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THE CONSTRUCTION OF THE SCOTTISH LIFE TABLES (1980-1982)

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1. INTRODUCTION

The preliminary data relating to the proposed Scottish Life Tables (1980-1982) were made available to the author by courtesy of the Government Actuary. In this paper we describe briefly experiments in graduating the underlying crude mortality rates by cubic splines and give an outline of the principal features of the graduations finally adopted. A complete description of the Scottish Life Tables, including a considerable amount of comparative statistics, is given in reference 1.

As with the corresponding English Life Tables (see references 2 and 8), the data related to the calendar years 1980, 1981, and 1982. For each integer $x \geq 2$ the values of θ_x, the number of deaths during the investigation period, aged x last birthday at the time of death, and E^c_x the corresponding *central* exposure to risk were available for both sexes. The quotient θ_x/E^c_x gives the crude central death rate at exact age x. For $x \geq 5$ the value of E^c_x was calculated by the same quadratic-based formula as was used for the English Life Tables, based on the numbers in the home population at each age at the mid-point of each of the three calendar years. For $x = 2$, 3 and 4 the value of E^c_x was calculated by reference to both the number of deaths and the related births.

For both sexes experiments were carried out, using the method of variable-knot spline graduation (see reference 7). These led to 'best' graduations for each sex, which formed the bases for the resulting life tables.

2. MORTALITY AT YOUNG AGES

Special techniques, based on methods described in reference 4, were used for the measurement of mortality at ages 0 and 1. The relevant results are summarised in the following table, in which E_x denotes the *initial* (as opposed to central) exposure to risk at age x last birthday and ϕ_0 is the average age at death of those dying in the first year of life.

Table 2.1

	Scotland: 1980-82	
	Males	Females
θ_0	1,352	1,012
E_0	104,747	99,623
q_0	0.01291	0.01016
θ_1	97	68
E_1	103,795	98,239
q_1	0.00093	0.00069
ϕ_0	0.14	0.15

In comparison with the values from the previous Scottish Life Tables, which relate to the period 1970-1972, the reduction in q_0 is 40% for both sexes. The corresponding reductions in the value of q_1 are 33% for males and 37% for females. These are slightly greater percentage reductions than those for England and Wales over the same period.

3. GRADUATION AT AGES 2 AND ABOVE BY CUBIC SPLINES

For both sexes at the highest ages the observed crude rates of mortality formed an irregular series. This feature was not unexpected, in view of a paucity in the available data. Initially, therefore, use was made of the data for individual ages for ages 2 to 95 inclusive. The calculation of the graduated rates of mortality at the highest ages is described in §4 below.

Following reference 7, for a given number of knots, n say, we define the 'best' n-knot cubic spline to be that spline s(x) which minimises is the 'relative deviation' at age x.

$$\chi^2 = \sum_{x=2}^{95} (z_x)^2,$$

where

$$z_x = \frac{\theta_x - E_x^c \, s(x)}{[E_x^c \, s(x)]^{1/2}}$$

Note that, in determining the best-fitting n-knot cubic spline we consider the knot positions as free parameters. This means that a total of $2n + 4$ free parameters are required to define the spline (see reference 6). The resulting minimum value of χ^2 is denoted by $\chi^2(n)$. Since in this case the χ^2 value has been obtained as a sum of squares over 94 ages, empirically we regard it as arising from a χ^2 distribution for which the number of degrees of freedom is $94-(2n + 4) = 90-2n$ and consider also the test statistic $t(\chi^2(n))$, where

$$t(\chi^2(n)) = \sqrt{2\chi^2(n)} - \sqrt{2(90-2n)-1} \quad \text{(see reference 3)}$$

$$= \sqrt{2\chi^2(n)} - \sqrt{179-4n}$$

For small values of n the values of both $\chi^2(n)$ and $t\,(\chi^2(n)$ decrease rapidly as n increases. Although the values of $\chi^2(n)$ necessarily form a decreasing sequence, there is, however, a 'critical' point at which the value of $t\,(\chi^2(n))$ is first *greater* than the value of $t\,(\chi^2(n-1))$. The existence of this critical point indicates that an increase in the number of knots has not produced a significantly lower value of χ^2 and determines both the appropriate number of knots to be used in the graduation and the best fitting spline.

The table below shows, for each sex, the values of $\chi^2(n)$ and t $(\chi^2(n))$ relevant to the above discussion.

Table 3.1

No. of knots	Statistics for best-fitting cubic splines			
	Males		Females	
n	$\chi^2(n)$	$t(\chi^2(n))$	$\chi^2(n)$	$t(\chi^2(n))$
.	.	.	.	.
.	.	.	.	.
5	110.17	2.23	96.21	1.26
6	93.78	1.24	91.46	1.08
7	85.85	0.82	84.70	0.73
8	85.15	0.93	84.53	0.88
.	.	.	.	.
.	.	.	.	.

The values in the above table show that for both males and females we obtain no significant improvement by increasing the number of knots from 7 to 8. This means that it is appropriate to use a 7-knot graduation for each sex. (It is perhaps of interest to note that in the construction of the English Life Tables the number of knots required was 10 for males and 11 for females.)

The positions of the knots in the best graduations for each sex are indicated in the table below.

Table 3.2

i	Knot position x_i	
	Males	Females
1	16.01	19.67
2	16.05	19.97
3	16.91	19.97
4	18.01	24.38
5	39.03	57.47
6	59.25	70.86
7	85.90	85.13

(*Note:* In the above table the knot-positions are given to only two decimal places. In the actual calculation of the graduated rates of mortality greater accuracy was retained. The difference is of no practical significance.)

The closeness of the knots in the late teens (for both sexes) is a reflection of the relatively large increase in mortality over these ages (see figure 5.1 below). In particular, for females there is a clear 'peak' in the rate of mortality at age 20. For males there are three knots between ages 16 and 17 and for females three knots (including one repeated knot) between ages 19 and 20.

4. ADJUSTMENT TO THE GRADUATION FOR FEMALES

For females the spline graduation produced a sharp "spike" between ages 19 and 20 in the curve of the graduated central death rates. There was a similar peak in the corresponding experience for England and Wales (see reference 2), but it was considerably less pronounced (and based on far more deaths).

It was difficult to justify such an extreme spike in the Scottish graduation and, accordingly, between ages 18 and 25 the spline values of the graduated central death rates for females were modified slightly.

The modification was carried out by an interpolating cubic defined over the age range [18,25]. To ensure reasonable smoothness, this cubic was chosen to have its value at age 18 and its first derivative at that age the same as the corresponding values (at that age) of the cubic through the spline values of m_x at ages 15, 16, 17 and 18. Similarly the interpolating cubic had its value at age 25 that of the spline value at age 25 and its first derivative at age 25 that (at that age) of the cubic through the spline values of m_x at ages 25, 26, 27 and 28.

The modified set of central death rates was completely satisfactory and produced smoother rates than those of the original spline graduation.

5. EXTRAPOLATION OF THE GRADUATED RATES OF MORTALITY TO THE HIGHEST AGES

In order to construct the life tables it was necessary to extrapolate the rates of mortality (arising from the spline graduations) to the highest ages. After several experiments, for each sex an acceptable extrapolation was obtained using a cubic polynomial. The extrapolating polynomial $f(x)$ was defined for $x \geq \alpha$ (where $\alpha = 89$ for males and 92 for females) by requiring f to have the same value and first and second derivatives at $x = \alpha$ as the quadratic defined by the spline values m_α, $m_{\alpha-1}$, and $m_{\alpha-2}$ and by letting the value of $f(107)$ be 0.95 for males and the value of $f(105)$ be 0.7 for females. (These last values were chosen after careful consideration of the resulting rates of mortality and the corresponding values for England and Wales.)

Summary statistics for the resulting graduations are given in the following tables.

Table 5.1

Graduation summary statistics
(over the age range 2 to 95 inclusive)

Statistic	Males	Females
$\Sigma\ E^c_x$	7,295,862	7,870,297
$\Sigma\ \theta_x$	93,442	94,664
$\chi^2 = \Sigma\ (z_x)^2$	88.46	86.23
No. of positive deviations	48	49
No. of sign changes in deviations	45	50
No. of relative deviations > 2	3	2
No. of relative deviations > 3	1	0
Total accumulated deviations	–54.82	–35.84

Note: The deviation at each age is defined to be $[\theta_x - E^c_x\, m_x]$, where m_x is the graduated central death rate at age x. The relative deviation is defined to be

$$z_x = [\theta_x - E^c_x\, m_x]/[E^c_x\, m_x]^{1/2}$$

6. CONSTRUCTION OF THE LIFE TABLES FROM THE GRADUATED RATES OF MORTALITY

The life tables were produced (on the basis of a limiting age of 113) from the following input parameters:

(i) q_0, q_1, and ϕ_0(see §2 above)

(ii) m_2, m_3, . . ., m_{111}, (the graduated central death rates).

The tables were constructed by the same quadratic based-approximations as were used for the English Life Tables No. 14 (see references 2 and 8).

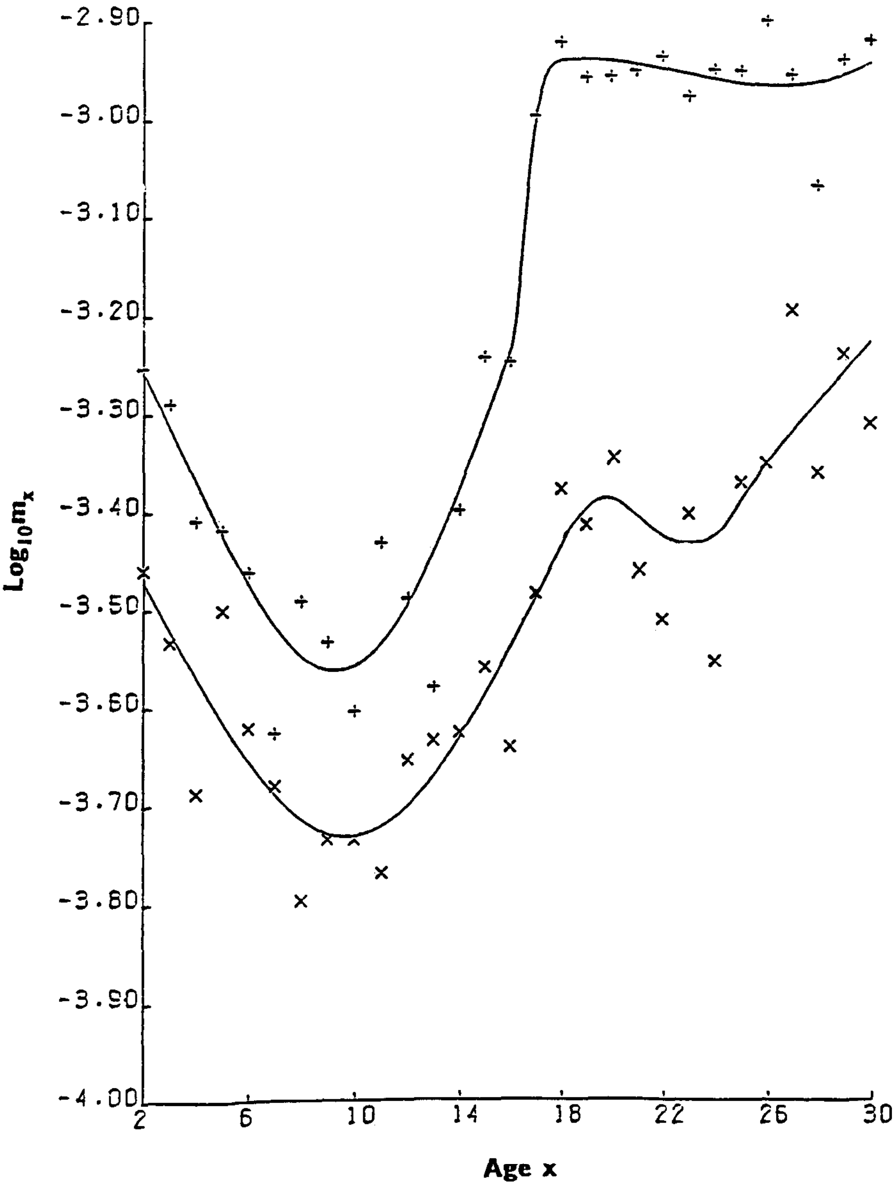
Figures 6.1 and 6.2 below show the logarithm (to the base 10) of the crude central death rates and the corresponding curve of the graduated mortality rates for both males and females over the age ranges 2 to 30 and 25 to 95 respectively. Figure 6.3 shows the ratio of the graduated central rate of mortality for males to the corresponding rate for females. The shape of this curve is of some interest.

Full details of the resulting life tables are given in reference 1. We give below abridged life tables extracted from the complete mortality tables.

Figure 6.1: Scottish Life Tables 1980-1982

+ Males

x Females



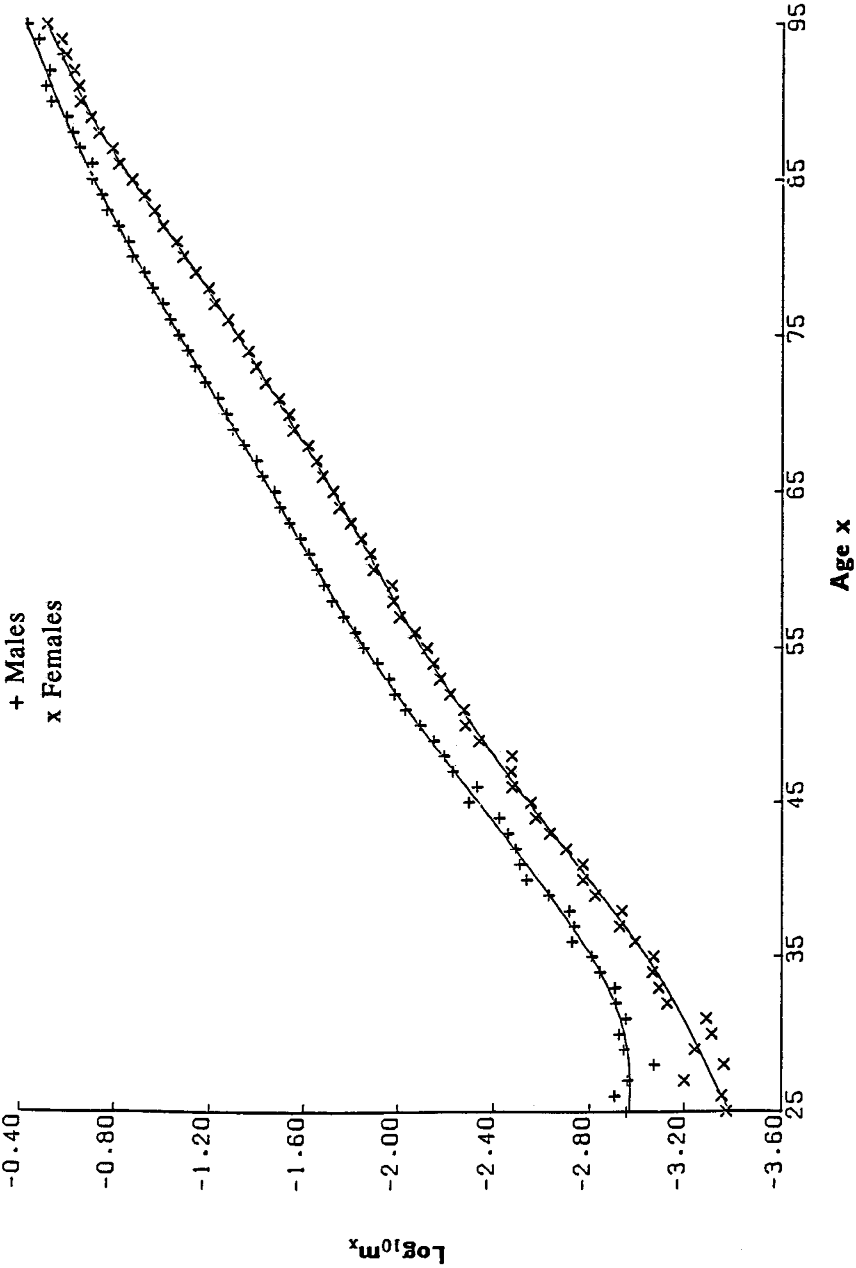


Figure 6.2: Scottish Life Tables 1980-1982

Figure 6.3: Scottish Life Tables 1980-1982

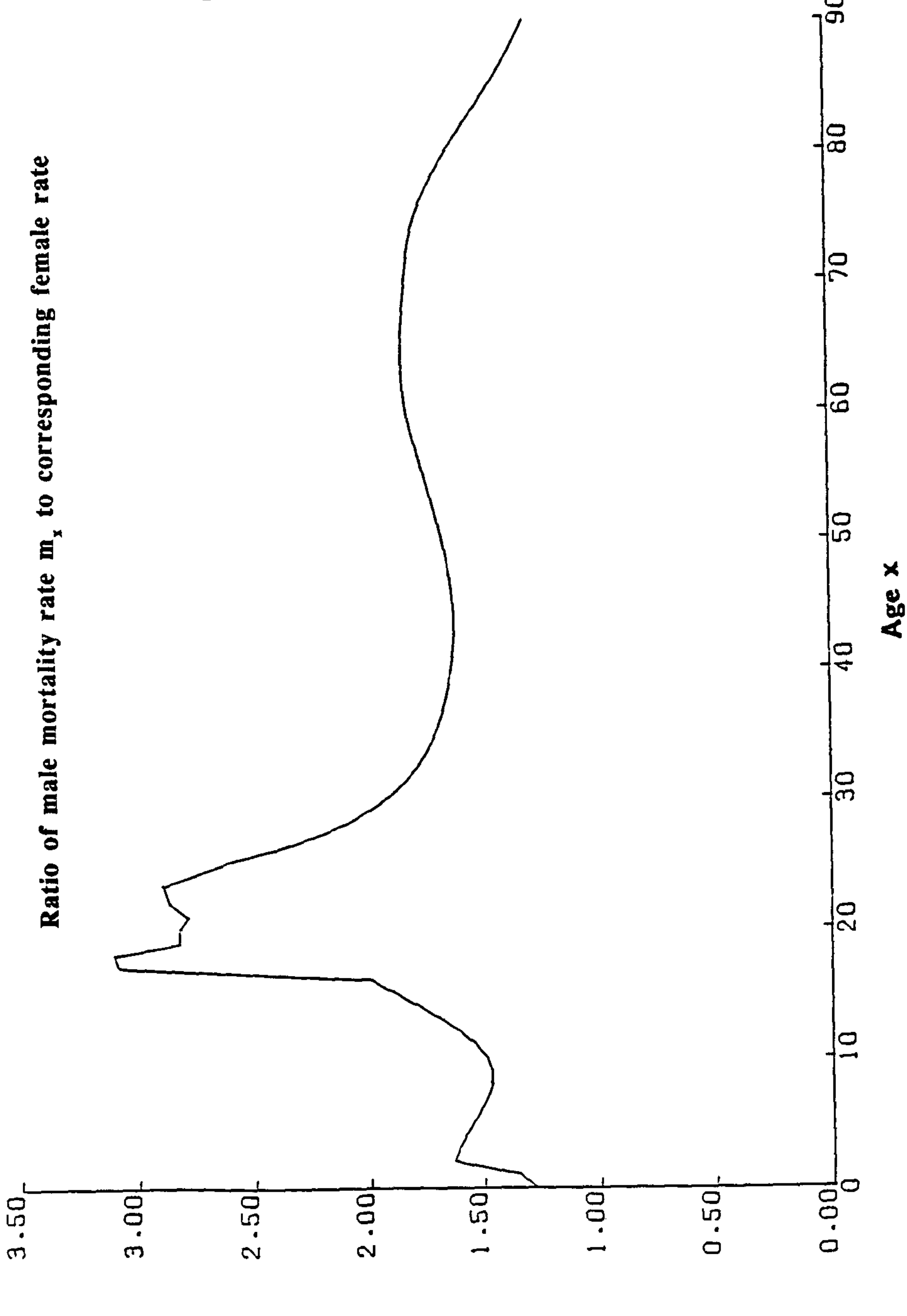


Table 6.1

Abridged form of the Scottish Life Tables 1980-1982

Males				Females			
x	ℓ_x	$10^5 q_x$	$\mathring{e}_x$	ℓ_x	$10^5 q_x$	$\mathring{e}_x$	x
0	100,000	1,291	69.14	100,000	1,016	75.32	0
1	98,709	93	69.04	98,984	69	75.09	1
5	98,473	38	65.20	98,826	24	71.20	5
10	98,318	28	60.30	98,722	19	66.28	10
15	98,154	49	55.40	98,621	26	61.34	15
20	97,729	114	50.63	98,460	40	56.44	20
25	97,186	108	45.90	98,269	41	51.54	25
30	96,662	113	41.13	98,034	59	46.66	30
35	96,052	156	36.38	97,692	91	41.81	35
40	95,137	260	31.70	97,131	159	37.04	40
45	93,577	461	27.18	96,144	285	32.39	45
50	90,860	814	22.91	94,430	489	27.93	50
55	86,315	1,379	18.98	91,624	792	23.70	55
60	79,274	2,209	15.43	87,347	1,220	19.73	60
65	69,293	3,429	12.27	81,159	1,883	16.04	65
70	56,136	5,329	9.54	72,331	2,971	12.67	70
75	40,348	8,195	7.27	60,212	4,724	9.69	75
80	24,197	12,247	5.48	44,657	7,731	7.17	80
85	11,187	17,611	4.12	27,071	12,618	5.20	85
90	3,630	23,956	3.14	11,860	19,168	3.85	90
95	771	31,018	2.38	3,505	25,221	2.93	95
100	91	41,099	1.72	661	34,212	2.07	100

ACKNOWLEDGEMENT

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UNDERWRITING MANUAL AND THE NUMERICAL RATING SYSTEM

The primary tool of the underwriter is an underwriting manual. This is a classification of all impairments arranged either by body system or alphabetically and, in accordance with the Numerical Rating System in general use, the manual will allot to each of these impairments a debit value reflecting the expected extra mortality. Such a manual can be placed in the hands of laymen and used to underwrite applications. All manuals have a basic table showing the expected mortality by build and probably all will have the expected mortality according to blood pressure findings. Specimen build and blood pressure tables are shown in the Appendix, as is also a specimen page from a manual covering numerical ratings for heart murmurs.

The Numerical Rating System, judging by the comments made both in the past and recently, still seems to be words " teeming with hidden meaning " for most British actuaries even though one of the co-authors of the system is a distinguished Fellow of the Faculty of Actuaries (see Rogers and Hunter, *T.A.S.A.* XX, p. 273.) The system is not a method of assessing extra premiums ; it is merely a convenient method of classification aimed at producing consistency in action on lives with similar prospects of longevity.

The Numerical Rating System was originally described as follows (Rogers and Hunter *loc. cit.*) :

> " The underlying principle in the numerical method of medical selection rests on the assumption that the average risk accepted by a company has a value of 100 per cent, and that each one of the factors which make up a risk shall be expressed numerically in terms of 100 per cent, and that, by the summation of them or by some modification of their summation, the value of any risk shall be determined and expressed with relation to that standard. . . . Wherever there is clear evidence that two factors are inter-dependent so that their addition is not sufficient or is distinctly too large, allowance is made for that inter-dependence."

The method assumes that the final result is consistent with good judgment. An accumulation of credits, for example, is not allowed to nullify a substantial and important debit. The method is not suitable nor necessary for impairments involving hazards which are essentially temporary.