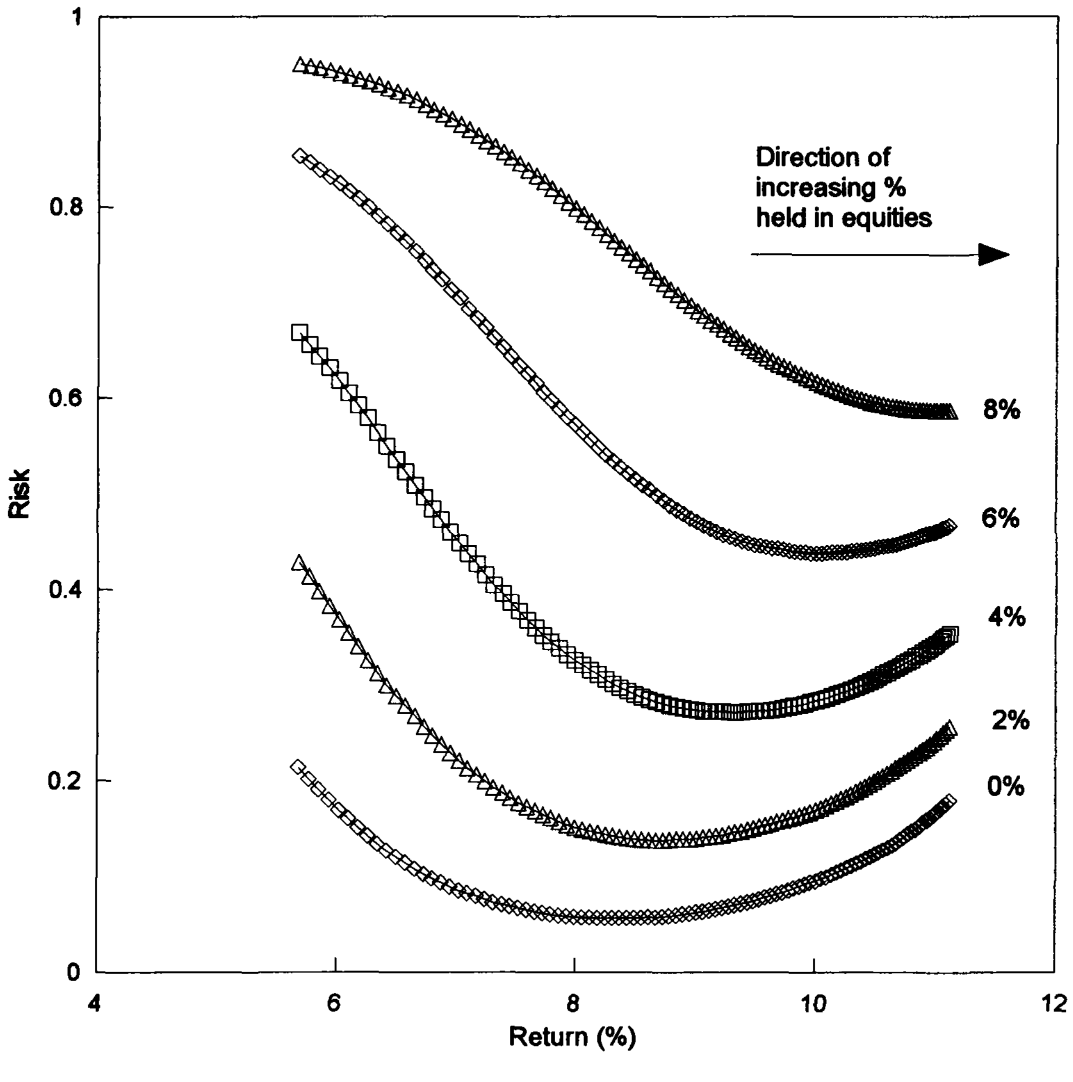

Figure 3. Equity/gilt risk/reward profiles. This graph plots "optimal" risk/reward points for the various possible combinations of equities and gilts under the various interest rate assumptions used in pricing the underlying liabilities. A curve is drawn through the points to create a risk/reward profile for each rate of interest.

The complete asset allocation model should incorporate the full range of assets available to the financial institution, which at the very least for a life office should include cash, property, and overseas equity and bond investment in addition to domestic equities and bonds. In order to hint at what effects the addition of extra asset categories may have, the model has been re-run with the addition of a single asset class, this being the commodity gold.

The cumulative probability distribution derived for this new asset class is also shown in section B of the appendix, and is again based on data for total returns going back to 1974 - this data has been kindly provided by Roger Murphy of the World Gold Council. The total returns are based on the price movement of gold over each calendar year expressed in terms of sterling. The equity and gilt distributions as previously defined have been maintained.

Figure 4 shows the plotted risk/reward points, using the same endowment liability profile as used in testing the two-dimensional equity/gilt model, with a rate of interest assumption of 0% (thus giving us a case very near to the traditional model of financial economics in that the nature of the liabilities will be relatively unimportant at such a low rate of interest). Each plotted point represents a combination of equities, gilts and gold in steps of 5% (eg 25% held in equities, with 30% in gilts, and 45% in gold). Again, the results are based on 50,000 runs, and again it is possible to construct an "efficient frontier" which represents a line which joins those points which are combinations of assets giving the minimum risk for any given rate of return. Points showing minimum risk for rates of return below approximately 9.1% are not "efficient" because it is possible to increase the rate of return from this level and simultaneously reduce the risk by choosing an alternative asset mix. Section G of the appendix summarises those asset mixes which may be regarded as "efficient" on this basis.

The shape of the curve in figure 4 is, not surprisingly, very similar to that shown for the 0% run in figure 3. For further illustration, the three-dimensional model has also been run for an 8% interest rate (see figure 5). Again, the shape of the curve is very similar to that shown for the 8% run in figure 3. Finally, to complete the picture, a run has been carried out at a 4% interest rate. Figure 6 shows the curve at the 4% interest rate and also the curves for the runs at 0% and 8%. These curves can be compared directly to the curves produced using the two-dimensional model as illustrated in figure 3. Comparison with figure 3 clearly brings out the similarity in curve shapes for liability profiles which are the same but also demonstrates that there is a general downshift in curve (ie a general reduction in risk) brought about by introducing the extra asset class. This general reduction in risk is not surprising as diversification into a wider range of asset categories should bring about an across-the-board reduction in risk.

How much credibility can be attached to the downward sloping curve produced by the 8% interest rate run? In practice it is unlikely that the actuary would use a rate of interest in pricing that was so high that this type of situation would be possible; the rate of interest is most likely to be pitched somewhere between the two extremes shown by the curves for the 0% and 8% runs.

The reader may be concerned to see that an 8% rate of interest assumption represents a highly optimistic pricing scenario. However, it should be born in mind that a fairly "tough" definition of risk has been adopted; a less conservative definition may place an 8% rate of interest assumption more towards the middle of our pricing expectation range, so that the more "traditional" upward sloping risk/reward curve should become apparent using the 8% basis.

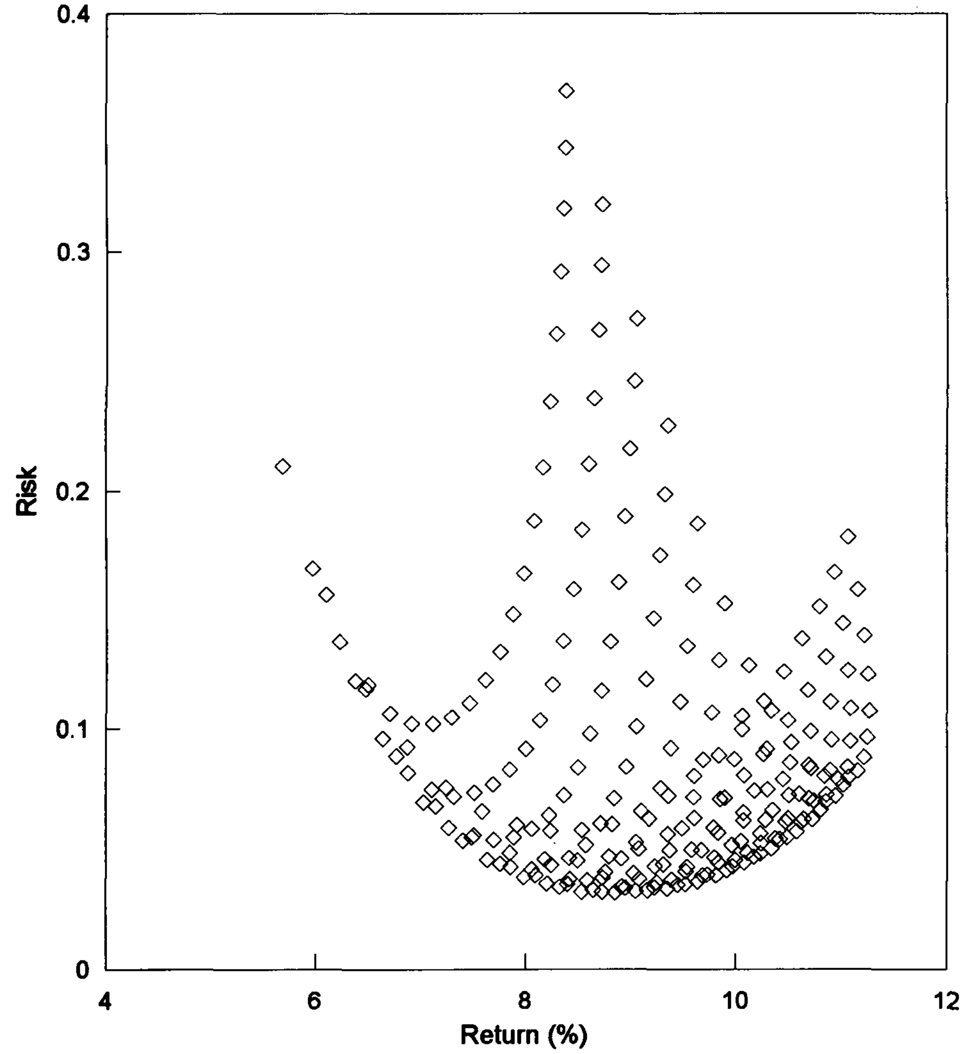

Figure 4. Equities/gilts/gold risk/reward profiles for i = 0 %. This graph plots each risk/reward point for combinations of equities, gilts and gold in steps of 5% (eg 25% held in equities, 30% held in gilts, and 45% in gold) where the interest rate assumption used in pricing the liabilities has been set to 0%. This run of the model gives a case very near to the traditional model of financial economics in that the nature of the liabilities becomes relatively unimportant at such a low rate of interest. (By reversing the x and y axis the conventional "efficient frontier" shape will be recognised.)

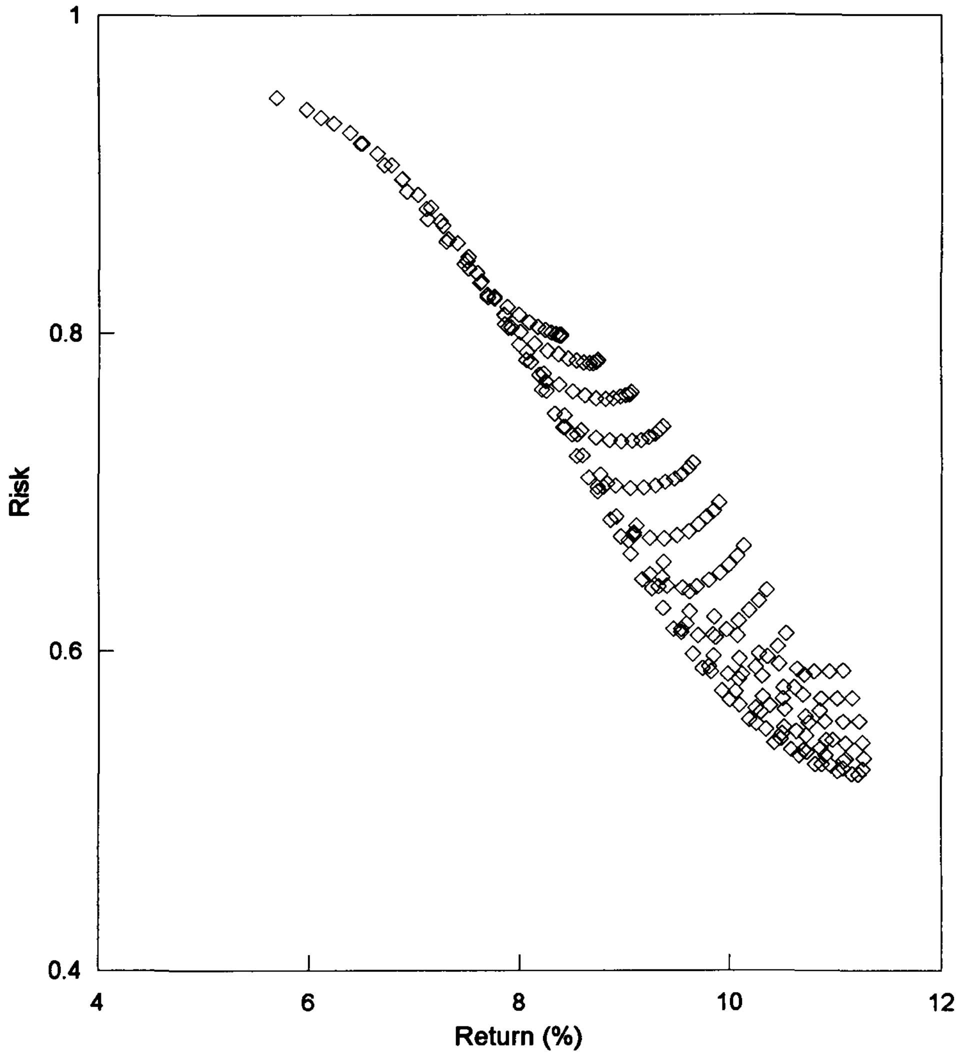

Figure 5. Equities/gilts/gold risk/reward profiles for i = 8%. This graph plots each risk/reward point for combinations of equities, gilts and gold in steps of 5% (eg 25% held in equities, 30% held in gilts, and 45% in gold) where the interest rate assumption used in pricing the liabilities has been set at 8%. At such a high rate of interest, the idea of an "efficient frontier" disappears. There are only one or two "efficient points", this being where 100%, or close to 100%, is held in equities since it is for this mix that maximum return and minimum risk is achieved simultaneously.

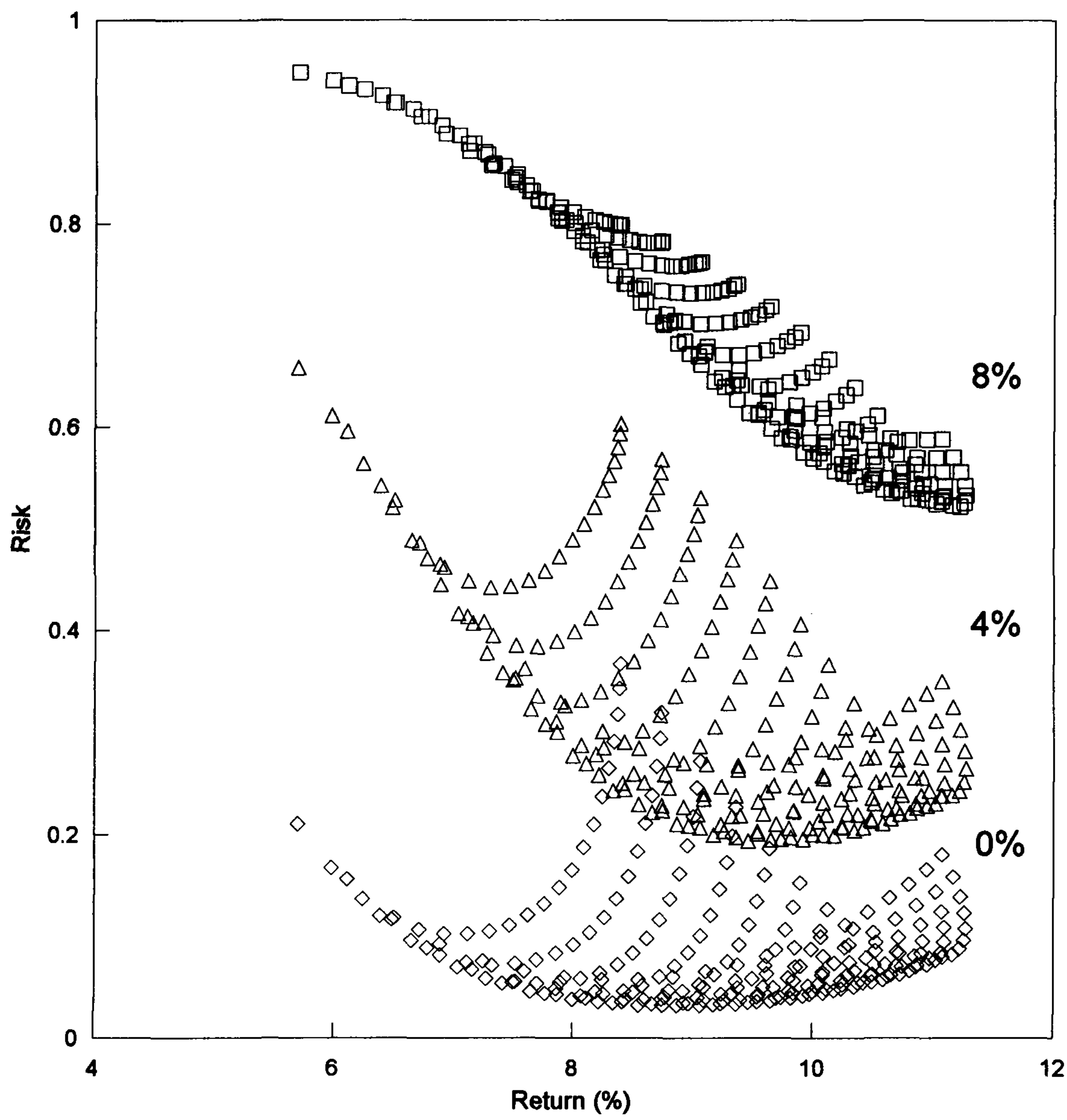

Figure 6. Equities/gilts/gold risk/reward profiles. This graph plots each risk/reward point for combinations of equities, gilts and gold in steps of 5% (eg 25% held in equities, 30% held in gilts, and 45% in gold) under the various interest rate assumptions used in pricing the underlying liabilities. The resulting "curves" are directly comparable to the curves produced by the two dimensional model as shown in figure 3. The comparison clearly brings out the similarity in curve shapes for the liability profiles which are the same but also demonstrates that there is a general downward shift in the curve (ie a general reduction in risk) brought about by introducing the extra asset class, evidence that diversification should bring about an across-the-board reduction in risk.

IX Summary and conclusion

This paper has described a new approach to asset allocation modelling for institutions that invest to meet liabilities. The model described has been shown to be consistent with conventional financial economics in that traditional risk/reward profiles become apparent where the nature of the liabilities is not considered or is relatively unimportant, but that such traditional risk/reward profiles may or may not become apparent once the nature of the liabilities is introduced. Thus, the traditional ideas of financial economics have been shown to be a special case of the more general asset allocation system using a true asset/liability model.

The paper has concentrated exclusively on the applications of an asset/liability model in the context of a life office issuing purely non-participating endowment life assurance. However, the principles can be applied equally to any type of financial institution which is concerned with "investing to meet liabilities".

The critical element of the model is in the definition of risk - it is not important that risk is taken as some measure of exposure to insolvency but rather that it somehow incorporates the liabilities.

Refinement of the model to incorporate the features of participating business should not be problematic; essentially this would be akin to lowering the rate of interest assumption used in pricing the liabilities which implicitly means a general reduction in the risk profile - and hence potentially greater freedom in investment policy.

The application of such a model to a pension fund poses some interesting issues, although these issues are rather specific to the particular country under consideration. On a general note, although it is recognised that pension funds can overcome deficit situations by increasing contribution rates from time to time, pension fund trustees may be interested to know whether a particular investment policy is more likely to lead to persistent deficits than another. Alternatively, if it is accepted that a primary objective of pension fund investment policy is to avoid unduly fluctuating contribution rates, then the asset/liability model could use a refined definition of risk - say, the probability of the fund falling outside a certain surplus or deficit range.

Incorporating the inflationary aspects of a pension fund model is problematic. However, some innovative research is currently being done by a number of actuaries whereby inflation is either linked to the yield curve in some way or alternatively stochastically modelled as an independent variable. It is not immediately obvious as to which of the two approaches may be more appropriate; perhaps, given the major uncertainties associated with inflation, there is no definitive model.

In concluding, there remains much exciting asset/liability modelling work to be done. Dramatic developments in the coming years can be expected as micro-computer processing power becomes more widely appreciated, making the type of stochastic model described in this paper a standard tool of financial analysis.

Appendices

A Annual total returns for equities and gilts 1923-1992 inclusive, and gold 1974-1992 inclusive (percent).

This appendix shows the basic raw data underlying all the calculations presented in this paper. The data for equities and gilts is taken from the appendices to the paper "The Expected Return on United Kingdom Equities and an Implication for Pension Fund Valuations" by P D Jones (JIA, 120, 253), and the data for gold has been provided by Roger Murphy of the World Gold Council.

Year	Equities	Gilts	Gold	Year	Equities	Gilts	Gold
1923	11.8	4.3		1958	47.9	17	
1924	21.9	7.3		1959	54.8	0.9	
1925	31.7	0.4		1960	1.8	-7	
1926	3.2	2.9		1961	1.7	-8.1	
1927	17.4	7.2		1962	0.4	24.7	
1928	20.8	5.7		1963	19.5	3.7	
1929	-12.2	-1.6		1964	-5.9	-2.3	
1930	-11.6	13.1		1965	12.4	4.4	
1931	-15.5	-0.4		1966	-5.5	4.2	
1932	35.2	40.1		1967	38.1	2.6	
1933	31.2	3.2		1968	41.7	-2.4	
1934	24.4	27.7		1969	-11.7	0.2	
1935	14	-3.1		1970	-1.9	3.6	
1936	19	0.2		1971	45.2	27.3	
1937	-12.8	-9.3		1972	21.7	-3.8	
1938	-10.2	-2.1		1973	-32.1	-8.9	
1939	2.1	0.9		1974	-49.4	-15.2	68.2
1940	-4.5	15.9		1975	149.6	36.8	6.1
1941	22.9	10.6		1976	-1.1	13.7	-5.5
1942	17.8	2.7		1977	57.2	44.8	23.6
1943	11.5	-0.5		1978	12.1	-1.8	19
1944	12.5	5.7		1979	9.7	4.1	48.7
1945	5.9	14.8		1980	34.1	20.9	75.2
1946	17.9	10.7		1981	12.2	1.8	-15.1
1947	-2.3	-14.3		1982	28.8	51.3	-2.8
1948	-3.8	0.7		1983	28.4	15.9	29.4
1949	-5.8	-8.9		1984	29.8	6.8	-3.5
1950	10.9	4		1985	21.6	11	-8.6
1951	8.5	-9.6		1986	26.4	11	0.7
1952	-0.1	-0.8		1987	8.4	16.3	9.7
1953	24.2	14		1988	12.8	9.3	-9.8
1954	48.6	6.1		1989	33.8	5.5	-5.1
1955	10.9	-10.1		1990	-6.6	4.6	-7.4
1956	-9	-3.2		1991	16.8	18	-4.9
1957	-1.1	-6.2		1992	2.8	13.7	-3.7

B Cumulative distributions for equity and gilt returns using historical total rates of return 1923-1992 inclusive, and for gold returns using historical rates of return 1974-1992 inclusive.

This section of the appendix shows the cumulative distributions for the three asset classes brought into the model in this paper, derived from the basic data of appendix A.

Equities

Rate of return (%)	Cumulative probability
-50	0.00%
-45	1.43%
-40	1.43%
-35	1.43%
-30	2.86%
-25	2.86%
-20	2.86%
-15	4.29%
-10	11.43%
-5	18.57%
0	28.57%
5	37.14%
10	42.86%
15	57.14%
20	65.71%
25	75.71%
30	81.43%
35	87.14%
40	90.00%
45	91.43%
50	95.71%
55	97.14%
60	98.57%
65	98.57%
70	98.57%
75	98.57%
80	98.57%
85	98.57%
90	98.57%
95	98.57%
100	98.57%
105	98.57%
110	98.57%
115	98.57%
120	98.57%
125	98.57%
130	98.57%
135	98.57%
140	98.57%
145	98.57%
150	100.00%

Gilts

Rate of return (%)	Cumulative probability
-20	0.00%
-15	1.43%
-10	4.29%
-5	14.29%
0	30.00%
5	57.14%
10	68.57%
15	81.43%
20	88.57%
25	91.43%
30	94.29%
35	94.29%
40	95.71%
45	98.57%
50	98.57%
55	100.00%

Gold

Rate of return (%)	Cumulative probability
-20	0.00%
-15	5.26%
-10	5.26%
-5	31.58%
0	52.63%
5	57.89%
10	68.42%
15	68.42%
20	73.68%
25	78.95%
30	84.21%
35	84.21%
40	84.21%
45	84.21%
50	89.47%
55	89.47%
60	89.47%
65	89.47%
70	94.74%
75	94.74%
80	100.00%

C A mathematical description of the application of the Monte Carlo sampling method used in the model.

If the rate of return on a particular asset class is defined as a random variable, S, then the cumulative probability distribution of the rate of return on a particular asset class can be expressed as F(s) such that

$$F_u(s) = F_{u-1}(s_{u-1}) + f_u \cdot (s - s_{u-1}) \qquad s_{u-1} \leq s < s_u$$

for all $u = 1, \ldots, z$ where s_u, f_u are known constants

with $F_0(s_0) = 0$

and the probability density function f(s) is defined in a form of a step function as

$$f(s) = \begin{cases} 0 & s \leq s_0 \\ f_1 & s_0 \leq s < s_1 \\ \vdots & \vdots \\ f_z & s_{z-1} \leq s < s_z \\ 0 & \text{otherwise} \end{cases}$$

$$\text{where } f_u = \frac{f'_u}{s_u - s_{u-1}}$$

such that $\int f(s).ds = \sum \int f_u \,.\, dt = 1$

Having defined the distribution of the random variable, S, it is now possible to demonstrate how the random variable S is simulated (ie how samples of the observation of the variable S are generated).

Note that F(s) is a non-decreasing function.

Let Y have a uniform distribution on $0 \leq y \leq 1$.

Then $Y = F(S)$ or $S = F^{-1}(Y)$ implies that the random variable S has a distribution function F(s).

It is then possible to generate a sample of observations for variable Y with a uniform distribution from the computer (ie a sample of random numbers on [0,1]), and to then use $S = F^{-1}(Y)$ to obtain a corresponding sample of observed values for the variable S.

It is now necessary to define $F^{-1}(Y)$, ie the observed value s of S if $Y = y$.

If $F_{n-1}(s_{n-1}) < y < F_n(s_n)$ then, for all n = 1,...z,

$$s = s_{n-1} + \frac{y - F(s_{n-1})}{f_n} \qquad \text{———} \quad (1)$$

where $F_n(s_n)$, f_n, s_n are defined above for all n = 1, ...,z

Thus the simulation process generates a sample for the random numbers y_1, ...y_t, and a respective sample of observations s_1, ... s_t can be calculated readily from equation (1).

The below graphs illustrate the probability functions f(s) and F(s) for the rate of return on gilts (figures 7 and 8 respectively).

From the graphs, the density functions f(s) can be expressed as (in percentage value):

f(s)	=	0	s < -20
	=	0.286	-20 ≤ s < -15
	=	0.572	-15 ≤ s < -10
	=	2	-10 ≤ s < -5
	=	3.142	-5 ≤ s < 0
		|	|
	=	0.286	50 ≤ s < 55
	=	0	s ≥ 55

Note: For comparison with section B of the appendix, each f(s) shown here needs to be multiplied by 5.

Assume that a computer generated random number is 24.4896% then the simulated return is:

$$s = -5 + \frac{24.4896 - 14.29}{3.142} = -1.75\%$$

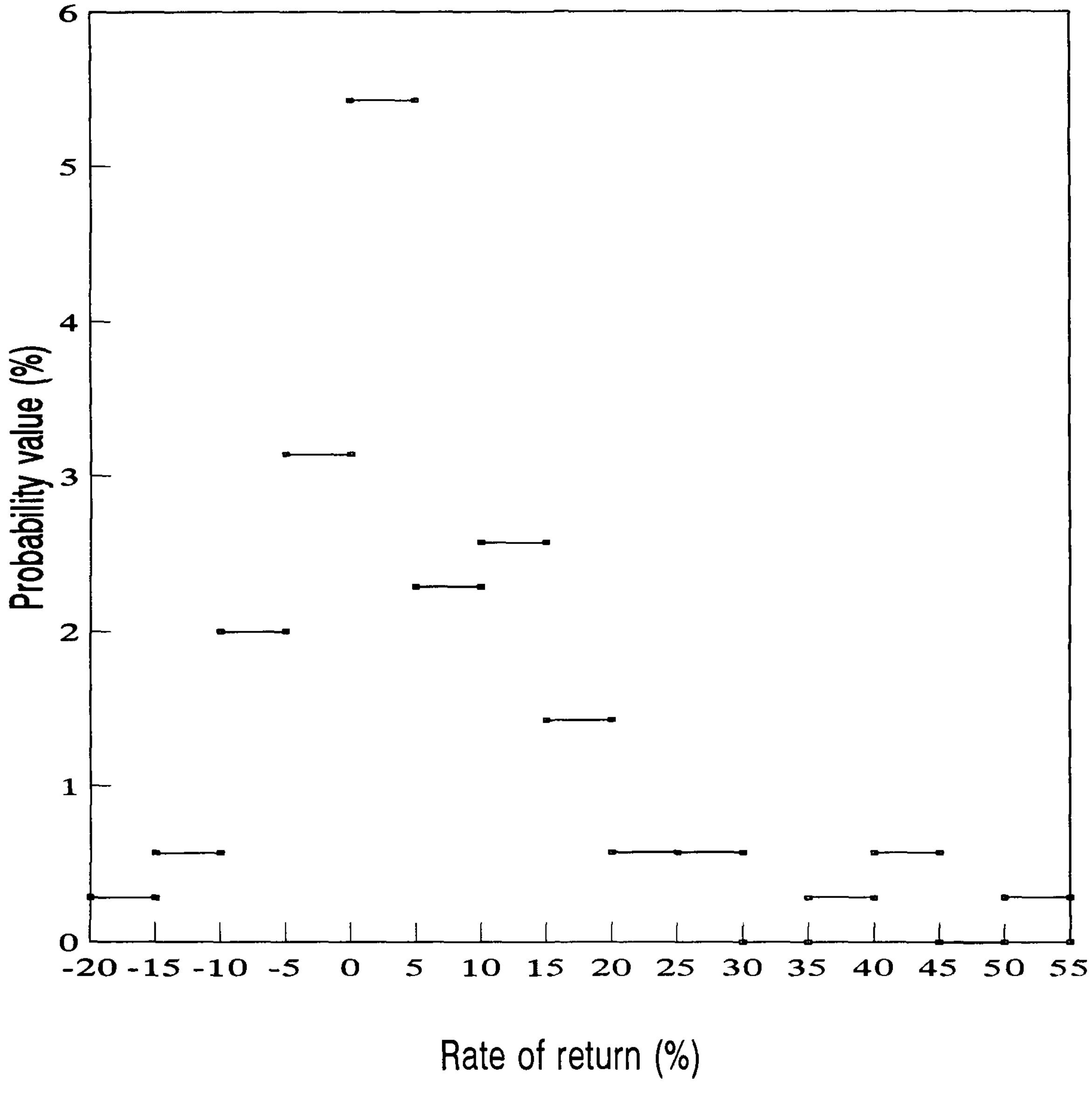

Figure 7. Probability density function f(s). The height of each step is $f'_u/(s_u-s_{u-1})$ where f'_u, s_u, s_{u-1} are defined (see figure 8).

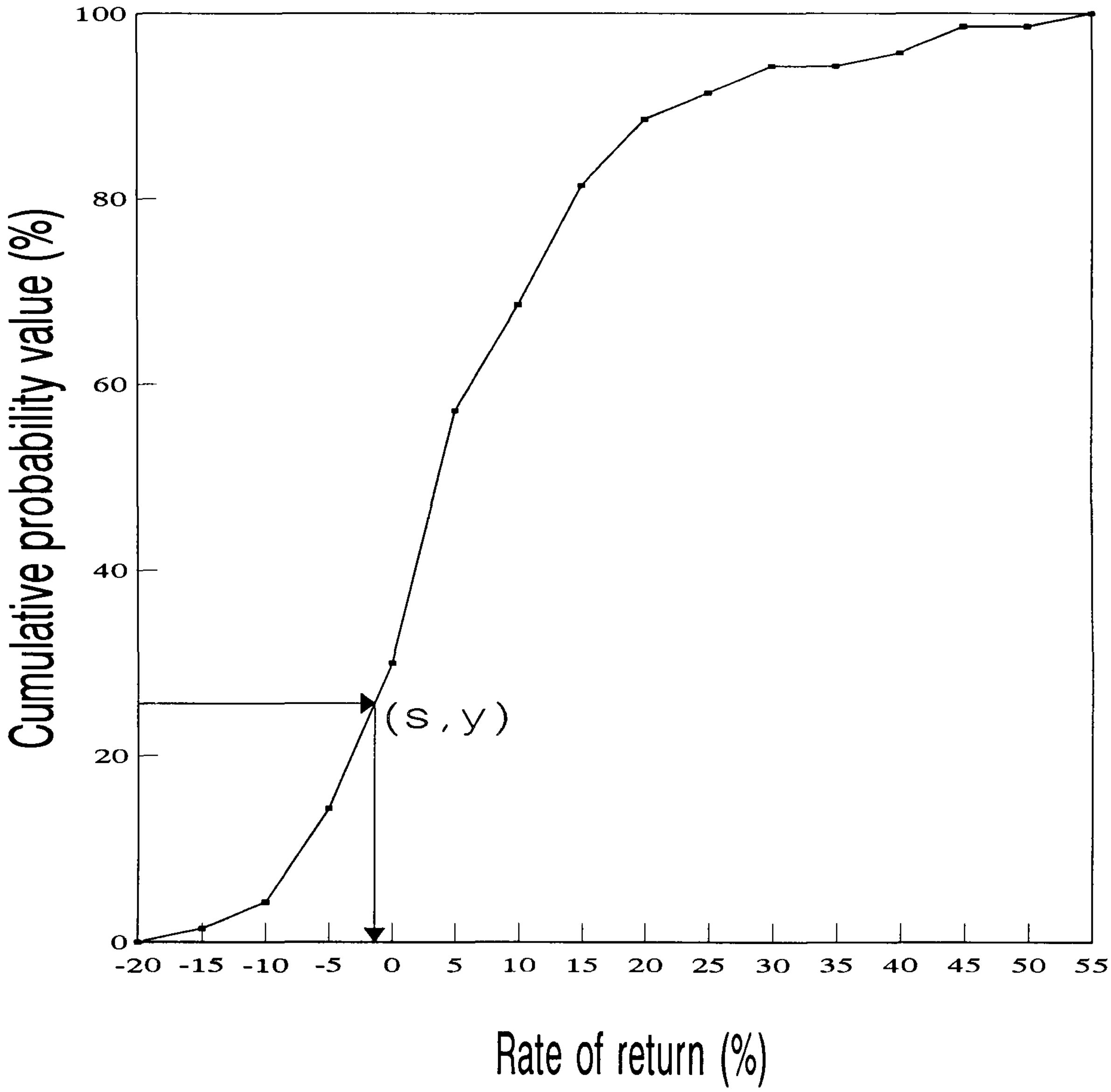

Figure 8. Probability distribution function F(s). For y, a computer generated random number,

$y = F_u(s)$ in [0,1]

$$s = s_{u-1} + \frac{y - F(s_{u-1})}{f_u}$$

with $s_{u-1} = -5\%$
$F(s_{u-1}) = 14.29\%$
$f_u = 3.142\%$

D Generation of a single projected investment scenario for equities and gilts using a Monte Carlo sampling method in conjunction with the derived cumulative probability distributions.

This section presents the results of a single simulation using a Monte Carlo sampling method in conjunction with the cumulative probability distributions derived for equities and gilts.

Equities			Gilts		
Year	Random Number	Derived rate of return (%)	Year	Random Number	Derived rate of return (%)
1	0.920635	45.74	1	0.138160	-5.24
2	0.015310	-34.65	2	0.232189	-2.16
3	0.724067	23.35	3	0.758383	12.83
4	0.313090	1.60	4	0.113507	-6.47
5	0.006140	-47.85	5	0.384087	1.55
6	0.270206	-0.77	6	0.989472	53.68
7	0.386301	6.30	7	0.469435	3.12
8	0.586671	15.89	8	0.572549	5.05
9	0.289659	0.23	9	0.051598	-9.57
10	0.213473	-3.61	10	0.822271	15.56
11	0.400113	7.51	11	0.391379	1.68
12	0.430702	10.07	12	0.710867	10.98
13	0.363476	4.54	13	0.863839	18.47
14	0.175675	-5.70	14	0.238886	-1.95
15	0.740600	24.17	15	0.098922	-7.20
16	0.621061	17.90	16	0.700823	10.59
17	0.210501	-3.76	17	0.372080	1.33
18	0.003005	-48.95	18	0.200743	-3.16
19	0.023485	-31.79	19	0.244896	-1.75
20	0.699254	22.11	20	0.417906	2.17
21	0.704270	22.36	21	0.455713	2.87
22	0.557279	14.51	22	0.622393	7.23
23	0.005115	-48.21	23	0.060257	-9.13
24	0.472407	11.53	24	0.940730	29.62
25	0.108943	-10.38	25	0.822261	15.56
26	0.215939	-3.49	26	0.871513	19.01
27	0.082465	-12.23	27	0.854089	17.79
28	0.513277	12.96	28	0.051033	-9.59
29	0.579305	15.46	29	0.507365	3.82
30	0.372050	5.06	30	0.268902	-0.99

E Mathematical justification for 50,000 simulations.

The rates of return of the various asset mixes can be determined mathematically which leaves the problem of estimating the value of the probability of insolvency, p. Thus the objective is to find:

$$p_1 = p \pm x\%p = (1 \pm x\%)p \quad \text{\_\_\_\_\_\_\_\_\_\_ (1)}$$

Assume that n simulations are performed, then if we denote the number of occasions that R = 1 by X, then each simulation produces a result of R = 1 with independent probability p and R = 0 with probability 1-p. This implies X has a Binomial distribution with parameters n and p [B(n,p) with mean = np and variance = np (1-p)]. Using the Central Limit Theorem, when n is large:

$X\text{-}np/\sqrt{np(1\text{-}p)}$ is asymptotically normally distributed with mean 0 and variance 1, i.e., N(0,1) ______ (2)

The simulation process estimates $p_1 = X/n$

Therefore equation (1) implies that the following is required:

$$P((1\text{-}x)p < X/n < (1+x)p) = 0.95 \quad \text{\_\_\_\_\_\_ (3)}$$

that is to say the objective is to estimate the probability of insolvency p=P(R=1) for a 95% confidence interval of x% either side of p.

Equation (2) and equation (3) imply that

$$n = (1.96/x)^2 \frac{(1\text{-}p)}{p}$$

Thus, if the probability of insolvency p lies in the range (0.1, 0.9) then the accuracy of the results (expressed in terms of x) lies in the range (0.3%, 3%) if the number of simulations is taken to be 50,000.

F Risk/return results using various interest rate assumptions (based on 50,000 simulated projected investment scenarios).

This section of the appendix shows the result of the model run for equities and gilts for the various liability profiles under consideration (ie for each rate of interest assumption).

Proportion of asset class		Rate of	Risk				
Equities	Gilts	return (%)	0%	2%	4%	6%	8%
100	0	11.1	0.179	0.256	0.354	0.467	0.587
99	1	11.1	0.176	0.253	0.351	0.465	0.587
98	2	11.1	0.173	0.250	0.349	0.464	0.587
97	3	11.0	0.170	0.247	0.346	0.462	0.587
96	4	11.0	0.167	0.244	0.343	0.461	0.587
95	5	11.0	0.164	0.241	0.341	0.460	0.587
94	6	11.0	0.162	0.238	0.339	0.458	0.587
93	7	10.9	0.158	0.235	0.336	0.457	0.587
92	8	10.9	0.156	0.232	0.333	0.456	0.587
91	9	10.9	0.152	0.229	0.331	0.455	0.587
90	10	10.8	0.149	0.227	0.329	0.453	0.587
89	11	10.8	0.147	0.224	0.327	0.452	0.588
88	12	10.8	0.144	0.221	0.324	0.450	0.588
87	13	10.7	0.141	0.219	0.322	0.449	0.589
86	14	10.7	0.138	0.216	0.320	0.448	0.590
85	15	10.7	0.135	0.213	0.317	0.447	0.590
84	16	10.6	0.132	0.210	0.315	0.446	0.591
83	17	10.6	0.130	0.207	0.313	0.445	0.592
82	18	10.6	0.128	0.205	0.311	0.445	0.593
81	19	10.5	0.125	0.202	0.308	0.444	0.593
80	20	10.5	0.123	0.199	0.306	0.443	0.595
79	21	10.5	0.121	0.197	0.304	0.442	0.596
78	22	10.4	0.119	0.194	0.301	0.441	0.597
77	23	10.4	0.116	0.191	0.299	0.441	0.599
76	24	10.3	0.114	0.188	0.297	0.441	0.600
75	25	10.3	0.112	0.186	0.296	0.440	0.602
74	26	10.3	0.109	0.184	0.293	0.439	0.604
73	27	10.2	0.107	0.181	0.292	0.438	0.606
72	28	10.2	0.104	0.179	0.290	0.438	0.608
71	29	10.1	0.102	0.176	0.288	0.439	0.610
70	30	10.1	0.099	0.174	0.286	0.438	0.612
69	31	10.1	0.098	0.171	0.285	0.437	0.614
68	32	10.0	0.096	0.169	0.284	0.437	0.617
67	33	10.0	0.093	0.167	0.282	0.437	0.619
66	34	9.9	0.092	0.165	0.281	0.438	0.622
65	35	9.9	0.089	0.164	0.279	0.439	0.625
64	36	9.8	0.087	0.162	0.278	0.439	0.627
63	37	9.8	0.085	0.160	0.277	0.440	0.630
62	38	9.7	0.083	0.159	0.275	0.441	0.633
61	39	9.7	0.081	0.157	0.275	0.442	0.636
60	40	9.6	0.080	0.156	0.274	0.443	0.639
59	41	9.6	0.078	0.154	0.273	0.444	0.642
58	42	9.5	0.076	0.152	0.273	0.445	0.646
57	43	9.5	0.075	0.151	0.272	0.446	0.650
56	44	9.4	0.073	0.149	0.272	0.448	0.654
55	45	9.4	0.072	0.148	0.271	0.450	0.658
54	46	9.3	0.070	0.146	0.271	0.452	0.663
53	47	9.3	0.069	0.145	0.271	0.454	0.667
52	48	9.2	0.068	0.144	0.272	0.457	0.672
51	49	9.2	0.066	0.142	0.272	0.460	0.677

Proportion of asset class		Rate of return (%)	Risk				
Equities	Gilts		0%	2%	4%	6%	8%
50	50	9.1	0.065	0.142	0.272	0.463	0.681
49	51	9.1	0.064	0.141	0.273	0.467	0.686
48	52	9.0	0.063	0.140	0.274	0.470	0.691
47	53	9.0	0.062	0.140	0.274	0.474	0.697
46	54	8.9	0.061	0.139	0.276	0.479	0.702
45	55	8.8	0.060	0.139	0.277	0.483	0.708
44	56	8.8	0.059	0.138	0.279	0.487	0.714
43	57	8.7	0.058	0.138	0.280	0.493	0.720
42	58	8.7	0.058	0.138	0.282	0.498	0.726
41	59	8.6	0.058	0.138	0.284	0.504	0.732
40	60	8.6	0.057	0.138	0.287	0.509	0.738
39	61	8.5	0.057	0.139	0.290	0.515	0.745
38	62	8.4	0.057	0.140	0.293	0.521	0.751
37	63	8.4	0.057	0.141	0.297	0.527	0.758
36	64	8.3	0.057	0.143	0.301	0.534	0.764
35	65	8.2	0.057	0.144	0.306	0.540	0.770
34	66	8.2	0.057	0.146	0.311	0.548	0.778
33	67	8.1	0.058	0.147	0.316	0.556	0.784
32	68	8.1	0.058	0.149	0.321	0.564	0.792
31	69	8.0	0.058	0.152	0.326	0.572	0.798
30	70	7.9	0.059	0.154	0.332	0.580	0.804
29	71	7.9	0.060	0.157	0.338	0.588	0.812
28	72	7.8	0.061	0.161	0.345	0.597	0.819
27	73	7.7	0.062	0.165	0.352	0.606	0.826
26	74	7.7	0.064	0.168	0.360	0.616	0.832
25	75	7.6	0.066	0.172	0.367	0.626	0.838
24	76	7.5	0.067	0.177	0.377	0.634	0.845
23	77	7.5	0.069	0.182	0.386	0.644	0.851
22	78	7.4	0.071	0.187	0.395	0.653	0.857
21	79	7.3	0.073	0.193	0.404	0.663	0.863
20	80	7.2	0.077	0.199	0.414	0.674	0.868
19	81	7.2	0.079	0.206	0.426	0.683	0.874
18	82	7.1	0.082	0.213	0.436	0.693	0.880
17	83	7.0	0.085	0.220	0.448	0.704	0.886
16	84	7.0	0.089	0.229	0.459	0.713	0.891
15	85	6.9	0.093	0.237	0.472	0.724	0.896
14	86	6.8	0.098	0.247	0.484	0.733	0.901
13	87	6.7	0.103	0.256	0.496	0.743	0.906
12	88	6.7	0.108	0.267	0.509	0.753	0.911
11	89	6.6	0.114	0.278	0.523	0.763	0.916
10	90	6.5	0.121	0.288	0.536	0.772	0.920
9	91	6.4	0.128	0.300	0.549	0.782	0.924
8	92	6.3	0.134	0.313	0.563	0.790	0.928
7	93	6.3	0.143	0.326	0.579	0.799	0.931
6	94	6.2	0.151	0.340	0.592	0.808	0.934
5	95	6.1	0.160	0.355	0.605	0.816	0.937
4	96	6.0	0.169	0.368	0.618	0.824	0.940
3	97	5.9	0.180	0.383	0.632	0.831	0.943
2	98	5.8	0.191	0.398	0.644	0.838	0.946
1	99	5.8	0.201	0.413	0.656	0.846	0.948
0	100	5.7	0.214	0.428	0.668	0.853	0.949

G Optimal asset mixes at a 0% interest rate assumption for equities, gilts and gold (based on 50,000 simulated investment scenarios).

This section of the appendix shows the results of the model run for equities, gilts and gold for a liability profile where liabilities have been priced on a 0% rate of interest assumption.

Proportion of asset class (%)			Rate of return (%)	Risk
Equities	Gilts	Gold		
30	45	25	9.1	0.032
35	40	25	9.4	0.033
35	35	30	9.5	0.035
40	35	25	9.6	0.036
40	30	30	9.8	0.039
45	30	25	9.9	0.041
40	25	35	10.0	0.043
45	25	30	10.1	0.044
50	25	25	10.2	0.047
45	20	35	10.2	0.048
50	20	30	10.3	0.050
55	20	25	10.4	0.054
50	15	35	10.5	0.055
55	15	30	10.6	0.057
50	10	40	10.6	0.061
55	10	35	10.7	0.062
60	10	30	10.8	0.066
55	5	40	10.9	0.070
60	5	35	10.9	0.072
65	5	30	11.0	0.077
60	0	40	11.1	0.080
65	0	35	11.2	0.082
70	0	30	11.2	0.088
75	0	25	11.3	0.096
80	0	20	11.3	0.107

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