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To Err is Human; to Measure the Error, Divine

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19 June 2014

Introduction – Examples of Model Output

- Range of pension deficits over time
- Range of cash contribution requirements to meet a deficit
- How much to contribute each year to target a level of pension
- Likelihood of a deficit being eliminated within a given period
- Likelihood of insolvency / pension scheme entering PPF
- Likelihood of individual's fund exceeding a certain level at retirement
- Meeting performance conditions for share incentive plans
- etc!



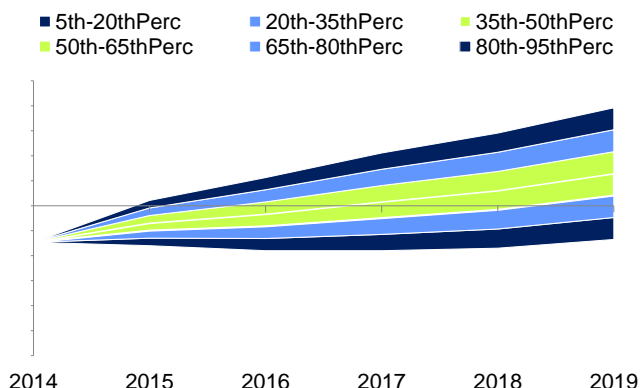
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Example: pension scheme funding position

- Asset Liability Modelling often used to illustrate range of outcomes
- Can be used to 'prove' that the downside risk is manageable / affordable across a long time horizon
- Attempts to quantify the risks of investing in risky asset classes



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Introduction: The problem

- Typical modelling / Monte Carlo approach:
 - We estimate a model with parameters
 - Produce a large numbers of scenarios all based on a single model and single set of parameters
 - Assign probabilities to outcomes and compare to risk appetite
- This supposes that we 'know' the true parameters and underlying distribution / model
- However, these parameters are subject to uncertainty
- There's another layer of risk associated with a lack of knowledge. Lee and Wilkie (2000)¹ gave suggestions to deal with parameter uncertainty. However, these are not widely used.



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¹Lee, P. J., and A. D. Wilkie. *A comparison of stochastic asset models*. (2000)

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How do YOU feel about uncertainty?

Volunteer. Possible cash prizes!



How do YOU feel about uncertainty?

- Container has 30 red, blue and yellow balls
- There are 10 red balls. Each of the remaining 20 balls is either blue or yellow (you don't know how many of each)
- A ball is to be drawn randomly. You're offered a gamble on the outcome
- Consider the following choice of gambles:

| Gamble | |
|----------------|-----------------|
| A | B |
| Win 100 if red | Win 100 if blue |

- Which gamble do you choose?



How do YOU feel about uncertainty?

- Same container.
(Recap: 10 red balls; each of the remaining 20 is either blue or yellow)
- Now choose between the following gambles:

| Gamble 2 | |
|--------------------------|---------------------------|
| C | D |
| Win 100 if red or yellow | Win 100 if blue or yellow |

- What is your preference?



The Ellsberg Paradox

- Most people prefer gambles A and D (and are not simply indifferent between the alternative options)

| Gamble 1 | | Gamble 2 | |
|---------------|----------------|-------------------------|--------------------------|
| A | B | C | D |
| 100 for a red | 100 for a blue | 100 for a red or yellow | 100 for a blue or yellow |
| 1/3 | ? | ? | 2/3 |

- Is it consistent to prefer A over B and D over C?
- People don't make decisions based on a single probability law
- Most people demonstrate ambiguity aversion



The Ellsberg Paradox

- The Ellsberg Paradox highlights the distinction between stochastic risk and model ambiguity (also called Knightian uncertainty)
- We like to know our odds... and we are willing to put a premium on that privilege
- The underlying variance of investment returns constitutes risk. Our inability to estimate this variance precisely constitutes ambiguity.



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Pensions Example

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Simple investor problem

- Investor makes one-off lump sum investment
- Expected log return and volatility are estimated using (finite) historical data
- A model, using these parameters, then projects the future returns in order to estimate their distribution
- Require 90% likelihood that lump sum will exceed £1,000 at the end of a fixed timeframe (e.g. 1, 10, 20, 50 years)
- How can we allow for the errors in the parameter estimates when assessing the confidence level?



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Investor problem- the model

- Consider investing a lump sum at time 0 for time T
- Assume investment returns follow Geometric Brownian motion
- If $S(t)$ is the value of the sum at time t then we can write:

$$\ln(S(t+1)) = \ln(S(t)) + \mu + \sigma Z(t)$$
- Log returns: geometric mean μ , variance σ^2
- The $Z(t)$ are independent identically distributed with mean zero and standard deviation 1.



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Investor problem- the model

- Estimate $\hat{\mu}$ and $\hat{\sigma}$ from historical data
- Given a stable underlying return distribution, and estimated parameters, we seek:

$$\Pr\left(\ln\left[\frac{S(T)}{S(0)}\right] \leq T\hat{\mu} - k\sqrt{T}\hat{\sigma}\right) = 0.1$$

- What k satisfies the above equation for different underlying return distributions?



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Investor problem model- methodology

- 3 Variables:
 - Underlying return distribution
 - Number of historical years to estimate parameters (5, 10, 50 or infinite)
 - Number of years to invest into the future (1, 10, 20 or 50)



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Investor problem model- methodology

- For each combination of variables:
 - A Monte Carlo simulation is performed with 100,000 runs
 - For each run i , k_i is then calculated as

$$k_i = \frac{\ln(S_i(T)) - T\hat{\mu}_i}{\hat{\sigma}_i \sqrt{T}}$$

- $S_i(T)$ is the fund value at the end of the investment period for run i (initial value $S_i(0) = 1$)
- The 10th percentile is then taken to find k such that

$$\text{Prob} \left[-k > \frac{\ln(S(T)) - T\hat{\mu}}{\hat{\sigma} \sqrt{T}} \right] = 0.1$$



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Investor problem model- methodology

$$k_i = \frac{\ln(S_i(T)) - T\hat{\mu}_i}{\hat{\sigma}_i \sqrt{T}}$$

- k_i (and subsequently k) is a function of parameters estimated from a finite quantity of historical data
- Therefore, there will be some error incorporated into $\hat{\mu}_i$ and $\hat{\sigma}_i$; the error is worse if there are fewer data points.



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Tackling the problem analytically

- When the underlying log returns distribution is normal, then the distribution of k follows a Student T distribution
- If n is the investment horizon, and m is the number of historical years of data, then k follows the distribution:

$$k \sim \sqrt{\left(1 + \frac{n}{m}\right)} T_{m-1}$$

- As m tends to infinity, k tends towards (the underlying) normal distribution
- No corresponding formula known for other distributions



Investor problem model- The Results (table of k)

- The results below are shown for the Gaussian distribution for the set of parameters listed below

| p=10% | | μ=5% | | | σ=10% |
|--------------------|----|------------------|------|------|----------|
| | | Years of history | | | |
| | | 5 | 10 | 50 | Infinite |
| Investment horizon | 1 | 1.68 | 1.45 | 1.31 | 1.28 |
| | 10 | 2.65 | 1.96 | 1.42 | 1.28 |
| | 20 | 3.44 | 2.41 | 1.54 | 1.28 |
| | 50 | 5.04 | 3.41 | 1.84 | 1.29 |



Investor problem model- The Results

- Alternatively, we can derive the amount needed now to be 90% sure of having >£1,000 at the end of the investment period

$p=10\%$

$\mu=5\%$

$\sigma=10\%$

Years of history

| | 5 | 10 | 50 | Infinite |
|--------------------|-------|-------|-------|----------|
| Investment horizon | | | | |
| 1 | 1,114 | 1,096 | 1,083 | 1,081 |
| 10 | 1,335 | 1,108 | 949 | 910 |
| 20 | 1,567 | 1,044 | 728 | 651 |
| 50 | 2,345 | 852 | 300 | 204 |

Time Value of Money

Impact of parameter uncertainty



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What does k represent?

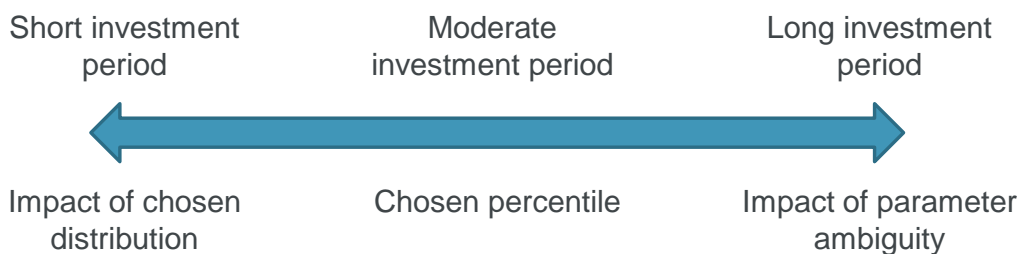
- Does k represent the level of prudence needed to be 90% sure of realising a target return?
- Yes in model framework, no in reality
- Our approach corrects for parameter uncertainty, but there are still other heroic assumptions that assume away risk (for example, IID returns)
- We cannot quantify all types of uncertainty (in particular model ambiguity) - we should keep this in mind when interpreting results



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Investor problem model- Important factors across different timeframes



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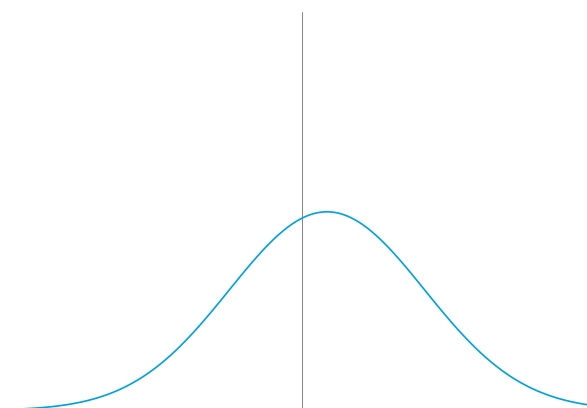
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Distributions- a recap

- Normal Distribution
 - Widely used
 - Arguably has unrealistically thin tails for modelling extreme tail risks

CDF: $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$$



distribution of log returns



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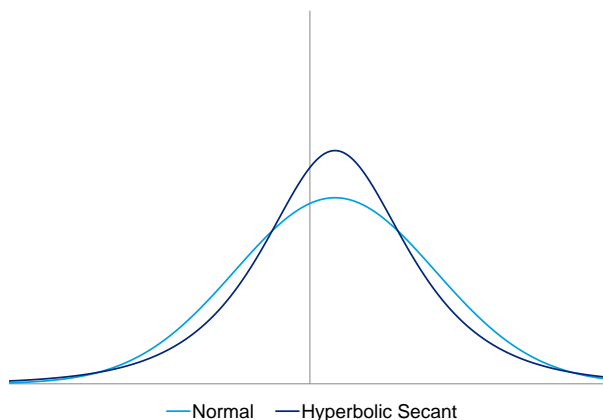
Distributions- a recap

- Hyperbolic Secant Distribution

- Alternatively known as the inverse-cosh distribution
- Has a more acute peak and heavier tails than the normal distribution

CDF:

$$F(x) = \frac{2}{\pi} \arctan \left[\exp \left(\frac{\pi}{2} \left(\frac{x - \mu}{\sigma} \right) \right) \right]$$



distribution of log returns



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Distributions- a recap

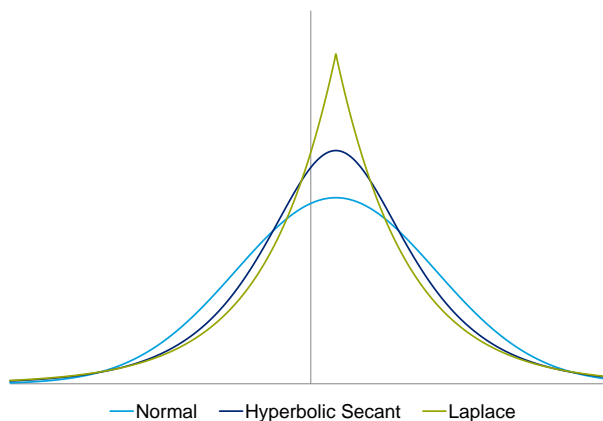
- Laplace Distribution

- Fatter tails than the normal distribution
- Laplace density is expressed in terms of absolute distance from the mean

CDF:

$$F(x) = \frac{1}{2} + \frac{1}{2} \operatorname{sgn}(x - \mu) \left(1 - e^{-\frac{|x - \mu|}{b}} \right)$$

- b is a scaling factor



distribution of log returns



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Alternative Distribution CDF Crossover points

- Intuitively fatter tails would require a higher buffer
- However, for a given variance, fat tails corresponds to thin 'shoulders'
- Therefore for moderate percentiles, fat tail distributions have a lower k than the normal distribution
- When p is increased, at some stage fat tail distributions will have a higher k than the normal distribution
- At what point does assuming a normal distribution cease to be prudent?



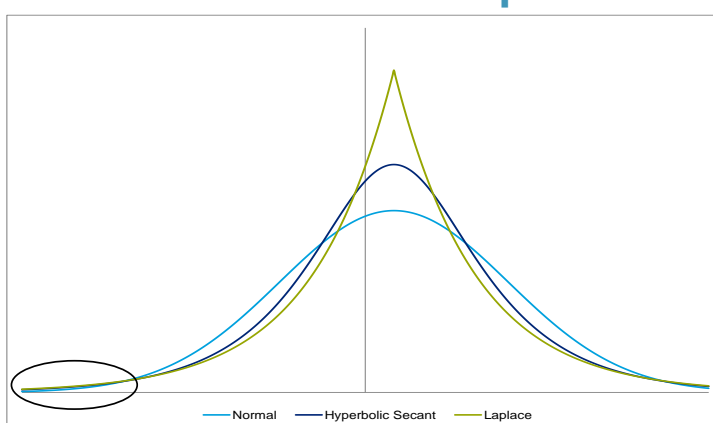
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Alternative Distribution CDF Crossover points

- Downside risk is higher in the Laplace and hyperbolic secant distributions as they exhibit higher levels of kurtosis



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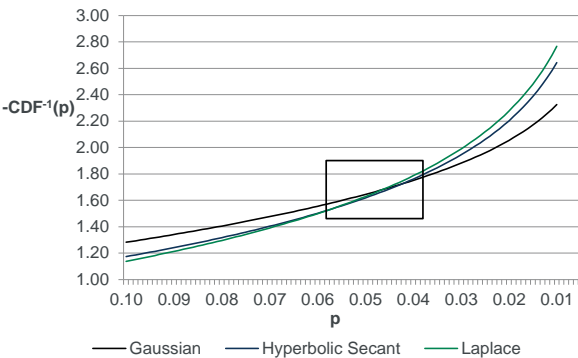
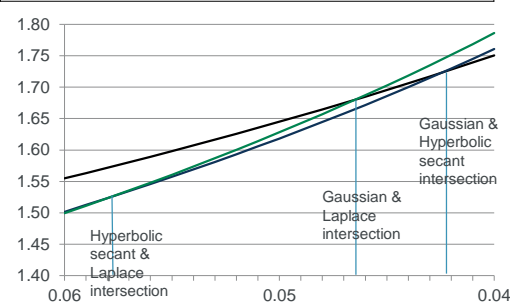
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Alternative Distribution CDF Crossover points

The crossover points are as follows:

| Distribution 1 | Distribution 2 | p* | inverse CDF(p*) |
|-------------------|-------------------|------|-----------------|
| Gaussian | Hyperbolic Secant | 4.2% | -1.72 |
| Gaussian | Laplace | 4.6% | -1.68 |
| Hyperbolic Secant | Laplace | 5.8% | -1.53 |



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Alternative Distribution CDF Crossover points

| Distribution 1 | Distribution 2 | p* | Inverse CDF(p*) |
|-------------------|-------------------|------|-----------------|
| Gaussian | Hyperbolic Secant | 4.2% | -1.72 |
| Gaussian | Laplace | 4.6% | -1.68 |
| Hyperbolic Secant | Laplace | 5.8% | -1.53 |

- When considering 95% one year Value at Risk, assuming a normal distribution would generate slightly more prudent k values than the alternative distributions (since the crossover p^* points of Gaussian with the other distributions occur at $<5\%$)



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Investor problem model- Discussion

- For shorter estimation periods, and longer investment periods, a significant increase in the standard deviation is required in order to capture parameter error impact
- In our problem, the impact of the chosen distribution over long investment periods is limited
 - This is a consequence of the central limit theorem by which compound log returns converge to normality regardless of annual return distributions
 - Distribution choice is more critical for short horizon problems, such as one-year value-at-risk calculations



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Impact of investment horizon

- As the investment horizon increases, the impact of parameter uncertainty increases.
- When looking at a 20-year investment horizon, the lump sum needed when estimating from 10 years of historical data is 1.6x as big as if the underlying parameters were known
- When looking at a 50-year investment horizon, this factor increases to around 4



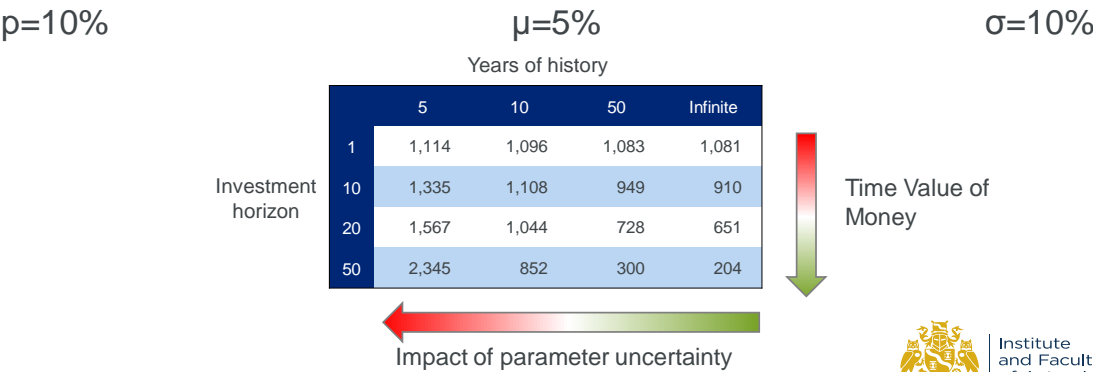
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The Results (recap)

- The amount needed now to be 90% sure of having >£1,000 at the end of the investment period

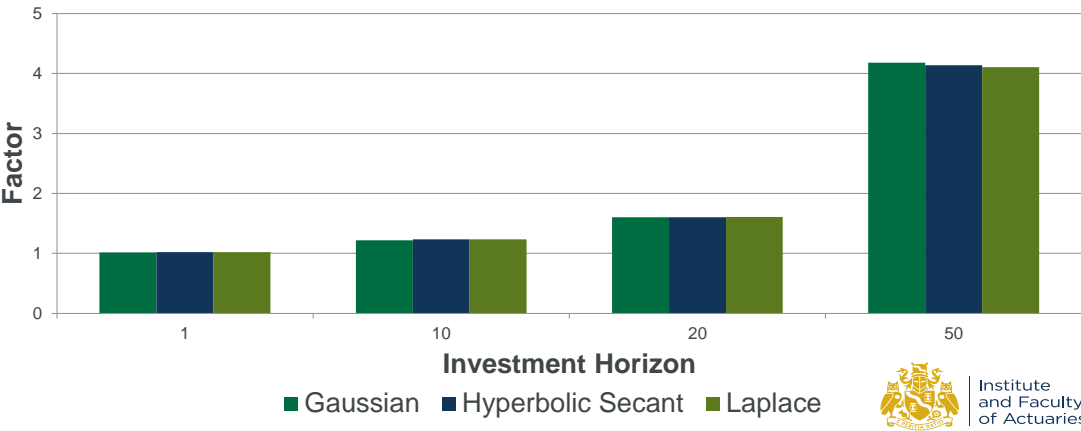


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Impact of Investment Horizon

- 20 years of historical data



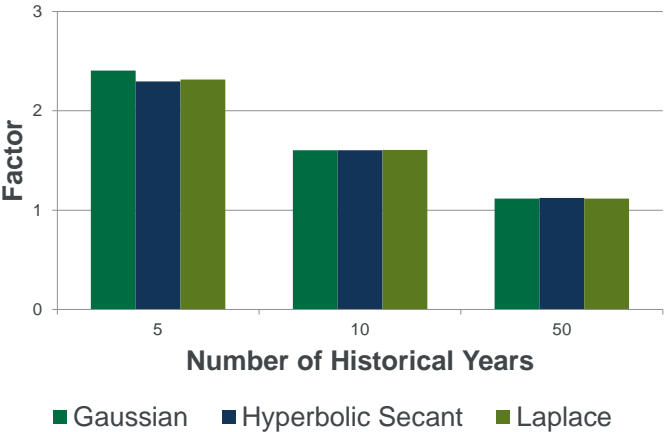
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Impact of Historical Data Length

- With 50 years of data or more, there is enough data to estimate fairly accurately into the future, and the impact of parameter uncertainty is limited if the investment period is <20 years
- As quantity of historical data decreases, the impact of parameter uncertainty becomes very significant

Impact of Historical Data Length



- Factors for a 20-year investment horizon

Impact of Historical Data Length

- When looking at a 50-year investment horizon, 4.5x bigger lump sum is needed when estimating from 10 years of data vs known parameters (based on $\sigma = 0.1$)
- When the data is reduced to 5 years, this factor increases to 11.5
- What implication does this have for new asset classes with limited historical data (i.e. infrastructure)?
 - We shouldn't treat them like another asset class and try to guess parameters, as not knowing the underlying parameters is itself a key risk



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Impact of Chosen Percentile

- Increasing the required probability of reaching the funding target increases the lump sum required
- Looking at a percentile further in the tail amplifies the effect of the two other sensitivities described earlier
 - Moderately amplifies the impact of the investment horizon
 - Significantly increases the impact of the quantity of historical data available



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Limitations and Exclusions

- Identical, independently distributed returns
 - If this assumption is removed, Central Limit Theorem may not 'save' us over the long term and the underlying distribution would have a more significant impact across long investment horizons
- Stationarity of underlying population distribution
 - Are investment returns 50 years ago reflective of investment returns today?
 - If not, can we tell where these 'breaks' occur, and what is the impact of including invalid data in our historical time series when estimating parameters?



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Limitations and Exclusions

- Ignoring survivorship bias in collected data
- Ignoring data-mining bias (asset selection)
- When any or all of these assumptions are relaxed and taken into account, they will increase our estimates of the impact of parameter uncertainty



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What about the assumptions you can't make stochastic?

- Possible approaches include:
 - Subjective probability
 - Limitations and exclusions
 - Use judgement to “correct” unexpected results
 - Trust the model less
 - Mould the system such that if there are unanticipated shocks, it benefits us, for example buying options (antifragility)
 - Head in the sand

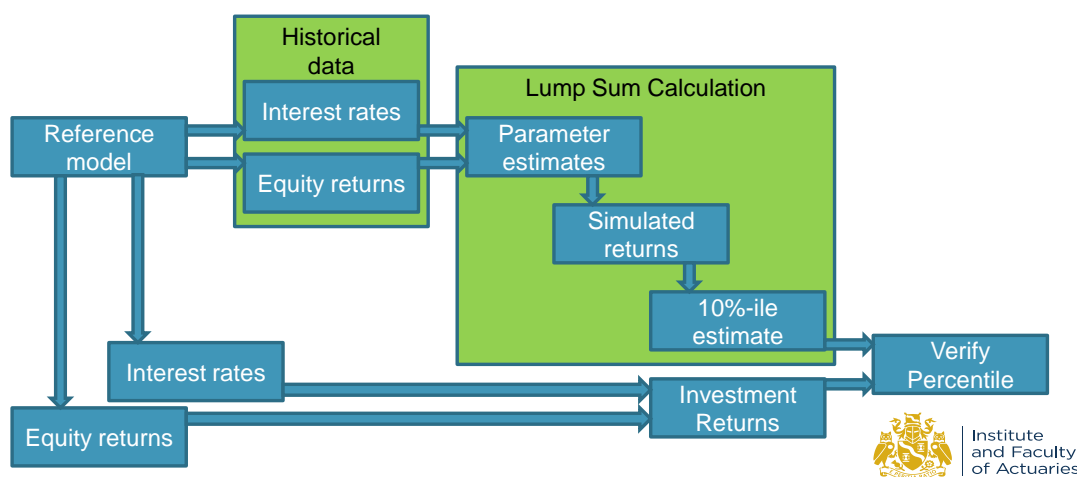


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Methodology - summary



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Model extensions and testing

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Possible model extensions

- Incorporate cash flows into the model
- Incorporate mortality into the model
 - An individual makes contributions for a set timescale, and then draws down income over the rest of his / her life
 - Given an assumed mortality distribution and stochastic investment returns, calculate the level of contributions needed in order to ensure the probability of not running out of money equal to a particular threshold (i.e. 90%)
 - The underlying investment return distribution is estimated from a finite quantity of historical data



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Testing the models

| | Benchmarking Approach | Consistency Tests | Robustness Tests |
|--------------------------|---|---|--|
| Description | Comparing the outputs of different models calibrated to available historic data. | Generating random data from a model, and feeding that data back into the calibration process to see if you recover the parameters you started with. | Taking random data from one model, using it to fit a different model, and seeing how good the predictions are relative to the first model. |
| What it tells you | The range of different experts' estimates given the data. | The likely accuracy of parameter estimates, both in terms of bias and variability. | How wrong your inference could be if you pick the wrong model. |
| What it doesn't tell you | How much the results might be distorted by random fluctuations in the observed history. | What happens if the model specification is incorrect? | How your fitting techniques behave on real data. |



Conclusions

Warnings

- Real life is not like classroom examples of urns containing a given number of coloured balls, or fair dice and unbiased coins, because the true model is unknown.
- The more you need to look into the tail of a distribution, and the less data you have, the more model choice matters.
- We are averse to ambiguity and we should not just ignore it
- Risk can be tackled quantitatively; ambiguity cannot be so easily dealt with, as by its very nature we cannot specify it precisely



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Questions for Discussion

- LDI approaches use matching arguments that mitigate dependence on expected return / risk assumptions, but introduce other assumptions (for example basis risks, cost of collateralising swap trades)
- Is ambiguity aversion a legitimate basis to favour LDI?
- Is it naïve to suggest stochastic modelling ‘takes all risks into account’ if we ignore the ambiguity in the model itself?
- By ignoring parameter misspecification, are we providing false comfort?



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The views expressed in this presentation are those of the presenters.

