JIA 119 (1992) 107-113

J.I.A. **119**, I, 107–113

COMMENTS ON 'SOME RESULTS ON THE GOMPERTZ AND HELIGMAN AND POLLARD LAWS OF MORTALITY'

By A. E. Renshaw, B.Sc., Ph.D.

(of The City University, London)

ABSTRACT

The consequences of graduating truncated English Life Table data by either the Gompertz or truncated Heligman and Pollard 'laws of mortality', recently advocated by Thatcher (1990), are examined in greater detail.

KEYWORDS

English Life Tables; Graduation

In a recent paper, Thatcher (1990) has reviewed the parallel mathematical properties of two 'laws of mortality', that due to Gompertz (1825) and a truncated version of that due to Heligman & Pollard (1980). These two 'laws of mortality' are characterised by the equations:

$$\log(-\log(1-q_x)) = \alpha_1 + \beta_1 x \tag{1}$$

and

$$\log\left(\frac{q_x}{1-q_x}\right) = \alpha_2 + \beta_2 x \tag{2}$$

where q_x denotes the probability of death at age x and α_i and β_i denote model parameters.

It is of interest to note that the 'mortality law' defined by equation (2) was first proposed by Wilkie and used at his suggestion in the construction of seven out of the eight q_x-graduations presented in *CMIR* **2** (1976). The structure defined by equation (2) is identical to that associated with the $\text{LGM}_x(0,2)$ model described in Forfar, McCutcheon & Wilkie (1988), in which a possibly different method of estimating the parameters α_2 and β_2 is described to add to the four methods discussed in Thatcher (1990), although it is not made clear to the reader what form the likelihood takes in Method C of Section 3 in Thatcher (1990). In addition, Renshaw (1991) has described how the two 'laws of mortality' characterised by equation (1) and equation (2) are members of a much more general class of 'mortality laws' based on the concepts of generalised linear and non-linear models. One consequence of this is to facilitate a direct comparison between the goodness-of-fit of the two competing models.

This note concerns the fundamental issue of the quality-of-fit of these two

Table 1. Statistical tests for the regraduation of the E.L.T. No. 13 data, males, ages 50–90 years

	q_x-graduation	μ_x-graduation
Formula	LGM(0, 2)	3 knot spline
Deviance & (d.f.)	405·91 (39)	29·74 (38)
Parameter estimates:	$\hat{\beta}_0 = -9{\cdot}6400$ s.e. = 0·01519 t = −634·7 $\hat{\beta}_1 = 0{\cdot}09687$ s.e. = 0·0002147 t = 451·2	$\hat{\alpha}_0 = -10{\cdot}0843$ s.e. = 0·03059 t = −329·64 $\hat{\alpha}_1 = 0{\cdot}1037$ s.e. = 0·0004807 t = 215·76 $\hat{\beta}_1 = -0{\cdot}00004442$ s.e. = 0·0000001400 t = −31·73
Signs test:		
Number of +	14	16
Number of −	27	25
p-value	2·12%	7·99%
Runs test:		
Number of runs	5	18
p-value	0·000%	20·16%
Kolmogorov–Smirnov test:		
Max. deviation	0·0085	0·0011
KS-statistic	3·06	0·385
Serial correlation test:		
lag 1		
r_1	0·918	0·161
t-ratio	5·80	1·02
lag 2		
r_2	0·854	−0·036
t-ratio	5·33	−0·23
lag 3		
r_3	0·755	−0·227
t-ratio	4·65	−1·34
Chi-square test:		
χ^2	404·59	29·74
d.f.	39	38
p-value	0·000%	82·5%

'laws' when used to describe the underlying mortality associated with the E.L.Ts. Nos. 10–14 data sets for both males and females in the age range 50–90 years as advocated by Thatcher (1990). For purposes of illustration, it is sufficient to concentrate on refitting the male E.L.T. No. 13 data to the model structure defined by equation (2), say, since the other cases are found to exhibit similar behaviour patterns as is readily verified.

Thatcher makes no reference to an examination of residuals once the model is fitted, nor to the application of basic graphical and formal diagnostic checks on the residuals. Focus first on his use of least-squares estimation, Method A in Section 3 of Thatcher (1990). It is remarked that the models are fitted to (transformations of) the smoothed q_x-graduations of the E.L.T.s, and not to (transformations of) the raw data which comprise the actual number of deaths and the exposures to the risk of death, thereby constituting a rather unusual type of two-stage smoothing process which is entered upon without comment by the author. This notwithstanding, the parameter estimates quoted in Table 1 of Section 3 of Thatcher (1990), *viz*:

$$\hat{G}=7{\cdot}7250\times 10^{-5} \qquad \hat{H}=1{\cdot}0980$$

are readily confirmed using a standard statistical package. Additionally, however, the residuals associated with the fit, plotted against age, are reproduced in Figure 1. Next consider the graduation of the raw data this time, using maximum likelihood estimation as described in Forfar *et al.* (1988), which leads to parameter estimates:

$$\hat{G}=6{\cdot}5071\times 10^{-5} \qquad \hat{H}=1{\cdot}1017.$$

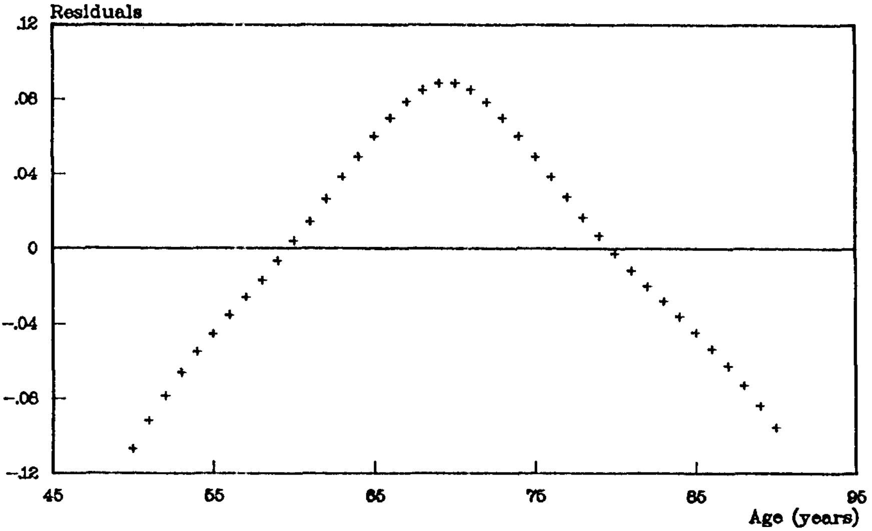

Figure 1. Residuals against Age. Regraduation of E.L.T. No. 13 (males), Ages 50–90.

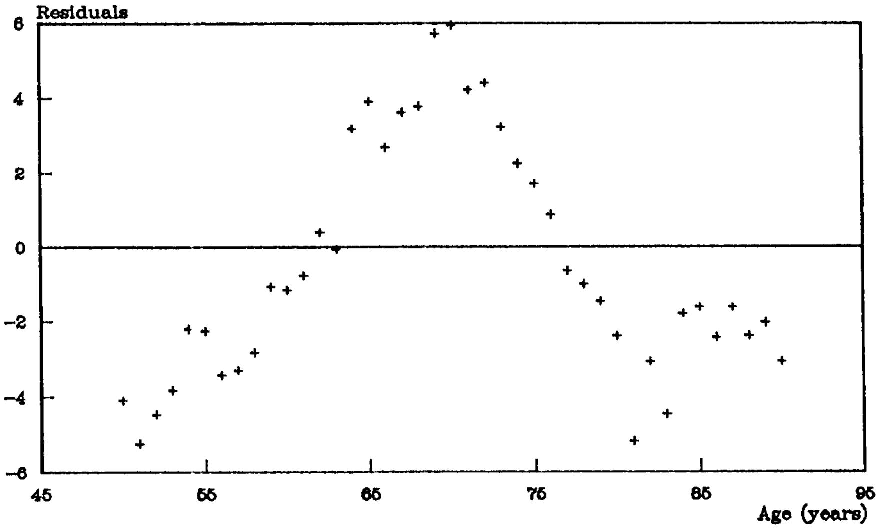

Figure 2(a). Residuals against Age. $q(x)$ graduation of E.L.T. No. 13 (males), Ages 50–90.

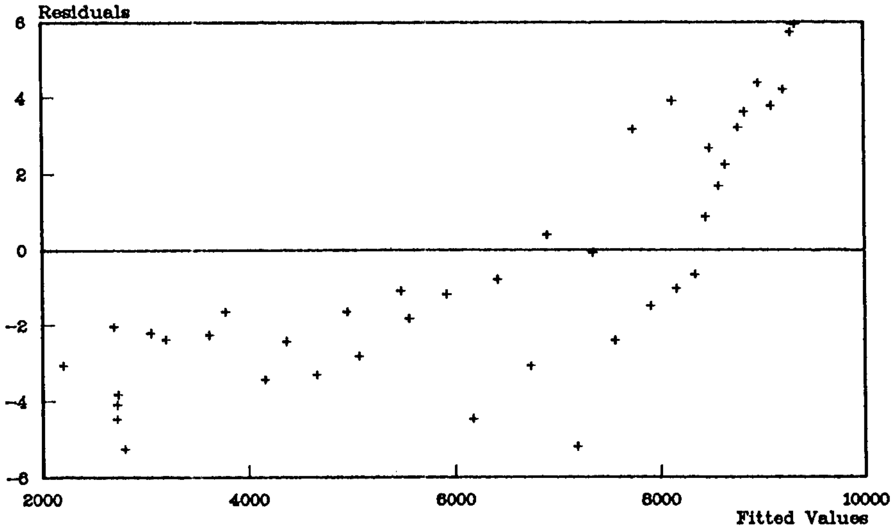

Figure 2(b). Residuals against Fitted Values. $q(x)$ graduation of E.L.T. No. 13 (males), Ages 50–90.

The residual plots for this fit are reproduced in Figure 2. While neither the method of estimation, nor indeed, the issue of whether the already smoothed E.L.T. q_x-values should be further graduated in preference to the raw data, are in contention here; such pronounced patterns in both sets of residual plots give a clear indication that the model structure defined by equation (2) has failed to account for all the underlying systematic variation in mortality with age for this particular data set. If further evidence is needed to augment that of Figure 2, the more formal statistical tests of a graduation, discussed in detail and used extensively in Forfar *et al.* (1988) and elsewhere, are given in the second column of Table 1 for the alternative graduation based on the raw data. Similar unsatisfactory test statistics, which are not reproduced here, are associated with Figure 1 and the Thatcher graduation of these data, as may be readily verified. In addition, distinctive unsatisfactory patterns are observed in the residual plots when graduating each of the truncated E.L.T. data sets under consideration, using either of the models defined by equations (1) or (2). Again this is readily verified.

In conclusion, the emergence of these distinctive residual patterns when fitting the model structures defined by equations (1) and (2) to truncated E.L.T. data would appear to cast serious doubts on some of the concluding remarks in Thatcher (1990). They would appear rather to vindicate the choice of more complex 'mortality laws' in the graduation of the E.L.T. data sets, such as those based on the variable knot cubic spline as advanced by McCutcheon (1987). As further evidence of this, the residual plots and formal statistical tests for the μ_x-graduation of the male E.L.T. No 13 data in the age range 50–90 years using, for example, the less sophisticated fixed knot spline with three equally spaced knots at ages 60, 70 and 80 years, are reproduced in Figure 3 and in the third column of Table 1. Comparison of these with Figures 1 and 2 and with the second column of Table 1 respectively indicate a dramatic improvement in the graduation. Here μ_x-graduation is selected in keeping with Forfar *et al.* (1988) since the E.L.T. data are based on central exposures. In fact the model used is identical to the Forfar *et al.* (1988) $GM_x(0, r)$ model in which the exponentiated polynomial of degree $(r-1)$ has been replaced by the exponentiated cubic spline:

$$\alpha_0 + \alpha_1 x + \sum_{j=1}^{n} \gamma_j (x - x_j)^3_+$$

with $n = 3$ fixed knots $\{x_j\}$ at ages 60, 70 and 80 years, where:

$$\begin{aligned} x^3_+ &= x^3, \qquad x \geqslant 0 \\ &= 0, \qquad x < 0. \end{aligned}$$

Following Chapter 16 of Benjamin & Pollard (1980), the following reparameterisation of the cubic spline is used:

$$\alpha_0 + \alpha_1 x + \sum_{j=1}^{n-2} \beta_j \Psi_j(x)$$

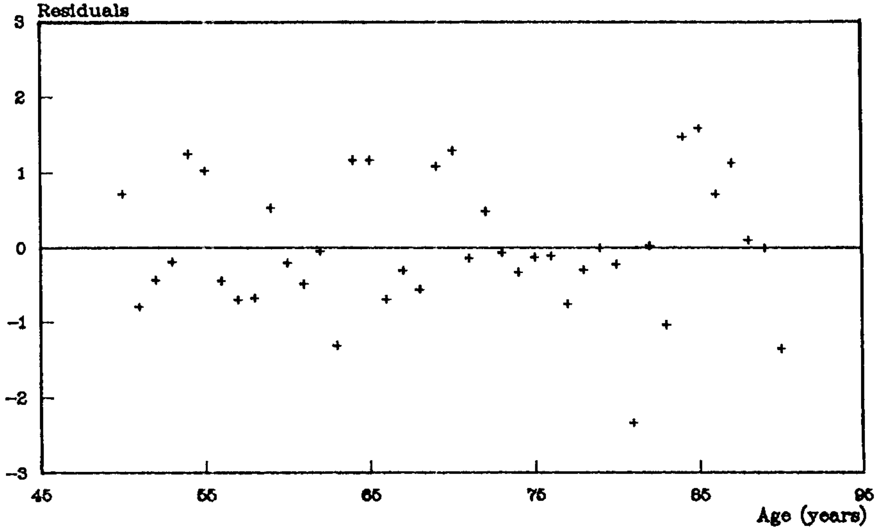

Figure 3(a). Residuals against Age. $\mu(x)$ graduation of E.L.T. No. 13 (males), Ages 50–90.

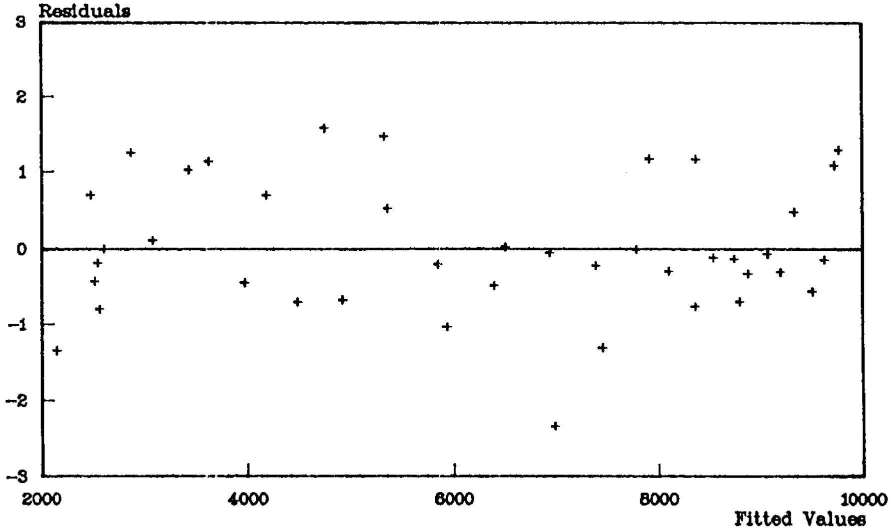

Figure 3(b). Residuals against Fitted Values. $\mu(x)$ graduation of E.L.T. No. 13 (males), Ages 50–90.

where:

$$\Psi_j(x) = (x - x_j)_+^3 - \frac{x_n - x_j}{x_n - x_{n-1}}(x - x_{n-1})_+^3 + \frac{x_{n-1} - x_j}{x_n - x_{n-1}}(x - x_n)_+^3 .$$

This leads to a reduction, by two, in the number of parameters which have to be estimated.

REFERENCES

BENJAMIN B. & POLLARD J. H. (1980). *Analysis of Mortality and other Actuarial Statistics.* Heinemann.

CMI COMMITTEE (1976). The Graduation of Pensioners' and of Annuitants' Mortality Experience 1967–70. *CMIR*, **2**, 57.

FORFAR, D. O., MCCUTCHEON, J. J. & WILKIE, A. D. (1988). On Graduation by Mathematical Formula. *J.I.A.* **115**, 1 and *T.F.A.* **41**, 97.

HELIGMAN, L. & POLLARD, J. H. (1980). The Age Pattern of Mortality. *J.I.A.* **107**, 49.

GOMPERTZ, B. (1825). On the Nature of the Function of the Law of Human Mortality, etc. *Phil. Trans. Roy. Soc.* **115**, 513.

MCCUTCHEON, J. J. (1987). Spline Graduation of the Crude Data for the English Life Table No. 14. *E.L.T. No. 14*, O.P.C.S., HMSO.

RENSHAW, A. E. (1991). Actuarial Graduation Practice and Generalised Linear & Non-Linear Models. *J.I.A.* **118**, 295.

THATCHER, A. R. (1990). Some Results on the Gompertz and Heligman and Pollard Laws of Mortality. *J.I.A.* **117**, 135.