CORRESPONDENCE.

ON THE CALCULATION OF PREMIUMS RETURNABLE AT DEATH OR WITHDRAWAL.

To the Editor of the Assurance Magazine.

SIR,—On looking over the pages of the last Number of the *Journal* I was struck with a paragraph in a letter signed J. W. Stephenson, wherein the writer professes to give a method of finding the single premium for a certain contingent benefit, with the condition that the premium shall be returnable (without interest) at death, and also in the event of the purchaser wishing to withdraw at any time before the benefit becomes payable.

Now, the determination of the premium required for the assurance of a given benefit, with the return of the premium *at death*, is a very simple matter; the latter contingency being perfectly susceptible of calculation. But as the contingency of having to pay a given sum on *withdrawal* (other than the surrender value of the policy) is not so, I was not a little curious to see how such a problem would be dealt with. The particular benefit discussed by Mr. Stephenson is a deferred annuity, and the following is, substantially, the reasoning by which he arrives at his solution.

Let P_x denote the single premium required; and suppose A, the intending purchaser, deposits this amount at interest in the hands of B, to be held at A's disposal until the time arrives at which the annuity is wanted—say at the expiration of n years. Let the yearly rate of interest which B allows A on his deposit be i per £1, which must also be the rate of interest assumed in the calculation.

With this yearly interest, amounting to $P_x i$, A is enabled to assure a deferred annuity (with forfeiture of premiums in the event of death) of $P_x i \cdot \frac{N_x - N_{x+n}}{N_{x+n}}$ per annum; and at the expiration of the period of n years

* See p. 176.

he can withdraw his deposit from B's hands, and with it purchase an immediate annuity of $P_x \cdot \frac{D_{x+n}}{N_{x+n}}$. The whole annuity thus acquired is $P_x \cdot \frac{(N_x - N_{x+n})i + D_{x+n}}{N_{x+n}}$, and equating to unity, the amount to be deposited in order to secure an annuity of £1 is expressed by the equation $P_x = \frac{N_{x+n}}{(N_x - N_{x+n})i + D_{x+n}}$, which is Mr. Stephenson's formula.

By the arrangement here supposed it is evident that in the event of A's death, before the expiration of n years, his representatives will receive from B (at the end of the year of death) the sum of $P_x(1+i)$, or P_x with one year's interest upon it. Now, the object of this letter is to show that the return of this sum in the event of death, and the payment of the annuity in the event of survivance, are the only contingencies really provided for by the formula above deduced; or, in fact, that the ordinary method of valuation would have conducted Mr. Stephenson to precisely the same result as he has arrived at by his mode of solution.

The value of the deferred annuity alone is $\frac{N_{x+n}}{D_x}$, and the value of $P_x(1+i)$, payable in the event of death, is $P_x(1+i)\frac{M_x - M_{x+n}}{D_x}$. Therefore,

$$P_x = \frac{N_{x+n}}{D_x} + P_x(1+i)\frac{M_x - M_{x+n}}{D_x};$$

whence

$$P_x\left\{1 - \frac{(M_x - M_{x+n})(1+i)}{D_x}\right\} = \frac{N_{x+n}}{D_x},$$

and

$$P_x = \frac{N_{x+n}}{D_x - (M_x - M_{x+n})(1+i)}.$$

By substituting for $M_x(1+i)$ and $M_{x+n}(1+i)$ their equivalents $D_x - N_x i$ and $D_{x+n} - N_{x+n} i$, we have

$$P_x = \frac{N_{x+n}}{(N_x - N_{x+n})i + D_{x+n}},$$

which is the formula previously obtained.

This proves conclusively that the option of withdrawal does not enter in any way in the calculation of the premium; and indeed a little reflection will show that it cannot—for the sum which the policyholder is entitled to receive in the event of withdrawal does not admit of being fixed arbitrarily (as Mr. S. assumes), but can necessarily be no other than the surrender value of the policy (whatever it may be) determined according to the usual methods of calculation. No wonder then that (as Mr. Stephenson naively remarks "no method of deducing premiums returnable *at the option*, as well as on the death of a purchaser, has hitherto been published in any work on life annuities;" nor, it must be added, has Mr. Stephenson yet succeeded in supplying the omission.

It is true that, under the supposed arrangement between A and B, the former would have the option of withdrawing his deposit from B's hands at any time before the expiration of n years, and he would be entitled in

addition to an allowance from the Office for the surrender of the deferred annuity secured by the annual interest. This, however, merely shows, that in assurances of this description the value of the policy always exceeds the premium paid upon it—a circumstance which does not depend upon the mode of computing the premium, but arises from the nature of the contingency itself.

As it is P_x and not $P_x(1+i)$ that the representatives of A are to receive in the event of his death, the proper formula for the proposed benefit will be

$$P_x = \frac{N_{x+n}}{D_x-(M_x-M_{x+n})} = \frac{N_{x+n}}{(N_{x-1}-N_{x+n-1})(1-v)+D_{x+n}},$$

This formula may also be deduced by Mr. Stephenson's method, by supposing B to pay the interest at the beginning instead of the end of the year; the annual interest per £1 being in this case $\frac{i}{1+i}$, or $1-v$, instead of i.

Although Mr. Stephenson's claim to a solution of a new and impossible problem cannot be allowed, yet I think he is fairly entitled to the credit of having treated an old and perfectly practicable one in an original and striking manner.

I am, Sir,

Your very obedient servant,

London, 10*th May*, 1865. W. M. MAKEHAM.

THE D, N, &c., COLUMNS OF THE EQUITABLE EXPERIENCE.

(TABLE A, INTEREST 3 PER CENT.)

To the Editor of the Assurance Magazine.

SIR,—In looking over some of the early Numbers of the *Assurance Magazine*, I have found some tables in volume iii., page 366, constructed by the late Mr. Peter Hardy from the table of mortality known as the Equitable Experience; and as, in introducing these, you observe that space will be afforded to those contributors who may have authentic and original tables to offer, I am induced to send you the enclosed, in case you may consider any of them worthy of insertion.

The D, N, &c., columns have not, that I know of, appeared in print before.

The tables of annuities and assurance premiums will be found to vary, between the ages of about 85 to 93, from those of Mr. Hardy, who has not tabulated all the values between those ages quite correctly.

I am, Sir,

Your obedient servant,

London. W. MORGAN.

Preparatory Table for finding the Value of Annuities, &c., according to the Equitable Experience. (*Table A*, 3 *per Cent.*)

Age.	D.	N.	S.	M.	R.
10	3720·470	86696·263	1741952·276	1086·97296	37046·758017
11	3586·100	83110·163	1655256·013	1060·96579	35959·785059
12	3456·401	79653·762	1572145·850	1035·71611	34898·819271
13	3331·214	76322·548	1492492·088	1011·20186	33863·103163
14	3210·389	73112·159	1416169·540	987·40162	32851·901305
15	3093 775	70018·384	1343057·381	964·29459	31864·499687
16	2981·231	67037·153	1273038·997	941·86057	30900·205099
17	2872·618	64164·535	1206001·844	920·07998	29958·344531
18	2767·804	61396·731	1141837·309	898 93377	29038·264553
19	2666·657	58730·074	1080440·578	878·40347	28139·330785
20	2569·610	56160·464	1021710·504	859·02482	27260 927317
21	2476·490	53683·974	965550·039	840·74814	26401·902499
22	2387·137	51296·837	911866 065	823·52567	25561·154361
23	2300·888	48995·949	8(0569·228	806·80486	24737 628673
24	2217·638	46778·311	811573 279	790 57105	23930 823815
25	2137·285	44641·026	764794·968	774·81007	23140·252767
26	2059·269	42581·757	720153 942	759·04445	22365·442699
27	1983·984	40597·773	677572·185	743·73802	21606·398251
28	1911·337	38686·436	636974·412	728·87741	20862·660233
29	1841·239	36845·197	598287·976	714·44963	20133·782825
30	1773 603	35071·594	561442·779	700 44208	19419 333197
31	1707·945	33363 649	526371·185	686·44253	18718·891119
32	1644·608	31719·041	493007·536	672 85073	18032·448591
33	1583·133	30135·908	461288·494	659·27778	17359·597863
34	1523·479	28612·429	431152·586	645·73412	16700·320085
35	1465·601	27146 828	402540·157	632·22955	16054·585967
36	1409·803	25737·025	375393·329	619·11831	15422·356419
37	1355·676	24381 349	349656·304	606·05398	14803·238111
38	1303·181	23078·168	325274·955	593 04493	14197·184133
39	1251·963	21826·205	302196·787	579·78328	13604·139205
40	1202·316	20623·889	280370 582	566·60133	13024·355927
41	1154·499	19469·390	259746·693	553·80333	12457·754599
42	1108·159	18361·231	240277·303	541·08912	11903·951271
43	1063·539	17297·692	221916·072	528·74523	11362·862153
44	1020·577	16277·115	204618·380	516 76087	10834 116925
45	978·9519	15298·163	188341·264	504 86113	10317·356057
46	938·3722	14359·791	173043·101	492·79451	9812·494929
47	899·3256	13460·465	158683·310	481·07935	9319·700421
48	861·5158	12598·949	145222·845	469·46341	8838·621073
49	824·9104	11774·039	132623·895	457·95085	8369·157665
50	789·4787	10984·560	120849·856	446·54550	7911·206817
51	754·9682	10229·592	109865·296	435 02942	7464·661319
52	721·1532	9508·4391	99635·7035	423·20372	7029·631901
53	688·0410	8820·3981	90127·2644	411·09620	6606·428183
54	655·4355	8164·9626	81306 8663	398·53065	6195·331985
55	623·7520	7541·2106	73141·9037	385 93755	5796·801337
56	592·9761	6948·2345	65600·6931	373·32917	5410·863789
57	562·7218	6385·5127	58652·4586	360·34614	5037·534621
58	532·8267	5852·6860	52266 9460	346·84090	4677·188483
59	503·4963	5349 1897	46414·2600	333 02972	4330·347585
60	474·5739	4874·6158	41065·0703	318·77214	3997·317867
61	446·2498	4428·3660	36190·4545	304·27067	3678·545729
62	418·8531	4009·5129	31762·0885	289·87159	3374·275061
63	392·5185	3616·9944	27752·5757	275·73657	3084·403473
64	367·0610	3249·9334	24135·5813	261·71164	2808·666905
65	342·4606	2907·4728	20885 6479	247·80238	2546·955267
66	318·2711	2589·2017	17978·1751	233·58750	2299·152889
67	294·5103	2294·6914	15388·9734	219·09660	2065·565391

Preparatory Table for finding the Value of Annuities, &c. (continued).

Age.	D.	N.	S.	M.	R.
68	271·4615	2023·2299	13094·2820	204·62580	1846·468793
69	249·1152	1774·1146	11071·0522	190·18622	1641·842995
70	227·3352	1546·7795	9296·93748	175 66202	1451·656777
71	206·6127	1340·1668	7750·15800	161·56086	1275·994759
72	186·9044	1153·2624	6409·99122	147·87041	1114·433901
73	168·1689	985·0935	5256·72884	134·57871	966·563493
74	150·3662	834·72728	4271·63535	121 67415	831 984785
75	133·4579	701·26938	3436·90807	109 14545	710·310637
76	117·5128	583·75658	2735·63869	97·08743	601·165189
77	102 8967	480·85938	2151·88211	85 89408	504·077761
78	89·43116	391·42872	1671·02222	75·425548	418·183683
79	77·04997	314·37875	1279·59350	65 649109	342·758135
80	65·78398	248·59477	965·21475	56 627307	277·109026
81	55·38262	193·21215	716·61998	48·141997	220 481719
82	45·79711	147·41504	523·40782	40·169578	172 339722
83	37·15302	110·26202	375·99278	32·859378	132·170144
84	29·14061	81·12141	265·73076	25·929089	99·310766
85	22·37407	58·74734	184·60935	20·011309	73·381617
86	16·92143	41·82591	125·86201	15·210346	53·370368
87	12·60798	29 21793	84·036094	11·389746	38·16002265
88	9·124926	20·093006	54·818162	8·273917	26 77027700
89	6·410280	13·682726	34·725155	5 825047	18·49636035
90	4·685162	8·997564	21·042429	4·286636	12·67131370
91	3·326662	5·670902	12·044865	3·064597	8·38467805
92	2·306977	3·363925	6·3739623	2·141806	5·32008140
93	1·535852	1 828073	3 0100369	1·43787345	3·17827575
94	·9940790	·8339944	1·1819635	·94083405	1·74040230
95	·5328830	·2911114	·3479691	·51859185	·79956825
96	·2342537	·0568577	·0568577	·22577475	·28097640
97	·0568577	·0000000	·0000000	·05520165	·05520165

Table showing the Value of an Annuity upon a Single Life according to the Equitable Experience. (*Table A*, 3 *per Cent.*)

Age.	Annuity.	Age.	Annuity.	Age.	Annuity.	Age.	Annuity.
10	23·3025	32	19·2867	54	12·4573	76	4 9676
11	23·1756	33	19·0356	55	12·0901	77	4·6732
12	23·0453	34	18·7810	56	11·7176	78	4·3769
13	22·9113	35	18·5226	57	11·3475	79	4·0802
14	22·7736	36	18·2558	58	10·9842	80	3·7790
15	22·6320	37	17·9846	59	10·6241	81	3·4887
16	22·4864	38	17·7091	60	10·2716	82	3 2189
17	22·3366	39	17·4336	61	9·9235	83	2·9678
18	22·1825	40	17·1535	62	9·5726	84	2·7838
19	22·0238	41	16·8639	63	9 2148	85	2·6257
20	21·8556	42	16·5691	64	8·8539	86	2·4718
21	21·6774	43	16·2643	65	8·4899	87	2 3174
22	21·4883	44	15·9489	66	8 1352	88	2·2020
23	21·2944	45	15·6271	67	7·7915	89	2·1345
24	21·0938	46	15·3029	68	7·4531	90	1·9204
25	20·8868	47	14 9673	69	7·1217	91	1·7047
26	20·6781	48	14·6242	70	6·8040	92	1·4581
27	20·4628	49	14·2731	71	6·4864	93	1·1902
28	20·2405	50	13 9137	72	6·1703	94	·83896
29	20·0111	51	13·5497	73	5·8578	95	·53623
30	19·7742	52	13·1850	74	5·5513	96	·24272
31	19·5344	53	12·8196	75	5·2546		

Single Premiums for the Assurance of £1 upon a Single Life according to the Equitable Experience. (Table A, 3 per Cent.)

Age.	Single Premium.	Age.	Single Premium.	Age.	Single Premium.	Age.	Single Premium.
10	·29216	32	·40912	54	·60804	76	·82619
11	·29585	33	·41644	55	·61874	77	·83476
12	·29965	34	·42385	56	·62958	78	·84339
13	·30355	35	·43138	57	·64036	79	·85203
14	·30756	36	·43915	58	·65094	80	·86081
15	·31169	37	·44705	59	·66143	81	·86926
16	·31593	38	·45507	60	·67170	82	·87712
17	·32029	39	·46310	61	·68184	83	·88443
18	·32478	40	·47126	62	·69206	84	·88979
19	·32940	41	·47969	63	·70248	85	·89440
20	·33430	42	·48828	64	·71299	86	·89888
21	·33949	43	·49716	65	·72359	87	·90338
22	·34498	44	·50634	66	·73393	88	·90674
23	·35065	45	·51572	67	·74394	89	·90870
24	·35649	46	·52516	68	·75379	90	·91494
25	·36252	47	·53493	69	·76345	91	·92122
26	·36860	48	·54493	70	·77270	92	·92840
27	·37487	49	·55515	71	·78195	93	·93621
28	·38134	50	·56562	72	·79116	94	·94644
29	·38803	51	·57622	73	·80026	95	·95526
30	·39493	52	·58684	74	·80919	96	·96380
31	·40191	53	·59749	75	·81783	97	·97087

Annual Premiums for the Assurance of £1 upon a Single Life according to the Equitable Experience. (Table A, 3 per Cent.)

Age.	Annual Premium.	Age.	Annual Premium.	Age.	Annual Premium.	Age.	Annual Premium.
10	·012022	32	·020167	54	·045183	76	·138445
11	·012238	33	·020785	55	·047268	77	·147140
12	·012462	34	·021427	56	·049505	78	·156856
13	·012695	35	·022096	57	·051862	79	·167717
14	·012937	36	·022806	58	·054317	80	·180125
15	·013189	37	·023548	59	·056902	81	·193656
16	·013452	38	·024324	60	·059593	82	·207904
17	·013725	39	·025123	61	·062419	83	·222904
18	·014010	40	·025960	62	·065458	84	·235159
19	·014307	41	·026853	63	·068771	85	·246684
20	·014627	42	·027792	64	·072356	86	·258911
21	·014970	43	·028797	65	·076248	87	·272313
22	·015340	44	·029875	66	·080340	88	·283179
23	·015728	45	·031017	67	·084619	89	·289904
24	·016135	46	·032213	68	·089174	90	·313288
25	·016563	47	·033502	69	·094001	91	·340603
26	·017003	48	·034877	70	·099014	92	·377683
27	·017466	49	·036348	71	·104450	93	·427439
28	·017954	50	·037926	72	·110337	94	·514659
29	·018468	51	·039604	73	·116694	95	·621817
30	·019010	52	·041371	74	·123515	96	·775561
31	·019573	53	·043235	75	·130756	97	·970873