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CAN WE EXPRESS $\mathring{e}_{xxx\ldots(n)}$ AND $1/\mu_x$ BY THE ORDINATES OF FREQUENCY CURVES?

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Many years ago it was suggested—I cannot remember by whom—that instead of graduating l_x or q_x or some such function it might be preferable to graduate annuity values. When I was fitting frequency curves to many kinds of statistics some fifty years ago, I noticed that the expectation of life from birth onwards looked something like the ordinates of a very skew frequency curve with a mode at age 4 or thereabouts. Partly because it seemed difficult to bring expectations of life within a definition of frequency distributions and partly because the example was neither a straightforward graduation nor helpful to the work on which I was then engaged, the notes I made were put aside and have long since been destroyed. From time to time, however, I have wondered whether life annuity values and expectations of life could be expressed roughly by one of the algebraic expressions used for frequency curves, and whether there would be advantages in working from joint expectations of life for n lives of the same age rather than for one life. Such a device, or the use of annuity values, may be taken to mean that we are using a series of values that could represent an expectation of life from a mortality table with heavier but related mortality. Approximations even if they are as rough as I had to anticipate in this case may be useful in unexpected ways, and that thought supplied me with some sort of justification for taking up the subject again and making some fresh calculations. Another aspect of expectations may be mentioned; as $\bar{e}_x$* tends to $1/\mu_x$ as μ_x tends to a constant, it seems that an examination of the values of $1/\mu_x$ may afford a possible supplement to any work on expectations.

2. There is some attraction in working directly on annuity values because in most of his calculations an actuary is concerned with premiums, annuities or valuations which depend on annuity values to such an extent that he seems only to be indirectly concerned with the more elementary functions. On the other hand, annuity values do not arise directly from a mortality experience, and when calculated to, say, four significant figures will give probabilities of death that may be unreliable even to two significant figures. This would hardly have been deemed of importance in the past by actuaries accustomed to probabilities that were based on whole numbers at the tail of an l_x column which started with 10,000 or even 100,000—the erratic figures at the end of the Carlisle table will still be remembered by many actuaries! Be that as it may, I do not see any likelihood of annuities or expectations becoming the function that actuaries would seek to graduate except, perhaps, when, in attempting a forecast, they have worked in terms of annuity values rather than in terms of probabilities of death.

* $\bar{e}_x$ seems a more natural symbol than $\mathring{e}_x$ when we want to express the limit of $\bar{a}_x$ when $i=0$.

3. If, having made a table of annuity values or expectations, we wish to proceed to probabilities it would seem to be advantageous to start with a_x or e_x rather than $\bar{a}_x$ or $\bar{e}_x$ because the former lead to exact formulae for p_x. The latter however is, I think, a preferable function in the sort of theoretical work with which we are at present concerned and the formula

$$p_x \doteq (\bar{e}_x - \tfrac{1}{2})/(\bar{e}_{x+1} + \tfrac{1}{2})$$

is only less than the true value by $(\mu_x - p_x\mu_{x+1})/12(\bar{e}_{x+1} + \tfrac{1}{2})$ approximately, or about ·00006 at age 70 by King's Text Book Table, where $q_{70} = {\cdot}06410$.

4. In order to give an idea of what annuities and expectations look like as possible ordinates of frequency curves Table 1 has been prepared. The values are taken from King's Text Book Tables merely because he gives expectations and annuities to four lives of equal ages and for a wide range of rates of interest. Our table also includes values of $1/10\mu_x$, the $10\mu_x$ being used instead of μ_x to give the figures something like the magnitude of the other columns. It is not pretended that King's Table is the most suitable, but, though a 'manufactured' table, it is convenient and was, in fact, the only table with wide range available to me when I restarted my calculations.

Table 1. Values of $1/10\mu_x$, $\bar{e}_{xxx\ldots(n)}$ and $\bar{a}_{xxx\ldots(n)}$ at 3% and 6%

(King's Text Book Table)

x	$1/10\mu_x$	$\bar{e}_{xxx\ldots(n)}$				$\bar{a}_{xxx\ldots(n)}$ 3%				$\bar{a}_{xxx\ldots(n)}$ 6%			
		$n=1$	$n=2$	$n=3$	$n=4$	$n=1$	$n=2$	$n=3$	$n=4$	$n=1$	$n=2$	$n=3$	$n=4$
0	·6	47·8	31·2	20·8	15·7	21·6	15·9	12·0	9·3	13·0	10·2	8·0	6·4
5	8·8	53·4	41·5	34·4	29·4	25·1	21·7	19·3	17·5	15·2	13·9	12·9	12·0
10	23·4	50·3	39·4	33·0	28·6	24·6	21·5	19·3	17·6	15·2	14·0	13·1	12·3
20	18·2	42·1	32·2	26·6	22·7	22·6	19·1	16·8	15·0	14·5	13·0	11·9	11·0
30	13·0	34·7	26·2	21·5	18·3	20·4	16·9	14·7	13·0	13·8	12·1	10·9	9·9
40	10·1	27·4	20·2	16·3	13·9	17·7	14·2	12·1	10·6	12·6	10·7	9·5	8·5
50	6·5	20·3	14·4	11·4	9·5	14·4	11·0	9·1	7·8	10·9	8·8	7·5	6·6
60	3·4	13·8	9·3	7·1	5·8	10·7	7·7	6·1	5·1	8·7	6·5	5·3	4·5
70	1·6	8·5	5·3	3·9	3·1	7·1	4·7	3·6	2·9	6·2	4·2	3·3	2·7
80	·7	4·7	2·7	1·9	1·5	4·2	2·5	1·8	1·4	3·8	2·4	1·8	1·4
90	·3	2·3	1·3	·9	·7	2·3	1·2	·9	·7	2·1	1·2	·9	·7

Table 2. Modal ages of joint life annuities

(King's Text Book Table)

No. of lives	Rate of interest (%)				
	0 (expectation)	3	4	5	6
1	2·8	5·4	6·2	6·7	7·3
2	4·7	6·6	7·0	7·4	7·8
3	5·7	7·1	7·5	7·8	8·1
4	6·4	7·5	7·8	8·1	8·3

5. It will be seen that the mode tends to shift to an older age as the number of lives or as the rate of interest increases. Table 2 gives the modal ages

estimated by an arithmetical process which assumes that the mode is approximately where $\Delta f(x) = \frac{1}{2}\Delta^2 f(x)$. The method is not exact, but as it has been used consistently the relationship of the values is unlikely to be disturbed. The modal age of $1/\mu_x$ lies between ages 13·5 and 13·6 and the modal value of $\bar{e}_{xxx...(10)}$ is 16·5 at age 8·1 approximately.

6. I do not think that we should be able to say on inspection of Table 1 that the figures could not be represented by the ordinates of a frequency curve, but we should say that we prefer to start with $\bar{e}_{xxxx}$ rather than $\bar{e}_x$ because we know that the more skew a curve is, the more difficult does it become to fit accurately, at any rate by moments. My first shot was made by using the figures at every eighth age without any adjustment and the result is given in Table 3. (I originally chose these ages because I saw that the mode was at age 8 for $\bar{a}_{xxxx}$ at 6%, and I hoped, foolishly, that the arrangement might help to bring the maximum of the curve somewhere near to this age.)

Table 3. Comparison of values of $\bar{e}_{xxxx}$ (King's Text Book Table) with the ordinates of a Type I frequency curve (using figures at every eighth age only)

Age x	$\bar{e}_{xxxx}$	Ordinates of curve	Col. (2)-Col. (3)
(1)	(2)	(3)	(4)
0	15·7	20·4	−4·7
8	29·5	26·7	+2·8
16	25·0	25·4	− ·4
24	20·9	21·9	−1·0
32	17·5	17·7	− ·2
40	13·9	13·6	+ ·3
48	10·3	10·0	+ ·3
56	7·2	7·0	+ ·2
64	4·6	4·6	—
72	2·7	2·8	− ·1
80	1·5	1·6	− ·1
88	·8	·8	—
96	·4	·4	—

The curve started at 3·22 years before birth, the mode was later than it should be and the modal figure was too low but the figure at age 0 too high. The brackets have been inserted in the table to draw attention to a certain amount of distortion due to difficulties in fitting a curve or to the use of an unsuitable curve. It is easy to see that by altering the start of the curve we could improve the value at age 0, but before attempting anything further in that direction I had a look at $\bar{a}_{xxxx}$ at 6%. The mode came out at age 14 by the fitted curve and was of about the right figure for that age, but as with $\bar{e}_{xxxx}$ similar distortion occurred elsewhere.

7. I knew that I had put a rather severe strain on the method of moments by the use of the particular ages, and after making a few further trials I thought rather more favourable results might be obtained by fixing the start of the curve and the mode. This meant a great saving in arithmetic as only the mean

and second moment would be wanted. As the formulae required are not actually given elsewhere they may be set out here.*

8. If μ_1' and μ_2' are the moments about the assumed start of the curve then

$$m_1 = a_1(3\mu_2' - 4\mu_1'^2)/(2\mu_1'^2 a_1 + \mu_1'\mu_2' - 3\mu_2' a_1),$$

$$m_1 + m_2 = (3\mu_2' - 4\mu_1'^2)/(2\mu_1'^2 - \mu_2' - a_1\mu_1'),$$

$$a_1 + a_2 = (m_1 + m_2)\, a_1/m_1.$$

a_1 is the distance from the start to the mode and is assumed to be known; $a_1 + a_2$ is the range.

9. It may happen that the curve should be Type VI. This will be shown if m_1 or m_2 has a large negative value by the above formulae. In such circumstances it is best to move the origin to the start of the curve which becomes $y'x^{q_2}(x+a)^{-q_1}$ and writing M for the known mode:†

$$q_1 - q_2 = (4\mu_1'^2 - 3\mu_2')/(2\mu_1'^2 - M\mu_1' - \mu_2'),$$

$$a = (3M\mu_2' - \mu_1'\mu_2' - 2M\mu_1'^2)/(2\mu_1'^2 - M\mu_1' - \mu_2'),$$

$$q_2 = (q_1 - q_2)\, M/a.$$

10. Before some further examples are considered attention may be drawn to a few difficulties that can occur in such calculations. The function with which we are dealing is a system of ordinates, not a distribution of N cases in a system of areas with equal bases, but the algebraic expressions we must use to find the constants are the result of integrations. The total of the system of ordinates need not be equal to the total of 'graduation ordinates' but the difference cannot be large. It is convenient to avoid using the values of the expectations at every age unless an adding machine is available, and even then some abridgment saves work. Accurate results can be obtained either by grouping the expectations in 5-year age-groups and then pretending they are areas, to which they closely correspond, or by choosing equidistant values and making an adjustment to arrive at areas by means of some formula of approximate summation (e.g. as in *F.C. and C.* pp. 26 et seq.). This last method gave figures for the first two moments of e_{xxxx} close to those obtained by using all the ages. When we fix the start of the curve an adjustment to the moments is needed in every case except where the curve is assumed to start six months before birth. Thus, if we assume the start of the curve to be three months before birth we are treating the area corresponding to the expectation at birth as if it related to 9 months, whereas if we calculate the arithmetical moments using the full value of the expectation and assume that it is situated 1 year before the position of the expectation at age 1, we shall be treating it as if it related to a base of a full year. A reduction in the expectation and an adjustment of its position is strictly speaking necessary before we calculate the arithmetical moments. As we require an ordinate of the curve we fit at each age we must have a curve that starts before birth, and the most comfortable starting point would be 6 months before birth because the adjustments

* They derive directly from formulae given on p. 64 of *Frequency Curves and Correlation*, 4th ed. (Cambridge University Press). y', the only other constant required, is given on p. 59. The notation for moments is that of *F.C. and C.*; an actuarial reader will not, I think, confuse them with the force of mortality especially as no moments about the mean occur so that all moments in this paper are accented.

† See *F.C. and C.* p. 79.

to which we have just referred will not then be needed. My attempts with this assumption, however, gave too high a value at birth, and showed that if the expectation at birth were to be reproduced closely the curve must start nearer birth.

11. We may now give two examples—the first is the ordinary fitting of a Type I curve by four moments without fixing the start or the mode; the second is a Type I curve with the mode fixed at age 6·4 and the start fixed at ·1 of a year before birth so that only two moments were used. The mode in the former case is at 7·75835, but the value found for age 0 is too high and thereafter too low for several ages—the curve being flatter than the statistics imply. The other curve is of a better shape up to age 10 and a little worse at the later ages as will be seen from Table 4. This 'improved' curve was the last of a number of trials and it might be further improved by persistent trial and error, with small variations in the positions of the start and the mode or

Table 4. Comparisons of values of $\bar{e}_{xxxx}$ (King's Text Book Table) with the ordinates of Type I frequency curves

Age	Value from table	Values from curve		Col. (2)-Col. (3)	Col. (2)-Col. (4)
		(*a*)	(*b*)		
(1)	(2)	(3)	(4)	(5)	(6)
0	15·7	19·0	16·2	−3·3	− ·5
5	29·4	27·3	29·1	+2·1	+ ·3
6·4	29·7 say	27·7	29·2	+2·0	+ ·5
10	28·6	27·6	28·6	+1·0	—
20	22·7	23·8	23·9	−1·1	−1·2
30	18·3	18·6	18·1	− ·3	+ ·2
40	13·9	13·5	13·3	+ ·4	+ ·6
50	9·5	9·2	9·1	+ ·3	+ ·4
60	5·8	5·8	5·7	—	+ ·1
70	3·1	3·3	3·3	− ·2	− ·2
80	1·5	1·6	1·7	− ·1	− ·2
90	·7	·6	·7	+ ·1	—

Equations to curves

(*a*) $\log y = \bar{6}{\cdot}68883 + {\cdot}24726 \log x + 3{\cdot}19605 \log (117{\cdot}8478 - x)$, origin at ·7587 before birth; mode at 7·75835.

(*b*) $\log y = \bar{6}{\cdot}48387 + {\cdot}18451 \log x + 3{\cdot}30682 \log (122{\cdot}9941 - x)$, origin at ·1 before birth; mode at 6·4.

even in the moments. It may be mentioned that in a number of the trials the range of the curve was shorter and the values of m_1 and m_2 in the ordinary notation less than the figures shown in Table 4, but most of these efforts were on e_{xxxx} and not on $\bar{e}_{xxxx}$. The curves gave disagreements with the tabular values similar to those shown by Table 4.

12. I had made some rather unsuccessful attempts with e_x and $\bar{e}_x$ by King's Text Book Table, but encouraged by the figures given in Table 4 I assumed that the curve would start at age −·1 and would have a mode at age 2·8 and reached the figures given in Table 5. I have no doubt that with much 'trial and error' this result could be improved. It serves to show that we can reproduce the expectations at the early ages.

Table 5. Comparison of values of $\bar{e}_x$ (King's Text Book Table) with the ordinates of a Type I frequency curve

Age	Value from table	Value from curve	Col. (2)-Col. (3)
(1)	(2)	(3)	(4)
0	47·8	47·4	+ ·4
2·8	53·9 say	53·9	+ —
5	53·4	53·3	+ ·1
10	50·3	50·2	+ ·1
20	42·1	42·7	− ·6
30	34·7	34·4	+ ·3
40	27·4	26·9	+ ·5
50	20·3	22·1	−1·8
60	13·8	14·0	− ·2
70	8·5	9·0	− ·5
80	4·7	6·2	−1·5
90	2·3	2·1	+ ·2

Equation to curve

$$\log y = \bar{3}\cdot 90766 + \cdot 05246 \log x + 1\cdot 188315 \log (106\cdot 9953 - x).$$

Origin at ·1 before birth; mode at 2·8.

13. It may appear strange that the range of the curve for $\bar{e}_{xxxx}$ starting at ·1 before birth is approximately 123 years, whereas the range of the curve for $\bar{e}_x$ is only 107 years. In ordinary curve-fitting to statistics, range is known to have a relatively large standard deviation and appreciable variations in the range can be made without seriously disturbing the general fit of the curve. I worked out, therefore, a curve for $\bar{e}_x$ fixing, not only the mode and start, but also the range at 122·9941. Only one moment was involved. The curve gave values that were about 5% too high at the early ages, but from age 30 to age 70 the values were definitely too low. If exact agreement in the range for any number of lives were essential we might alter the curves for $\bar{e}_{xxxx}$ and $\bar{e}_x$ by reducing the range of the former to 115, say, and increasing the range of the latter to the same figure but such agreement in the range would be purchased at the cost of the fit of the curves and the work required to pick the best common range was more than I could undertake. It does not, however, seem to me of immediate importance.

14. It has been suggested in paragraphs 11 and 12 that curves might be found to give ordinates that would agree more nearly with the tabular values than those I have given but, even though I regard this as probable, I cannot believe that a result would be obtained giving a close agreement to, say, two decimal places as I take the view that there are definite signs of distortion somewhere. This does not mean that we have not reached a rough approximation but it does mean that we must not expect to reach satisfactory values for p_x, q_x or μ_x from the ordinates of a single Type I curve which represents expectations of life fairly well. Such a result would, perhaps, have been too much to hope for, but even if I could have reached it with the mortality table I used I should have suspected that the table was unduly in my favour. With almost all tables of mortality covering the whole of life there is a heaping up of mortality at the early adult ages. There may be a distinct maximum in the mathematical sense, or there may be a quick rise followed by nearly level

mortality before the run of the rates of mortality seems to revert to the course followed in youth. King's table indicates such an effect so slightly that one might almost say it does not exist. Such a maximum (or tendency thereto) must be reflected in the expectation of life, though the calculation of that function tends to disguise minor variations and, even with a table such as the English Life No. 10 where there is a distinct maximum, it is hard to detect its position from values of $\bar{e}_x$; it is a little less disguised with $\bar{e}_{xxxx}$ though it can hardly be described as obvious. But it is there and must add to the difficulties of fitting a single curve to expectations of life.

15. The maximum mentioned in the previous paragraph was attributable in many cases to tubercular disease and to accidents. Tubercular mortality has decreased, but the accident risk becomes more obvious owing to the light mortality from other causes at the ages implied. So far as the curve of mortality is affected by accident we may be tempted to regard it as unnatural and thus justify ourselves in its elimination, but such a view seems to overlook the inherent qualities of venturesome youth and early manhood—adventure, whether physical or intellectual, is a young man's game and for the sake of its qualities we must accept its tragedies. The statistical effect of such aspects of mortality may be reduced by graduation and if a summation method with a large number of terms is used the heavy mortality may be spread out and be hard to detect. May this be taken as a warning against using graduated figures alone to find the meaning of statistics? Graduation is useful and necessary but may hide essential variations.

16. We may explore the statistical effect of the mortality at the early adult ages a little further. If $1/\mu_x$ be plotted for the English Life No. 10 (Males) table it will be seen that there is a maximum at age 12 and another at age 27. After age 35 there is a smooth run to extreme old age, the shape being like the down grade of a bell-shaped or J-shaped frequency curve. In fact—as can be seen from the graph—it looks as if a couple of frequency curves might lead to a reasonable fit. But the only way to find them would be by trial and error—the method used by Karl Pearson in splitting up the deaths column of a life table* and by me in splitting up a similar sickness column.† If the two curves were 'normal' we could use the methods that have been devised for finding them by moments; but this division is out of the question and we are left with a method over which it is possible to spend a few months of one's spare time before being satisfied with or despairing of the result! Such an effort is beyond my powers, though I might have enjoyed it fifty years ago! The reader will bear in mind when examining a graph of $1/\mu_x$ that it represents the inverse of mortality; it might be called vitality.

17. An inspection of a graph suggested two ways in which $1/\mu_x$ might be split and, ignoring other possibilities, I made some trials with them. One of the ways was to continue the downward slope from age 12 in a sweep till the curve could rejoin the original graph at about age 65. The piece cut off looked like a slightly skew frequency curve as can be seen from Diagram 1. The other way (see Diagram 2) assumed that the values of $1/\mu_x$ from about age 28 represented a part of a frequency curve having a maximum at or near that age. The

* See 'Skew variation in homogeneous material', *Phil. Trans.* A, vol. 186, pp. 343–414 or the later *Chances of Death*.

† See 'Graduation and analysis of a sickness table'. *Biometrika*, vol. 2, pp. 260–72.

earlier part of the curve could be estimated with a wide range of choice; but if it was assumed to start at about age 0 the remainder might look like another frequency curve with a maximum at about age 12, but with a more tractable shape elsewhere. My method was to make some drawings of what I thought might be possible dissections and then to try to fit one or both parts. I found that sometimes the deviation from my guessed figures were such as to produce unsatisfactory results when I added the two parts together; or if I got a fitted curve for one part I was left with a double hump for the other. This kind of trouble is not unexpected with the sort of problem I had set myself but it makes the work wearisome.

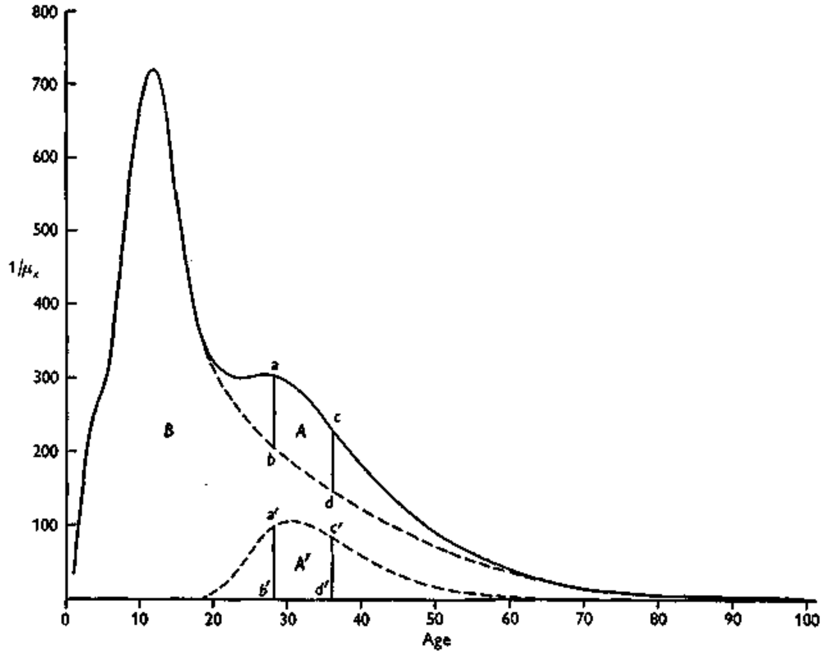

Diagram 1. English Life No. 10 (Males) table. Graph of $1/\mu_x$. The solid line gives $1/\mu_x$. If the part marked A be removed above the broken line it will be found to be expressed by A' and the area left, B, is something like a very skew frequency curve. A' is also like a frequency curve. The transfer of A to A' is easily followed by comparing the lines ab and cd with $a'b'$ and $c'd'$.

18. Taking Diagram 1 first, I found that the piece I had assumed to be cut off could be represented by a Type V frequency curve or, possibly, if I modified my drawing, by a symmetrical curve. But when I subtracted from the values of $1/\mu_x$ the figures I had found for the piece cut off I failed to find a suitable curve for the remainder and my efforts were sufficiently unsatisfactory to cause me to discard this kind of dissection. I give my drawing on the off chance that someone else may be able to do better than I have done with it. The very pronounced peak at age 12 seems to combine badly with the long slope at old age.

19. The plan shown in Diagram 2 was more successful but my early guesses at the left hand side of the curve that was to describe by its right hand side $1/\mu_x$ for most of life, were so poor that I found that I had got my maximum in the wrong place and, when the curves were combined with possible curves for the earlier ages, I had eliminated the small maximum at age 27 which I much wanted to retain. However, after a number of trials, I reached the one that is shown on Diagram 2. I used a Type III curve with its origin at the mode (fixed at age 27), a starting point at age 5 and a modal value of 305; p was 3·52. I could possibly have improved this by using a Type I curve but one of the unsuccessful trials had given a satisfactory fit of a Type III curve from about age 32 onwards, though the mode was in the wrong place and it

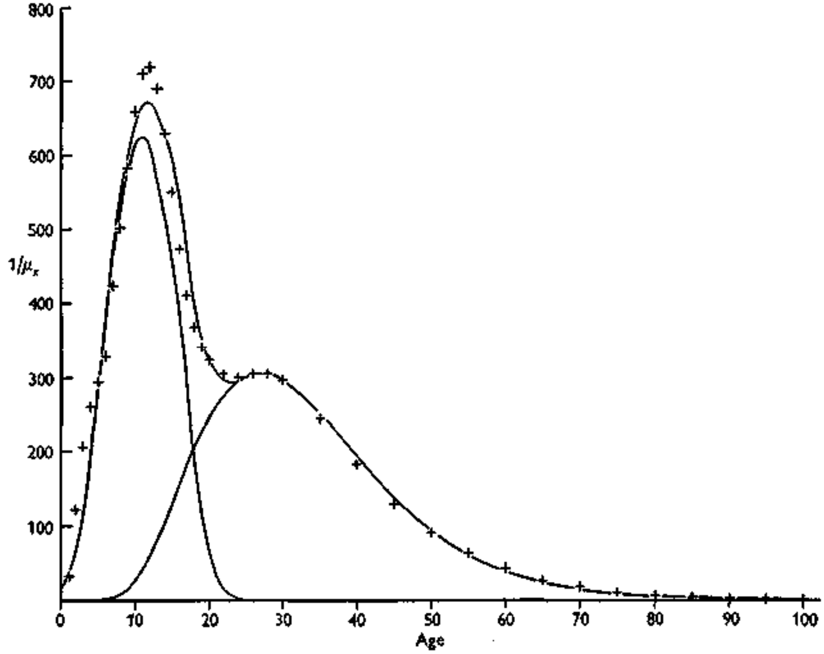

Diagram 2. English Life No. 10 (Males) table. Graph of $1/\mu_x$ showing dissection into a Type II and a Type III frequency curve.

The tabular values of $1/\mu_x$ are shown by crosses. The curves and the sum of them are shown by solid lines.

failed to help with the dissection; I thought I might retain that type of curve because it used one less constant than a Type I curve and because I liked the idea of a curve having an exponential in its make-up. If, when he was modifying Gompertz's hypothesis, Makeham had tried the effect of multiplying the exponential by a factor depending on the age he might have reached the formula for a Type III curve! The Type III curve I used left me with a balance looking something like an isosceles triangle with its apex near age 12. To keep down the number of constants I used a Type II curve with origin at 11; m was ·59672, a was 14·4056 and the modal value was 637. Strictly speaking I should have used a slightly skew Type I curve but the result of combining my Type III and Type II curves was good enough to indicate the possibility of splitting $1/\mu_x$ into two frequency curves and that was all I really wanted. A little change in the constituent curves might well remove some of the deficiencies. Some figures are given in the Appendix.

20. The fact that it requires the sum of the ordinates of two frequency curves to express $1/\mu_x$ with this sort of approximation tends to confirm the remark made in paragraph 14 that we can only look to rough approximations to $\bar{e}_{xxx\ldots(n)}$ by a single curve. The dissection of $\bar{e}_{xxx\ldots(n)}$, would probably present considerable arithmetical difficulties even though we know of a possible dissection of $1/\mu_x$.

21. The few numerical examples given in this paper must not be regarded as anything but an indication of a possibility and though I should have liked to have added to them I cannot now hope to undertake the large amount of arithmetical work that would be involved. I have therefore given the paper a title implying doubt.

22. In any investigation into mortality some of our difficulties may arise because the tables of mortality against which we try our theories are unsuitable; if, for instance, a theory is built up on the idea of the wearing out of the human frame it should be tested against a table of the kind that traces a batch of individuals throughout life. Similarly if we are working on expectations of life it would appear that the same kind of table should be used. These tables may be called 'generation' tables; and we may describe as 'census' tables those tables of mortality that seek rates of mortality at each age in a given year or in a few years. I do not much like the designations but I cannot think of better. In general the 'census' table is what we make, and 'generation' tables in modern times have been evolved approximately from a succession of 'census' tables. An algebraic expression worked out to describe a function based on an l_x found by tracing, say, 10,000 babies throughout life may do it well but its failure to describe a series of q_x based on censuses and deaths may make it look wrong; whereas the apparent failure may be because we try to make it do something that should not necessarily be expected of it.

23. It is interesting that when a table of 'census' type has been made and monetary tables have been made from it actuaries use them as if they were based on a single table of the 'generation' type. This can be defended if it is assumed that mortality does not and will not vary, but the unfortunate thing is that the assumption has no likeness to the evidence. A more logical attitude lies behind the attempts to make forecasts suitable for use in annuity business where actuaries are provided with values applicable to each age at a given moment of time; the rates of mortality that can be evolved from them, or on which they were calculated, are not the rates of any particular moment of experience. May I, perhaps, be allowed to regard all this as an excuse for trying to express expectations of life at various ages by an algebraic formula?

24. I am grateful for help in some of my calculations and with the diagrams from members of the staff of the Equitable and particularly from G. E. Wallas. He provided me with the diagram on which I worked, with a table of $1/\mu_x$ in a form in which my possible needs were foreseen, and he saw to the calculation of the ordinates from some of the curves for which I had obtained constants. The diagrams in this paper are also his.

APPENDIX

The following table gives the figures on which Diagram 2 is based. As they might serve as a starting point to obtain a better dissection of $1/\mu_x$, an alternative, giving somewhat similar results is also shown.

In examining the figures, or in any further work, it will be seen that there must be difficulty at the early ages as $1/\mu_x$ runs unevenly. The values of μ_x are the result of approximate calculation and need not be regarded as sacrosanct especially, perhaps, at the early ages.

Age x	English Life No. 10 (Males) $1/\mu_x$	Ordinates used in Diagram 2			Alternative dissection			Differences	
		Type II curve	Type III curve	(3)+(4)	Normal curve	Type VI curve	(6)+(7)	(2)−(5)	(2)−(8)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	32	46	—	46	47	—	47	−14	−15
3	207	127	—	127	120	—	120	+80	+87
5	284	294	—	294	252	—	252	−10	+32
7	426	463	2	465	433	1	434	−39	−8
9	581	572	13	585	597	16	613	−4	−32
11	709	627	41	668	667	42	709	+41	—
13	690	572	81	653	597	82	679	+37	+11
15	549	463	130	593	433	129	562	−44	−13
17	410	294	179	473	252	178	430	−63	−20
19	340	127	223	350	120	225	345	−10	−5
21	311	46	260	306	47	259	306	+5	+5
23	300	7	285	292	14	285	299	+8	+1
25	301	1	300	301	3	300	303	—	−2
27	305	—	305	305	1	305	306	—	−1
29	300	—	298	298	—	300	300	+2	—
31	290	—	273	273	—	287	287	+17	+3
37	217	—	231	231	—	230	230	−14	−13
42	162	—	173	173	—	173	173	−11	−11
47	111	—	120	120	—	122	122	−9	−11
52	80	—	80	80	—	83	83	—	−3
57	55	—	52	52	—	55	55	+3	—
62	36	—	33	33	—	36	36	+3	—
67	22	—	20	20	—	22	22	+2	—
72	14	—	11	11	—	14	14	+3	—
77	8·7	—	7	7	—	8·9	8·9	+2	−·2
82	5·6	—	5	5	—	5·7	5·7	+1	−·1
87	3·8	—	3	3	—	3·4	3·4	+1	+·4
92	2·7	—	1·3	1·3	—	2·1	2·1	+1·4	+·6
97	1·9	—	·5	·5	—	1·2	1·2	+1·4	+·7

Equations to curves

For Diagram 2:

Col. (3) $y = 627\,(1 - x^2/(14{\cdot}4056)^2)^{\cdot 59672}$ origin at age 11.
Col. (4) $y = 305\,(1 + x/22)^{8\cdot 52}\, e^{-8\cdot 52x/22}$ origin at age 27.

For alternative dissection:

Col. (6) $y = 667\, e^{-x^2/9\cdot 307}$ origin at 11.
Col. (7) $y = (\log^{-1} 62{\cdot}5762)\,(x - 35{\cdot}0453)^{4\cdot 7015}\, x^{-39\cdot 5445}$ in units of 5 years; origin 39·7836 units before age 27; mode at 27.

ABSTRACT OF THE DISCUSSION

The President (Mr J. F. Bunford) said that it was his great honour and pleasure to introduce to the meeting Sir William Elderton—as if Sir William needed any introduction! He used the words 'honour and pleasure' deliberately, for he was one of the many young men whose youthful actuarial footsteps Sir William had guided. Sir William was tremendously interested in young men, and his interest had continued through generations of young men as time had slipped by.

In the Council Chamber there were two replicas of Gold Medals, the only two which had ever been awarded by the Institute and the Faculty jointly. One of those had been awarded to George James Lidstone, and the other to William Palin Elderton, in recognition of his distinguished services to actuarial science. Other honours which had been awarded to Sir William for work in wider fields were acknowledged and were represented by the initials which followed his name on the paper. All those dignities he wore with such modesty that the President was almost afraid to mention them, for fear of incurring Sir William's displeasure. Their very existence, however, indicated the great reputation which Sir William enjoyed outside the actuarial profession, and indeed outside Great Britain, and it emphasized too the special honour which he did them in bringing a paper to them in the year of their return to Staple Inn.

When he offered the paper they naturally accepted it with delight, but that evening he brought not only a paper but also a visitor whom they were specially delighted to see, Lady Elderton.

Sir William Elderton, in introducing his paper, said he had been asked why he did not give the constants to the Type V curve in the cut-off piece of Diagram 1. The dissection was a failure, and anyone who wanted to have another try at that dissection had better start afresh. The origin was at 26·709 years before birth, the unit was 6 years, and the curve was given by the expression $x^{-59\cdot 069}e^{-568\cdot 79/x}$. The ordinates were negligible before age 10 and after age 75.

He had not made any attempts with other systems of curves than Karl Pearson's. He did not think that the Charlier group, or the interesting translation curves given by Dr N. L. Johnson (*Biometrika*, vol. 36) would have yielded materially different results.

In the paper he had given two excuses for writing it. In §1 he said that rough approximations might be useful in unexpected ways, and in §23 he referred to forecasting and suggested that perhaps in some ways forecasting annuity values might on certain occasions be more useful than forecasting rates of mortality. He was, however, under no illusion about it; those were not the real reasons for writing the paper. He had figured many times in the old Hall of Staple Inn, and he had a feeling that he would like to do so in its successor. That was the real reason for the paper, and he hoped that he would be forgiven for it.

Mr M. T. L. Bizley, in opening the discussion, said that while it would be comparatively easy to comment on a poem by a poet of average merit, to criticize a work by Shakespeare or Milton or Shelley would be a much more difficult task, though a great privilege. It was just that sense of privilege and that sort of difficulty which he experienced in being called upon to open the discussion on a paper by one who at a recent meeting had been described as the greatest living actuary.

In 1934 the author had given to the Students' Society a talk (*J.S.S.* 4, 211) which undoubtedly ranked as one of the most stimulating and inspiring addresses that had ever been delivered to the actuarial profession. It was a talk about research, and it was concluded with a number of items of wise advice, one of which was 'Do not be in a hurry to rush into print.' Presumably in recommending a delay between the conception of an idea and its publication Sir William had in mind a period of perhaps half a year but certainly not the half-century which had elapsed in the case of the paper under discussion. It was remarkable that after a lapse of fifty years that particular piece of research had still remained to be carried out and that it came to them that day as fresh and as unspoiled by anticipation as if it had flowed from the author's fertile pen in the early 1900's.

The delay was a matter at once for sadness and for gladness. For sadness, because many of those who might have enjoyed reading and discussing the paper fifty years earlier were no longer present; for gladness, because the paper would enjoy a wider audience and, he ventured to say, because it would be better understood. He presumed that fifty years earlier not very many actuaries had been familiar with Karl Pearson's frequency curves, whereas subsequently they had been incorporated in the examination syllabus, so that many actuaries looked on them, if not as old friends, at any rate as old enemies. It should be remembered, however, that they were indebted for that most valuable education, as for so much of their statistical training, primarily to the author of that admirable book *Frequency Curves and Correlation,* who was none other than the author of the paper under discussion.

It was possible to discuss the paper from either of two quite different standpoints. On the one hand, the question in the title could be accepted as indicating the scope of the research, and discussion would then be restricted to the methods used and the conclusions reached. On the other hand, taking a wider view, it might be asked why the functions considered should be represented by mathematical curves at all and, if they were to be so represented, why Pearson's system of frequency curves should be used for that purpose.

From the first point of view he had found the paper of great methodological interest, and he thought that it should be studied from that aspect by all students of curve-fitting and graduation. He had been particularly struck by the ingenious way in which the data in Diagrams 1 and 2 had been dissected so that they could be represented as the sum of two frequency curves. With the help of the auxiliary lines in those diagrams it was easy to see how the dissection had been made, and it might be thought fairly obvious how the dissection was to be carried out; but, of course, everything was obvious when it was explained. As a matter of fact, however, it was difficult to appreciate the problems which the author had had to face in the paper, for the very reason that he had made it so easy to see how they had been solved. The idea of representation by the sum of two curves was reminiscent of Fourier analysis, and he could not help wondering whether some technique akin to that might be evolved, using frequency curves instead of the usual periodic functions and employing more than two of them if the analysis warranted it.

The author had extended his work over the whole range of life, and had not shirked the inclusion of the youngest ages, where the fitting problem was usually so troublesome. Table 3 showed rather an unsatisfactory fit at those ages, and so did column (*a*) of Table 4; but in column (*b*) of Table 4 that feature had been eliminated—with what labour and what patience could only be guessed. The author mentioned that persistent trial and error might further improve the fit in column (*b*) of Table 4, but in the form in which it was presented it was already a memorial to the author's own persistence. Table 5 was also strikingly successful in representing single-life expectations by means of a Type I curve. Moreover, credit was due to the author for even more success in fitting than his tables showed, because he strongly suspected that Sir William had carried out all the arithmetic himself, and that only the fact that he had naturally to stop somewhere had prevented him from going on to still better results.

Turning to the other aspect, and looking at the paper from the outside, as it were, he said that the scope of the research was summed up in its title. The object of the paper was to express single and joint life expectations and the reciprocal of the force of mortality by means of frequency curves. The author mentioned that $1/\mu$ was the expectation of a single life under constant mortality; but, as he was dealing also with joint-life expectations, it was worth noting that the joint-life expectation of n lives on that assumption was also proportional to $1/\mu$, and was in fact $1/n\mu$.

Some reasons were given in the paper why it might be useful to have mathematical representations of the functions considered, including annuities, and the speaker found it easy to imagine other applications. Outside the sphere of human life, for example, the time during which a machine would work until its first breakdown might be regarded as the joint-life expectation of its several independent components, and there were other applications of a much more homely kind. For instance, did not Lidstone's Z-method of valuation depend for its validity upon the fact that the temporary life annuity could be represented approximately by a simple curve of the form $\alpha - \beta c^x$?

Having said so much, however, he confessed that he found it much less easy to see why, if those functions were to be represented by algebraic curves, Pearson's frequency curves should be used; for the functions with which the author had dealt, although in a sense observables, were not frequencies. The very fact that the author was such a master of the art and science of curve-fitting might raise a doubt whether or not the curves which he had fitted were intrinsically suitable. It was rather like hearing a fine piece of music played on a violin and deciding that the violin could only be a Stradivarius or an Amati; but if it was known that the player was a Heifetz or a Menuhin there were less grounds for such an assumption, because such a master would be able to draw wonderful sounds from a much more modest instrument. So great was his faith in the author's curve-fitting ability that he felt that even if the author had chosen to fit a parabola, or indeed a parallelogram, to the data he would have produced very passable results. The suggestion that there might be other curves more suited to the author's purpose, or better suited than those which he had used, was in no sense a criticism of the objects of the paper, for all research was valuable. Still less was it a criticism of the paper itself: in any event, was not one of the most desirable objects of research its reproductive ability to beget a long line of further research?

In that connexion he would make a general observation on curve-fitting and graduation. There were, of course, occasions when it was desired to fit one particular type of curve to data. A shining example was to be found in Beard's Centenary paper (*Proc. Cent. Assembly Inst. Actuaries*, II, 89) and in a later paper (*J.I.A.* **78**, 341) in which he fitted a Type III curve to the data, not because he thought that it was the best curve to fit but because, if a reasonable approximate fit could be obtained, it would make possible the use of existing extensive tables for the calculation of actuarial functions and the labour of commutation columns would be avoided. Generally speaking, however, there was no special reason to use one curve rather than another, and where that was so the usual procedure was, after careful inspection of the data, to select a mathematical curve which was known by experience to be likely to fit, or to think of a new curve which had not been tried before. The numerical values of the parameters would then be determined by reference to the data, by the method of moments or otherwise; the curve chosen ensured smoothness, and the fitting process ensured a reasonable degree of fidelity to the data.

It seemed to him that to impose the form of the curve on the problem at the outset was to impose an unnecessary restriction, and that there could be a fundamentally different approach. Could not they let the data tell them which of all possible mathematical curves, including many which they had never even thought of, was the best for their purpose? They might postulate first that the moments, or other suitable statistics, of the data should agree with those of the as yet unknown form of curve which they were going to use. That ensured a measure of fidelity to the data; but all sorts of curves would be admissible so far which would be hopeless as graduations because they might twist and turn in order to pass through every point of the data. From those many curves, therefore, it was necessary to choose the one best suited to the purpose in view by a criterion of smoothness. One way of doing that might be to find the curve with the minimum length between the first and last ordinates, or from the starting point if that was to be fixed. That would ensure smoothness, just as pulling taut a piece of string straightened out the kinks; but they would not obtain a straight line—the moment equations would prevent that—and a better analogy than the string might be the unsuccessful attempt to pull taut an iron chain in a strong gravitational field.

To determine the form of curve in question involved the solution of an isoperimetric problem of the type treated in the Calculus of Variations; but it would not be necessary, of course, to solve such a problem every time it was desired to fit a curve. The problem could be solved, at any rate in part, in terms of arbitrary moments or points, and the particular curve for a given case would then follow by assigning appropriate values deduced from the data. In putting forward that new method of curve-fitting, he did not suggest the criterion of minimum length as being necessarily the best, or as a necessary part thereof, because there were other possible criteria. One alternative might be that the second differential coefficient of the curve over its whole length, d^2y/dx^2, should change sign as few times as possible, together with suitable restrictions on higher differential coefficients. But, whatever criterion was used, it seemed more logical to postulate first that the curve should fit the data and then to choose the form of the curve by a criterion of smoothness, rather than to choose the form of the curve at the outset and to force that particular form to fit.

Towards the end of the paper, the author made some most important comments on 'generation' and 'census' tables. The speaker regarded those remarks as of the first importance, because he believed that there was a very real danger inherent in the derivation of any actuarial instrument, whether a mortality table or a salary scale, by one method and then the use of it according to a fundamentally different method. At the end of §23, the author asked to be allowed to offer those reasons as an excuse for putting forward the paper. While he (the speaker) would like to see the rest of those paragraphs (§22–23) printed in heavy type, he would like to see that sentence printed in the smallest type possible, because he did not think that the author needed any excuse for putting the paper forward.

The author had set himself a difficult problem and had solved it brilliantly, and had provided actuaries with a piece of research which they had not seen before. No more was needed to ensure a welcome for the paper in a profession which went from strength to strength only because the spirit of research burned in it like an unquenchable fire.

He felt that he could not close his remarks without paying a brief tribute to the style in which the paper had been written. It was not often possible to read a technical paper from beginning to end without once having to look back over the lines to find out what it was that the author was trying to say; but it was possible to read Sir William's paper straight through, because the exposition was characterized throughout by that lucidity which they had come to expect of its author and which in any profession was the hall-mark of the master—and by that he meant not only the master of exposition but the master of his subject-matter. In reading his writings they did not feel merely that they were able to follow the author, which all too often meant that they followed at a great and ever-increasing distance, but rather that they were journeying with him, all the way.

Mr H. A. R. Barnett welcomed the opportunity of speaking immediately after the opener, because he wished to follow up some of his remarks. He spoke, however, with considerable diffidence, for two reasons. In the first place, some of his remarks might be critical, and he hesitated to criticize the author in any respect; secondly, like the author he enjoyed doing the kind of work described in the paper. He felt inclined to answer the question posed by the title of the paper by saying 'Why should we?' The opener had suggested that they should examine all the possible curves which might be fitted and then choose the best. At a previous Institute discussion it had been suggested by one speaker that any curve would do if it fitted, and by another that it did not matter if it did not fit. He himself thought that the choice of curves was neither as wide as that, nor as wide as the opener had suggested, but that they should try to fit a curve only if there was some underlying reason for fitting that curve.

If they were to fit frequency curves, he preferred the approach adopted by Ogborn in a paper submitted to the Institute in 1953 (*J.I.A.* **79**, 170). There had been some rationale in Ogborn's approach, although he (Ogborn) had given it only in his reply to the discussion. Phillips's paper (*J.I.A.* **80**, 289) abounded in rationale, and there was some reason in the Perks family of curves if they were regarded as a modification of Makeham's law. No doubt Sir William, had he desired to do so, could have given a reason for wanting to fit frequency curves to the two functions with which he dealt, but he had not done so. That was a gap which he felt detracted from the value of the paper, and he would humbly invite him to fill that gap in replying to the discussion.

Mr G. E. Wallas said that he had had the privilege of doing some work for the author in connexion with the paper. What appealed to him most was the author's use of $1/\mu$, and he would like to compare that with the more usual employment of μ itself. They were both ratios and could both be obtained from the data, and he firmly believed that they were equally valid. It was true that the reciprocal of μ was not directly useful to them, but it gave, to his mind, a more vivid picture. The statement that 1 out of 200 died was more vivid than the statement that $\mu = \cdot 005$. The graph of $1/\mu$ was more like a conventional frequency curve and probably more tractable to work with. The tails did not try to take control, and, he suggested, the important part of the data at the central ages had most influence on the curve.

It might be objected that it was improper to try to express those functions by frequency curves, because they were not frequencies. He was not of that opinion. In the first place, he thought that a fair case could be made out for arguing that they were frequency curves of rather a special type, but he took his main stand on the contention that a frequency curve did not have to apply only to frequency distributions. To apply it to anything where it could give useful results was not just expediency, it was common-sense. That enabled them to try the large family of Pearson frequency curves (or any others) with the general advantages of a uniform system and method of fitting.

The sophistication of human mortality could scarcely be expressed by a single curve, and the author had used two curves. That was at least as justifiable as Makeham's addition of a constant to Gompertz's expression for the force of mortality. For assurance tables, the range was shorter, and it was sometimes expedient to smooth out the hump in mortality at the early ages. In that case he thought it might be possible to use a single curve.

The author did not set out to consider the application of frequency curves directly to the data, but he (the speaker) thought that that could be done. The normal method of fitting a frequency curve by moments required that there should be data for the whole range. In the case under consideration they did not have that, but it was possible by considering the differentials of the logarithms of the data to fit a frequency curve to part of the range. There were difficulties in doing that, and a large volume of data was necessary, but as far as his arithmetic had gone it should be possible to graduate in that manner.

Mr Barnett had answered the question posed in the title of the paper by asking 'Why should we?' That was not the spirit of research; the more appropriate response was 'Why shouldn't we? Let's have a try!', and that was what the author had done. In §21 the author used the phrase 'an indication of a possibility'. To the speaker the paper had been an indication of a possibility which was both stimulating and interesting.

Mr L. V. Martin was glad to see that in the section of the paper dealing with expectations and annuity values the author had not dodged the question of trying to get a value at age 0, which always caused difficulty when an attempt was made to fit a mortality function over the whole of life. That was partly the result of the unsatisfactory nature of their usual conception of 'age' at that period of life. A speaker on the wireless, by a slip of the tongue, once said 'The average age at birth is increasing.' It should be remembered that the babies that they described as being at age 0 were a mixture of babies of full term and babies in various stages of prematurity, the proportions probably varying with advances in medical treatment. That meant that μ_0 was connected with a very heterogeneous set of data, and if the values of μ_0 for infants of various ages from conception could be known, it might be found that the difficulties would be less; certainly the expectation for full-term infants would be higher than the usual one calculated for age 0. That, if it were possible, would have helped the author to make a better fit.

The author had done most of his work on $1/\mu$ with the English Life Table No. 10, which was the last one published but was already more than twenty years out-of-date. The speaker had thought that it would be of interest to see what the shape of the $1/\mu$ curve would be with later data. For that purpose he had taken the 1950–52 census data. Looking at the younger ages first, he had found that the peak was at about age 11, and instead of being about 700 the maximum was nearly 2000. The top of the curve was rather flatter, and from ages 3 to 19 the curve was remarkably symmetrical. He thought it followed that a dissection of either of the two types shown in the appendix would be more difficult, since the Type III or the Type VI curve would have to cut off very sharply at the lower end. With the more recent mortality the bump in the curve at about age 27 was much less pronounced and there was no actual maximum in that range. The values over the range of ages 20–40 were more than twice those in Diagram 1, but from age 40 onwards the curve rapidly approached the old one, and from age 55 onwards there was little difference.

As regards the curve as a whole, it seemed that the area A in Diagram 1 would be relatively smaller, while the long downward slope of area B would have a sharper elbow between ages 40 and 60. The rather broader peak about age 11 and the steeper down-slope between the peak and age 55 seemed to make it more likely that a frequency curve could be fitted to area B, though he had not tried to do it.

The paper was a very useful reminder that there were functions other than q, m and μ which could be experimented with, and it was a pleasing thought that they were dealing with what the author called an 'index of vitality', for the author's own index was still remarkably high.

Dr N. L. Johnson added his congratulations to the author to those of other speakers and made some comments on the fitting of the data on expectation of life. He and Dr Moore, he said, had divided the paper between them so far as commenting on it was concerned, and Dr Moore would deal with the $1/\mu_x$ part.

With regard to Table 3, he asked how the fitting by moments had actually been applied. Had the moments been obtained by assuming discrete 'lumps' for ordinates at the points 0, 8, 16 and so on, or had some smoothing device been employed? Another question was whether any abruptness corrections had been made. He had never applied them in his life, but it looked to be the sort of case where they might be applied, and they would make a considerable difference. That applied to the data in Tables 3, 4 and 5.

He had made a few trials to see whether he could improve on the author's work in fitting the curves, and had soon found that it was not so easy as it looked; however, some of the results which he had obtained in regard to the tails of the curves might be of interest. If $f(x)$ represented the values of $\bar{e}_{xxxx}$ in column (2) of Table 3, it would be found that $\dfrac{f(x-8)f(x+8)}{[f(x)]^2}$ was practically constant (in other words the second differences of log $\bar{e}_{xxxx}$ were nearly constant), and that implied that at any rate in the tail the function would be of the form $Ae^{-b(x/8)^2}$. The numerical values $A=33\cdot3$ and $b=0\cdot031$ exactly reproduced the values shown in the table from age 56 onwards, and there were increasing discrepancies, the formula being in excess, in going down towards zero; at ages 0, 8, 16, ... 48 the differences were 17·6, 2·8, 4·4, 4·2, 2·8, 1·4 and 0·6 respectively.

All attempts to fit the curves seemed to meet with difficulty in fitting the initial ages. The fitted curves were all in excess at age 0 and deficient at age 8.

He had also tried a similar attack on the values of $\bar{e}_x$ in Table 5. In that case the same function, the product of the values at $x-8$ and $x+8$ divided by the square of the value at x, gave a slowly decreasing series which seemed to indicate a function of the type $x^r e^{-kx}$.

Trying a more theoretical approach, he had attempted to obtain an approximate formula for $\bar{e}_x$, assuming the Makeham law of mortality. With the usual notation,

$$\bar{e}_x = \int_0^\infty s^t g^{c^x(c^t-1)}\, dt$$

$$\fallingdotseq \int_0^\infty e^{-mt^2+nt}\, dt,$$

where
$$m = -\tfrac{1}{2}c^x(c-1)^2 \log g,$$

$$n = \log s - \tfrac{1}{2}c^x(c-1)(c-3)\log g.$$

The integral could be evaluated, in terms of a partial Normal integral, as

$$e^{-\frac{n^2}{4m}}\sqrt{\frac{\pi}{m}}\cdot\frac{1}{\sqrt{(2\pi)}}\int_{-\frac{n}{\sqrt{(2m)}}}^{\infty} e^{-\frac{1}{2}t^2}dt.$$

The method, however, gave poor results except at the more advanced ages.

The few experiments that he had made had convinced him more than ever of the value of the author's work.

Dr P. G. Moore, dealing with the second part of the paper, on fitting $1/\mu_x$, said it should be noted that the curves which had been fitted were truncated at the lower end; effectively therefore the curve in Diagram 2 implied not 8 parameters, but 9. That held for the alternative dissection given in the appendix; there were again 9 parameters effectively, since the Type VI curve had one more parameter than the Type III, so that there was in fact one more parameter than appeared to be shown.

In answer to Mr Barnett, he would say that one type of use of fitting $1/\mu_x$ would be that if it were shown to work well it would make the calculation of certain functions easier. If it were found that $1/\mu_x$ was approximately the same as a Type VI curve, and μ_x was therefore approximately the same as a Type I curve, the function $\log l_x = -\int \mu_x dx$ could be calculated using the incomplete Beta-function ratio tables with a suitable change in scale and origin.

In the first part of the paper the author had used King's Text-Book Table based on Makeham's law of mortality from age 28 onwards with a suitable set of constants. He (the speaker) had then gone to the second part of the paper, thinking that it also was based on King's Tables, and had produced some results before noticing that it was based on English Life Table No. 10. Those results, however, were interesting to a certain degree, as showing that had it followed Makeham's law the agreement would have been even better than it was in the actual table.

For instance, Makeham's law was $\mu_x = A + Bc^x$. Writing $y = \dfrac{1}{\mu_x}$ it was found that the slope of the curve of $1/\mu_x$ was

$$\frac{dy}{dx} = y\left[\frac{A\log c}{A+Bc^x} - 1\right].$$

For the Type III distribution, taking the simplest form $y = x^{k-1}e^{-x}/\Gamma(k)$, it followed that

$$\frac{dy}{dx} = y\left[\frac{k-1}{x} - 1\right].$$

Those two were of very similar form, and by changing the origin and scale of the Type III it was possible to get the two forms to agree almost exactly. That was particularly the case because c was usually fairly near 1, so that to a first order of approximation c^x could be written as $1+x(c-1)$ and if c was unity they would agree almost exactly. With Makeham the Type III distribution could be expected to give a very good fit. It would be noticed that as x became larger the values of dy/dx tended to $-y$, because the first term contained x in the denominator. That, of course, was analogous to the pure exponential distribution of the form $y=e^{-x}$, and he had tried as a matter of interest to see whether for large values of x the values corresponded to an exponential distribution. Using the author's values for μ_x for English Life Table No. 10, and writing $y=ke^{-tx}$, then if y_1, y_2, y_3 were the values of y corresponding to three equidistant values of x namely x_1, x_2 and x_3, it would be found that $y_1y_3/y_2^2=1$. The first 10 values of that expression from age 37 onwards were ·92, 1·05, ·95, ·95, ·93, 1·04, ·98, 1·04, 1·05 and 1·05, the average being 1·00 to two decimal places, but there was some variation about unity.

A variety of distributions could be fitted by inspection. He had tried $y=5443\,e^{-\cdot 052819x}$. He would not go into his reasons for choosing those constants, but the calculated results from age 42 onwards were quite good, and taking the average of the moduli of the errors, i.e. eliminating plus and minus signs, he found that the average error came out at 2·1, as compared with 3·2 for the Type III fitting. Below age 42 the divergencies became much greater. Needless to say, if something very simple, like an exponential, could be assumed for the higher values, the various functions involving μ_x could be calculated very simply.

He could not end without emphasizing the President's remarks about the author's great kindness to young men. As one of the younger people present, Dr Moore said that he had in the past, over some six years, received from Sir William many kindnesses in one way and another, and the help and consideration which he had received were all the more surprising in view of the fact that he had nothing to do with life assurance in his daily work and had never shown much interest in it as a means of earning his livelihood.

Mr H. E. Elderton said that he came from what was an older school of actuaries, of which he hoped that there were other representatives present, to whom so much talk of frequency curves at such an advanced level was positively frightening. However, he had enjoyed reading the paper, which he thought served a very useful purpose indeed. In § 16 the author pointed out that $1/\mu_x$ represented the inverse of mortality and might be called vitality. The speaker felt that that struck a responsive chord, because ten years earlier he had experimented in the hope of finding a law of mortality, as no doubt everyone did at some stage, and it had occurred to him to ask himself 'Why are we so concerned about the rate of death? Could not we think of something more in the nature of a force which was endeavouring to impose death on us, and that it was various degrees of vitality which managed to keep us going?'

The author had reproduced very closely, with the use of two curves, an almost complete representation of $1/\mu_x$. He had perhaps made life more difficult than he need, because he would be justified in using more than two curves. He (the speaker) submitted, without any mathematics being involved at all, that mortality might be made up of a factor representing environment, which might be favourable or otherwise, a basic vitality which from birth onwards would tend to increase, and thirdly, right from the inception of life, an element of decay which eventually became the predominating influence.

There were two other functions to be taken into consideration. One was intelligence. The child had, to begin with, little intelligence but enjoyed the benefit of parental control. Later, the parents gradually relinquished control, which was taken over by the child itself. He did not believe, however, that it was taken over in a continuous way. At age 21 the child got 'the key of the door', and that was probably the explanation of the hump which occurred so frequently in mortality tables.

Coming down to mathematics, there were only four basic exponential functions represented by the four combinations of signs in the expression $\pm Aa^{\pm x}$. He thought that a combination of those, such as $Aa^x - Bb^x + Cc^{-x} - Dd^{-x}$, with an additive constant k which might be negative or positive, would give a fair approach to a representation of the force of mortality from 0 to the limit of life. The last speaker had referred to 9 parameters. He had not followed where those 9 parameters came from, but his own formula included 9 unknowns, and if they could be given appropriate values he believed useful results might be obtained.

Mr M. E. Ogborn, speaking as one of the (he hoped) still young members of the Institute who had been inspired by the author, recalled the frequent occasions on which he had walked down Coleman Street with him. No doubt others had had the same experience and had found that his stride always seemed just a little longer than their own. In dealing with such a paper as that under discussion they were in the same kind of difficulty. The author was apt to proceed from the statement of a problem to its solution in one gigantic stride, leaving everybody else trotting behind and trying to find how the author did it. That was the joy of the paper; it went straight to the answer to see what form was taken by the curve of mortality.

To critics who said that the type of approach in the paper was wrong the speaker put the question: was the method by which they graduated statistics more important, or was the form of the curve which they used more important? He thought that undoubtedly if it were decided to adopt a particular form of curve then, as the opener had said, some constraints were imposed on the statistics, so that it was important to start with the right form. He did not think that there was any defence of the present form in any philosophical sense. There was no reason why a frequency curve should represent the particular statistics, but it was interesting to try, when it was seen that arithmetically there was some correspondence between frequency curves and mortality functions.

First of all, the opener had suggested that they should let the form arise from the data. That was exactly what the speaker had been trying to do in his paper of 1953 (*J.I.A.* 79, 170). The force of mortality, μ_x, did not exist by itself but was merely the differential coefficient of $\log l_x$. In the attempt to wring from mortality statistics some rate or force of mortality it was necessary to postulate some relationship which would give the rate of mortality. There were two sets of statistics, the exposed-to-risk and the deaths, and he had assumed that there was a continuous relationship between them. If the relationship were taken in terms of moments, it led to a ratio of polynomials such as he had found. Geometric moments led to a similar ratio but of polynomials with a geometric variable; and he was encouraged when the Mortality Committee adopted just such a ratio of polynomials in geometric form for the latest table (i.e. A 1949–52).

He would join issue with the author with regard to §22 of the paper. Frequently the criticism was made that if the formula did not fit the particular statistics it was the statistics which were wrong. That was an approach which the speaker resented; he felt that the statistics were there, and that the graduator had to seek to learn from them what he could. If the formula did not fit the statistics, it was unfortunate, but it was the formula which was wrong.

The modal point of the expectation of life was at the age when the expectation was equal to $1/\mu$, and the fact that the expectation was increasing in the early years explained why it took a form similar to a frequency curve. The form which the author had obtained on pages 89–90 for the expectation of life looked a little complicated, but its nature could be better seen by taking the differential coefficient of the formula underneath the tables. It would be found that the logarithmic derivative $(1/y)\,(dy/dx)$ was of the form $a/x-b(\omega-x)$, where ω was the limit of life for the particular curve. The parameters ω, a and b, of course, had different values in the two tables.

The differential coefficient of log $\bar{e}_x$ was not μ_x itself, but $\mu_x - 1/\bar{e}_x$ and it was interesting that $a/x-b(\omega-x)$ was similar to the form which he had obtained for μ_x, though in his formula there had been a quadratic added as well.

In §13 the author mentioned the paradox that the curve for the joint expectation for four lives ended at a later age than the expectation for a single life. Although that seemed to be a paradox, the speaker wondered whether it was, perhaps, a necessary consequence of the use of the particular frequency curve. The expectation for four lives never reached so high a level as that for a single life, and therefore ran down less steeply. It might be just a matter of arithmetic that it had to be carried down to a later age at the end.

Diagrams 1 and 2 were diagrams of the reciprocal of μ. It was of interest that the earliest form of statement of mortality had not been as a rate, i.e. a small fraction, but in the form of the number of lives out of which one death would occur; it was stated as 1 in 200 or 1 in 300 and not as ·005 or ·003. If the force of mortality was assumed to take a mathematical form, it did not seem to matter whether calculations were made on μ itself or on its reciprocal. In dissecting the curves, the author had tried to add two frequency curves. The approach in §19 seemed to be preferable to that in §§17–18, because it led to the thought that if two would not do, three might. The speaker, however, instead of adding frequency curves preferred to multiply the ordinates. That was not so easy to visualize, but it did lead to an easier form in the end, because the differential coefficients were then added and the mathematical form was, in fact, simpler.

The form of mortality at the older ages was comparatively well known, but at the younger ages it had not been much studied, because it was not usually of financial importance. It was, however, of interest, and if there were any actuaries in industrial assurance offices who had statistics of mortality at the younger ages it would be an interesting study to concentrate on the mortality from childhood through adolescence to early manhood. So far their information had been derived from national statistics, which were collected at infrequent intervals and were open to some objections, and he thought that more reliable results could probably be obtained from the experience of the industrial assurance offices.

With reference to the older ages, the formulae at the foot of the table in the Appendix related, of course, to the reciprocal of μ. The implied formula for μ, derived from the col. (4) formula at the foot of the table, was simply an exponential divided by a function of x, as the author had noted. That was a corrective to the usual approach by way of adding functions of x to the Gompertz geometric curve. Similarly, with the Type VI curve of col. (7), the origin being changed to birth, the implied formula for μ was simply $\frac{k(x+a)^n}{(x-b)^n}$, which was like the ratio of polynomials with which he had been working.

There was one point which he would like to mention with regard to § 22. He thought that there had been far too much talk in the past about things like homogeneity and heterogeneity when trying to fit curves. They ought to accept the statistics which they had and see what could be done with them and what could be learnt from them. What was essential, to his mind, was that there should be continuity; i.e. it should be possible to say that μ_x existed as a continuous relationship between the two sets of statistics. If there was a break, the assumption was unjustified. The author's work showed that if the force of mortality was made more complicated by bringing in other ideas, a simpler mathematical form might be obtained. Because there was heterogeneity in the statistics, it did not necessarily mean that the mathematical form was more complicated; it might be simplified, and that simplification might have helped Gompertz in his day. He had higher mortality and less reliable statistics, and it had, perhaps, been easier to find a simple mathematical form.

The speaker's interest in the paper lay in the various suggestions about the mathematical form of mortality functions rather than in new techniques. It might be forgotten just how the author had arrived at his results, but it was to be hoped that people who had to fit mortality statistics would come back to the paper as a quarry for suggestions of forms which might be used.

Mr R. Ll. Gwilt welcomed the opportunity of saying a few words, having known the author for a long time and having received many kindnesses from him. When they first met, about thirty years earlier, the speaker had been a very recently qualified Fellow. The interest that the author had always taken in young men was well known and had been referred to earlier that evening; he was at all times ready to go out of his way to help them. From 1930 he had been closely associated with Sir William on mortality matters until Sir William retired from the chairmanship of the Mortality Committee.

He would have liked to be able to offer some comments of value on the interesting experiments which were described in the paper, but unfortunately, having been preoccupied with other matters during the short period for which the paper had been in his hands, he had not been able to study it as carefully as he would have wished, nor had he been able to make any calculations himself; and on a matter of the kind in question, as the author would be the first to admit, a bit of arithmetic helped a lot, or perhaps it would be more correct to say that a lot of arithmetic helped a bit.

The paper served as a reminder, as others had said, that the Pearson system of curves, although they had originally been designed to represent frequency distributions, could be of considerable help in fitting mathematical formulae to series of values of other kinds. It was interesting to reflect that the author's first contribution to the *Journal*, made as far back as 1903, was a paper on temporary assurances in which use was made of the Pearson curves in graduating the temporary assurance experience. At that time the Pearson curves were not known to many members of the actuarial profession. That evening the author had again presented a paper in which those curves were used. It might almost be said that he had returned to his first love, were it not quite clear that he had never deserted her; it was evident from all he had written and said over the past

50 years that he had always had an intense interest in the possible applications of those curves.

Many of those who, like himself, were studying for the actuarial examinations about the end of the first world war—nearly forty years earlier—were introduced to mathematical statistics by work of the author—his notes in the early numbers of *J.S.S.*, the *Primer of Statistics* which he wrote in collaboration with his sister, and his masterly work *Frequency Curves and Correlation*. There had been little on the subject of mathematical statistics, or indeed on statistics at all, in the syllabus at that time, which might seem strange to the modern generation of students, who were faced with a heavy syllabus of statistics and had no lack of textbooks. He could remember clearly, just after the first world war, the thrill of learning about the Pearson system of curves from *Frequency Curves and Correlation*, which had been written at the request of the Council because few actuaries at that time knew of the curves and methods and a systematic exposition of the subject was desired.

In Sir William's preface to the first edition of that book, he wrote 'There are few subjects which offer a richer field for original work than statistical mathematics and its applications.' How right he had been, and how extensively that field had been tilled! There had become available numerous textbooks on mathematical statistics, including the Institute's own textbooks.

Reference had been made by other speakers to §§22–23 of the paper. The speaker liked those sections, which were a reminder that a formula built up on general considerations to represent the progression of human mortality according to age was only properly applicable to a generation table, and there was no particular reason to expect it to apply to what the author called a 'census' table, because of the changes in mortality over time. That was one of the reasons why the speaker personally had never felt that it mattered very much how a table of the 'census' type was graduated—he said 'graduated' but perhaps it would be better to say 'smoothed'—so long as a result was obtained which was sufficiently smooth for practical purposes.

The opener had referred to the masterly brevity and clarity of the paper, and that had reminded him of a remark made by Sir Edward Appleton, the Principal of Edinburgh University, when President of the British Association. Speaking to a meeting of scientists, Sir Edward had said 'If you can't put it in simple terms, you really haven't got to the bottom of it yourself.' He felt that the author would agree whole-heartedly with that remark.

Mr J. M. MacLeod emphasized that the fitting of curves to statistics was not an end in itself, as some previous speakers had implied. If all the data were examined a curve might be found to underlie them which in its turn would lead to new knowledge. The astronomers' observations of the motion of the planets, which were found to form ellipses, led to the formulation of the law of gravitation, with all its vast philosophical and religious implications, and the existence of the planet Neptune had been forecast by a process of curve-fitting. By continuing the practice of curve-fitting, therefore, some important principle might be discovered.

Mr C. M. O'Brien, speaking as one of the many young men whom the author had led into the actuarial profession, said he suspected that there was a fourth reason for the presentation of the paper, in addition to those given by him in the paper and in his introductory remarks, namely that it should be an invitation, and perhaps even a challenge, to others to continue what he had done and to do more of it; as the opener had said, research was self-reproducing.

There was one sentence in the paper which he wished to take, somewhat improperly, out of its context. The author had been talking about the effect of graduation in hiding, wholly or in part, the peak at age 27, and then he said 'May this be taken as a warning against using graduated figures alone to find the meaning of statistics?' The speaker thought that graduations were used in two ways. On the one hand they were used, as the author implied, in finding a meaning for statistics, but at the same time they did not absolve the user from the need to look at the actual data. The graduation would hide things; any graduation involved assumptions about the form of the function graduated. On the other hand, their second use was to produce a table for a particular practical purpose. Referring again to the maximum kink at age 27, its financial implications might be small, and in looking for a suitable table for the calculation of premiums an actuary might say 'I do not want a kink; I want a smooth curve.'

In almost any investigation actuaries made they implicitly used graduation in those two distinct ways. They used the first to try to find a meaning in the statistics and what was significant in them, and the second to produce a practical table. There was a danger in confusing those two uses, since, on account of the labour involved, one graduation was often made to serve both purposes.

It was important, however, to bear in mind the two different aspects of graduation, because they might well lead to two different methods of attack, or, what was perhaps more important, two different answers.

Mr B. Benjamin, in closing the discussion, said he was old enough and young enough to indulge in reminiscences about the author. More than twenty years earlier, browsing in a library, he had come across a book called *Frequency Curves and Correlation* by W. Palin Elderton. At that time he had been reading physics, and his ideas about statistical error were firmly tied to the theory of least squares, so that it had been an adventure to read about the method of moments and to find that it was not only easier but of wider application than the method of least squares, and even mathematically respectable. He particularly remembered finding at the beginning of the book a key to actuarial terms, and it was there that he had seen his first commutation function. Soon after that he gave up his ambition to become a physicist and became a student of the Institute. To what extent the second followed the first he was not clear, but at least he had become an actuary, and the author was partly to blame.

Although he had almost forgotten what to do with a commutation function, he was from time to time involved in graduation and curve-fitting, and, since it was usually from sheer necessity, he had learnt to depend less on the text-books and more on experiment as a means of achieving his particular objectives with the least possible effort. The author had written in the preface to his book that the reader who went through a book on a practical subject and did not work out examples was as certain to encounter imaginary and to miss real difficulties as he was to fail to attain any satisfactory knowledge of the subject, and Mr Gwilt had underlined that even more strongly. It was for that reason that when colleagues came to him (the speaker) and asked him whether he thought it would be useful to employ such-and-such a procedure he replied 'Why don't you try it and see?' That was what he had thought when he read the title of the paper, and that was precisely what the author had done.

The next question was not 'Can you?' but 'Do you want to?' The two principal reasons for curve-fitting, in his own experience, were (i) to achieve smoothness, and (ii) to find out a little more than was already known about the behaviour of the variable under observation. It was essential that the first purpose should not be allowed to obscure the second, and he appreciated the warning given in § 15 of the paper. It was important that the actuary should take note of what was already known about the variable, for that was in fact part of the data. It ought to be taken for granted that adherence to the data, even in the wider sense, was something which could be controlled by simple techniques, such as the precaution of plotting the observed values with their associated confidence limits, and, not least, common sense. What was the next step? Was it to draw a freehand curve and polish the differences, to use a summation formula or to fit a mathematical curve? The first was sufficient in many circumstances and was widely used. In what circumstances should there be used a mathematical curve or curves, which might be a laborious procedure in so far as persistent trial and error, about which a warning was given in § 11 of the paper, might be necessary?

There might be very good reasons indeed. The classical advantages were perfect smoothness, more precision in reading graduated values, mathematical treatment (e.g. the calculation of derived functions), the provision of indications as to the law of progression of the dependent variable in terms of the independent variable, and finally forecasting by extrapolation of parameters, as in the recent work by Prawitz on Swedish mortality. That last advantage had indeed become important as those who studied mortality endeavoured to encompass in the one operation both the analysis of mortality rates into component curves and a consideration of prospects for changes in future in those curves. Practical applications had been suggested in the discussion of the A 1949–52 mortality table. The scope for research had been exemplified in Mr Barnett's paper (*J.I.A.* **81**, 105) and had been underlined by the exercises in dissection discussed that evening; but it was the more analytical approach to graduation and forecasting which had been the recurring theme of recent papers on mortality.

It was right that some of their oldest techniques should be re-examined, and it was clearly right that, as the doctors told them more about the ways in which people died, they should study the implications for the shapes of those mortality curves about which they might have become too conservative.

He agreed with other speakers that it was good to find that no less an authority than Sir William emphasized, in § 22, that it was hardly possible to expect a perfect fit from a curve of death rates constructed on a 'census' basis. Of all the many sources of heterogeneity, the mixture of generations was not the least. They constructed life tables from observations knowing that the latter would never be reproduced. That should be borne in mind when they talked about adherence to data.

It had been a great pleasure to be in their new-old hall and listen to the new-old Sir William reading a new-old paper.

The President, in proposing a vote of thanks to the author, also thanked Mr Benjamin for his remarks, and particularly for that last happy phrase, and congratulated Mr Bizley on the way in which he had opened the discussion—his speech had been a real *tour de force.*

Referring to the question of research, which had been mentioned during the discussion, the President said that a science such as theirs needed growth and development in order to survive. Over the hundred years or more of the life of the Institute there had never been any lack of men ready to investigate new lines of thought in theory and in practice. Some men had been outstanding in the wealth of their contributions to actuarial science and to the development of new ideas. It was the duty of them all to see that the spirit of research and inquiry did not flag, in spite of the demands which fell on them in their post-graduate years. Not all of them were tutors or examiners, and there were many inviting lines awaiting research. His predecessor, Mr Gardner, had mentioned a number of such avenues of exploration in some remarks which he made at the Annual General Meeting in 1953. There was no need to repeat them again; they were in the record. The experiments which formed the basis of Sir William's paper, however, would serve as an example to them all of the delights of exploration and of their duty to add something to the sum of actuarial knowledge.

Sir William had been a Fellow of the Institute for almost exactly one half of its existence, and it was even eighteen years since he had received the Gold Medal given by the Institute and the Faculty, but he was present that evening more youthful than ever and still an example to them all.

Lady Elderton must have had an enjoyable evening hearing something of the love which they bore for her husband and hearing him speak to them.

Addressing Sir William, the President said 'You have by this paper put us in your debt more than ever. We are deeply grateful to you for the stimulation of your example. I hope you feel that the discussion which we have had this evening is a suitable acknowledgement of the efforts which have gone into the preparation of this paper.' Addressing all those present, he added 'Ladies and gentlemen, members and visitors, this occasion of a paper presented by our well-loved friend calls for some special acknowledgement. I therefore suggest that we stand to give him our enthusiastic applause.'

Sir William Elderton, in reply, said he felt most embarrassed, but wished to express his thanks. He would ask the Editors to allow him to reply to the technical parts of the discussion at his leisure, when he could see them in their more or less final form. He might perhaps mention that he had in the past—though he was afraid that the papers had gone the way of all flesh, or rather of all papers—tried the expectation of life with parabolas and with Fourier series, and had not been satisfied with either. That perhaps lent colour to what had been said, that after all there might be some justification for using the group of curves he had employed rather than some of the other functions, the justification being that perhaps they worked. He was not sure that a defence could not be made out in an extended definition of frequency distributions which might rope in expectations, but that was a detail.

He was very grateful to all the speakers for the kind things that they had said; in fact, he had been so touched by all their kindness that it was very difficult for him to reply. He was most grateful to them all, and would specially thank his friend Mr Gwilt for referring to his first love and his return to it when he might so easily have referred to Proverbs xxvi. 11!

Sir William Elderton subsequently wrote:

I think some of my kind critics when saying that $\bar{e}$ and $1/\mu$ are not frequencies were really objecting to the title of the paper which describes a group of curves in words that I thought would identify them most easily for actuaries. If I had said 'Can we express $\bar{e}_{xx\ldots(n)}$ and $1/\mu_x$ by $f(x)$, where $\frac{d}{dx}\log f(x) = \frac{a_0+a_1x}{b_0+b_1x+b_2x^2}$?'

their objections might have been diminished or even have disappeared. Surely no one thinks that, because a curve has been used successfully for a particular purpose, it is therefore excluded from other employment. While Wallas, MacLeod and some others had no such thought, a few remarks by other speakers seemed to get fairly near to it. I was asked by Barnett for 'a reason for wanting to fit frequency curves to the two functions'. My reason was that the shapes of the two functions invited it; their appearance led me to think it possible. But I suspect that that is not the sort of reason that Barnett and a few others have in mind for they are thinking in terms of the applicability of a 'frequency curve', while I am thinking in terms of the applicability of a mathematical expression. And the disagreement, such as it is, seems to be my fault because I used the phrase 'frequency curve'. If we want to express mathematically some arithmetical numbers that take a shape like a semicircle we need not refrain from using that form because we do not see an underlying reason why such a curve should do it. Some day we may evolve a theory in explanation—anyway we should have added to our knowledge by noticing that a semicircle could describe those figures. I am a bit sceptical about a philosophical argument in favour of a particular kind of curve; there is a risk that we may be disguising an inherent arbitrariness. Like Cupid, whose poor shots are notorious, we may feel content with a happy marriage between statistics and a curve, hoping that it may produce something in the future as useful as may be evolved from *a priori* hypotheses, however entrancing. I wonder whether $\mu_x = a + bc^x$ is regarded as being based on a good hypothesis or a sound reason. If so, does the use of $\phi(x)$ for x make it arbitrary? Personally I regard $\mu_x = a + bc^x$ as a useful approximation, sometimes accurate though limited in range. That seems to me a good defence for it; an excellent reason for its existence. Did Makeham notice that the form $a + bc^x$ fitted mortality better than bc^x and subsequently look for reasons to justify or excuse its use; or did he evolve the reason first? I do not know. Would it have been any less convincing, any less useful, if no one had ever given a justification, reason, hypothesis, excuse or whatever you like to call it on which $a + bc^x$ could stand?

Johnson asks me about the actual fitting of the curves. There was no adjustment to the moments leading to Table 3, col. 3, and §6 was meant to imply this. It was admittedly rough; what sculptors might regard as a trial piece. With the later shots however I had summations and accumulated summations of $\bar{e}$ to guide me and used some adjustments (see §10). It may be worth mentioning that the trial and error process for the mode and start seems to compensate in some way for minor inaccuracies in the moments. Johnson's remarks were very interesting, so were those of Moore especially when he pointed to possible developments. But I cannot make the number of parameters for the curves in the Appendix so many as nine.

Martin's remarks about the mortality in 1950–2 would be of even greater interest if some trials could be made of dissections of $1/\mu$. I needed a mortality table with a fairly obvious hump in μ in early manhood; it was easier thereby to show what I was trying to do. Many suggestions were thrown out by other speakers which I hope they may develop successfully.

§22 had a slightly mixed reception. It does not say that, if a formula which fits a 'generation' set of rates of mortality fails to fit a 'census' set, then the statistics of the latter are 'wrong', nor are they even suspect. It merely says that we ought not to be surprised if a formula fails to do a second task different from the one in which it was successful. Gwilt understood what I had in mind but if I was misunderstood it is probably my own fault: the paragraph was compressed from some dozen pages of manuscript; I like compression and may have overdone it.

Many suggestions have been made in the recent past as to how the problem of representing mortality mathematically could be solved and they were reflected in the discussion. But whether we set out with an attempt to sub-divide the deaths according to cause, or with an examination of the exposed-to-risk and deaths or with any of the other ways that it has been a pleasure to study, our path is 'beset with pitfall and with gin'. Each plan has its own dangers but whatever our approach we must distinguish between 'generation' and 'census' mortality and ask ourselves whether the rates of mortality which will emerge are to be connected age by age because they relate to individuals in common (i.e. q_x and q_{x+t} come out of the same set of individuals) or because the calendar year to which they relate is common to all ages though the individuals at the various ages are independent (i.e. q_x and q_{x+t} relate to the same calendar year but to entirely different sets of people).

Even though, as I have shown, I do not agree with everything every speaker said, I enjoyed listening to them all from the President and opener to the closer and President—how could I help doing so when there was such a transparent conspiracy to give pleasure to me—and to my wife.