ON THE COEFFICIENTS IN THE EXPANSION OF $e^{\lambda e^t}$

BY M. T. L. BIZLEY, F.I.A., F.S.S., F.I.S.

R. E. BEARD (*J.I.A.* LXXVI, 152) has derived the expansion of e^{e^t} and e^{-e^t} and has shown that the coefficients which he obtains can be expressed in terms of Stirling's numbers of the second kind ('reduced differences of zero').

The following demonstration derives the expansion of $e^{\lambda e^t}$, or $(e^{e^t})^\lambda$, directly in this form from first principles, thus embracing e^{e^t} and e^{-e^t} as special cases.

By expanding $(e^t-1)^n$ by the binomial theorem the coefficient of t^r in $(e^t-1)^n$ is seen at once to be

$$\left\{n^r-\binom{n}{1}(n-1)^r+\binom{n}{2}(n-2)^r-\ldots\right\}\Big/ r!$$

$$=(E-1)^n\, 0^r/r!$$

$$=\Delta^n\, 0^r/r!.$$

Hence the coefficient of t^r in $\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}(e^t-1)^n$ is $\frac{1}{r!}\sum_{n=1}^{\infty}\frac{\lambda^n\Delta^n 0^r}{n!}$, where, since $\Delta^n 0^r=0$ for $n>r$, the upper limit ∞ may be replaced by r.

But $\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}(e^t-1)^n=\exp\{\lambda(e^t-1)\}-1$, so that the coefficient of t^r in $e^{\lambda(e^t-1)}$ is

$$\frac{1}{r!}\sum_{n=1}^{r}\frac{\lambda^n\Delta^n 0^r}{n!},$$

i.e. the coefficient of t^r in $e^{\lambda e^t}$ is $\frac{e^\lambda}{r!}\sum_{n=1}^{r}\lambda^n\frac{\Delta^n 0^r}{n!}$.

We have thus obtained the expansion of $e^{\lambda e^t}$ for all λ. Setting $\lambda=1$ and $\lambda=-1$ respectively we have the expansions of e^{e^t} and e^{-e^t} in terms of reduced differences of zero, as given by Beard (*loc. cit.* p. 154).