Correlation & Dependency Structures

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Why are we interested in correlation/dependency?

- Risk management
- Portfolio management
- Reinsurance purchase
- Pricing

How does dependency arise in the insurance world?

Related Events

- insurance cycle
- economic factors
- physical events
- social trends
- reinsurance failure

Items to be covered:

1. Simulations of correlated variables

2. Problems with the traditional approach

3. Copulas

Statement of problem

Suppose we have *n* classes of business with each class of business having its own marginal distribution.

How do we model the portfolio?

Traditionally this problem is tackled using correlation as the measure of dependence.

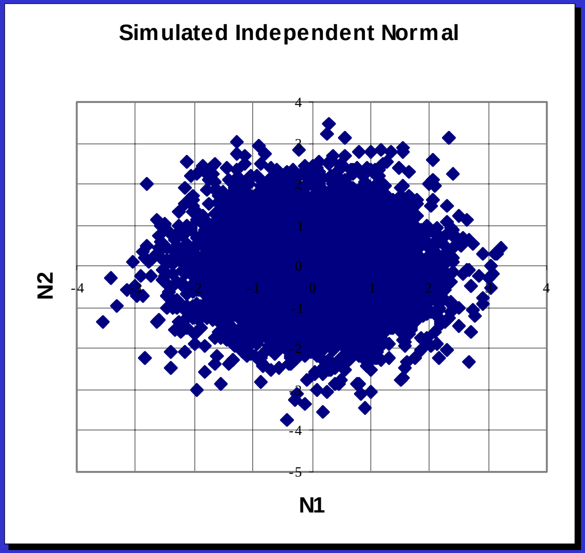
- The mean does not tell everything about a distribution.
- The coefficient of correlation does not tell everything about the dependency structure.

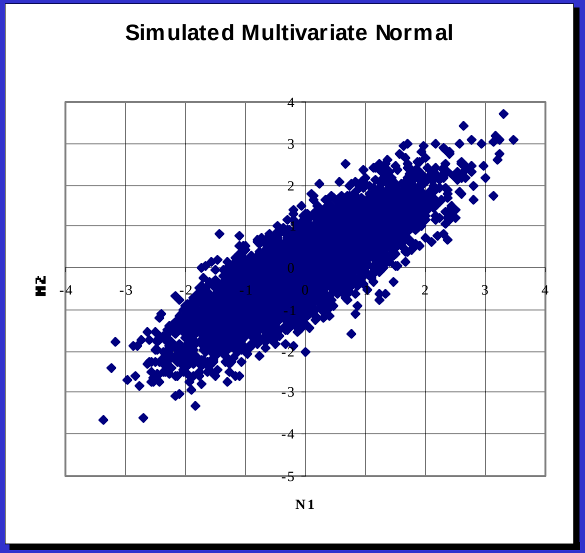
Standard Simulation Technique

Step 1

Simulate an n-dimensional multivariate normal distribution with correlation matrix ρ

Simulate Multivariate Normal Variables





Standard Simulation Technique

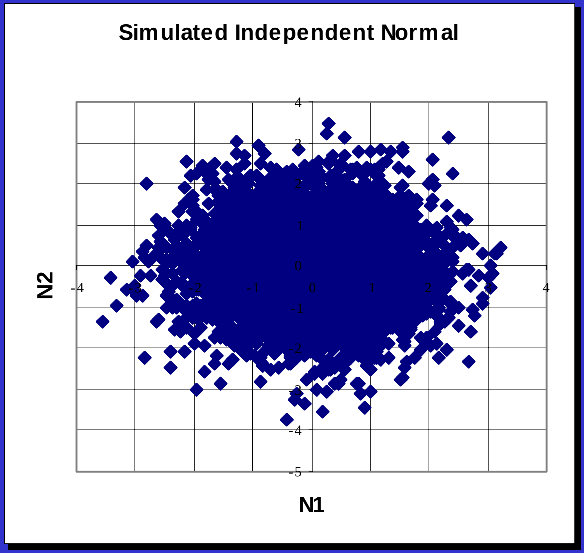
Step 2

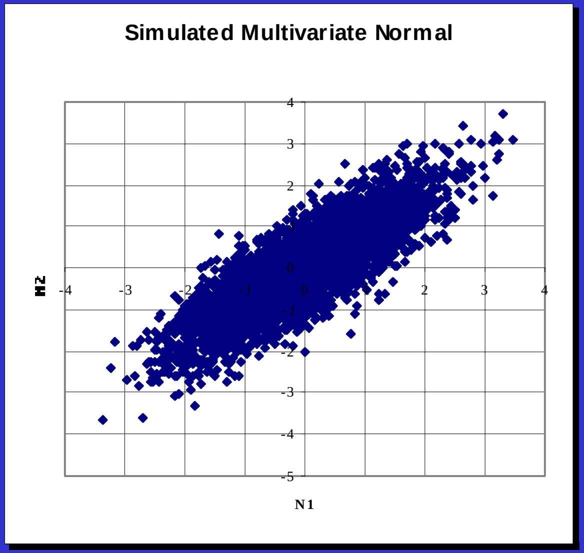
generate *n* series of the appropriate marginal distributions and put into a matrix

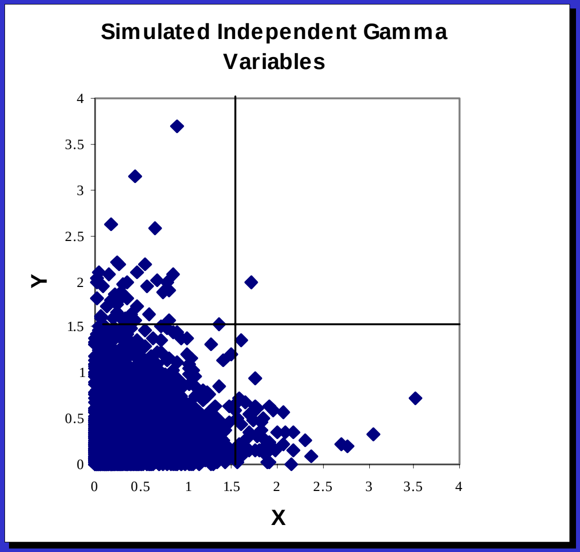
Step 3

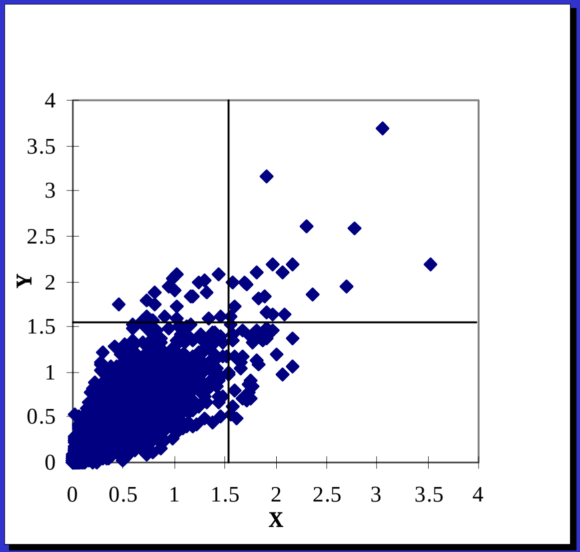
Apply Normal multivariate dependency structure from step 1

Standard Simulation Technique





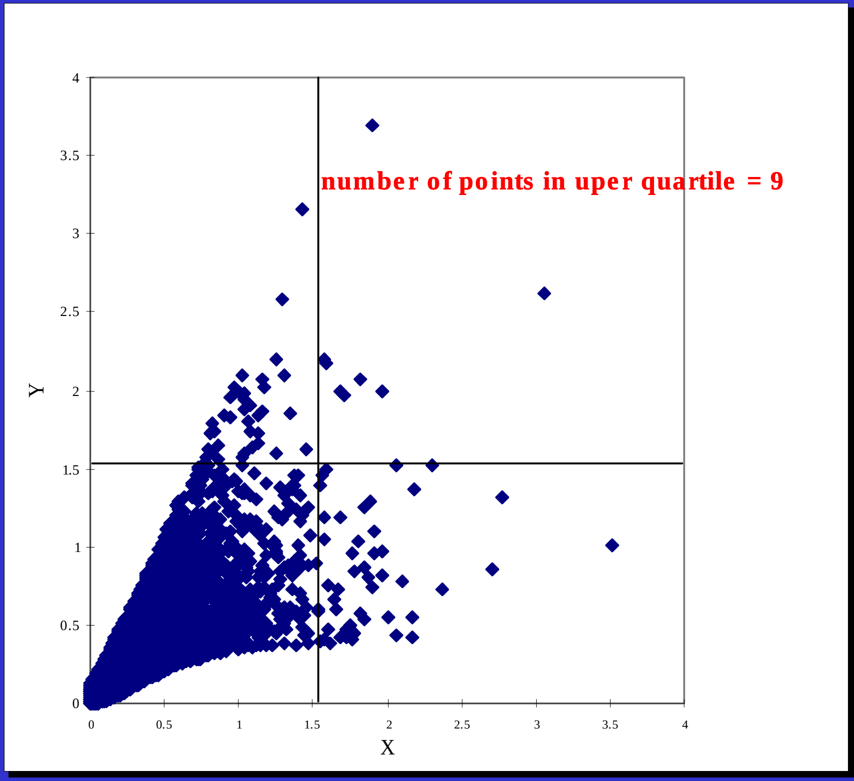


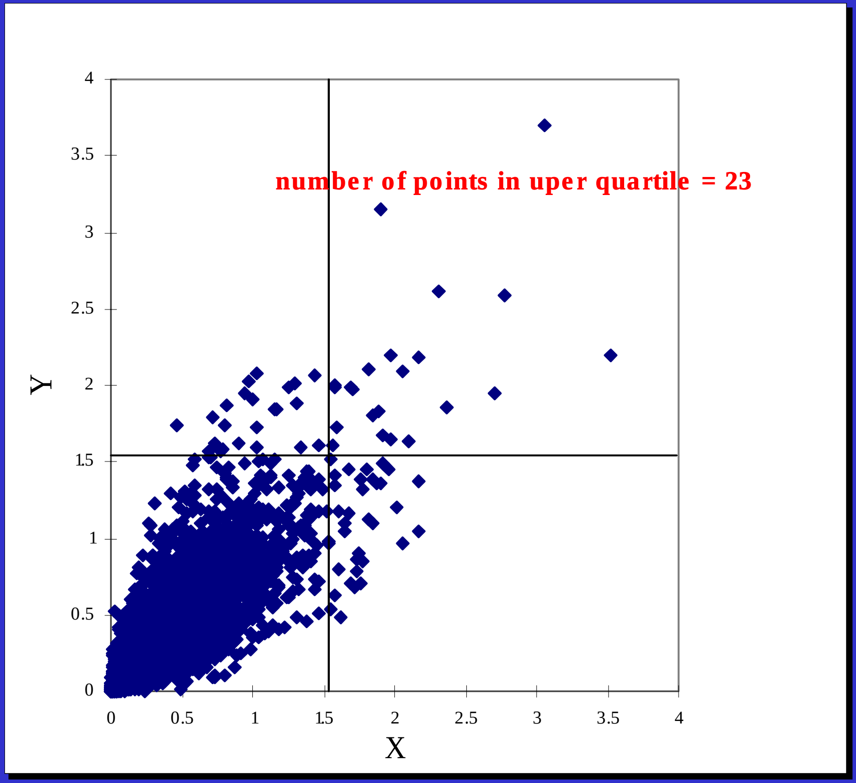


Observations

- The Spearman rank correlations are matched rather then the Pearson correlations
- The solution is not unique
- In particular use of normal distribution influences the tail of the modelled portfolio

The same Correlation, but Different Dependency Structures





Observation

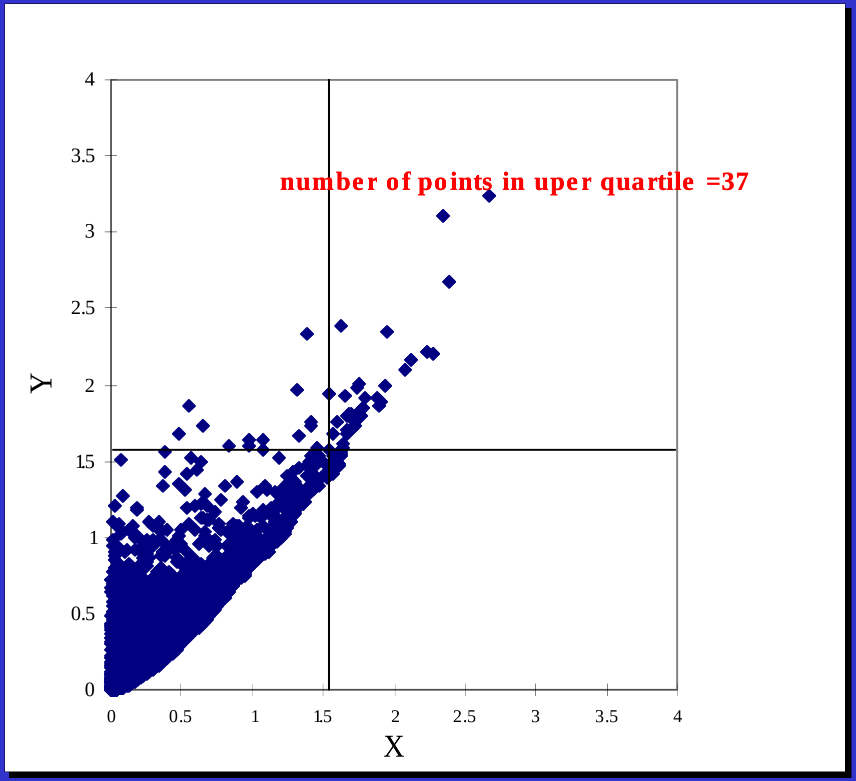
In the majority of applications a symmetric approach is used in determining the dependency.

However in practice this need not be the case.

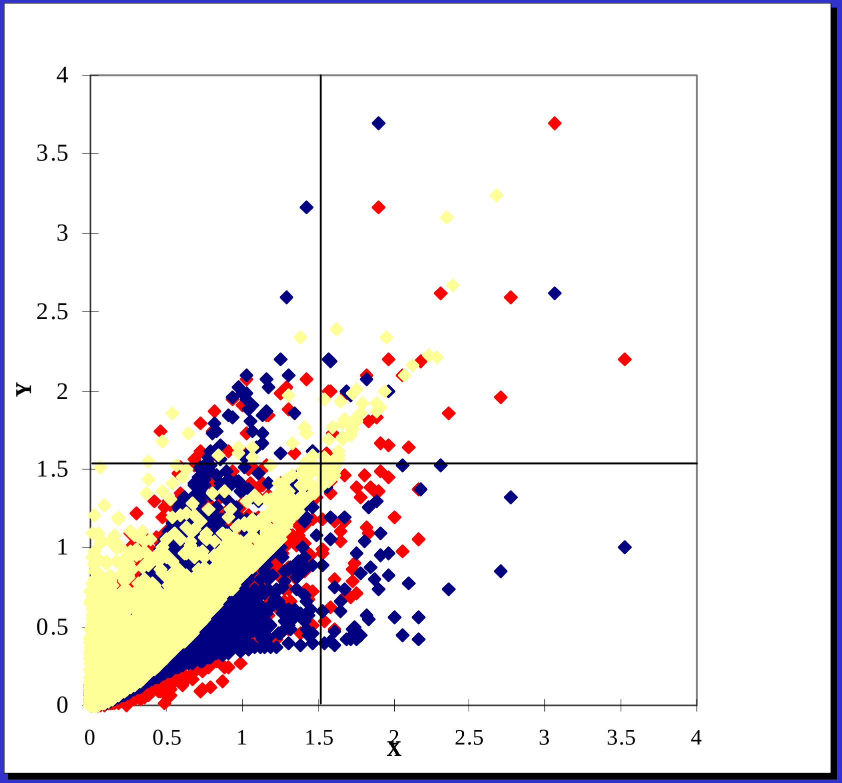
Example: An earthquake may cause a stockmarket crash but a stockmarket crash will not cause an earthquake.

Skewed distributions may be used to get around this problem.

Asymmetric Dependency Structure



The same Correlation, but Different Dependency Structures



Two Fallacies

Fallacy 1

Marginal distributions and correlations determine the joint distribution

Fallacy 2 (not for rank correlation)

F_1, F_2 marginal distribution for X_1, X_2

$\forall \rho \in [-1,1] \; \exists F$ such that F is the joint distribution and X_1, X_2 have Pearson correlation ρ

Other problems with Pearson correlation

Problem 1

- A correlation of zero does not indicate independence of risk.

Problem 2 (not for rank)

- Correlation is not invariant under transformations of the risk.

Problem 3

- Correlation is not an appropriate dependence measure for very heavily-tailed distributions.

Why dependency structures?

We need to amend our concept of dependency to allow for desirable features

- In particular we have to introduce non-linear dependency. For example:

$$X \sim \mathrm{u}[-1,1], \quad Y = X^2, \quad \rho = 0$$

- Need to reflect special features of tail dependence.
- In general, single numeric measures of dependency are insufficient.

The Copula approach

- Operates by separating marginal distributions from dependency structures
- Combination of copula and marginal will yield original distribution exactly
- No longer have the problems associated with correlation

Definition of Copula

For m-variate distribution F with j th univariant margin F_j the copula associated with F is a distribution function

$$\boxed{\begin{aligned} &C:[0,1]^m \to [0,1] \text{ that satisfies} \\ &F(\underline{X}) = C(F_1(X_1),..,F_m(X_m)) \end{aligned}}$$

(Note: if F is a continuous m-variate distribution the copula associated with F is unique)

This can also be represented via a density function

$$\boxed{c(u,v) = \frac{\partial C(u,v)}{\partial u\,\partial v}, \qquad 0 < u,v < 1}$$

Construction of a Copula

- Construction of a copula
  - parametric
  - non-parametric
- parametric form needed for higher dimension problem

Example of Parametric Copula (1)

Independent Copula

$$C(u,v) = uv, \ \ 0 \le u,v \le 1$$

$$c(u,v) = 1, \ \ 0 \le u,v \le 1$$

Example of Parametric Copula (2)

Normal copula

$$c(u,v,\rho)=(1-\rho^2)^{-\frac{1}{2}}\exp\{-\frac{1}{2}(1-\rho^2)^{-1}[x^2+y^2-2\rho xy]\}.\exp\{\frac{1}{2}[x^2+y^2]\}$$

where

$x=\Phi^{-1}(u),\ y=\Phi^{-1}(v)$ Φ is $N(0,1)$

and ρ is the correlation

Example of Parametric Copula (3)

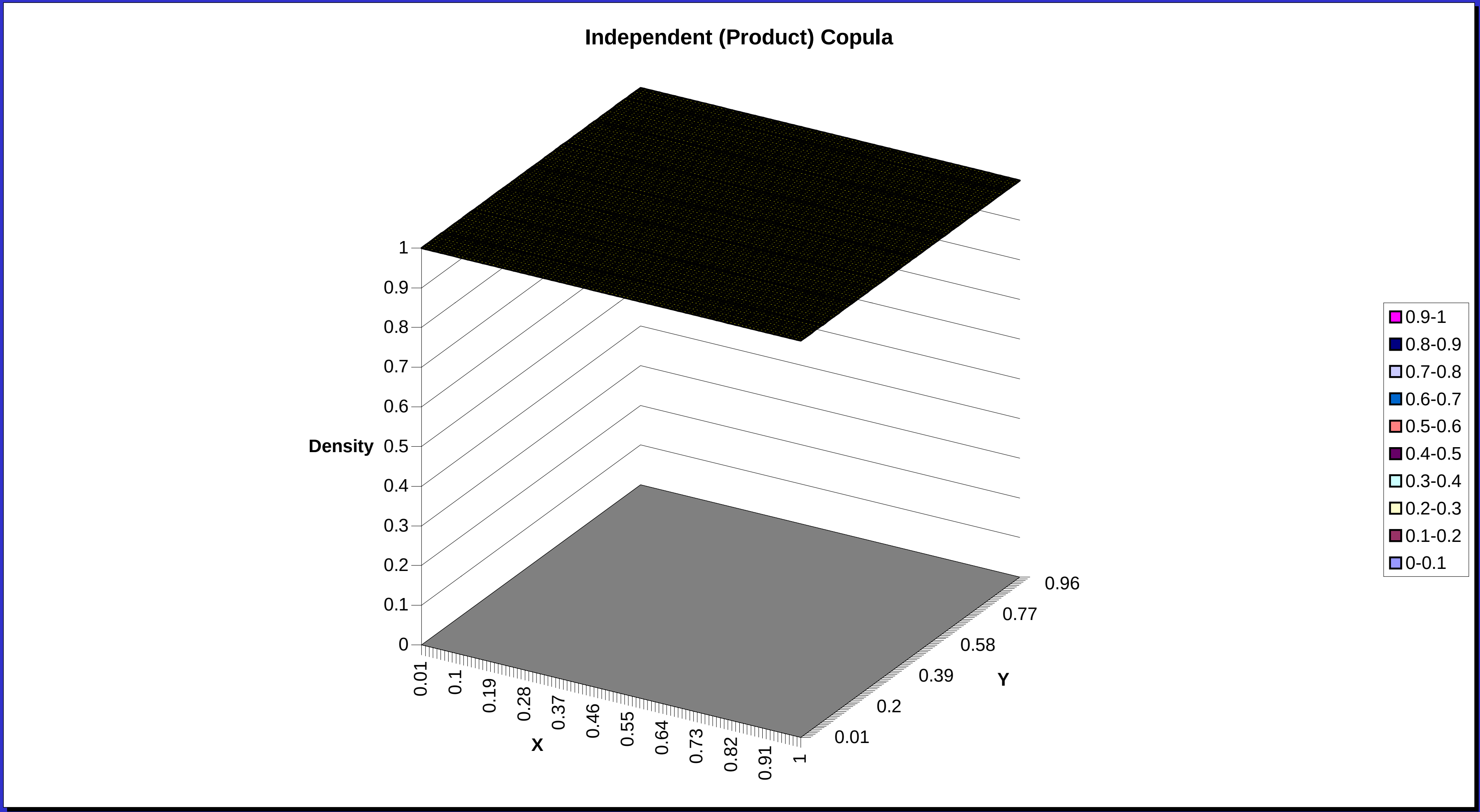
Gumbel Copula

For $1 \leq \delta < \infty$

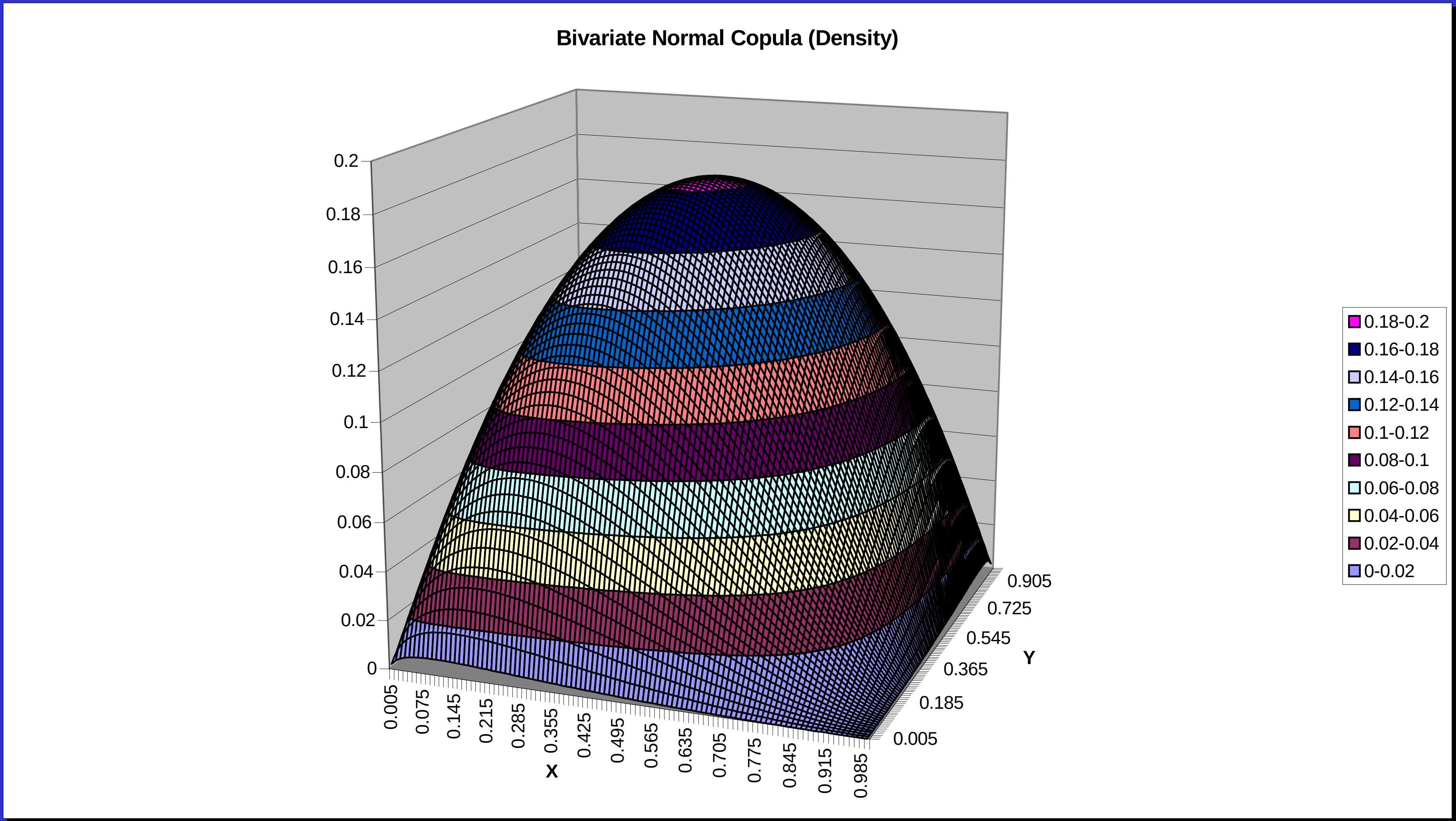
$$C(u,v;\delta) = \exp\left\{-\left((-\log u)^{\delta} + (-\log v)^{\delta}\right)^{1/\delta}\right\}, \quad 0 \leq u,v \leq 1$$

Note: this is an extreme value copula

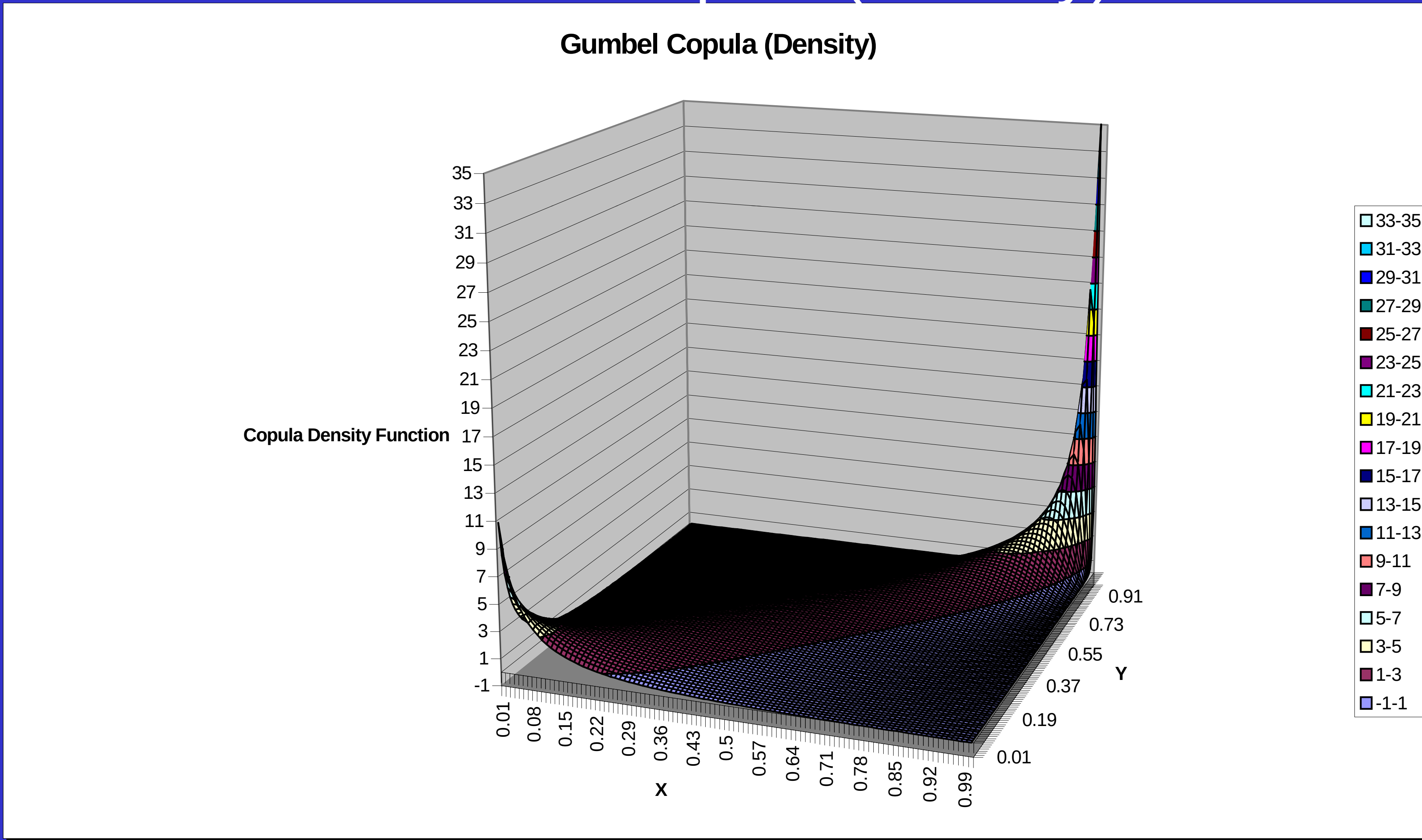
Independent (Product) Copula Density



Bivariate Normal Copula (Density)



Gumbel Copula (Density)



Simulation of a copula

- simulate a value u_1 from U (0,1)
- simulate a value u_2 from $C_2\ (u_2|u_1)$
- simulate a value u_n from $C_n\ (u_n|u_1 \ldots\ u_{n-1})$

where $C_i = C(u_1,\ldots,u_i,1,\ldots,1)$ for $i=2,\ldots,n$

Example: Reinsurance Pricing

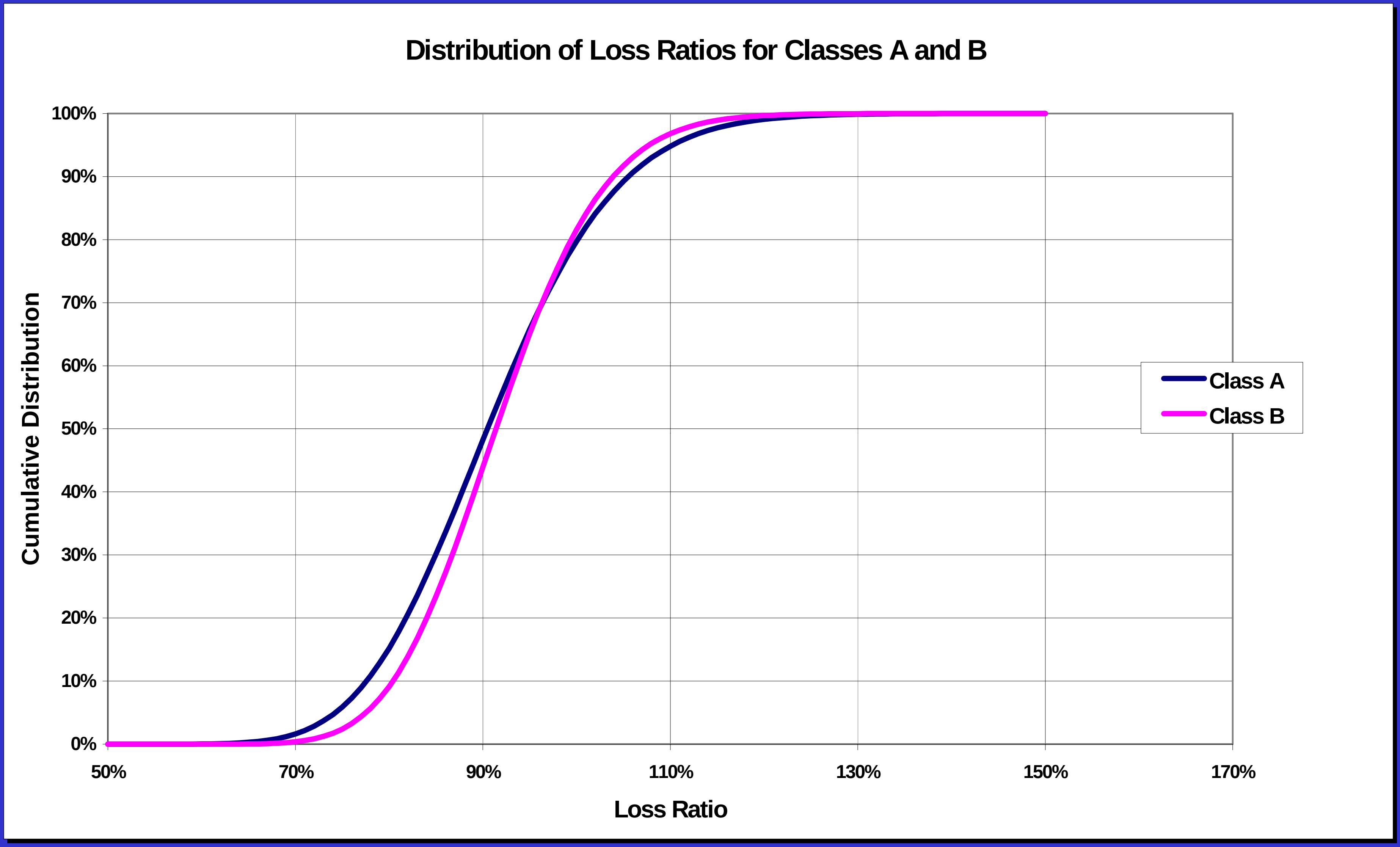
- Stop Loss on two classes of business
- Assumptions

	Class A	Class B
mean L/R	91.14%	91.85%
st. dev. of L/R	10.98%	9.21%
Premium	100	100

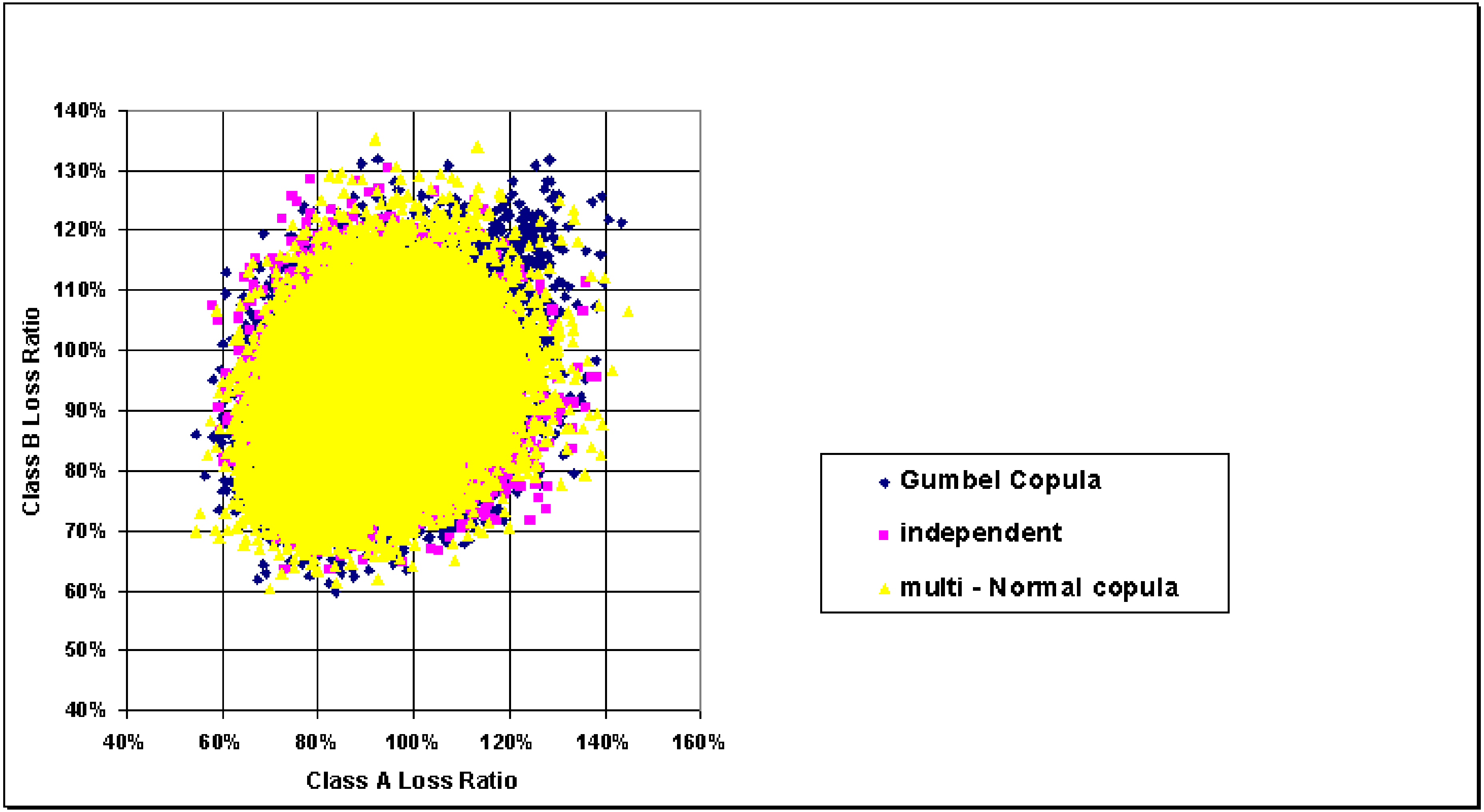
gumbel delta	1.2
rank correlation	0.25

- Limit: 15%
- Deductible: 107.5%
- Expected L/R: 91.5%

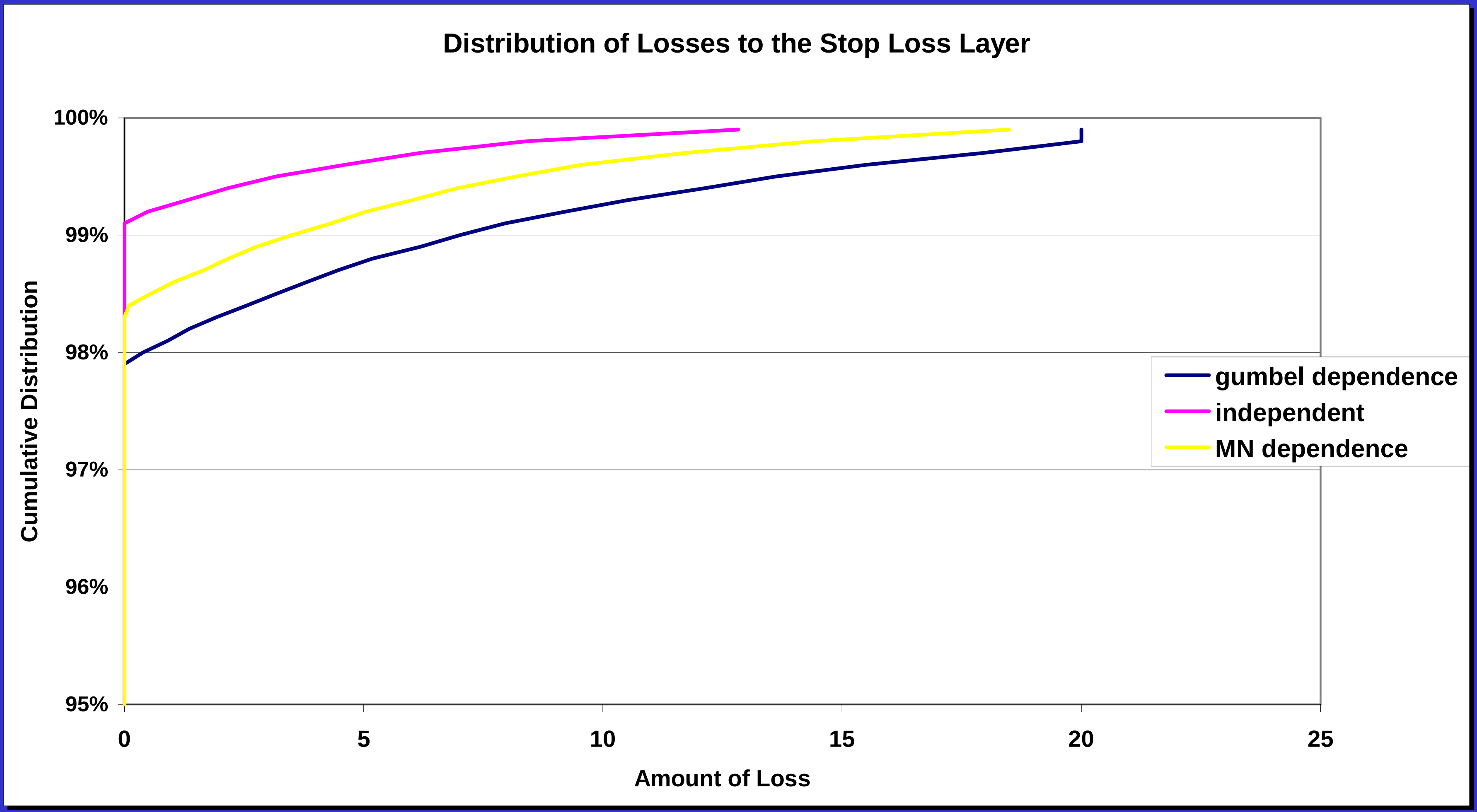
Loss Ratios are assumed to be lognormally distributed



The same rank correlation (ρ =0.25), but different dependency structures especially at the tail



Distribution of Losses to the Stop Loss Layer



Results

Expected amount of loss to the layer is underestimated approximately by 50%, if a multivariate Normal dependency structure is used.

Losses to the Stop Loss	Gumbel	Independent	Mult.Normal
mean	0.325	0.116	0.220
standard deviation	2.297	1.169	1.735
Rate on Line	1.1%	0.4%	0.7%

Conclusions

- Correlation is not a sufficient measure of dependence.
- Opportunities may be missed by remaining in the correlation framework.
- True dependency reflected in the copula approach by separating marginal distribution from dependency structures.