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CONTINGENT ASSURANCE FUNCTIONS

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It is now many years since any functions suitable for the calculation of contingent assurance premiums have been published. (Indeed, when a complicated quotation involving more than two lives is required, a special calculation on a 'modern' basis can produce results which are embarrassingly inconsistent with simple two-life quotations based on a very much out-of-date published table!) The appended Tables 7 and 8 giving the functions $\ddot{a}_{mf}$ and $\bar{A}^1_{mf}$ are put forward, together with various adjustments detailed below, to meet this, admittedly small, need.

Basis

The main calculations have been done on the following basis:

Failing (or male) life: C.M.I. Assured Lives 1947/48, ultimate mortality (see *J.I.A.* 77, 117).

Counter (or female) life: *a*(55) ultimate mortality.

Interest: $2\frac{1}{2}\%$.

It was considered appropriate to use an up-to-date table for the failing life, rather than the A1924–29 table, particularly since many years are likely to elapse before any further calculations of this type are made. It is not suggested that office premiums should at present be based on these tables, as they stand. Approximations to other bases can, however, be made with sufficient accuracy.

Method of calculation

Values of $\bar{a}_{mf}$ and $\bar{A}^1_{mf}$ were calculated by using Simpson's approximate integration formula at intervals of three years. $\ddot{a}_{mf}$ was then calculated by the formula

$$\ddot{a}_{mf}=\bar{a}_{mf}+\tfrac{1}{2}+\tfrac{1}{12}(\mu_m+\mu_f+\delta).$$

Specimen values so obtained are compared in Table 1 with values calculated by the formulae

$$\ddot{a}_{mf}=\frac{1}{D_{mf}}\sum_{t=0}^{\omega} D_{m+t:f+t}$$

and

$$\bar{A}^1_{mf}=\frac{v^{\frac{1}{2}}}{D_{mf}}\sum_{t=0}^{\omega} D_{f+t}\,(1-\tfrac{1}{2}q_{f+t})\,d_{m+t}.$$

From these figures it is clear that the errors of the approximate integration method, even at extreme ages, are negligible. Some divergence between the values of $\bar{A}^1_{mf}$ at extreme ages of the failing life is only to be expected, partly because of the difficulty of calculating μ at these ages and partly because the assumptions implied in the two calculations are not identical.

Mortality

The *a*(55) female table has been used throughout for the counter life.

The A1924–29 standard table might be regarded as a suitable basis for the failing life in this particular class of policy at the present time. Tables 2 and 3 show the decrease per mille in the annuity value and the increase per

Table 1

Ages		$\ddot{a}_{mf}$		$\bar{A}^{1}_{mf}$	
Male	Female	Approx. integration	Formula	Approx. integration	Formula
21	21	27·277	27·277	·2120	·2120
39	39	21·178	21·178	·3122	·3122
57	57	13·102	13·102	·4455	·4454
75	75	5·678	5·678	·5549	·5545
84	84	3·377	3·375	·5714	·5698
90	90	2·468	2·444	·5676	·5595
21	63	14·915	14·915	·0266	·0266
33	75	9·406	9·405	·0222	·0222
42	84	5·946	5·946	·0233	·0234
48	90	4·256	4·254	·0255	·0256

mille in the assurance value respectively if C.M.I. 1947/48 mortality is replaced by A 1924–29 mortality.

It is clear that the annuity values can be obtained with great accuracy. The assurance function changes relatively rapidly, but interpolation to second differences gives satisfactory results.

Rate of interest

Any of the usual devices for approximating to values at other rates of interest could probably be used. Satisfactory results were obtained for annuity values by comparing with single-life annuity values based on the $a(55)$ female mortality table and for the assurance values by comparing with values of v^n where n was taken as three quarters of the lower complete expectation of life of the two lives involved.

Selection

It may well be thought suitable to use select mortality tables for both lives, since the effect of selection may be considerable in a class of assurance which is very sensitive to changes in mortality.

Table 4 sets out the increase per mille of the ultimate annuity value to give the select value. The select rates of mortality for the male life were calculated from the ultimate rates of the C.M.I. 1947/48 table by the same formulae as were used in the A 1924–29 table.

It was found that the function $(\bar{A}^{1}_{mf} - \bar{A}^{1}_{[m][f]})$ was nearly independent of the age f. Select values of the assurance function can therefore be obtained from Table 5.

Tests of approximations

To test the accuracy of the final result when these approximations are combined, various values have been calculated by these means on the basis of A 1924–29 standard mortality for the failing life, interest at 3 %, and select mortality for both lives. These values are shown in Table 6, together with the corresponding values calculated directly on this basis using Simpson's formula.

The ages chosen were intended to include some in which the approximations would be difficult because of the rapid change of value of $\bar{A}^{1}_{mf}$.

Table 2. Value of $1000\left(1-\frac{\ddot{a}_{mf}\,(1924/29)}{\ddot{a}_{mf}\,(1947/48)}\right)$

Age of male	Age of female																								Age of male
	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	84	87	90	
21	28	—	—	—	22	—	—	—	17	—	—	—	12	—	—	—	7	—	—	—	4	—	—	—	21
24	—	28	—	—	—	23	—	—	—	17	—	—	—	11	—	—	—	7	—	—	—	3	—	—	24
27	—	—	29	—	—	—	23	—	—	—	17	—	—	—	11	—	—	—	6	—	—	—	3	—	27
30	—	—	—	31	—	—	—	24	—	—	—	17	—	—	—	11	—	—	—	6	—	—	—	3	30
33	—	—	—	—	32	—	—	—	25	—	—	—	17	—	—	—	11	—	—	—	6	—	—	—	33
36	—	—	—	—	—	33	—	—	—	25	—	—	—	17	—	—	—	11	—	—	—	6	—	—	36
39	—	—	—	—	—	—	34	—	—	—	25	—	—	—	16	—	—	—	10	—	—	—	6	—	39
42	—	—	—	—	—	—	—	34	—	—	—	24	—	—	—	15	—	—	—	8	—	—	—	4	42
45	—	—	—	—	—	—	—	—	34	—	—	—	24	—	—	—	14	—	—	—	7	—	—	—	45
48	—	—	—	—	—	—	—	—	—	36	—	—	—	24	—	—	—	13	—	—	—	6	—	—	48
51	—	—	—	—	—	—	—	—	—	—	39	—	—	—	26	—	—	—	13	—	—	—	6	—	51
54	—	—	—	—	—	—	—	—	—	—	—	43	—	—	—	28	—	—	—	14	—	—	—	6	54
57	—	—	—	—	—	—	—	—	—	—	—	—	47	—	—	—	31	—	—	—	15	—	—	—	57
60	—	—	—	—	—	—	—	—	—	—	—	—	—	53	—	—	—	35	—	—	—	16	—	—	60
63	—	—	—	—	—	—	—	—	—	—	—	—	—	—	59	—	—	—	38	—	—	—	18	—	63
66	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	67	—	—	—	45	—	—	—	21	66
69	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	74	—	—	—	51	—	—	—	69
72	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	74	—	—	—	52	—	—	72
75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	64	—	—	—	43	—	75
78	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	53	—	—	—	32	78
81	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	49	—	—	—	81
84	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	45	—	—	84
87	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	34	—	87
90	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	32	90

Table 3. Value of $1000\left(\dfrac{\bar{A}^{1}_{mf}\,(1924/29)}{\bar{A}^{1}_{mf}\,(1947/48)}-1\right)$

Age of male	Age of female																								Age of male
	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	84	87	90	
21	150	—	—	—	230	—	—	—	340	—	414	—	492	—	560	—	589	—	—	—	575	—	—	—	21
24	—	144	—	—	—	223	—	—	—	329	—	404	—	489	—	562	—	603	—	—	—	583	—	—	24
27	—	—	139	—	—	—	216	—	—	—	321	—	394	—	481	—	563	—	620	—	—	—	606	—	27
30	—	—	—	134	—	—	—	209	—	—	—	309	—	381	—	471	—	566	—	641	—	—	—	656	30
33	—	—	—	—	129	—	—	—	201	—	—	—	295	—	365	—	453	—	554	—	647	—	—	—	33
36	—	—	—	—	—	122	—	—	—	191	—	—	—	275	—	337	—	420	—	520	—	629	—	—	36
39	—	—	—	—	—	—	115	—	—	—	179	—	—	—	251	—	303	—	374	—	460	—	556	—	39
42	—	—	—	—	—	—	—	108	—	—	—	166	—	—	—	221	—	258	—	306	—	369	—	429	42
45	—	—	—	—	—	—	—	—	101	—	—	—	155	—	—	—	197	—	219	—	247	—	279	—	45
48	—	—	—	—	—	—	—	—	—	97	—	—	—	148	—	—	—	182	—	194	—	210	—	227	48
51	—	—	—	—	—	—	—	—	—	—	95	—	—	—	145	—	—	—	174	—	182	—	192	—	51
54	—	—	—	—	—	—	—	—	—	—	—	93	—	—	—	144	—	—	—	170	—	173	—	177	54
57	—	—	—	—	—	—	—	—	—	—	—	—	93	—	—	—	146	—	—	—	173	—	177	—	57
60	—	—	—	—	—	—	—	—	—	—	—	—	—	93	—	—	—	149	—	—	—	177	—	178	60
63	—	—	—	—	—	—	—	—	—	—	—	—	—	—	93	—	—	—	152	—	—	—	184	—	63
66	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	95	—	—	—	162	—	—	—	209	66
69	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	95	—	—	—	170	—	—	—	69
72	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	86	—	—	—	160	—	—	72
75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	70	—	—	—	125	—	75
78	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	56	—	—	—	96	78
81	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	54	—	—	—	81
84	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	46	—	—	84
87	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	44	—	87
90	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	35	90

Table 4. $1000(\ddot{a}_{[m]\,[f]}/\ddot{a}_{mf}-1)$

Age of male	Age of female						
	21	33	45	57	69	75	81
21	1	2	2	3	9	15	25
33	—	2	2	4	9	15	25
45	—	—	3	5	10	17	26
57	—	—	—	10	15	22	31
69	—	—	—	—	31	36	45
75	—	—	—	—	—	53	60
81	—	—	—	—	—	—	89

Table 5

Age of male	$(\bar{A}^{1}_{mf}-\bar{A}^{1}_{[m]\,[f]})$
21	·0008
33	·0006
45	·0013
57	·0037
69	·0089
75	·0130
81	·0190

Table 6

Age		$\ddot{a}_{[m]\,[f]}$		$\bar{A}^{1}_{[m]\,[f]}$	
Male	Female	Correct	Approximation	Correct	Approximation
22	74	9·654	9·655	·0220	·0222
26	46	19·890	19·883	·1131	·1121
46	74	9·285	9·277	·0881	·0888
56	64	11·195	11·153	·3405	·3392
71	77	5·977	6·003	·4409	·4443

It is suggested that Table 6 shows that the values obtained by these methods of approximation are sufficiently accurate to be used for obtaining office premiums.

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Table 8. $\bar{A}^{1}_{mf}$ at $2\frac{1}{2}\%$

Age of male	Age of female						Age of male
	21	24	27	30	33	36	
21	·2120	·1954	·1774	·1585	·1397	·1214	21
24	—	·2255	·2075	·1880	·1675	·1471	24
27	—	—	·2401	·2206	·1994	·1773	27
30	—	—	—	·2561	·2350	·2120	30
33	—	—	—	—	·2733	·2505	33
36	—	—	—	—	—	·2922	36
	39	42	45	48	51	54	
21	·1044	·0889	·0752	·0633	·0531	·0445	21
24	·1274	·1089	·0922	·0774	·0645	·0535	24
27	·1552	·1338	·1139	·0958	·0798	·0659	27
30	·1881	·1642	·1411	·1196	·1001	·0828	30
33	·2257	·1999	·1740	·1491	·1259	·1049	33
36	·2675	·2407	·2129	·1850	·1581	·1331	36
39	·3122	·2856	·2567	·2267	·1967	·1678	39
42	—	·3335	·3049	·2738	·2415	·2091	42
45	—	—	·3555	·3247	·2912	·2564	45
48	—	—	—	·3779	·3447	·3087	48
51	—	—	—	—	·4005	·3647	51
54	—	—	—	—	—	·4233	54
	57	60	63	66	69	72	
21	·0374	·0315	·0266	·0226	·0192	·0163	21
24	·0443	·0366	·0303	·0251	·0209	·0174	24
27	·0541	·0442	·0360	·0293	·0238	·0194	27
30	·0678	·0551	·0445	·0357	·0285	·0228	30
33	·0863	·0702	·0565	·0451	·0358	·0282	33
36	·1104	·0904	·0731	·0585	·0463	·0364	36
39	·1408	·1164	·0950	·0765	·0608	·0479	39
42	·1780	·1490	·1229	·0999	·0802	·0636	42
45	·2216	·1882	·1571	·1292	·1047	·0838	45
48	·2712	·2338	·1980	·1648	·1350	·1090	48
51	·3259	·2856	·2455	·2071	·1718	·1402	51
54	·3848	·3430	·2998	·2569	·2161	·1786	54
57	·4455	·4041	·3593	·3131	·2674	·2241	57
60	—	·4669	·4224	·3745	·3252	·2769	60
63	—	—	·4867	·4391	·3881	·3359	63
66	—	—	—	·5047	·4539	·3997	66
69	—	—	—	—	·5210	·4672	69
72	—	—	—	—	—	·5377	72
	75	78	81	84	87	90	
21	·0139	·0118	·0099	·0083	·0069	·0057	21
24	·0145	·0121	·0101	·0084	·0069	·0057	24
27	·0158	·0129	·0106	·0087	·0071	·0058	27
30	·0182	·0145	·0116	·0093	·0075	·0061	30
33	·0222	·0174	·0136	·0107	·0084	·0067	33
36	·0284	·0221	·0171	·0132	·0103	·0080	36
39	·0374	·0291	·0224	·0173	·0133	·0103	39
42	·0500	·0389	·0302	·0233	·0180	·0140	42
45	·0663	·0520	·0405	·0315	·0244	·0190	45
48	·0870	·0686	·0537	·0419	·0326	·0255	48
51	·1129	·0898	·0707	·0554	·0432	·0338	51
54	·1454	·1168	·0928	·0732	·0575	·0452	54
57	·1847	·1500	·1204	·0958	·0757	·0599	57
60	·2313	·1902	·1542	·1239	·0988	·0787	60
63	·2849	·2373	·1946	·1578	·1270	·1019	63
66	·3446	·2912	·2418	·1980	·1606	·1297	66
69	·4101	·3525	·2970	·2460	·2013	·1635	69
72	·4812	·4218	·3622	·3053	·2535	·2084	72
75	·5549	·4968	·4361	·3757	·3186	·2668	75
78	—	·5675	·5079	·4464	·3861	·3297	78
81	—	—	·5698	·5093	·4480	·3891	81
84	—	—	—	·5714	·5104	·4497	84
87	—	—	—	—	·5676	·5082	87
90	—	—	—	—	—	·5676	90

Table 7. $\ddot{a}_{mf}$ at $2\frac{1}{2}\%$

Age of male	Age of female						Age of male
	21	24	27	30	33	36	
21	27·277	26·956	26·557	26·073	25·497	24·824	21
24	—	26·443	26·097	25·669	25·149	24·532	24
27	—	—	25·540	25·169	24·709	24·154	27
30	—	—	—	24·564	24·166	23·676	30
33	—	—	—	—	23·511	23·087	33
36	—	—	—	—	—	22·380	36
	39	42	45	48	51	54	
21	24·055	23·192	22·237	21·195	20·072	18·874	21
24	23·815	22·998	22·084	21·078	19·985	18·810	24
27	23·496	22·736	21·873	20·912	19·857	18·714	27
30	23·085	22·390	21·588	20·682	19·675	18·574	30
33	22·567	21·943	21·212	20·371	19·424	18·376	33
36	21·932	21·385	20·731	19·966	19·090	18·105	36
39	21·178	20·708	20·136	19·454	18·659	17·750	39
42	—	19·912	19·423	18·828	18·122	17·300	42
45	—	—	18·598	18·092	17·479	16·750	45
48	—	—	—	17·253	16·733	16·103	48
51	—	—	—	—	15·883	15·351	51
54	—	—	—	—	—	14·493	54
	57	60	63	66	69	72	
21	17·607	16·282	14·914	13·525	12·134	10·764	21
24	17·562	16·251	14·895	13·513	12·126	10·760	24
27	17·492	16·201	14·861	13·490	12·112	10·753	27
30	17·386	16·124	14·805	13·452	12·086	10·735	30
33	17·232	16·008	14·720	13·390	12·043	10·705	33
36	17·018	15·842	14·594	13·297	11·976	10·658	36
39	16·732	15·616	14·420	13·165	11·878	10·587	39
42	16·362	15·319	14·186	12·986	11·743	10·488	42
45	15·904	14·946	13·889	12·754	11·567	10·357	45
48	15·355	14·492	13·523	12·466	11·346	10·191	48
51	14·706	13·946	13·076	12·109	11·068	9·981	51
54	13·952	13·299	12·536	11·671	10·722	9·715	54
57	13·102	12·556	11·904	11·149	10·303	9·387	57
60	—	11·728	11·186	10·543	9·808	8·994	60
63	—	—	10·392	9·861	9·239	8·535	63
66	—	—	—	9·117	8·606	8·013	66
69	—	—	—	—	7·907	7·425	69
72	—	—	—	—	—	6·755	72
	75	78	81	84	87	90	
21	9·443	8·195	7·042	6·004	5·092	4·315	21
24	9·441	8·193	7·041	6·003	5·092	4·315	24
27	9·436	8·191	7·039	6·002	5·092	4·315	27
30	9·425	8·184	7·036	6·000	5·091	4·314	30
33	9·406	8·172	7·028	5·996	5·088	4·313	33
36	9·373	8·150	7·013	5·986	5·082	4·309	36
39	9·323	8·115	6·990	5·971	5·072	4·302	39
42	9·251	8·064	6·955	5·947	5·055	4·291	42
45	9·156	7·996	6·907	5·913	5·033	4·276	45
48	9·034	7·909	6·846	5·871	5·004	4·256	48
51	8·879	7·797	6·767	5·817	4·967	4·231	51
54	8·680	7·651	6·663	5·743	4·916	4·197	54
57	8·431	7·467	6·529	5·649	4·851	4·151	57
60	8·128	7·240	6·363	5·530	4·767	4·093	60
63	7·768	6·966	6·161	5·384	4·663	4·020	63
66	7·353	6·647	5·922	5·210	4·539	3·933	66
69	6·874	6·270	5·636	4·999	4·388	3·827	69
72	6·312	5·815	5·280	4·730	4·190	3·685	72
75	5·678	5·283	4·848	4·390	3·931	3·491	75
78	—	4·748	4·402	4·029	3·647	3·273	78
81	—	—	3·998	3·698	3·383	3·069	81
84	—	—	—	3·377	3·123	2·864	84
87	—	—	—	—	2·853	2·646	87
90	—	—	—	—	—	2·468	90