JIA 76 (1950) 0152-0163

ON THE COEFFICIENTS IN THE EXPANSION OF e^{e^t} AND e^{-e^t}

By R. E. BEARD, M.B.E., F.I.A., F.S.S.

Assistant General Manager of the Pearl Assurance Company, Ltd.

THIS investigation has arisen out of D. C. Fraser's note on the Gompertz Table (*J.I.A.* Vol. LXXIII, p. 423), and although of limited actuarial application the results are believed to be new and worth publication. Fraser expresses some difficulty in regard to the determination of the coefficients $A_0, A_1, \ldots$ defined in his note, and in a personal letter Dr E. Michalup has pointed out a method of deriving a recurrence formula for them without utilizing the 'abacus' but involving certain analytical properties of the Gamma function. Consideration of Dr Michalup's letter suggested that the recurrence formula could be obtained by elementary algebraic methods, and these are used below. In the course of subsequent correspondence Dr J. C. P. Miller pointed out certain related studies and made some suggestions which prompted me to further analysis, the main results of which are now presented.

2. Using the conventions adopted by Fraser, but replacing $x\lambda c$ in his notation by t, the expression for l_x becomes e^{-e^t} and the problem is to expand this in ascending powers of t. The allied problem of expanding e^{e^t} has been studied fairly deeply by Epstein,* who also gives a number of references to earlier work on this topic.

3. Now if
$$e^{e^t}=e\left(1+K_1t+K_2\frac{t^2}{2!}+\ldots\right) \tag{1}$$

and
$$e^{-e^t}=e^{-1}\left(1+A_1t+A_2\frac{t^2}{2!}+\ldots\right), \tag{2}$$

then, since $e^{e^t}e^{-e^t}=1$, the two series in brackets are reciprocal and the K_r and A_r are related by the formula

$$K_r+\binom{r}{1}K_{r-1}A_1+\binom{r}{2}K_{r-2}A_2+\ldots+A_r=0, \tag{3}$$

so that given the K_r the A_r can be found and vice versa.

4. Now
$$\begin{aligned} e^{-e^t}&=1-\frac{e^t}{1!}+\frac{e^{2t}}{2!}-\ldots\\ &=1-\frac{1}{1!}\sum_{r=0}\frac{t^r}{r!}+\frac{1}{2!}\sum_{r=0}\frac{(2t)^r}{r!}-\ldots\\ &=1-\sum_{r=0}\frac{t^r}{r!}\left(\frac{1^r}{1!}-\frac{2^r}{2!}+\frac{3^r}{3!}-\ldots\right)\\ &=1-\sum_{r=0}\frac{t^r}{r!}C_r \text{ say;} \end{aligned} \tag{4}$$

* Epstein, L. F., *J. Math. Phys.* Vol. XVIII, pp. 153–73, 1939.

thus
$$e^t e^{-e^t} = \sum_{s=0} \frac{t^s}{s!}\left(1 - \sum_{r=0} \frac{t^r}{r!} C_r\right), \tag{5}$$

also
$$\begin{aligned} e^t e^{-e^t} &= e^t - \frac{e^{2t}}{1!} + \frac{e^{3t}}{2!} - \dots \\ &= \sum_{r=0} \frac{t^r}{r!}\left(1 - \frac{2^r}{1!} + \frac{3^r}{2!} - \dots\right) \\ &= \sum_{r=0} \frac{t^r}{r!} C_{r+1}. \end{aligned} \tag{6}$$

Equating powers of t^{r+s} in (5) and (6) we find

$$\frac{C_{r+s+1}}{(r+s)!} = -\left\{\frac{C_{r+s}}{(r+s)!} + \frac{C_{r+s-1}}{1!(r+s-1)!} + \dots + \frac{C_0 - 1}{(r+s)!}\right\}.$$

Putting $r+s=k$, we finally have

$$C_{k+1} = 1 - \sum_{r=0}^{k} \binom{k}{r} C_r. \tag{7}$$

If we now put $C_r = -A_r e^{-1}$ for $r>0$ and note that $C_0 = 1 - e^{-1}$, we have

$$e^{-e^t} = e^{-1}\left(1 + A_1 t + A_2 \frac{t^2}{2!} + \dots\right), \tag{8}$$

where
$$A_r = e \sum_{n=0} (-)^n \frac{n^r}{n!} \quad (r=1, 2, \dots), \tag{9}$$

with the symbolic relationship

$$-A_{k+1} = (1+E)^k A_0. \tag{10}$$

5. The values of A_r, which by (10) are all integral, have been calculated as far as $r=26$ and are given below. For $0 \leqslant r \leqslant 20$ they were given by Michalup and for $20 < r \leqslant 26$ by Miller and have been verified by myself.

r	A_r	r	A_r	r	A_r
0	+ 1	9	+ 267	18	− 2784 75061
1	− 1	10	+ 413	19	− 25409 56509
2	0	11	− 2180	20	− 98168 60358
3	+ 1	12	− 17731	21	+ 2 71722 88399
4	+ 1	13	− 50533	22	+ 72 55030 33401
5	− 2	14	+ 110176	23	+ 559 25431 75252
6	− 9	15	+ 1966797	24	+ 1582 35875 07881
7	− 9	16	+ 9938669	25	− 16839 26105 36153
8	+ 50	17	+ 8638718	26	− 2 84811 54971 32448

6. Similarly, the function $e^{e^t} = e(1 + K_1 t + \dots)$ may be considered and the following symbolic relationship found for the coefficients K_r:

$$K_{k+1} = (1+E)^k K_0, \tag{11}$$

where
$$K_r = e^{-1} \sum_{n=0} \frac{n^r}{n!} \quad (r=1, 2, \dots). \tag{12}$$

Values of K_r for $0 \leqslant r \leqslant 20$ have been given by Epstein* and are given below, together with the values for $20 < r \leqslant 26$ which have been calculated by myself:

r	K_r	r	K_r	r	K_r
0	1	9	21147	18	68 20768 06159
1	1	10	1 15975	19	583 27422 05057
2	2	11	6 78570	20	5172 41582 35372
3	5	12	42 13597	21	47486 98161 56751
4	15	13	276 44437	22	4 50671 57384 47323
5	52	14	1908 99322	23	44 15200 58550 84346
6	203	15	13829 58545	24	445 95886 92948 05289
7	877	16	1 04801 42147	25	4638 59033 22299 99353
8	4140	17	8 28648 69804	26	49631 24652 36187 56274

7. Now, whilst the A_r and K_r can be calculated from the relationships (10) and (11) respectively, they can also be derived from a table of the differences of zero. Thus from (9)

$$\begin{aligned} A_r &= e \sum_{n=0} (-)^n \frac{n^r}{n!} \quad (r = 1, 2, \ldots) \\ &= e \sum_{n=0} (-)^n \frac{E^n 0^r}{n!} \\ &= e \sum_{n=0} (-)^n \frac{(1+\Delta)^n 0^r}{n!} \\ &= e \sum_{n=0} (-)^n \left\{ \frac{\Delta 0^r}{(n-1)!} + \frac{\Delta^2 0^r}{2!(n-2)!} + \ldots + \frac{\Delta^n 0^r}{n!} \right\} \\ &= -\frac{\Delta 0^r}{1!} + \frac{\Delta^2 0^r}{2!} - \frac{\Delta^3 0^r}{3!} + \ldots . \end{aligned} \tag{13}$$

The numbers $\Delta^p 0^r$ are the differences of zero and the numbers $\frac{\Delta^p 0^r}{p!}$ are usually referred to as Stirling's numbers of the second kind or reduced differences of zero. The A_r can thus be found by cross-addition of a table of $\frac{\Delta^p 0^r}{p!}$ with alternating signs (as is pointed out by Michalup), and similarly the K_r can be obtained by cross-addition without alternating the signs.

8. The 'abacus' referred to by Fraser is a table of Stirling's numbers of the second kind and, as he points out, one of its uses is to express x^m in terms of $x^{(m)}, x^{(m-1)}, \ldots$, the property used in § 7 above. The law of formation of the 'abacus' numbers $\frac{\Delta^p 0^r}{p!}$ can be very simply established by consideration of the equation

$$s^r = s^{(r)} + a_1 s^{(r-1)} + \ldots + a_{r-1} s^{(1)},$$

multiplying each side by s and expressing the right-hand side in terms of $s^{(r+1)}, s^{(r)} \ldots$, etc.

* Epstein, L. F., *J. Math. Phys.* Vol. XVIII, pp. 153–73, 1939.

9. References to tabulations of $\frac{\Delta^p 0^n}{p!}$ and $\Delta^p 0^n$ are given in the *Index of Mathematical Tables*, §§4.9241 and 4.9242. An extensive table is given in *Statistical Tables* by Fisher and Yates covering the range $n=2(1)25$, $p=2(1)n$, and this has been used for the calculation of A_r and K_r. Unfortunately, this table contains a number of serious errors of which a list has recently been published in *Mathematical Tables and other Aids to Computation*, Vol. IV, No. 9, p. 27, January 1950. An extension of the table to $n=26$ is given in the Appendix to this note.

10. It is of some interest to note that an explicit expression for the numbers $\frac{\Delta^p 0^n}{p!}$ can be found in the form

$$\frac{n^{p-1}-\binom{n-1}{1}(n-1)^{p-1}+\binom{n-1}{2}(n-2)^{p-1}-\dots}{(n-1)!}, \tag{14}$$

and the coefficients A_r can be expressed as

$$\begin{aligned} -A_r = {} & 1+1+\frac{1}{2!}+\dots+\frac{1}{(r-1)!} \\ & -\frac{2^{r-1}}{1!}\left\{1+1+\dots+\frac{1}{(r-2)!}\right\} \\ & +\frac{3^{r-1}}{2!}\left\{1+1+\dots+\frac{1}{(r-3)!}\right\} \\ & \vdots \\ & (-)^{r-1}\frac{r^{r-1}}{(r-1)!}, \end{aligned} \tag{15}$$

but these are not suitable for numerical calculation.

11. No reference has been found to the previous tabulations of the coefficients A_r, but references to the K_r are given in the *Index of Mathematical Tables*, §§ 4.676 and 5.216.

12. The rapid increase of the coefficients K_r is interesting, as is the oscillating nature of the A_r, and suggests that an investigation into their behaviour for large r would be of value; the remainder of this note is devoted to these problems. Some investigations for K_r are given by Epstein,* but an alternative approach has been developed and some considerable improvement made. Furthermore, it has been found possible to extend the analysis to the A_r.

13. For the purpose of this analysis it was found desirable to have values of K_r and A_r for a fairly large value of r, and the following values were found by summation of the series (9) and (12)

$$\log_{10} K_{100} = 115{\cdot}677476,$$

$$\log_{10} A_{100} = 104{\cdot}599421 \text{ (+ve sign).}$$

To calculate the value of A_{100} to eight significant figures it was necessary to calculate the value of the individual terms from twenty-figure logarithms.

* Epstein, L. F., *J. Math. Phys.* Vol. XVIII, pp. 153–73, 1939.

14. The following lemma* will be used for the investigation of K_r and A_r.

LEMMA. *If* $\frac{d^n}{dt^n}f(t)=0$ *for* $t=\infty$ *then*

$$\tfrac{1}{2}f(0)+\sum_{t=1}^{\infty}f(t)=\int_0^{\infty}f(t)\,dt+2\sum_{s=1}^{\infty}\int_0^{\infty}f(t)\cos 2\pi st\,dt. \tag{16}$$

The Euler-Maclaurin formula gives

$$\sum_{t=0}^{\infty}f(t)=\int_0^{\infty}f(t)\,dt+\tfrac{1}{2}f(0)-\frac{B_1}{2!}f'(0)+\frac{B_2}{4!}f'''(0)\ldots+(-)^m\frac{B_m}{(2m)!}f^{2m-1}(0)\ldots.$$

Therefore $\tfrac{1}{2}f(0)+\sum_{t=1}^{\infty}f(t)-\int_0^{\infty}f(t)\,dt=\sum_{m=1}^{\infty}(-)^m\frac{B_m}{2m}a_{2m-1}$,

where $\quad a_{2m-1}=f^{2m-1}(0)/(2m-1)!$,

so that by Taylor's theorem $\quad f(y)=\sum_{t=0}^{\infty}a_ty^t.$

Since $$B_m=\frac{\zeta(2m)(2m)!}{2^{2m-1}\pi^{2m}}, \tag{17}$$

where $\zeta(2m)$ is the Riemann zeta function, we have

$$\begin{aligned}
\sum_{m=1}^{\infty}(-)^m\frac{B_m}{2m}a_{2m-1}&=\sum_{m=1}^{\infty}(-)^m\int_0^{\infty}\frac{x^{2m-1}}{2^{2m-1}\pi^{2m}(e^x-1)}a_{2m-1}\,dx\\
&=\sum_{t=0}^{\infty}\left\{\int_0^{\infty}\frac{x^t a_t}{(e^x-1)(2\pi i)^{t+1}}dx+\int_0^{\infty}\frac{x^t a_t}{(e^x-1)(-2\pi i)^{t+1}}dx\right\}\quad\text{where } t=2m-1\\
&=\sum_{t=0}^{\infty}\left\{\int_0^{\infty}x^t\sum_{s=1}^{\infty}\frac{e^{-sx}a_t}{(2\pi i)^{t+1}}dx+\int_0^{\infty}x^t\sum_{s=1}^{\infty}\frac{e^{-sx}a_t}{(-2\pi i)^{t+1}}dx\right\}\\
&=\sum_{t=0}^{\infty}\sum_{s=1}^{\infty}\frac{t!}{(2\pi si)^{t+1}}a_t+\sum_{t=0}^{\infty}\sum_{s=1}^{\infty}\frac{t!}{(-2\pi si)^{t+1}}a_t\\
&=\sum_{t=0}^{\infty}\sum_{s=1}^{\infty}\int_0^{\infty}y^te^{-2\pi siy}a_t\,dy+\sum_{t=0}^{\infty}\sum_{s=1}^{\infty}\int_0^{\infty}y^te^{2\pi siy}a_t\,dy\\
&=2\sum_{s=1}^{\infty}\int_0^{\infty}\cos 2\pi sy\sum_{t=0}^{\infty}y^ta_t\,dy\\
&=2\sum_{s=1}^{\infty}\int_0^{\infty}\cos 2\pi sy f(y)\,dy.
\end{aligned}$$

* Formula (16) was found, in the first place, for the special function under discussion, but its elegance and the fact that it was capable of easy generalization suggested that it could not have escaped the attention of mathematicians. In fact, it was found to be a special case of a formula known as Poisson's Summation Formula. Hardy (*Divergent Series*, p. 330, Oxford University Press, 1949) indicates that Poisson arrived at it in his investigation of the Euler-Maclaurin formula and gives the finite form

$$\tfrac{1}{2}f(a)+f(a+\omega)+\ldots+\tfrac{1}{2}f(b)=\frac{1}{\omega}\int_a^b f(t)\,dt+\frac{2}{\omega}\sum_{l=1}^{\infty}\int_a^b f(t)\cos\frac{2\pi l(t-a)}{\omega}\,dt.$$

An alternative form with various proofs and special cases together with various references is given by Titchmarsh (*Introduction to Theory of Fourier Integrals*, p. 60, 1937) as follows:

$$\sqrt{\beta}\left\{\tfrac{1}{2}F_c(0)+\sum_{n=1}^{\infty}F_c(n\beta)\right\}=\sqrt{\alpha}\left\{\tfrac{1}{2}f(0)+\sum_{n=1}^{\infty}f(n\alpha)\right\},$$

where $\quad \alpha\beta=2\pi,\quad \alpha>0\quad$ and $\quad F_c(x)=\sqrt{\frac{\pi}{2}}\int_0^{\infty}f(t)\cos xt\,dt.$

It was thought that the proof given here would be of interest to actuaries.

15. Taking $f(t)=\frac{t^r}{t!}$, we have

$$eK_r=\int_0^\infty \frac{t^r}{t!}dt+2\sum_{s=1}^{\infty}\int_0^\infty \frac{t^r}{t!}\cos 2\pi st\,dt. \tag{18}$$

16. Considering now the coefficients A_r, we have

$$A_r e^{-1}=\sum_{t=0}^{\infty}(-)^t\frac{t^r}{t!} \quad (r>0)$$

$$=2\sum_{t=0}^{\infty}\frac{(2t)^r}{(2t)!}-\sum_{t=0}^{\infty}\frac{t^r}{t!}.$$

Applying the lemma to $\frac{(2t)^r}{(2t)!}$, we have

$$2\sum_{t=0}^{\infty}\frac{(2t)^r}{(2t)!}=2\int_0^\infty\frac{(2t)^r}{(2t)!}dt+4\sum_{s=1}^{\infty}\int_0^\infty\frac{(2t)^r}{(2t)!}\cos 2\pi st\,dt$$

$$=\int_0^\infty\frac{t^r}{t!}dt+2\sum_{s=1}^{\infty}\int_0^\infty\frac{t^r}{t!}\cos\pi st\,dt.$$

Hence

$$A_r e^{-1}=2\sum_{s=1}^{\infty}\int_0^\infty\frac{t^r}{t!}\cos\pi st\,dt-2\sum_{s=1}^{\infty}\int_0^\infty\frac{t^r}{t!}\cos 2\pi st\,dt$$

$$=2\sum_{s=1}^{\infty}\int_0^\infty\frac{t^r}{t!}\cos(2s-1)\pi t\,dt. \tag{19}$$

From (18) and (19) we see that the problem of approximating to K_r and A_r reduces to the evaluation of

$$\int_0^\infty\frac{t^r}{t!}dt \quad \text{and} \quad \int_0^\infty\frac{t^r}{t!}\cos\pi st\,dt.$$

17. First considering $\int_0^\infty\frac{t^r}{t!}dt$ let m denote the value of t for which $t^r/t!$ is a maximum, so that $r=m\left(\frac{d}{dt}\ln t!\right)_{t=m}$.

Write

$$t^r/t!=(m+\delta)^r/(m+\delta)!. \tag{20}$$

Taking logarithms and expanding in powers of δ, we have

$$\ln(t^r/t!)=r\ln m-\ln(m!)-\frac{\delta^2}{2m^2}\{r+m^2\psi'(m)\}+\frac{\delta^3}{3!\,m^3}\{2r-m^3\psi''(m)\}-\dots, \tag{21}$$

where

$$\psi(z)=\frac{d}{dz}\ln(z!) \quad \text{(digamma function)},$$

$$\psi'(z)=\frac{d^2}{dz^2}\ln(z!) \quad \text{(trigamma function)},$$

$$\dots\dots\dots\dots\dots\dots$$

and

$$m\psi(m)=r.$$

If terms in δ^3 and higher powers of δ are ignored we may write

$$\int_0^\infty \frac{t^r}{t!}dt \doteqdot \frac{m^r}{m!}\int_{-m}^\infty \exp\left[-\frac{\delta^2}{2m^2}\{r+m^2\psi'(m)\}\right]d\delta$$

$$\doteqdot \frac{m^{r+1}\sqrt{(2\pi)}}{m!\sqrt{[r+m^2\psi'(m)]}}, \tag{22}$$

provided r is large.

18. Alternatively, we may write

$$\int_0^\infty \frac{t^r}{t!}dt = \frac{m^r}{m!}\int_{-m}^\infty \left(1+\frac{\delta}{m}\right)^r e^{-\delta\psi(m)} \exp -\left\{\frac{\delta^2}{2!}\psi'(m)+\frac{\delta^3}{3!}\psi''(m)+\ldots\right\}d\delta,$$

and if terms in { } are ignored we have the approximation

$$\int_0^\infty \frac{t^r}{t!}dt \doteqdot \frac{m^{r+1}e^r r!}{m!\,r^{r+1}}. \tag{23}$$

19. Finally, we may write

$$\int_0^\infty \frac{t^r}{t!}dt = \frac{m^r}{m!}\int_{-m}^\infty \left(1+\frac{\delta}{m}\right)^{r+m^2\psi'(m)} e^{-\delta/m\{r+m^2\psi'(m)\}}$$

$$\times \exp\left[-\frac{\delta^3}{3!\,m}\{2\psi'(m)+m\psi''(m)\}+\ldots\right]d\delta, \tag{24}$$

and if terms in [ ] are ignored we have

$$\int_0^\infty \frac{t^r}{t!}dt \doteqdot \frac{m^{r+1}e^p p!}{m!\,p^{p+1}}, \tag{25}$$

where $p = r + m^2\psi'(m)$.

20. The following table sets out the closeness of the approximations (22), (23) and (25):

r	$\log_{10} K_r$	Approximation by formula		
		(22)	(23)	(25)
1	·000000	·024692	·217599	·042070
2	·301030	·305180	·454695	·315029
3	·698970	·701166	·832034	·708136
5	1·716003	1·717182	1·829365	1·721644
10	5·064364	5·064658	5·157492	5·067059
20	13·713693	13·713697	13·791921	13·714970
26	19·695755	19·695716	19·769329	19·696715
100	115·677476	115·67742	115·73296	115·67770

Consideration of these results shows that the errors from formula (22), whilst smaller than those from (23) or (25), change sign between $r=10$ and $r=20$. On the other hand, the errors from formula (25) are all of the same sign and decrease steadily with increasing r. The errors in formula (22), apart from the Euler-Maclaurin terms, arise from two sources, namely, substitution of $-\infty$ for $-m$ as the lower limit of the integral and the neglect of terms in δ^3, etc., whilst those in formula (25) arise only from the terms neglected.

21. We can, however, find an asymptotic formula for the integral by retaining the neglected exponential terms in (24), expanding them in powers of δ and integrating the resulting series of terms of the form $\int_{-m}^\infty \left(1+\frac{\delta}{m}\right)^p e^{-\delta p/m}\delta^s\,d\delta$.

After expressing the trigamma, tetragamma, ... functions by their asymptotic expansions the following formula results after a considerable amount of reduction:

$$\int_0^\infty \frac{t^r}{t!}dt \doteqdot \frac{m^{r+1}e^p p!}{m!\,p^{p+1}}$$
$$\times\left(1-\frac{m}{3p^2}+\frac{5m^2}{24p^3}+\frac{m}{12p^3}+\frac{m^2}{18p^4}-\frac{35m^3}{72p^5}+\frac{1}{72p^2m}-\frac{5}{72p^3}-\frac{m}{60p^4}+\dots\right). \qquad (26)$$

Values calculated from the leading terms in (26) are as follows:

r	$\log_{10} K_r$	Formula (26)
5	1·716003	1·715928
10	5·064364	5·064353
20	13·713693	13·713694
26	19·695755	19·695755
100	115·677476	115·67748

It is seen that the Euler-Maclaurin adjustment terms are negligible to the above order of accuracy for $r \geqslant 10$.

22. The reduction of $\int_0^\infty \frac{t^r}{t!}\cos \pi st\,dt$ proved to be more troublesome, the difficulty partly arising from an observation that the signs of the values of A_r for $r \leqslant 26$ were reproduced by $\cos\overline{p+1}\left(\tan^{-1}\frac{\pi m}{p}\right)$. Subsequent analysis showed this to be partly coincidence, but not until some fruitless investigations had been made. Following the analysis of § 19 we may write, considering the case $s=1$,

$$\int_0^\infty \frac{t^r}{t!}\cos \pi t\,dt = \frac{m^r}{m!}\int_{-m}^\infty \left(1+\frac{\delta}{m}\right)^p e^{-\delta p/m}\cos \pi(m+\delta)$$
$$\times \exp\left[-\frac{\delta^3}{3!\,m}\{2\psi'(m)+m\psi''(m)\}+\dots\right]d\delta. \qquad (27)$$

If the terms in [ ] are ignored the integral may be evaluated as

$$\int_0^\infty \frac{t^r}{t!}\cos \pi t\,dt \doteqdot \frac{m^{r+1}e^p p!}{m!\,p^{p+1}}\cos(p+1)\theta\cos^{p+1}\theta, \qquad (28)$$

where $\theta = \tan^{-1}\frac{\pi m}{p}$, and we may write

$$A_r e^{-1} \doteqdot 2K_r e\cos(p+1)\theta\cos^{p+1}\theta. \qquad (29)$$

23. As an indication of the accuracy of formula (28) the following values are given—in all cases the signs of the terms are correct:

r	$\log_{10} \mid A_r \mid$	Calculated from formula (29)	Error
5	·3010	·6062	+·3052
10	2·6160	2·7960	+·1800
15	6·2938	6·5946	+·3008
20	9·9920	10·3374	+·3454
25	14·2263	14·4957	+·2694
100	104·5994	104·9574	+·3580

The approximate values start by being roughly twice the true values, showing a very slow convergence to the true values.

24. Inclusion of the term $\exp\left[-\frac{\delta^3}{3!\,m}\{2\psi'(m)+m\psi''(m)\}+\dots\right]$ in formula (27), expanding in powers of δ and integrating term by term leads to expressions of the form

$$\frac{m^{s+1}e^p p!\cos^{p+1}\theta}{p^{p+1}}\left[\frac{(p+s)^{(s)}}{p^s}\cos^s\theta\cos(p+s+1)\theta-\dots\right].$$

By noting that

$$\cos(p+s+1)\theta\cos^s\theta=\cos(p+1)\theta\,\tfrac{1}{2}\left\{\frac{1}{(1+it)^s}+\frac{1}{(1-it)^s}\right\}$$
$$+\sin(p+1)\theta\,\frac{1}{2i}\left\{\frac{1}{(1+it)^s}-\frac{1}{(1-it)^s}\right\},$$

where $t=\tan\theta$, and that the terms in [ ] in formula (27) can be expressed as

$$m\left[\left(\frac{\delta}{m}+2\right)\log\left(\frac{\delta}{m}+1\right)-2\frac{\delta}{m}+\frac{1}{12m^2}\left\{2\log\left(\frac{\delta}{m}+1\right)-\frac{\delta}{m}\frac{\delta+2m}{\delta+m}\right\}+\dots\right],$$

we finally find that, after considerable reduction,

$$\int_0^\infty\frac{t^r}{t!}\cos\pi t\,dt=\frac{m^{r+1}e^p p!\cos^{p+1}\theta}{m!\,p^{p+1}}e^{-m\alpha}\{X\cos(\overline{p+1}\,\theta+m\beta)$$
$$+Y\sin(\overline{p+1}\,\theta+m\beta)\},\qquad(30)$$

where $\alpha=(2-\sin^2\theta)\log\cos\theta+2\sin^2\theta-\theta\sin\theta\cos\theta+\text{terms in }\frac{1}{m^2},$

$$\beta=\theta(\sin^2\theta-2)-\sin\theta\cos\theta\log\cos\theta+2\sin\theta\cos\theta+\text{terms in }\frac{1}{m^2},$$

$$X=1+\frac{\cos^2\theta}{p}\{m(\theta-\tan\theta)\tan\theta-m\log\cos\theta\}$$
$$+\frac{(p+2)\cos^4\theta}{2!\,p^2}[(1-\tan^2\theta)\{m^2(\log\cos\theta)^2-m^2(\theta-\tan\theta)^2-m\tan^2\theta\}$$
$$-2\tan\theta\{2m^2(\theta-\tan\theta)\log\cos\theta-m\tan\theta\}]$$
$$+\dots\dots,$$

and $Y=\frac{\cos^2\theta}{p}\{m\tan\theta\log\cos\theta+m(\theta-\tan\theta)\}$

$$+\frac{(p+2)\cos^4\theta}{2!\,p^2}[(\tan^2\theta-1)\{2m^2(\theta-\tan\theta)\log\cos\theta-m\tan\theta\}$$
$$-2\tan\theta\{m^2(\log\cos\theta)^2-m^2(\theta-\tan\theta)^2-m\tan^2\theta\}]$$
$$+\dots\dots\qquad(31)$$

25. The expressions for X and Y are asymptotic, and for very low values of r the successive terms change sign but diverge before reaching a small enough magnitude for the formula to be of value other than for assigning limits. However, for larger values of r the terms converge fairly rapidly, although some irregularities exist due to the periodic nature of the functions.

The following table shows the degree of approximation (*a*) for the formula

$$A_r \doteqdot 2e^2 \frac{m^{r+1} e^p p!}{m!\, p^{p+1}} \cos^{p+1}\theta\, e^{-m\alpha} \cos(\overline{p+1}\,\theta + m\beta), \tag{32}$$

and (*b*) for the formula (31), where X and Y are taken to two further terms than shown above:

r	$\log_{10} \lvert A_r \rvert$	(*a*) Formula (32)	(*b*) Formula (31)
5	·3010	·2442	·2217
10	2·6160	2·6647	2·6205
20	9·9920	10·0257	9·9897
26	15·4546	15·4421	15·4526
100	104·5994	104·5765	104·5990

The nature of the convergence for a fairly low value of r is shown in the following figures for $r=10$ which give the approximate values at successive stages in (*b*) above.

No. of terms	$\log_{10} \lvert A_{10} \rvert$	Approx. − True
1	2·6647	+·0487
2	2·5542	−·0618
3	2·6236	+·0076
4	2·5890	−·0270
5	2·6205	+·0045
True value	2·6160	—

26. Some part of the error is due to neglect of the terms in $1/m^2$, etc., but the calculations have not been extended to include these. Consideration of the order of magnitude of these terms and of the closeness of the results (*b*) in the preceding paragraph shows their effect to be small for larger values of m. The values of m and p for representative values of r are given below:

r	m	p
5	3·561266	8·107366
10	5·551247	15·081080
20	8·916780	28·435425
26	10·743336	36·258823
100	29·423186	128·92885

27. A further error arises from neglect of terms for $s>1$ in formula (19). An order of magnitude comparison can be obtained from formula (28) because we can write

$$I_1 = \int_0^\infty \frac{t^r}{t!} \cos \pi t\, dt \doteqdot \frac{m^{r+1} e^p p!}{m!\, p^{p+1}} \cos \overline{p+1}\,\theta \frac{1}{\left(1+\dfrac{\pi m}{p}\right)^{\frac{1}{2}(p+1)}}$$

and
$$I_3 = \int_0^\infty \frac{t^r}{t!} \cos 3\pi t\, dt \doteqdot \frac{m^{r+1} e^p p!}{m!\, p^{p+1}} \cos \overline{p+1}\,\theta_1 \frac{1}{\left(1+\dfrac{3\pi m}{p}\right)^{\frac{1}{2}(p+1)}},$$

where
$$\theta_1 = \tan^{-1}\left(\frac{3\pi m}{p}\right),$$

whence
$$I_3/I_1 \doteqdot \frac{\cos\overline{p+1}\,\theta_1\left(1+\frac{\pi m}{p}\right)^{\frac{1}{2}(p+1)}}{\cos\overline{p+1}\,\theta\left(1+\frac{3\pi m}{p}\right)^{\frac{1}{2}(p+1)}}.$$

For $r=20$ $\quad \frac{\pi m}{p}=\cdot 985$, so that $\frac{\left(1+\frac{\pi m}{p}\right)^{\frac{1}{2}(p+1)}}{\left(1+\frac{3\pi m}{p}\right)^{\frac{1}{2}(p+1)}}=3\cdot 92\times 10^{-5}$,

$r=100$ $\quad \frac{\pi m}{p}=\cdot 717$, so that $\frac{\left(1+\frac{\pi m}{p}\right)^{\frac{1}{2}(p+1)}}{\left(1+\frac{3\pi m}{p}\right)^{\frac{1}{2}(p+1)}}=7\cdot 42\times 10^{-18}$.

Unless therefore $\cos\overline{p+1}\,\theta$ happens to approximate to zero the terms with $s>1$ will be negligible relative to the term with $s=1$.

28. The above analysis has been carried through on the assumption that r is positive integral, but clearly the approximations are true for positive real r. For complex r both K_r and A_r are complex quantities, but no analysis has been made in this region. Similarly, this note has been largely relieved of various analytical considerations which arise at a number of points.

APPENDIX

r	Values of $\frac{\Delta^r 0^n}{r!}$ for $n=26$
2	335 54431
3	42 36107 50290
4	18722 63569 46265
5	12 23019 61602 92565
6	224 59518 69741 25331
7	1631 85379 79910 16600
8	5749 62225 19456 64950
9	11201 51678 09551 25625
10	13199 55537 28468 48005
11	10029 07834 09984 76760
12	5149 50735 38569 58820
13	1850 56857 42535 50060
14	477 89861 83962 88260
15	90 44903 01911 04000
16	12 72587 72424 82560
17	1 34373 17953 78830
18	10702 55461 01760
19	643 38390 18750
20	29 06228 64675
21	97591 04355
22	2389 29405
23	41 26200
24	47450
25	325
26	1