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CENTRAL DIFFERENCE FORMULAE OBTAINED BY MEANS OF OPERATOR EXPANSIONS

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1. INTRODUCTION AND SUMMARY

INTERPOLATION and other formulae involving advancing differences are frequently developed, or at least conveniently reproduced from memory, from the familiar operational identities

$$E \equiv 1 + \Delta \equiv e^{D}, \tag{1·1}$$

where $\quad Eu_x = u_{x+1}, \quad \Delta u_x = u_{x+1} - u_x, \quad Du_x = du_x/dx.$

Although operator expansions have been used to develop formulae involving central differences (notably by Steffensen, *Interpolation*, § 18), operational methods are not generally used in the case of central differences and means of employing such methods of development are not widely known. In this note the fundamental identities

$$\delta \equiv 2 \sinh \tfrac{1}{2}D, \tag{1·2}$$

$$\mu \equiv \cosh \tfrac{1}{2}D, \tag{1·3}$$

are used to develop systematically and unify processes for expanding and synthesizing formulae involving central differences. Various algebraic artifices for manipulating the basic identities are discussed and the methods are illustrated by application to a wide variety of formulae.

No attempt is made here to justify or prove the legitimacy of operational methods. It is assumed that any operator or operators may be divorced from their operand and manipulated as purely algebraic quantities (with certain restrictions on the commutability of inverse operators), and then may be reunited in their manipulated form to the operand to give a new operator expansion. In an Appendix all the expansions of the hyperbolic functions, sinh x, cosh x, etc., and their inverses which are used in this note are developed.

2. BASIC IDENTITIES INVOLVING CENTRAL DIFFERENCE OPERATORS

Lemma. Taylor's theorem,

$$u_{a+x} = u_a + xDu_a + \frac{x^2}{2!}D^2u_a + \ldots,$$

where D represents the differential operator d/da, may be expressed in the condensed (operational) form

$$u_{a+x} = [e^{xD}]\, u_a. \tag{2·1}$$

The right-hand term of (1·1) is derived from this representation.

The central difference operators δ and μ are defined by $\delta u_x = u_{x+\frac{1}{2}} - u_{x-\frac{1}{2}}$, $\mu u_x = \frac{1}{2}(u_{x+\frac{1}{2}} + u_{x-\frac{1}{2}})$. With the aid of (2·1) we can derive from these definitions the following operational identity:

$$\begin{aligned}\delta u_x = u_{x+\frac{1}{2}} - u_{x-\frac{1}{2}} &= [e^{\frac{1}{2}D} - e^{-\frac{1}{2}D}]\, u_x \\ &= [2\sinh \tfrac{1}{2}D]\, u_x \quad \text{[see (A 1·1)]}.\end{aligned}$$

[References to formulae in the Appendix are prefaced by the letter 'A'.]

Detaching the operators from their operands, this may be expressed in the form:

$$\delta \equiv e^{\frac{1}{2}D} - e^{-\frac{1}{2}D} \equiv 2 \sinh \tfrac{1}{2}D; \tag{2·2}$$

and inversely [see (A 1·6)]

$$D \equiv 2 \sinh^{-1} \tfrac{1}{2}\delta \equiv 2 \log_e [\tfrac{1}{2}\delta + \sqrt{\{1 + \tfrac{1}{4}\delta^2\}}]. \tag{2·3}$$

Similarly, from the definition of the operator μ,

$$\mu u_x = \tfrac{1}{2}(u_{x+\frac{1}{2}} + u_{x-\frac{1}{2}}) = \tfrac{1}{2}[e^{\frac{1}{2}D} + e^{-\frac{1}{2}D}]\, u_x$$
$$= [\cosh \tfrac{1}{2}D]\, u_x \quad \text{[see (A 1·2)]},$$

leading to the operational identity

$$\mu \equiv \cosh \tfrac{1}{2}D \equiv \sqrt{\{1 + \tfrac{1}{4}\delta^2\}}, \tag{2·4}$$

[since $\cosh \frac{1}{2}D \equiv \sqrt{\{1 + \sinh^2 \frac{1}{2}D\}} \equiv \sqrt{\{1 + \frac{1}{4}\delta^2\}}$ from (A 1·3) and (2·2)].

It is from these basic identities that all the processes used will be developed.

3. DIFFERENCE OPERATORS REFERRING TO INTERVALS OTHER THAN UNITY

In § 2 the tacit assumption was made that the difference table applied to intervals of unity in the argument, x. Where finite difference operators apply to tabular intervals, h, other than unity, the notation δ_h, μ_h, etc. will be used. The reader may confirm, either by development of fundamental expansions analogous to (1·1), (2·2), (2·3), (2·4) *ab initio*, or by replacing the argument x by the new argument $y = hx$, that the following rules enable any formula which has been developed for interval of differencing unity to be generalized to refer to interval of differencing h:

Any formula involving differential and finite difference operators which has been developed for differencing interval unity may be generalized to refer to interval h by:

(i) adding a subscript h to finite difference operators to signify that they refer to interval of differencing h,

(ii) replacing D by (hD),

(iii) replacing $\int \dots dx$ by $\frac{1}{h} \int \dots dx$.

For example, (2·2), (2·3) and (2·4) become

$$\delta_h \equiv e^{\frac{1}{2}hD} - e^{-\frac{1}{2}hD} \equiv 2 \sinh \tfrac{1}{2}hD, \tag{3·1}$$

$$hD \equiv 2 \sinh^{-1} \tfrac{1}{2}\delta_h \equiv 2 \log_e [\tfrac{1}{2}\delta_h + \sqrt{\{1 + \tfrac{1}{4}\delta_h^2\}}], \tag{3·2}$$

$$\mu_h \equiv \cosh \tfrac{1}{2}hD \equiv \sqrt{\{1 + \tfrac{1}{4}\delta_h^2\}}. \tag{3·3}$$

4. NUMERICAL DATA AVAILABLE FROM THE DIFFERENCE TABLE

Unlike advancing differences, central differences of all orders are not available for every tabular value of the argument. For values of the argument at which a function is tabulated, central differences of even order only are available; mean central differences of odd order ($\mu\delta^{2n+1} u_x$) may be obtained. Corresponding to values of the argument mid-way between tabular values, central differences of odd order are available; mean central differences of even order ($\mu\delta^{2n} u_{x+\frac{1}{2}}$) may be obtained.

It is this peculiarity, imposed by the definition and notation of central differences, that renders generalized operational methods of obtaining *usable* central difference formulae more complex than the corresponding methods for advancing or backwards difference formulae.

5. PRINCIPLE OF METHODS EMPLOYED IN EXPANDING CENTRAL DIFFERENCE OPERATORS

In every case the fundamental operator representing the requirement of the formula we wish to deal with will be expressed as a function of the operator D [generally by means of the operational form of Taylor's theorem (2·1)]. Then, by means of the identities (2·2), (2·3) and (2·4), this function of D will be transformed to a function of δ representing operators which are usable in the sense of § 4 when applied to an operand which exists in the form of tabular data. It is therefore expedient to examine methods of predetermining the nature of the δ-operator, and of manipulating it when necessary into a form which when expanded will be usable.

Since $\sinh x$ (and therefore $\sinh^{-1} x$) is an odd function of x, it follows from (2·2) and (2·3) that the transformation of an odd or even function of D will be an odd or even function of δ, and conversely. For, if

$$\mathrm{F}(-\mathrm{D}) \equiv -\mathrm{F}(\mathrm{D}) \quad [\mathrm{F}(\mathrm{D}) \text{ being an odd function of D}],$$

i.e. $$\mathrm{F}(-2\sinh^{-1}\tfrac{1}{2}\delta) \equiv -\mathrm{F}(2\sinh^{-1}\tfrac{1}{2}\delta),$$

i.e. $$\mathrm{F}[2\sinh^{-1}(-\tfrac{1}{2}\delta)] \equiv -\mathrm{F}(2\sinh^{-1}\tfrac{1}{2}\delta),$$

say $$\mathrm{G}(-\delta) \equiv -\mathrm{G}(\delta), \quad \text{i.e. } \mathrm{G}(\delta) \text{ is an odd function of } \delta.$$

Similarly, if $$\mathrm{F}(-\mathrm{D}) \equiv \mathrm{F}(\mathrm{D}), \quad \text{an even function of D},$$

its transformation $$\mathrm{G}(-\delta) \equiv \mathrm{G}(\delta), \quad \text{is an even function of } \delta.$$

6. METHOD OF INTRODUCING THE OPERATOR μ TO PRODUCE USABLE OPERATORS

If the expression in D (or a part of it) which we obtain in our initial step in deriving our central difference formula is an odd function of D, it follows from the preceding paragraph that its transformation in δ which we proceed to obtain will be an odd function of δ. If the resulting operator is to be applied to an operand u_x at a tabular value, x, of the argument, the resulting terms $\delta^{2n+1} u_x$ will not be of the form appearing in the difference table.

Suppose $\mathrm{F}(\mathrm{D}) \equiv \mathrm{H}(\delta)$ is the operator (or part of an operator) which is an odd function of δ. Then, since $\cosh x$ is an even function of x,

$$\mu \equiv \cosh\tfrac{1}{2}\mathrm{D} \equiv \sqrt{\{1+\tfrac{1}{4}\delta^2\}}$$

is an even function of D and δ. Also

$$\frac{\mu}{\cosh\frac{1}{2}\mathrm{D}} \equiv \frac{\mu}{\sqrt{\{1+\frac{1}{4}\delta^2\}}} \equiv 1.$$

Hence $$\mathrm{F}(\mathrm{D}) \equiv \frac{\mu\mathrm{F}(\mathrm{D})}{\cosh\frac{1}{2}\mathrm{D}} \equiv \frac{\mu\mathrm{H}(\delta)}{\sqrt{\{1+\frac{1}{4}\delta^2\}}}.$$

It follows from § 5 and the fact that $\cosh(-\tfrac{1}{2}\mathrm{D}) \equiv \cosh\tfrac{1}{2}\mathrm{D}$ that

$$\frac{\mu\mathrm{H}(-\delta)}{\sqrt{\{1+\frac{1}{4}(-\delta)^2\}}} \equiv -\frac{\mu\mathrm{H}(\delta)}{\sqrt{\{1+\frac{1}{4}\delta^2\}}},$$

i.e. the modified operator $\quad F(D) \equiv \dfrac{\mu H(\delta)}{\sqrt{\{1+\frac{1}{4}\delta^2\}}}$

is also an odd function of δ, and when expanded and applied to u_x has terms of the form $\mu\delta^{2n+1}u_x$ which are usable.

7. OTHER METHODS OF PRODUCING USABLE OPERATORS

An even function of δ applied to an operand u_x at tabular values of the argument (or an odd function of δ applied to an operand $u_{x+\frac{1}{2}}$ midway between two tabular values) is usable in the sense of §4. An odd function of δ applied to an operand at tabular values of the argument (or an even function at midway values) may be transformed into a usable operator by the artifice of §6.

In general the operator in D (and therefore in δ) will be neither an even nor an odd function. The even and odd terms of such an operator may be separated. Formally

$$F(D) \equiv G(D^2) + DH(D^2) \equiv J(\delta^2) + \delta K(\delta^2).$$

The term which is not of usable form may then be multiplied by

$$\mu/\sqrt{\{1+\tfrac{1}{4}\delta^2\}} \equiv 1$$

as in §6. An alternative treatment used in (for example) the Gauss central difference formula, is to convert the operator F(D) into an operator of even order operating upon u_x plus an operator of odd order operating upon $u_{x+\frac{1}{2}}$ or $u_{x-\frac{1}{2}}$. Formally

$$F(D)\,u_x = G(D^2)\,u_x + DH(D^2)\,u_{x+\frac{1}{2}},$$

whence $\quad F(D) \equiv G(D^2) + DH(D^2)\,e^{\frac{1}{2}D}.$

The problem then is to determine the form of the functions G and H in order that this identity may hold. One method of so doing is illustrated in Example IV below.

8. EXAMPLES

Three groups of examples are given: (i) those resulting from the expansion of an even (or odd) function of D which can be expanded immediately as a usable series of operators of the form δ^{2n} (or of the form $\mu\delta^{2n+1}$ using the artifice of §6); (ii) the central difference interpolation formulae where methods suggested in §7 are used; (iii) summation and quadrature formulae, involving inverse powers of the operators.

9. GROUP I

Example I. The differential coefficients of a tabulated function

We have, by (2·3), $\quad D^n \equiv [2\sinh^{-1}\tfrac{1}{2}\delta]^n. \qquad (9·1)$

If n is even (say $n=2m$), we can write [from (A 2·1)]

$$D^{2m} \equiv [2\sinh^{-1}\tfrac{1}{2}\delta]^{2m} \equiv \delta^{2m}[1 - \tfrac{1}{2}.\tfrac{1}{3}(\tfrac{1}{2}\delta)^2 + \tfrac{1}{2}.\tfrac{3}{4}.\tfrac{1}{5}(\tfrac{1}{2}\delta)^4 - \tfrac{1}{2}.\tfrac{3}{4}.\tfrac{5}{6}.\tfrac{1}{7}(\tfrac{1}{2}\delta)^6 + \ldots]^{2m}, \qquad (9·2)$$

and for any particular value of $2m$ the right-hand side can be calculated by actual involution; e.g. if $m=2$,

$$D^4 \equiv \delta^4[1 - \tfrac{1}{6}\delta^2 + \tfrac{7}{240}\delta^4 - \ldots], \qquad (9·3)$$

which when attached to an operand, say u_0, is to be interpreted as

$$\left[\frac{d^4u_x}{dx^4}\right]_{x=0} = \delta^4 u_0 - \tfrac{1}{6}\delta^6 u_0 + \tfrac{7}{240}\delta^8 u_0 - \dots.$$

If n is odd (say $n = 2m+1$), we can, following § 6, write

$$\begin{aligned}\mathrm{D}^{2m+1} &\equiv [2\sinh^{-1}\tfrac{1}{2}\delta]^{2m+1} \equiv \mu\,[2\sinh^{-1}\tfrac{1}{2}\delta]^{2m+1}(1+\tfrac{1}{4}\delta^2)^{-\frac{1}{2}} \\ &\equiv \mu\delta^{2m+1}[1 - \tfrac{1}{2}.\tfrac{1}{3}(\tfrac{1}{2}\delta)^2 + \tfrac{1}{2}.\tfrac{3}{4}.\tfrac{1}{5}(\tfrac{1}{2}\delta)^4 - \dots]^{2m+1} \\ &\qquad \times [1 - \tfrac{1}{2}(\tfrac{1}{2}\delta)^2 + \tfrac{1}{2}.\tfrac{3}{4}(\tfrac{1}{2}\delta)^4 - \dots],\end{aligned}$$

and perform the indicated algebraical multiplications. There is, however, another way of obtaining central difference expansions of odd order differential coefficients. Consider the differentiation of both sides of equation (9·1) with regard to δ:

$$\begin{aligned}\frac{d}{d\delta}[\mathrm{D}^{2m}] &\equiv 2m\,[2\sinh^{-1}\tfrac{1}{2}\delta]^{2m-1}(1+\tfrac{1}{4}\delta^2)^{-\frac{1}{2}} \quad [\text{see (A 1·8)}] \\ &\equiv \frac{2m}{\mu}\mathrm{D}^{2m-1},\end{aligned}$$

whence $$\mathrm{D}^{2m-1} \equiv \frac{\mu}{2m}\frac{d}{d\delta}\mathrm{D}^{2m}.$$

Applying this, for example, to (9·3), we get

$$\mathrm{D}^3 \equiv \frac{\mu}{4}\frac{d}{d\delta}[\mathrm{D}^4] \equiv \frac{\mu}{4}\frac{d}{d\delta}[\delta^4 - \tfrac{1}{6}\delta^6 + \tfrac{7}{240}\delta^8 - \dots],$$

i.e. $$\mathrm{D}^3 \equiv \mu\,[\delta^3 - \tfrac{1}{4}\delta^5 + \tfrac{7}{120}\delta^7 - \dots],$$

which, when applied to u_0, gives the usable expansion

$$\left[\frac{d^3u_x}{dx^3}\right]_{x=0} = \mu\delta^3 u_0 - \tfrac{1}{4}\mu\delta^5 u_0 + \tfrac{7}{120}\mu\delta^7 u_0 - \dots.$$

Example II. Subtabulation—differences in subdivided intervals. Assume that we are given a table of u_x at unit intervals of the argument x (whence the functions $\delta^{2m}u_0$, $\mu\delta^{2m+1}u_0$ may be calculated), and that it is required to construct a table for intervals of $1/h$.

Since $$\mathrm{D} \equiv 2\sinh^{-1}\tfrac{1}{2}\delta \equiv 2h\sinh^{-1}\tfrac{1}{2}\delta_{1/h}, \qquad \text{(see § 3)},$$

$$\delta_{1/h} \equiv 2\sinh[(1/h)\sinh^{-1}\tfrac{1}{2}\delta]. \tag{9·4}$$

Using expansion (A 3·1), replace n by $1/h$ and y by $\sinh^{-1}\tfrac{1}{2}\delta$ (so that $\sinh y = \tfrac{1}{2}\delta$). We then get

$$\delta_{1/h} \equiv \frac{\delta}{h}\left[1 + \frac{1-h^2}{3!}\left(\frac{\delta}{2h}\right)^2 + \frac{(1-h^2)(1-9h^2)}{5!}\left(\frac{\delta}{2h}\right)^4 + \dots\right]. \tag{9·5}$$

By raising both sides of this expansion to the power $2m$, we get a usable expansion for even-order subtabular central differences; e.g. $\delta^2_{1/h}u_0$, $\delta^4_{1/h}u_0$, ..., etc., are calculable therefrom.

To obtain a usable expansion for odd-order differences apply again the device of differentiating with regard to δ the $2r$th power of (9·4) to give

$$\frac{d}{d\delta}(\delta_{1/h})^{2r} \equiv 2^{2r}\frac{r}{h}\left[\sinh\left\{\frac{1}{h}\sinh^{-1}\tfrac{1}{2}\delta\right\}\right]^{2r-1}\cosh\left[\frac{1}{h}\sinh^{-1}\tfrac{1}{2}\delta\right](1+\tfrac{1}{4}\delta^2)^{-\frac{1}{2}}.$$

But $$2^{2r-1}\left[\sinh\left\{\frac{1}{h}\sinh^{-1}\tfrac{1}{2}\delta\right\}\right]^{2r-1}\equiv(\delta_{1/h})^{2r-1}\quad\text{from (9·4)},$$

and $$\cosh\left[\frac{1}{h}\sinh^{-1}\tfrac{1}{2}\delta\right]\equiv\cosh\frac{1}{2h}\mathrm{D}\equiv\mu_{1/h},$$

and $$(1+\tfrac{1}{4}\delta^2)^{-\frac{1}{2}}\equiv\mu^{-1}.$$

Hence $$\frac{d}{d\delta}(\delta_{1/h})^{2r}\equiv\mu_{1/h}\frac{2r}{h}(\delta_{1/h})^{2r-1}/\mu,$$

or $$\mu_{1/h}(\delta_{1/h})^{2r-1}\equiv\mu\frac{h}{2r}\frac{d}{d\delta}(\delta_{1/h})^{2r}. \tag{9·6}$$

For example, putting $h=10$, and raising (9·5) to the fourth power we get

$$\delta^4_{1/10}\equiv\cdot0001\delta^4-\cdot0000165\delta^6+\ldots.$$

Applying (9·6) to this expansion, we obtain

$$\mu_{1/10}\delta^3_{1/10}\equiv\cdot001\mu\delta^3-\cdot0002475\mu\delta^5+\ldots.$$

By means of such expansions we may obtain the following (unbracketed) differences:

Argument x	Function u_x	Differences			
0	u_0	$\mu_{1/h}\delta_{1/h}.u_0$ $[\delta_{1/h}.u_{1/2h}]$	$\delta^2_{1/h}.u_0$	$\mu_{1/h}\delta^3_{1/h}.u_0$ $[\delta^3_{1/h}.u_{1/2h}]$	$\delta^4_{1/h}.u_0$

The bracketed terms follow from the fundamental relation

$$\delta^{2r-1}_{1/h}u_{1/2h}=\mu_{1/h}\delta^{2r-1}_{1/h}u_0+\tfrac{1}{2}\delta^{2r}_{1/h}u_0,$$

and with this start the 'sub-table' may be constructed by building up differences.

Example III. The summation operator $[m]$. A derivation of the expansion for the operator $[m]$ on precisely the basis employed in this note has already been given by Aitken [*J.I.A.* Vol. LX (1929), p. 339] and, earlier, by Henderson [*Transactions of the Actuarial Society of America*, Vol. IX (1906), p. 211].

The operational form is

$$[m]\equiv\frac{\sinh\tfrac{1}{2}m\mathrm{D}}{\sinh\tfrac{1}{2}\mathrm{D}},$$

which may be expanded in terms of δ by using (A 3·1) divided throughout by $\sinh y$, writing $y=\tfrac{1}{2}\mathrm{D}$ therein and using (2·2) in the right-hand side.

10. GROUP II

Example IV. The derivation of the operational formula for one interpolation formula only—Stirling's—will be examined in detail. The general form of the others, which may be derived by similar processes will be stated without proof.

It is required to obtain a formula of the following nature:

$$u_x=[e^{x\mathrm{D}}].u_0=[(\text{even function of }\delta)+\mu\,(\text{odd function of }\delta)].u_0,$$

i.e. to find two functions F and G such that

$$e^{x\mathrm{D}}\equiv\mathrm{F}(x,\delta^2)+\mu\delta.\mathrm{G}(x,\delta^2). \tag{i}$$

In (i) replace D by $-$D, whence δ becomes $-\delta$ (cf. § 5), but μ, being an even function of D and δ, is unaltered.

$$e^{-xD} \equiv F(x, \delta^2) - \mu\delta . G(x, \delta^2). \qquad \text{(ii)}$$

Solving (i) and (ii) as simultaneous equations in F and δG gives

$$F \equiv \cosh xD, \quad \delta G \equiv \frac{\sinh xD}{\cosh \frac{1}{2}D},$$

whence is obtained the operational form of Stirling's central difference interpolation formula:

$$e^{xD} \equiv \cosh xD + \mu \frac{\sinh xD}{\cosh \frac{1}{2}D}.$$

Using (A 3·2) and (A 4·1), $\cosh xD$ and $\sinh xD/\cosh \frac{1}{2}D$ may be expanded in terms of $\sinh \frac{1}{2}D \equiv \frac{1}{2}\delta$ to give

$$e^{xD} \equiv \left[1 + \frac{x^2}{2!}\delta^2 + \frac{x^2(x^2-1)}{4!}\delta^4 + \dots\right]$$
$$+ \left[x\delta + \frac{x(x^2-1)}{3!}\delta^3 + \frac{x(x^2-1)(x^2-2^2)}{5!}\delta^5 + \dots\right]\mu.$$

When attached to the operand u_0 the right-hand side of this expansion gives Stirling's formula for u_x.

General form of the expressions for the other central difference interpolation formulae

(*a*) Gauss 'forwards' formula: $$e^{xD} \equiv \frac{\cosh(x-\frac{1}{2})D}{\cosh\frac{1}{2}D} + \frac{\sinh xD}{\cosh\frac{1}{2}D}e^{\frac{1}{2}D}.$$

(*b*) Gauss 'backwards' formula: $$e^{xD} \equiv \frac{\cosh(x+\frac{1}{2})D}{\cosh\frac{1}{2}D} + \frac{\sinh xD}{\cosh\frac{1}{2}D}e^{-\frac{1}{2}D}.$$

(*c*) Bessel's formula (to be applied to the operand $u_{\frac{1}{2}}$):

$$e^{(x-\frac{1}{2})D} \equiv \frac{\cosh(x-\frac{1}{2})D}{\cosh\frac{1}{2}D}\mu + \sinh(x-\tfrac{1}{2})D.$$

(*d*) Everett's formula: $$e^{xD} \equiv \frac{\sinh xD}{\sinh D}e^{D} + \frac{\sinh \xi D}{\sinh D},$$

where $\xi = 1 - x$.

To expand the right-hand side of (*d*), express $\sinh D$ in the form

$$\sinh(\tfrac{1}{2}D + \tfrac{1}{2}D) \equiv 2\sinh\tfrac{1}{2}D\cosh\tfrac{1}{2}D \equiv \delta\cosh\tfrac{1}{2}D.$$

The operational form of Everett's formula then becomes

$$e^{xD} \equiv \frac{1}{\delta}\frac{\sinh xD}{\cosh\frac{1}{2}D}e^{D} + \frac{1}{\delta}\frac{\sinh\xi D}{\cosh\frac{1}{2}D},$$

and the hyperbolic terms may be expanded to give a usable central difference formula by means of (A 4·1) and (A 4·2).

11. GROUP III

Interpretation of the inverse operators $\mu\delta^{-1}$, D^{-1}

The indefinite integrals (finite and infinitesimal) $\mu\delta^{-1}$ and D^{-1} leave undefined an arbitrary constant of integration. By operating upon each side of the following expansions with δ and D respectively, it may be confirmed that they are representations of the inverse operators in the sense that they are nullified by the direct operators:

$$\mu\delta^{-1}u_x = \tfrac{1}{2}u_x + u_{x-1} + u_{x-2} + \ldots + C,$$

$$D^{-1}u_x = \int u_x\,dx + C.$$

When, as in the following expansions, these operators are taken between definite limits, the arbitrary constant of integration vanishes, and we get

$$[\mu\delta^{-1}u_x]_a^b = \tfrac{1}{2}u_b + u_{b-1} + \ldots + u_{a+1} + \tfrac{1}{2}u_a,$$

$$[D^{-1}u_x]_a^b = \int_a^b u_x\,dx.$$

Similarly,
$$[\mu_h\delta_h^{-1}u_x]_a^b = \tfrac{1}{2}u_b + u_{b-h} + \ldots + u_{a+h} + \tfrac{1}{2}u_a.$$

Example V. The summation and quadrature formulae. The summation formula is the central difference analogue of Lubbock's formula. It will be developed by applying the artifice of differentiating with respect to an operator.

First note that

$$\frac{d}{dD} \equiv \frac{d\delta}{dD}\frac{d}{d\delta} \equiv \frac{d\,[2\sinh\tfrac{1}{2}D]}{dD}\frac{d}{d\delta} \equiv \cosh\tfrac{1}{2}D\,\frac{d}{d\delta} \equiv \mu\,\frac{d}{d\delta}. \qquad (11\cdot1)$$

Writing formula (9·5) for interval m in place of $1/h$ to get

$$\delta_m \equiv m\delta\left[1 + \frac{m^2-1}{3!\,2^2}\delta^2 - \frac{(m^2-1)(m^2-3^2)}{5!\,2^4}\delta^4 + \ldots\right],$$

we take the natural logarithm of both sides,

$$\log\delta_m \equiv \log m\delta + \log\left[1 + \frac{m^2-1}{3!\,2^2}\delta^2 - \frac{(m^2-1)(m^2-3^2)}{5!\,2^4}\delta^4 + \ldots\right],$$

i.e.
$$\log\delta_m \equiv \log m\delta + \frac{m^2-1}{24}\delta^2 - \frac{(m^2+11)(m^2-1)}{2880}\delta^4 + \ldots. \qquad (11\cdot2)$$

Operate upon the right- and left-hand sides of (11·2) with the right- and left-hand sides respectively of (11·1), noting that

$$\frac{d}{dD}\log\delta_m \equiv \frac{1}{\delta_m}\frac{d}{dD}(2\sinh\tfrac{1}{2}mD) \equiv m\delta_m^{-1}\cosh\tfrac{1}{2}mD \equiv m\mu_m\delta_m^{-1},$$

to get

$$m\mu_m\delta_m^{-1} \equiv \mu\frac{d}{d\delta}\left[\log m\delta + \frac{m^2-1}{24}\delta^2 - \frac{(m^2+11)(m^2-1)}{2880}\delta^4 + \ldots\right],$$

whence
$$\mu_m\delta_m^{-1} \equiv \frac{1}{m}\mu\left[\delta^{-1} + \frac{m^2-1}{12}\delta - \frac{(m^2+11)(m^2-1)}{720}\delta^3 + \ldots\right].$$

Applying this operational identity to the operand u_x, and taking the resulting expression between the limits a and b, we get

$$\tfrac{1}{2}u_b + u_{b-m} + \ldots + u_{a+m} + \tfrac{1}{2}u_a = \frac{1}{m}\left(\tfrac{1}{2}u_b + u_{b-1} + \ldots + u_{a+1} + \tfrac{1}{2}u_a\right)$$
$$+\frac{m^2-1}{12m}\left[\mu\delta u_b - \mu\delta u_a\right] - \frac{(m^2+11)(m^2-1)}{720m}\left[\mu\delta^3 u_b - \mu\delta^3 u_a\right] + \ldots.$$

Adding to each side $\frac{1}{2}(u_b+u_a)$, there results

$$\sum_{x=0}^{(b-a)/m} u_{a+mx} = \frac{1}{m}\sum_{x=0}^{b-a} u_{a+x} + \frac{m-1}{2m}(u_b+u_a) + \frac{m^2-1}{12m}\left[\mu\delta u_b - \mu\delta u_a\right]$$
$$-\frac{(m^2+11)(m^2-1)}{720m}\left[\mu\delta^3 u_b - \mu\delta^3 u_a\right] + \ldots, \qquad (11\cdot 3)$$

i.e. the practical central difference form of Lubbock's formula.

Whilst the central difference quadrature formula may be obtained operationally *ab initio* by similar means to that employed in deriving (11·3), it may also be obtained directly from (11·3). Multiply both sides of (11·3) by m, and take the limit as $m \to 0$. Since

$$\lim_{m\to 0} m \sum_{x=0}^{(b-a)/m} u_{a+mx} = \int_a^b u_x\,dx,$$

we get

$$\int_a^b u_x\,dx = \sum_{x=0}^{b-a} u_{a+x} - \tfrac{1}{2}\left[u_b+u_a\right] - \tfrac{1}{12}\left[\mu\delta u_b - \mu\delta u_a\right] + \tfrac{11}{720}\left[\mu\delta^3 u_b - \mu\delta^3 u_a\right] - \ldots,$$

i.e. Laplace's formula.

12. CONCLUSION

Although in general the methods used in this note are not original, an attempt has been made to unify and systematize methods of synthesizing central difference formulae by employing the identities of §2.

APPENDIX

Hyperbolic functions

A 1. The hyperbolic sine and cosine are defined by

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots, \qquad (A\,1\cdot 1)$$

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots. \qquad (A\,1\cdot 2)$$

From (A 1·1) and (A 1·2):

squaring and subtracting,

$$\cosh^2 x - \sinh^2 x = 1; \qquad (A\,1\cdot 3)$$

by direct differentiation,

$$d(\sinh x)/dx = \cosh x, \tag{A 1·4}$$

$$d(\cosh x)/dx = \sinh x. \tag{A 1·5}$$

If $y = \sinh x$, $z = \cosh x$, then $x = \sinh^{-1} y = \cosh^{-1} z$. Also,

$$y + z = e^x,$$

whence
$$x = \log_e(y+z).$$

Since
$$y = \sinh x = \sqrt{\{\cosh^2 x - 1\}} = \sqrt{\{z^2 - 1\}} \quad \text{[from (A 1·3)]}$$

and
$$z = \cosh x = \sqrt{\{\sinh^2 x + 1\}} = \sqrt{\{y^2 + 1\}},$$

the inverse functions may be written

$$\sinh^{-1} y = \log_e [y + \sqrt{\{y^2+1\}}], \tag{A 1·6}$$

$$\cosh^{-1} z = \log_e [z + \sqrt{\{z^2-1\}}]. \tag{A 1·7}$$

Also,
$$dy/dx = \cosh x = \sqrt{\{y^2+1\}};$$

therefore
$$dx/dy = d(\sinh^{-1} y)/dy = 1/\sqrt{\{y^2+1\}}. \tag{A 1·8}$$

Similarly
$$d(\cosh^{-1} z)/dz = 1/\sqrt{\{z^2-1\}}. \tag{A 1·9}$$

A 2. *The expansion of sinh^{-1}y in terms of y.* From (A 1·8),

$$d(\sinh^{-1} y)/dy = (1+y^2)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}y^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}y^4 - \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}y^6 + \dots.$$

Integrating both sides with respect to y between the limits 0 and y,

$$\sinh^{-1} y = y - \frac{1}{2}\frac{y^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{y^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{y^7}{7} + \dots, \tag{A 2·1}$$

since from (A 1·1) $\sinh^{-1} 0 = 0$.

A 3. *The expansions of* $\sinh ny$ *and* $\cosh ny$ *in terms of* $\sinh y$. Let $\sinh y = z$, so that $y = \sinh^{-1} z$, and $\sinh ny = \sinh n(\sinh^{-1} z) = f(z)$, say. Then $f(0) = 0$. Differentiating this expression twice with respect to z we get the differential equation

$$(1+z^2) f''(z) + z f'(z) - n^2 f(z) = 0. \tag{i}$$

[From the first differentiation,

$$f'(z) = n[\cosh n(\sinh^{-1} z)](1+z^2)^{-\frac{1}{2}},$$

we get $f'(0) = n$.]

Differentiating (i) m times (using Leibnitz's theorem), and then putting $z = 0$, we get the recurrence relation

$$f^{(m+2)}(0) = (n^2 - m^2) f^{(m)}(0). \tag{ii}$$

Relation (ii), together with the initial values of $f(0)$ and $f'(0)$, enable the successive values of the coefficients in the Maclaurin expansion for $f(z)$ to be obtained. It is seen that all values of the even derivatives of zero vanish. Putting

$$m = 1, \quad f^{\text{iii}}(0) = (n^2 - 1) f'(0) = n(n^2 - 1),$$

$$m = 2, \quad f^{\text{v}}(0) = (n^2 - 3^2) f^{\text{iii}}(0) = n(n^2 - 1)(n^2 - 3^2), \text{ etc.}$$

Now $$f(z) = f(0) + zf'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f^{\text{iii}}(0) + \ldots;$$

hence, since $z = \sinh y$, $f(z) = \sinh ny$,

$$\sinh ny = n(\sinh y) + \frac{n(n^2-1)}{3!}(\sinh y)^3 + \frac{n(n^2-1)(n^2-3^2)}{5!}(\sinh y)^5 + \ldots. \tag{A 3·1}$$

If we initially put $f(z) = \cosh ny = \cosh n(\sinh^{-1} z)$, we get the same differential equation (i) for $f(z)$, but with the initial values $f(0) = 1$, $f'(0) = 0$. We then get the same recurrence relation (ii), leading this time to zero differential coefficients of zero of odd order, with the following values for even orders:

$$f''(0) = n^2 f(0) = n^2,$$
$$f^{\text{iv}}(0) = (n^2 - 2^2) f''(0) = n^2(n^2 - 2^2),$$
$$f^{\text{vi}}(0) = (n^2 - 4^2) f^{\text{iv}}(0) = n^2(n^2 - 2^2)(n^2 - 4^2), \text{ etc.,}$$

whence

$$\cosh ny = 1 + \frac{n^2}{2!}(\sinh y)^2 + \frac{n^2(n^2-2^2)}{4!}(\sinh y)^4 + \frac{n^2(n^2-2^2)(n^2-4^2)}{6!}(\sinh y)^6 + \ldots. \tag{A 3·2}$$

A 4. *The expansions of* $(\sinh ny)/(\cosh y)$ *and* $(\cosh ny)/(\cosh y)$ *in terms of* $\sinh y$. Differentiating both sides of (A 3·1) and (A 3·2) respectively with respect to y, and then dividing throughout by $\cosh y$, we get:

$$\frac{\sinh ny}{\cosh y} = n(\sinh y) + \frac{n(n^2-2^2)}{3!}(\sinh y)^3 + \frac{n(n^2-2^2)(n^2-4^2)}{5!}(\sinh y)^5 + \ldots, \tag{A 4·1}$$

$$\frac{\cosh ny}{\cosh y} = 1 + \frac{(n^2-1)}{2!}(\sinh y)^2 + \frac{(n^2-1)(n^2-3^2)}{4!}(\sinh y)^4 + \ldots. \tag{A 4·2}$$