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THE COCKTAIL PROBLEM: THE RELATIONSHIP BETWEEN COMPOSITE YIELD AND COMPONENT YIELDS

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ABSTRACT

When two or more independent or component loans are consolidated to form a single composite loan the result is often referred to as a cocktail loan. In this paper we analyse properties of a composite loan induced by properties of the component loans. Of particular interest are the properties of the composite yield in terms of the yields and other characteristics of the component loans. Some upper and lower bounds are also established for the composite yield in terms of the component yields. Of some importance is a sensitivity analysis of the composite yield with respect to various parameters of the component loans.

1. INTRODUCTION

It frequently happens that borrowers of funds obtain parts of their total loan from a variety of lenders. This may occur due to rationing on the part of lenders, as in the case of limited housing loans being available at preferential (below market) rates of interest as a result of statutory limits on the rate permitted on such loans. Alternatively, the situation may occur as a result of the prudential requirements of the lender(s) or following the accumulation of debt by the borrower over a period of time. In any event it is common for borrowers to find themselves paying off loans which have different balances owing, different times to maturity, different repayment amounts and of course different rates of interest. Some of these loans may be from the same lender or they may all be from different lenders. These loan situations are variations of what are commonly referred to as composite or 'cocktail' loans.

An obvious question to ask in relation to any cocktail loan is:

> Given the yields on the component loans, what is the yield on the composite or cocktail loan?

This paper addresses this question and a number of related questions. We will present the mathematical analysis for cocktail loans which comprise just two component loans. However the majority of our results have obvious generalizations to cocktail loans comprising more than two component loans. Given the loan parameters for each of the component loans, we wish to determine the characteristics of the yield on the composite or consolidated loan. The two cases of major focus in this paper concern component loans that are pure discount bonds (zero coupon bonds) or annuities.

A number of simplifying assumptions will be made throughout the paper. First, the impact of late repayments will not be considered here; we are primarily interested in establishing results on the assumption that the various loan conditions are satisfied. Second, we will not specifically incorporate up-front or periodic loan costs into our analysis. However, with regard to this point, it should be appreciated that up-front or establishment costs associated with a loan can be accommodated by deducting the total up-front costs from the notional loan amount to obtain the net loan amount. Also, any periodic or recurrent loan costs (management or other fees, for example) can be amortized over the gross repayment amounts to give the net repayment amount.

The net price of the loan and net repayment amount, together with the time to maturity, then imply the true rate of interest on the loan as compared with the notional rate of interest on such.

The major results of our analysis are equally valid when both of the component loans are either pure discount bonds or annuities. In both cases for a two component loan we show that the composite yield:

- increases as the relative amount of the higher yielding component loan increases,
- increases as either of the component yields increases,
- increases as the time to maturity of the higher yielding component loan increases.

Many of the results that we obtain for the two-component composite loan generalize to an arbitrary number of composite loans. However, the degree of generalization varies from result to result, so we shall present the analysis for a two-component loan and indicate the generalization in each case.

We also establish various upper and lower bounds for the composite yield in terms of the component loan parameters. In addition, power series approximations for the composite yield are determined. Of particular interest is the result that the composite yield for loans of common maturity can in general be expected to exceed the weighted average of the component yields, the weighing factor being the proportion of the component loans in the cocktail. Some generalizations and interesting special cases of this result are also established and discussed.

2. COCKTAIL LOAN COMPRISING TWO PURE DISCOUNT BONDS

In this section we will analyse the properties of the yield on a cocktail loan comprising two pure discount bonds, that is, two component loans each having a single balloon repayment. Suppose the price of the loans are $\$P_1$ and $\$P_2$, repayable after n_1 and n_2 periods, using repayments of $\$R_1$ and $\$R_2$ respectively. If the yields on these loans are 100 r_1% per period and 100 r_2% per period respectively, and we put $x_i = 1 + r_i$, $i = 1,2$, then

$$R_i = P_i x_i^{n_i}, i = 1,2. \tag{1}$$

As noted in the introduction, if P_1, P_2, R_1 and R_2 include establishment and recurrent costs then r_1 and r_2 will represent the true yields on each loan. In any event, r_1 and r_2 follow from P_1, R_1, n_1 and P_2, R_2, n_2 respectively. Indeed, any three of the four loan parameters P, R, n and r imply the fourth. It should also be noted that in considering the consolidated properties of a new loan and an existing loan, the balance outstanding, time to maturity and repayment amount should be used in the latter for P, n and R respectively.

The yield on the cocktail loan comprising a consolidation of the two loans above, $100r\%$ per period, is the solution of the equation

$$P_1 + P_2 = R_1 x^{-n_1} + R_2 x^{-n_2}, \tag{2}$$

where $x = 1 + r$.

Note that the solution for r will be unique as there is only one sign change in the cash flow sequence (regarding $P_1 + P_2$ as expenditure, say, and the redemption monies as income).

Eliminating R_1 and R_2 from equations (1) and (2) we obtain

$$\gamma \left(\frac{x_1}{x}\right)^{n_1} + (1-\gamma)\left(\frac{x_2}{x}\right)^{n_2} = 1, \tag{3}$$

where $\gamma = \beta\ (1 + \beta)$, and $\beta = P_1/P_2$.

Note that β represents the ratio of the loan prices whilst γ and $(1-\gamma)$ represent the proportions of the first and second component loans in the cocktail.

In the case of a many loan situation we shall write (3) as

$$\sum_{i=1}^{s} \gamma_i \left(\frac{x_i}{x}\right)^{n_i} = 1, \text{ where } \sum_{i=1}^{s} \gamma_i = 1. \tag{4}$$

It immediately follows from (3) that r always lies between r_1 and r_2, that is

$$r_1 \gtreqless r \text{ if and only if } r_2 \lesseqgtr r. \tag{5}$$

In the following analysis it will be convenient to assume, without loss of generality, that $r_1 > r_2$, that is the first loan is the higher yielding loan. The case in which $r_1 = r_2$ leads, as demonstrated above, to $r = r_1 = r_2$ and is of no further interest to us.

Theorem 2.1

The yield on the composite loan is an increasing function of γ.

Proof:

Differentiating (3) partially with respect to γ, and re-arranging gives

$$\frac{\partial x}{\partial \gamma}=\frac{\left(\dfrac{x_1}{x}\right)^{n_1}-\left(\dfrac{x_2}{x}\right)^{n_2}}{n_1\,\gamma\,x_1^{\,n_1}\,x^{-n_1-1}+n_2\,(1-\gamma)\,x_2^{\,n_2}\,x^{-n_2-1}}, \tag{6}$$

from which it immediately follows that $\partial x/\partial\gamma > 0$ since $x_2 < x < x_1$. The required results now follows readily.

The above result accords with our intuition, namely that the composite yield will increase as the relative amount of the higher yielding loan in the cocktail increases. Naturally, the composite yield decreases as the proportion of the lower yielding component loan increases.

The generalization of this result requires qualification. Suppose the weight γ_i of loan i changes in relation to the weight γ_j of loan j in such a way that $\gamma_i+\gamma_j$ and all other weights remain constant. Then the numerator of the corresponding result to (6) will consist of the difference

$$\left(\frac{x_i}{x}\right)^{n_i}-\left(\frac{x_j}{x}\right)^{n_j}.$$

Clearly the sign of this expression is determinate if x lies between x_i and x_j, but is indeterminate otherwise. If $r_i > r > r_j$, r increases as γ_i increases, and decreases as γ_j increases.

Theorem 2.2

The yield on the composite loan is an increasing function of the component yields, that is

$$\frac{\partial r}{\partial r_1} > 0, \qquad \frac{\partial r}{\partial r_2} > 0.$$

Proof:

Differentiating (3) partially with respect to x_1, and re-arranging gives

$$\frac{\partial x}{\partial x_1}=\frac{n_1\,\gamma\,x_1^{\,n_1-1}\,x^{-n_1}}{n_1\,\gamma\,x_1^{\,n_1}\,x^{-n_1-1}+n_2\,(1-\gamma)\,x_2^{\,n_2}\,x^{-n_2-1}},$$

from which it follows that r is an increasing function of r_1.

A similar analysis shows that $\partial r/\partial r_2 > 0$ also.

This result generalizes immediately to any number of component loans; that is the yield on the composite loan is an increasing function of any one component yield.

Theorem 2.3

The yield on the composite loan is an increasing function of the time to maturity of the higher yielding component loan and a decreasing function of the time to maturity of the lower yielding component loan.

Proof:

Consider n_1 as a continuous variable, and differentiate (3) partially with respect to n_1. This gives, after some re-arranging

$$\frac{\partial x}{\partial n_1} = \frac{\gamma \left(\frac{x_1}{x}\right)^{n_1} ln \left(\frac{x_1}{x}\right)}{n_1 \gamma x_1^{n_1} x^{-n_1-1} + n_2 (1-\gamma) x_2^{n_2} x^{-n_2-1}}. \tag{7}$$

Since $x_1 > x$, it follows that $\partial x/\partial n_1 > 0$, and hence that $\partial r/\partial n_1 > 0$.

A similar analysis also shows that $\partial r/\partial n_2 < 0$.

In the sense that paying a higher rate of interest for a longer period of time is more costly and vice-versa, this result is again as we would expect.

For a many component loan it is clear from the general expression corresponding to (7) that the yield on the composite loan is an increasing or decreasing function of the time to maturity of the component loan provided the yield on the component loan is, respectively, greater than or less than the composite yield.

Theorem 2.4

When the component loans have the same time to maturity, the composite yield

(i) is readily expressible in terms of the component yields, and

(ii) is greater than (less than) the weighted average of the component yields when the time to maturity exceeds (is less than) unity.

That is, $r > \gamma r_1 + (1-\gamma) r_2$ when $n_1 = n_2 > 1$; and $r < \gamma r_1 + (1-\gamma)r_2$ when $n_1 = n_2 < 1$.

Proof:

(i) When $n_1 = n_2 = n$, equation (3) can be written

$$\gamma x_1^n + (1-\gamma)x_2^n = x^n. \tag{8}$$

Taking the nth root of both sides gives r explicitly in terms of r_1 and r_2.

(ii) Let $g(x) = x^n$. Then g is a convex function when $n > 1$, so it follows from (8) that

$$\begin{aligned} g(x) &= \gamma\, g(x_1) + (1-\gamma)\, g(x_2) \\ &\geqslant g\,(\gamma x_1 + (1-\gamma)\, x_2), \end{aligned}$$

and since g is also increasing, we obtain

$$x \geqslant \gamma x_1 + (1-\gamma) x_2, \text{ and hence } r > \gamma r_1 + (1-\gamma)r_2. \tag{9}$$

Note as a special case that when $\beta = 1$, i.e. $\gamma = 1/2$,

$$r > \tfrac{1}{2}(r_1 + r_2).$$

This result tells us that when $P_1 = P_2$, that is, the loan amounts are equal, then the composite yield is greater than the arithmetic mean of the component yields.

When $n < 1$, the concavity of $g(x)$ reverses the sign of the inequality.

The generalization of (9) is immediate to a many-component loan situation, giving

$$r > \sum_{i=1}^{s} \gamma_i r_i, \quad \text{when } n_i > 1 \text{ for all } i. \tag{10}$$

It is interesting to compare the result of Theorem 2.4 with the corresponding result based on the flat or simple rates of interest. In this case we can establish:

Theorem 2.5

When the component loans have the same time to maturity, the equivalent flat rate of interest on the composite loan *is equal to* the weighted average of the component flat rates of interest. This result may be stated thus:

$$f = \gamma f_1 + (1 - \gamma) f_2,$$

where $100 f_1\%$ and $100 f_2\%$ are the flat rates of interest per annum on the component loans and $100\ f\%$ per annum is the flat rate of interest on the composite loan.

Proof:

This result follows immediately if we note that

$$R_1 = P_1\left(1 + f_1 \frac{n}{N}\right),\ R_2 = P_2\left(1 + f_2 \frac{n}{N}\right),\ R_1 + R_2 = (P_1 + P_2)\left(1 + f\frac{n}{N}\right),$$

where N is the number of periods per annum.

This result stresses the linearity of flat or simple rates of interest, in contrast to the non-linearity of nominal or effective rates of interest.

Once again, note that for $\beta = 1$, when the component loans are for the same amount, the result gives

$$f = \tfrac{1}{2}(f_1 + f_2),$$

that is, the composite flat rate is equal to the arithmetic mean of the component flat rates.

The result of this theorem generalizes in the obvious way.

When $n_1 \neq n_2$ it is possible to generalize Thereom 2.4 as follows:

Theorem 2.6

When $n_1 \neq n_2$, $n_1 > 1$ and $n_2 > 1$,

$$r > \alpha r_1 + (1 - \alpha) r_2,$$

where

$$\alpha = \frac{n_1 P_1}{n_1 P_1 + n_2 P_2} = \frac{\beta n_1}{\beta n_1 + n_2} = \frac{\gamma n_1}{\gamma n_1 + (1-\gamma) n_2}.$$

This is always a better lower bound than r_2.

Note that α and $(1-\alpha)$ represent the 'maturity weighted' component loan proportions. In other words α is similar to γ except that the loan prices in the former are first multiplied by their term to maturity.

Proof:

We apply the inequality $y^b > 1 + b(y-1)$, where y is positive and $b > 1$ is a constant, to each term of (3); see, for example, Hardy, Littlewood and Polya (1964). After simplification the desired result is obtained. Since $r_1 > r_2$, $\alpha(r_1 - r_2) + r_2 > r_2$ and the bound is stricter than r_2.

Applying the above inequality to (4) leads to the obvious generalization, namely

$$r > \sum_{i=1}^{s} \delta_i r_i,$$

$$\text{where } \delta_i = \frac{n_i \gamma_i}{\sum_{i=1}^{s} n_i \gamma_i}, \text{ and } n_i > 1 \text{ for all } i.$$

Power Series for Composite Yield

Useful approximations for the composite yield r can be obtained from a power series expansion for r in terms of the component yields r_1 and r_2. Assuming r has the form

$$r = a\, r_1 + b\, r_2 + c\, r_1^2 + d\, r_1 r_2 + e\, r_2^2 + \ldots,$$

substituting in (3) and equating coefficients in the normal way gives

$$r = \alpha\, r_1 + (1-\alpha)\, r_2 + \frac{\alpha(1-\alpha)}{2} (\alpha\, n_2 + (1-\alpha)\, n_1 - 1)\, (r_1 - r_2)^2 + \ldots$$

Terms of degree 3 and higher can be obtained if necessary by continuing the above procedure.

The power series expansion for r is valid for r_1 and r_2 sufficiently small; the precise ranges are not readily available from the analysis, but are given by the condition $|r| < 1$.

It is interesting to note from the power series expansion that the linear terms are precisely the lower bound for the composite yield as in Theorem 2.6.

Graphical representations

The dependence of the composite yield on the component yields and other component loan parameters can be clearly illustrated by means of appropriate figures. For example, Figure 1 plots the cocktail loan rate, r% per period, against $P_1/(P_1 + P_2)$ or γ.

Figure 1 is based on component yields of 1·25% per month for a 300 month (25 year) (housing) loan and 1·75% per month for (personal) loans of various durations. The contours are indicated for personal loans of 60, 120, 180, 240 and 300 month durations. A similar graphical presentation, using effective rates of interest, can be found in Hathaway (1986).

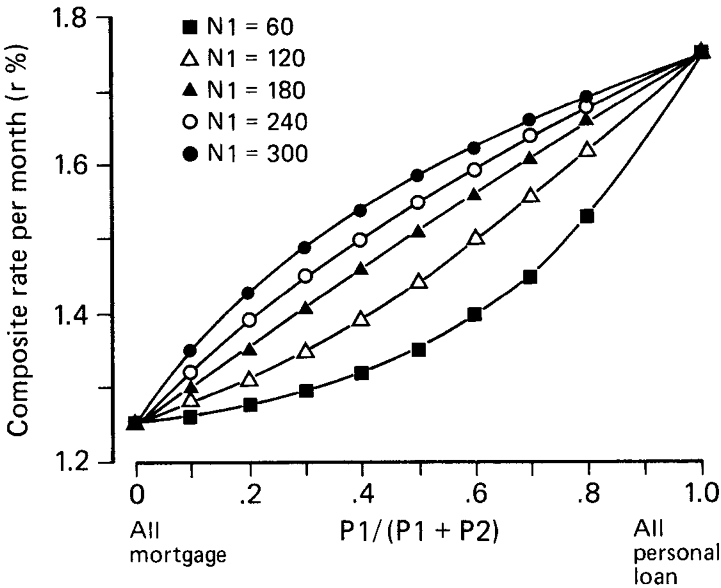

Figure 1. Composite rate—pure discount bond

3. COCKTAIL LOAN COMPRISING TWO ANNUITIES

In this section we analyse the properties of the yield on a cocktail loan comprising two ordinary annuities, sometimes referred to as credit foncier loans. Similar results emerge when the component loans are annuities due or deferred annuities.

We shall retain the notation of the previous section, except that we will now use R_1 and R_2 to denote the periodic repayment amount for each of the annuities, comprising n_1 and n_2 repayments respectively. The component yields are given by

$$P_1 = \frac{R_1}{r_1}(1-(1+r_1)^{-n_1}),\ P_2 = \frac{R_2}{r_2}(1-(1+r_2)^{-n_2}), \tag{11}$$

whilst the composite yield is the (unique) solution of the analogue of equation (3):

$$\gamma \frac{r_1}{r} \frac{(1-(1+r)^{-n_1})}{(1-(1+r_1)^{-n_1})} + (1-\gamma)\frac{r_2}{r} \frac{(1-(1+r)^{-n_2})}{(1-(1+r_2)^{-n_2})} = 1, \tag{12}$$

or, in general,

$$\sum_{i=1}^{s} \gamma_i f(r_i, n_i)/f(r, n_i) = 1,$$

where

$$f(r, n) = \frac{r}{1-(1+r)^{-n}} \tag{13}$$

is the *capital recovery factor* (that is, the reciprocal of the present value of an annuity), based on the interest rate 100 r% per period for n periods. In conventional actuarial notation, $f(r, n) = 1/a_{\overline{n}|}$ at rate r.

When convenient, $f(r, n)$ will be written as $f(r)$. In general, the results of § 2 extend to the case of annuities. Before proving the corresponding results it will be convenient to establish two preliminary results.

Theorem A

The expression

$$D \equiv D(\gamma, r, r_1, r_2, n_1, n_2)$$
$$= 1-\gamma\, n_1\, f(r_1, n_1)\,(1+r)^{-n_1-1} - (1-\gamma)\, n_2\, f(r_2, n_2)\,(1+r)^{-n_2-1} \tag{14}$$

is positive for all parameters.

The proof of this theorem is given in the Appendix.

Theorem B

The function

$$f(r) = \frac{r}{1-(1+r)^{-n}},$$

where $r > 0$ and n is a positive integer, is a monotonically increasing and convex function.

The proof of this theorem is also given in the Appendix.

The analogues of the theorems in § 2 and their generalizations will now be established. As in § 2 we present the analysis for a two-component loan, with $r_1 > r_2$.

Theorem 3.1

The yield on the composite loan is an increasing function of γ, that is

$$\frac{\partial r}{\partial \gamma} > 0.$$

Proof:

Multiplying (12) by r, and differentiating partially with respect to γ gives

$$(1-\gamma)\frac{\partial r}{\partial \gamma} = \frac{1}{D}[1-(1+r)^{-n_1}]\,[f(r_1, n_1)-f(r, n_1)],$$

where D is given in expression (14). Since the function $f(r)$ is monotonically increasing (see Theorem B) and $r_1 > r$, the numerator of $\partial r/\partial\gamma$ is positive. Since D is also positive, $\partial r/\partial\gamma$ is positive, as required.

The generalization of this theorem, like that for Theorem 2.1, requires qualification. If we assume $\gamma_i + \gamma_j$ and all other weights to be constant, then the sign of $\partial r/\partial\gamma_i$ depends upon the sign of

$$\frac{f(r_i, n_i)}{f(r, n_i)} - \frac{f(r_j, n_j)}{f(r, n_j)}. \tag{15}$$

From Theorem B, $f(r, n)$ is a monotonically increasing function of r, hence the sign of (15) is determinate if r lies between r_i and r_j, but indeterminate otherwise. Again, if $r_i > r > r_j$, r increases as γ_i increases, and decreases as γ_j increases.

Theorem 3.2

The composite yield is an increasing function of each component yield, that is

$$\frac{\partial r}{\partial r_1} > 0 \quad \text{and} \quad \frac{\partial r}{\partial r_2} > 0.$$

Proof:

Equation (12) can be written in the form

$$1 = \gamma\frac{f(r_1, n_1)}{f(r, n_1)} + (1-\gamma)\frac{f(r_2, n_2)}{f(r, n_2)}, \tag{16}$$

where again $f(r)$ is, by Theorem B, an increasing function or r.

Suppose that r_1 increases; then $f(r_1, n_1)$ increases, and since γ and $f(r_2, n_2)$ are independent of r_1, r cannot decrease. Consequently r increases when r_1 increases. A similar argument shows that r is also an increasing function of the other component yield r_2.

The generalization of this result is quite straightforward.

Theorem 3.3

The yield on the composite loan is an increasing function of the time to maturity of the higher yielding component loan and a decreasing function of the time to maturity of the lower yielding component loan.

Proof:

Treating n_1 as a continuous variable, differentiating (12) partially with respect to n_1, simplifying, and using Theorem A, we find that the sign of $\partial r/\partial n_1$ is the same as the sign of

$$x^{-n_1}(1-x_1^{-n_1})\, ln\, x - x_1^{-n_1}(1-x^{-n_1})\, ln\, x_1,$$

and this in turn has the same sign as

$$x^{-n_1} x_1^{-n_1}(x_1^{n_1}-1)(x^{n_1}-1)\left[\frac{ln\, x}{x^{n_1}-1}-\frac{ln\, x_1}{x_1^{n_1}-1}\right].$$

We now consider the function

$$h(x) = \frac{ln\, x}{x^n - 1},\ x > 1,$$

and show that $h(x)$ is a decreasing function of x. Clearly

$$h'(x) = \frac{x^n - 1 - n\, x^n\, ln\, x}{x(x^n-1)^2}. \tag{17}$$

Using the inequality

$$ln\,(1+x) \geqslant \frac{x}{1+\frac{1}{2}x},\ x \geqslant 0,$$

the numerator of (17) cannot exceed

$$\frac{(x+1)(x^n-1)-2\,n\,(x-1)\,x^n}{x+1}.$$

The numerator of the last expression has the same sign as

$$\begin{aligned}
& (1+x)(1+x+x^2+\ldots+x^{n-1})-2\,n\,x^n \\
= \;& 1+2\,x+2\,x^2+\ldots+2\,x^{n-1}+2x^n-2\,n\,x^n-x^n \\
\leqslant \;& 1+2\,n\,x^n-2\,n\,x^n-x^n, \text{ since } x>1. \\
= \;& 1-x^n \\
\leqslant \;& 0.
\end{aligned}$$

We have therefore shown that $h(x)$ is a decreasing function of x, and since $x < x_1$, it follows that $\partial r/\partial n_1$ is positive. A similar analysis will show that $\partial r/\partial n_2$ is negative.

For the general case a corresponding analysis shows that the yield on the composite loan is an increasing or decreasing function of the time to maturity of the component loan provided the yield on the component loan is, respectively, greater than or less than the composite yield.

We are now in a position to establish:

Theorem 3.4

When the component loans have the same time to maturity, and this time to maturity exceeds unity, the composite yield is greater than the weighted average of the component yields. In other words,

$$r > \gamma\, r_1 + (1-\gamma)\, r_2 \text{ when } n_1 = n_2 > 1. \tag{18}$$

Proof:

When $n_1 = n_2 = n$, equation (12) can be written in the form

$$f(r, n) = \gamma\, f(r_1, n) + (1-\gamma)\, f(r_2, n)$$

or

$$f(r) = \gamma\, f(r_1) + (1-\gamma)\, f(r_2) \text{ say.}$$

Since f is convex (Theorem B),

$$f(r) \geqslant f(\gamma\, r_1 + (1-\gamma)\, r_2),$$

and since f is also monotonically increasing it follows immediately that

$$r \geqslant \gamma\, r_1 + (1-\gamma)\, r_2,$$

as required.

Generalization of this result is obvious.

Further Bounds for the Annuity, and Power Series

Using the inequalities $1 \geqslant 1-(1+x)^{-n} \geqslant x/(1+x)$ for all $x \geqslant 0$ and for all $n \geqslant 1$, it follows from (12) that

$$\gamma f(r_1, n_1) + (1-\gamma)\, f(r_2, n_2) \geqslant r \geqslant \gamma\, f(r_1, n_1) + (1-\gamma)\, f(r_2, n_2) - 1$$

As for the discount bond a power series can be obtained for the composite yield r in terms of the component yields r_1 and r_2. Using similar techniques the following power series can be obtained under the condition $|r| < 1$:

$$r = \delta\, r_1 + (1-\delta) r_2 + \ldots,$$

$$\text{where } \delta = \frac{\beta(n_1+1)}{\beta(n_1+1)+n_2+1} = \frac{\gamma(n_1+1)}{\gamma(n_1+1)+(1-\gamma)\,(n_2+1)}.$$

Note that δ is the same as α except that n_1 and n_2 have been replaced by (n_1+1) and (n_2+1) respectively.

One can repeat the previous analysis (for the case of two pure discount bonds) of the power series approximation to assess how accurate the truncated form is when used to estimate the composite yield. Similar results are obtained to those presented in § 2; details are not give here.

Again, when $n_1 = n_2$, we observe that the linear terms are exactly the lower bound for r in (18).

Graphical Representation

As in the previous section for the case of component loans consisting of pure discount bonds, it is perhaps helpful to illustrate graphically how the composite yield varies with the component yields for the case of annuity loans.

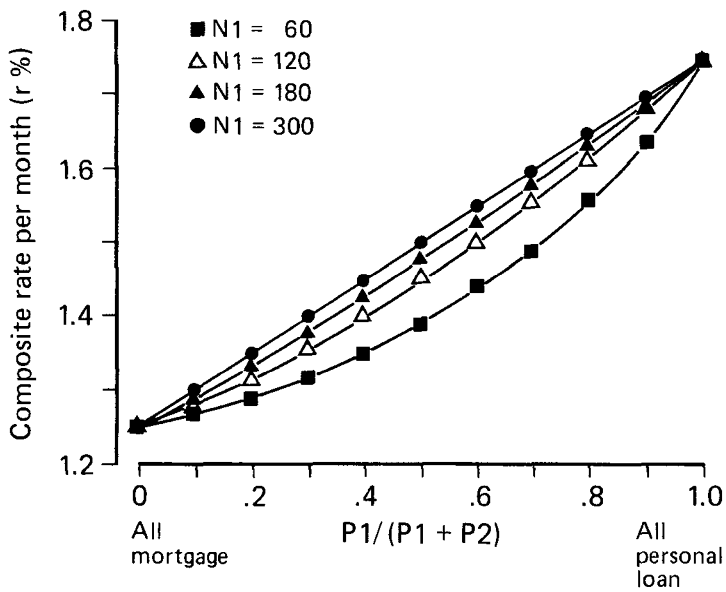

Figure 2. Composite rate—ordinary annuity

Figure 2 plots the cocktail rate, $r\%$ per period, against, γ, or $P_1/(P_1 + P_2)$ for the same range of contours as chosen in Figure 1.

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APPENDIX

In the Appendix we present the proofs of Theorems A and B.

Theorem A:

The expression

$$D = D(\gamma, r, r_1, r_2, n_1, n_2)$$
$$= 1 - \gamma\, n_1\, f(r_1, n_1)\,(1+r)^{-n_1-1} - (1-\gamma)\, n_2\, f(r_2, n_2)\,(1+r)^{-n_2-1} \qquad (14)$$

is positive for all parameters.

Proof:

Using (12), (14) can be written

$$D = \gamma\, f(r_1, n_1)/f(r, n_1) + (1-\gamma)\, f(r_2, n_2)/f(\mathrm{r}, n_2)$$
$$-\gamma\, n_1\, f(r_1, n_1)\,(1+r)^{-n_1-1} - (1-\gamma)\, n_2\, f(r_2, n_2)\,(1+r)^{-n_2-1}. \qquad \text{(A1)}$$

Combining the first and third terms of (A1) gives

$$\gamma\, f(r_1, n_1)/r\,[1-(1+r)^{-n_1} - n_1\, r\,(1+r)^{-n_1-1}],$$

and the expression in brackets can be written

$$\frac{(1+r)^{n_1+1} - 1 - r - n_1\, r}{(1+r)^{n_1+1}}.$$

The numerator of this last expression is seen to be positive by employing the Binomial Theorem for the term $(1+r)^{n_1+1}$.

A similar argument shows that the second and fourth terms of (A1) when combined are also positive. Consequently D is positive, and the theorem is proved.

A similar analysis to the above establishes a corresponding result for the general case of a many-component loan.

Theorem B:

The function

$$f(r) = \frac{r}{1-(1+r)^{-n}},$$

where $r > 0$ and n is a positive integer, is a monotonically increasing and convex function.

Proof:

It is readily shown that the derivative of $f(r)$ is

$$f'(r) = \frac{[(1 + r)^{n+1} - 1 - (n + 1)r]}{(1 + r)^{n+1}\,[1 - (1 + r)^{-n}]^2}\,. \tag{A2}$$

Using the Binomial Theorem to expand $(1 + r)^{n+1}$ shows that the numerator of $f'(r)$ is non-negative, and hence that $f(r)$ is monotonically increasing.

Differentiating (A2), and simplifying gives

$$\begin{aligned}[1 - (1 + r)^{-n}]^3 f''(r) &= n(n + 1)\, r\, (1 + r)^{-n-2}\, [1 - (1 + r)^{-n}] \\ &\quad - 2n\, (1 + r)^{-n-1}\, [1 - (1 + r)^{-n-1}\, (1 + (n + 1)r)],\end{aligned}$$

from which it is apparent that the sign of $f''(r)$ is the same as the sign of

$$n(n + 1)\, r\, [1 - (1 + r)^{-n}] - 2n\, (1 + r)\, [1 - (1 + r)^{-n-1}\, (1 + (n + 1)r)].$$

After further simplification it is clear that the last expression has the same sign as

$$(1 + r)^n\, [(n - 1)r - 2] + 2 + (n + 1)r.$$

After expanding $(1 + r)^n$ by the Binomial Theorem we find that the constant term and the coefficient of r are both zero, and that the coefficient of r^k for $k \geqslant 2$ is equal to

$$n(n - 1) \ldots (n - k + 2)\, (k - 2)\, (n + 1)/k! \geqslant 0.$$

Thus $f''(r) \geqslant 0$ when $r > 0$ and so $f(r)$ is convex.