

**CURRENCY RISK MODELS IN INSURANCE: A
MATHEMATICAL PERSPECTIVE**

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ABSTRACT. In this paper we consider a general two-country model of exchange rate dynamics in which interest rates are stochastic. Special cases of this model are also illustrated in which the interest rates are constant in each country and when interest rates are Gaussian in each country. Uncertainty in this global economy consists of the exchange rate between the two countries and the noise in interest rates in each local economy. This model provides an important tool for actuaries dealing with global risk which includes valuation of currency derivatives (forwards, futures, options and swaps), frequency/severity claims model with exchange rate risk in the claim size (i.e. the claims are paid in a foreign currency), valuation of individual insurance contracts written in foreign currency, and valuation of general interest sensitive claims in foreign currency. Typical examples of practical importance include marine insurance, health insurance, and life insurance. The simple case of the model for which interest rates are constant in each local economy is used to explain and analyse the "Siegel paradox" [32] (or lack thereof) which we describe because it has confused some readers of the 1970 era currency risk literature. We illustrate the model with the calculation of forward prices, currency options, a simple life insurance contract, and a marine insurance policy.

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1. THE BLACK-SCHOLES CURRENCY MODEL

1.1. Introduction. In this section we consider a simple two-country model in which interest rates are fixed. The only source of uncertainty is the exchange rate between the two countries. The model has been described elsewhere. Baxter and Rennie [1] give a good introduction. Garman and Kohlhagen [13] develop foreign currency option values for this model. Our aim is to write a self-contained account for actuaries which includes valuation of derivatives (forwards, futures, options and swaps) as well as valuation of insurance products with embedded currency risk. We show how to estimate model parameters from observations. The model explains the Siegel paradox [32] (or lack thereof); we describe this briefly because it has confused some readers of the 1970 era currency risk literature.

We assume that the reader is familiar with the Black/Scholes model for European options on a stock as described in Hull [16], Musiela and Rutkowski [20] or Cox and Rubinstein [10].

1.2. Model Description. Suppose the two currencies involved are US dollars and British pounds (also called sterling). These are merely convenient labels – we could call the currencies 1 and 2. There is nothing special (as far as the model is concerned) about the particular countries except that there is a well-developed dollar – pound currency market. The Black/Scholes model for stock options specifies two basic securities: a default free constant interest rate bond and a risky stock. All investors have the same information, skills, access to the markets, etc. Moreover trading takes place continuously and there are no transactions costs or taxes. Finally, there are no arbitrage opportunities in the market for the securities and their derivatives. We assume the same here, but for emphasis (and because this confuses some students) we say again there is only one type of investor in this international market. All traders have access to all securities whether they are denominated in dollars or pounds. In this model the terms “dollar investor” and a “pound investor” merely refer to the denomination of the security – dollars or pounds – that the investor is considering. There are no distinctions between domestic and foreign investors as far as the model is concerned.

The dollar bond is denoted $B(t) = e^{rt}$ where r is the continuously compounded default free interest rate for dollars. One dollar invested at time t is worth $B(T)/B(t)$ at time T . The sterling bond is denoted $D(t) = e^{ut}$ where u is the continuously compounded default free rate for pounds. The dollar value of 1 pound at time t is denoted $C(t)$ and we assume that $C(t)$ is a geometric Brownian motion. There are two

equivalent mathematical descriptions of $C(t)$. The first is to write

$$(1) \quad C(t) = C_0 \exp((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W(t))$$

where $\{W(t)|t \geq 0\}$ is a standard Brownian motion. The second is to require that $C(t)$ satisfies a stochastic differential equation

$$(2) \quad dC(t) = \mu C(t) dt + \sigma C(t) dW(t)$$

subject to an initial condition $C(t) = C_0$. In each description the parameters μ and σ are given constants, called the drift and volatility respectively. The two descriptions are equivalent; this follows from Itô's formula [22, page 37]. Baxter and Rennie discuss this at a less rigorous level [1, page 60].

We assume that everyone uses the same Brownian motion

$$\{W(t)|t \geq 0\}$$

and the same parameters σ and μ . There are two dollar denominated investments: $B(t)$ – the dollar bond and $C(t)D(t)$ – the dollar value of the sterling bond. Let $S(t) = C(t)D(t)$ denote the dollar value of the sterling bond. Similarly, there are two sterling denominated investments, $D(t)$ – the pound bond and $B(t)/C(t)$ – the pound value of the dollar bond. Let $P(t) = B(t)/C(t)$. The dollar value of the pound bond can be written in terms of the Brownian motion

$$(3) \quad S(t) = C_0 \exp((u + \mu - \tfrac{1}{2}\sigma^2)t + \sigma W(t))$$

or as a differential equation:

$$dS(t) = (u + \mu) S(t) dt + \sigma S(t) dW(t).$$

1.3. Brownian Motion. Here briefly are some of the facts about Brownian motion that we will use. The details are in Øksendal [22] and Karatzas and Shreve [17]. We can think of Brownian motion as the space $\mathcal{C}$ of all continuous functions $W : [0, T] \rightarrow R$ equipped with a probability measure P satisfying conditions (i) – (iv) below. An element W of $\mathcal{C}$ is called a sample path, or realization of the Brownian motion $\mathcal{C}$. An abuse of the notation is common – often we say W is a Brownian motion when in fact W is a sample path. Similarly, we use shorthand notation such as $W(t) \leq w$ to denote the set of sample paths $W \in \mathcal{C}$ for which $W(t) \leq w$. The σ -algebra $\mathcal{F}$ of subsets of $\mathcal{C}$ generated by sets of the form $W(t) \leq w$, for various t and w , are assumed to be P -measurable. For a given t , the sub-algebra generated by $W(t) \leq w$, for various w , is denoted by $\mathcal{F}_t$. These are the defining conditions for Brownian motion:

- (i) $W(t)$ is a continuous function of t for all t , $0 \leq t \leq T$ and $W(0) = 0$.

(ii) For all w and $t, 0 \leq t \leq T$,

$$P(W(t) \leq w) = \int_{-\infty}^w \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{u^2}{2t}\right) du.$$

This suggests another interpretation: $W(t)$ is a random variable with a normal distribution having mean zero and variance t .

(iii) For all $t > 0$,

$$P(W(t+s) - W(s) \leq w) = \int_{-\infty}^w \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{u^2}{2t}\right) du.$$

We can interpret this to mean that $W(s+t) - W(s)$ is a normal random variable with mean 0 and variance t .

(iv) For all $v, w, s \geq u > 0$ and $t > 0$, the events $W(t+s) - W(s) \leq w$ and $W(u) \leq v$ are P -independent. This means that the increment $W(t+s) - W(t)$ is independent of values the process takes at earlier times. Another way of saying this is that $W(t+s) - W(t)$ and $\mathcal{F}_u$ are P -independent for $u \leq t$ and $s > 0$.

Two measures P and Q on $\mathcal{C}$ are said to be equivalent if, for all $A \in \mathcal{F}$,

$$P(A) = 0 \text{ if and only if } Q(A) = 0.$$

A very useful and elegant theorem, called the Cameron-Martin-Girsanov Theorem, provides equivalent Brownian motions, i.e., equivalent measures on $\mathcal{C}$. We can make an equivalent measure by changing the drift. Let γ be a given constant and define $\widetilde{W}(t) = W(t) + \gamma t$ for $t \leq T$. This defines a transformation of the space $\mathcal{C}$ onto itself. The theorem says there is an equivalent measure Q for which the transformed sample paths satisfy the conditions (i) - (iv) with P replaced by Q . Moreover, the Q -expectation of an $\mathcal{F}_T$ -measurable random variable Y_T can be calculated in terms of the P -measure:

$$(4) \quad E_Q[Y_T] = E_P[Y_T \frac{dQ}{dP}]$$

where

$$\frac{dQ}{dP} = \exp(-\gamma W(T) - \tfrac{1}{2}\gamma^2 T).$$

The random variable

$$\frac{dQ}{dP}$$

is called the Radon-Nikodym derivative of Q with respect to P . There is a converse to the theorem: If Q is equivalent to P , then there is a

positive $\mathcal{F}$ -measurable function $\phi : \mathcal{F} \rightarrow R$ for which

$$Q(A) = \int_A \phi dP$$

for all $A \in \mathcal{F}$. When ϕ is given, then we can identify the Radon-Nikodym derivative – it is just ϕ – and when ϕ has the explicit form $\phi = \exp(-\gamma W(T) - \frac{1}{2}\gamma^2 T)$ then the process $\widetilde{W}(t) = W(t) + \gamma t$ is a Q -Brownian motion. Moreover, in this case Q -expectations can be calculated by the formula:

$$E_Q[Y_T | \mathcal{F}_t] = \frac{1}{E_P[\phi | \mathcal{F}_t]} E_P[\phi Y_T | \mathcal{F}_t]$$

Because $\{\exp(-\gamma W(t) - \frac{1}{2}\gamma^2 t) : t \geq 0\}$ is a P -martingale, this simplifies to

$$E_Q[Y_T | \mathcal{F}_t] = E_P[\exp(-\gamma(W(T) - W(t)) - \frac{1}{2}\gamma^2(T - t)) Y_T | \mathcal{F}_t]$$

We illustrate the change of measure with the dollar value of the pound bond $S(t)$. Using the Brownian motion representation (3) we can write

$$S(t) = S(0) \exp((u + \mu - \frac{1}{2}\sigma^2)t + \sigma W(T)).$$

Now substitute $W(t) = \widetilde{W}(t) - \gamma t$ where $\gamma = \sigma^{-1}(u + \mu - r)$. The new representation is

$$S(t) = S(0) \exp((r - \frac{1}{2}\sigma^2)t + \sigma \widetilde{W}(t))$$

which turns out to be convenient for calculations. For example, relative to the new measure Q we have

$$E_Q[S(T) | \mathcal{F}_t] = S(t) e^{r(T-t)} E_Q[\exp(-\frac{1}{2}\sigma^2(T-t) + \sigma(\widetilde{W}(T) - \widetilde{W}(t)))].$$

Since $\{\widetilde{W}(s) | s \geq 0\}$ is Brownian motion relative to Q , then using the moment generating function¹ of the normal distribution we obtain

$$E_Q[\exp(-\frac{1}{2}\sigma^2(T-t) + \sigma(\widetilde{W}(T) - \widetilde{W}(t)))] = 1.$$

Therefore, relative to the Q measure the expected time T value of 1 dollar invested in the pound bond at time t is

$$E_Q[S(T)/S(t) | \mathcal{F}_t] = e^{r(T-t)}$$

at time t . The P expectation is different.

¹Recall that for a normal random variable X , the moment generating function of X is

$$E[\exp(sX)] = \exp(sE[X] + \frac{1}{2}s^2\text{Var}[X]).$$

We frequently run into a another form with ϕ given by

$$\phi = \exp(\sigma\widetilde{W}(T) - \tfrac{1}{2}\sigma^2T)$$

where $\{\widetilde{W}(t) : t \geq 0\}$ is a Q -Brownian motion. In other words, we sometimes change measures twice in the same calculation. First we switch from the real-world (or physical) measure P to an equivalent pricing measure Q . Then we switch to another measure Q_T which is convenient for calculations regarding securities maturing at time T . This second change of measure formally is just substituting one Brownian motion for another and one parameter for another. In this case, the formula relating the expectations takes this form, letting Q_T denote the new measure:

$$E_{Q_T}[Y_T|\mathcal{F}_t] = E_Q[\exp(\sigma(\widetilde{W}(T) - \widetilde{W}(t)) - \tfrac{1}{2}\sigma^2(T-t))Y_T|\mathcal{F}_t]$$

The corresponding transformation of the Brownian motion is given by $\widetilde{\widetilde{W}}(t) = \widetilde{W}(t) - \sigma t$.

This is a good place to introduce a separation theorem – an application of the converse to the CMG theorem. The Q -expectation of the product $S(T)Y_T$ can be written as the product of the Q -expectation of $S(T)$ and the Q_T -expectation of Y_T where Q_T is the equivalent measure corresponding to $\phi = \exp(\sigma\widetilde{W}(T) - \tfrac{1}{2}\sigma^2T)$. Here is how this goes: First write $S(T)|\mathcal{F}_t$ as

$$\begin{aligned} S(T) &= S(0)e^{rT} \exp(\sigma\widetilde{W}(T) - \tfrac{1}{2}\sigma^2T) \\ &= S(t)e^{r(T-t)} \exp(\sigma(\widetilde{W}(T) - \widetilde{W}(t)) - \tfrac{1}{2}\sigma^2(T-t)) \end{aligned}$$

and then substitute in the expectation:

$$\begin{aligned} E_Q[S(T)Y_T|\mathcal{F}_t] &= \frac{1}{E_Q[\phi|\mathcal{F}_t]} E_Q[S(t) \exp(r(T-t))\phi Y_T|\mathcal{F}_t] \\ &= S(t) \exp(r(T-t)) E_{Q_T}[Y_T|\mathcal{F}_t] \end{aligned}$$

This is very useful because it allows us to calculate the Q_T -expectation by merely adjusting the drift of the Brownian motion. Here is a summary of the changes of measure in terms of the three representations of the dollar value of the sterling bond $S(T)$, conditional on $\mathcal{F}_t$:

$$\begin{aligned} S(T) &= S(t) \exp((u + \mu - \tfrac{1}{2}\sigma^2)(T-t) + \sigma(W(T) - W(t))) \\ (5) \quad &= S(t) \exp((r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma(\widetilde{W}(T) - \widetilde{W}(t))) \\ &= S(t) \exp((r + \tfrac{1}{2}\sigma^2)(T-t) + \sigma(\widetilde{\widetilde{W}}(T) - \widetilde{\widetilde{W}}(t))) \end{aligned}$$

In each representation $S(T)|\mathcal{F}_t$ is lognormal. The volatility parameter σ is the same for each representation. The drift parameters for the representations are, respectively,

$$u + \mu, r \text{ and } r + \sigma^2.$$

Apply this to calculate the conditional expectations:

$$\begin{aligned} E_P[S(T)|\mathcal{F}_t] &= S(t)e^{(u+\mu)(T-t)} \\ E_Q[S(T)|\mathcal{F}_t] &= S(t)e^{r(T-t)} \\ E_{Q_T}[S(T)|\mathcal{F}_t] &= S(t)e^{(r+\sigma^2)(T-t)} \end{aligned}$$

The Q -measure is sometimes called the dollar risk-neutral measure because an investment of 1 dollar in $S(t)$ is expected to accumulate to $e^{r(T-t)}$ using the Q measure. The equivalent differential equation representations of S are as follows:

$$\begin{aligned} (6) \quad dS(t) &= (u + \mu)S(t)dt + \sigma S(t)dW(t) && \text{(physical)} \\ &= rS(t)dt + \sigma S(t)d\widetilde{W}(t) && \text{(dollar risk-neutral)} \\ &= (r + \sigma^2)S(t)dt + \sigma S(t)d\widetilde{\widetilde{W}}(t) && \text{(T-maturity)} \end{aligned}$$

Similarly, the three Brownian motion representations of the exchange rate, conditional on $\mathcal{F}_t$, are as follows:

$$\begin{aligned} (7) \quad C(T) &= C(t) \exp((\mu - \tfrac{1}{2}\sigma^2)(T-t) + \sigma(W(T) - W(t))) \\ &= C(t) \exp((r - u - \tfrac{1}{2}\sigma^2)(T-t) + \sigma(\widetilde{W}(T) - \widetilde{W}(t))) \\ &= C(t) \exp((r - u + \tfrac{1}{2}\sigma^2)(T-t) + \sigma(\widetilde{\widetilde{W}}(T) - \widetilde{\widetilde{W}}(t))) \end{aligned}$$

We can work with the pound denominated investments rather than dollar denominated ones. Here is how turns out. The pound value of a dollar is $C(t)^{-1}$. Using the parameterization given by equation (1), we have

$$C(t)^{-1} = C_0^{-1} \exp(-(\mu - \tfrac{1}{2}\sigma^2)t - \sigma W(t))$$

which we re-write with parameters $\mu' = -\mu + \sigma^2$ and $\sigma' = -\sigma$. In the alternative representation, the dollar value of a pound is

$$\begin{aligned} C(t)^{-1} &= C_0^{-1} \exp((\mu' - \tfrac{1}{2}(\sigma')^2)t + \sigma' W(t)) \\ &= C_0^{-1} \exp((- \mu + \tfrac{1}{2}\sigma^2)t - \sigma W(t)) \end{aligned}$$

which is the same form as (1). Now we can simply write down from (6) the pound risk neutral representation of the pound value of a dollar

$B(t)C(t)^{-1}$, which we denote by $P(t)$:

(8)

$$\begin{aligned}
 dP(t) &= (\tau + \mu')P(t)dt + \sigma'P(t)dW(t) && \text{(physical)} \\
 &= (\tau - \mu + \sigma^2)P(t)dt - \sigma P(t)dW(t) \\
 &= uP(t)dt + \sigma'P(t)d\widehat{W}(t) && \text{(pound risk-neutral)} \\
 &= uP(t)dt - \sigma P(t)d\widehat{W}(t) \\
 &= (u + (\sigma')^2)P(t)dt + \sigma'P(t)d\widehat{\widehat{W}}(t) && (T\text{-maturity}) \\
 &= (u + \sigma^2)P(t)dt - \sigma P(t)d\widehat{\widehat{W}}(t)
 \end{aligned}$$

The pound risk neutral measure is obtained by the transformation $\widehat{W}(t) = W(t) + \gamma't$ where

$$\gamma' = (\sigma')^{-1}(\tau + \mu' - u) = \sigma^{-1}(u - \tau + \mu - \sigma^2).$$

We note that the two risk neutral measures generally have different expectations. This is the basis of the so-called Siegel paradox, which we discuss later. When it is necessary to distinguish the two risk neutral measures we use $Q\$$ for the dollar risk neutral measure is denoted and $Q\mathcal{L}$ for the pound risk neutral measure. When no confusion can arise, Q denotes the dollar risk neutral measure.

The CMG theorem and its converse simplify calculations. A process $\{f(\widehat{W}(t)) | t \geq 0\}$ where $f(w)$ is a given function can be made into a Q -martingale by changing from P to Q . This means that its drift is zero relative to Q . We will illustrate this shortly. We use the converse when we can identify the variable ϕ . We illustrate this below also.

Now we move on to analysis of derivatives in the Black/Scholes currency model.

2. DERIVATIVES

We discuss three derivative investments or securities in detail: forward currency contracts, European currency options, and currency swaps. We briefly discuss futures contracts and American options. The currency markets such as the Philadelphia Stock Exchange have European style options (as well as American style options). In addition the currency markets provide both exchange traded futures contracts as well as forward contracts arranged through financial intermediaries such as banks and insurance companies. We have real examples of published prices which can be used to calibrate the model.

2.1. Derivative Security Pricing. Prices which avoid arbitrage opportunities can be calculated as discounted expected values relative to an equivalent measure. Suppose we are working with dollar investments. For convenience, the basic discounted dollar securities $B(t)/B(t) = 1$ and $S(t)/B(t)$ should be martingales (i.e. have zero drift) relative to the new measure. The discounted dollar bond has zero drift for any measure, so we take a look at the other process

$$Z(t) = S(t)/B(t) = C_0 \exp((\mu + u - r - \frac{1}{2}\sigma^2)t + \sigma W(t)).$$

We see that if we define γ as

$$\gamma = \sigma^{-1}(\mu + u - r)$$

then the equivalent measure Q defined by the transformation

$$\begin{aligned} \widetilde{W}(t) &= W(t) + \gamma t \\ (9) \qquad &= W(t) + (\sigma^{-1}(\mu + u - r)) t \end{aligned}$$

makes $Z(t)$ a martingale. Just substituting for $W(t)$ gives the representation in terms of $\widetilde{W}(t)$:

$$\begin{aligned} Z(t) &= C_0 \exp((\mu + u - r - \frac{1}{2}\sigma^2)t + \sigma W(t)) \\ (10) \qquad &= C_0 \exp(-\frac{1}{2}\sigma^2 t + \sigma \widetilde{W}(t)) \end{aligned}$$

From this we see that $Z(t)$ is a Q -martingale. Let $f : [0, +\infty) \rightarrow R$ be a given function and consider a contract calling for a single payment of $f(S(T))$ at time T . Let $E(t) = E_Q[f(S(T))/B(T)|\mathcal{F}_t]$. Then $E(t)$ is also a Q -martingale and by the martingale representation theorem has the form $E(t) = E(0) + \int_0^t \phi(s) dZ(s)$ or $dE = \phi(t) dZ(t)$. Now following the Black/Scholes procedure, we make a portfolio of $\phi(t)$ dollars invested in $S(t)$ and $\psi(t) = E(t) - \phi(t)Z(t)$ invested in the dollar bond. The value of the portfolio is $V(t) = \phi(t)S(t) + \psi(t)B(t) = \phi(t)S(t) + E(t)B(t) - \phi(t)Z(t)B(t) = E(t)B(t)$. The portfolio replicates the derivative

$$V(T) = E(T)B(T) = f(S(T))$$

and is self-financing since

$$\begin{aligned} dV(t) &= d[(B(t)E(t))] = B(t)dE(t) + E(t)dB(t) \\ &= \phi(t)B(t)dZ(t) + (\psi(t) + \phi(t)E(t))dB(t) \\ &= \phi(t)d[B(t)Z(t)] + \psi(t)dB(t) \\ (11) \qquad &= \phi(t)dS(t) + \psi(t)dB(t). \end{aligned}$$

The self-financing equation shows that changes in the portfolio value are due entirely to changes in the market value of the two dollar investments and does not require dollars from other sources. Therefore,

to avoid arbitrage, the price of the portfolio at t is given by

$$(12) \quad V(t) = B(t)E(t) = e^{-r(T-t)}\mathbb{E}_Q[f(S(T)|\mathcal{F}_t].$$

If an investor pays some other price for the derivative, she or he provides an arbitrage profit (or obtains one) to (or from) another investor. The Q -measure is a convenience now that we see the pricing implications. We could always rewrite the expectation in terms of the original distribution if we preferred it. This price has nothing to do with our beliefs about the expected (relative to P) values of future exchange rates. Even if $\mu + u > r$ so that we expect $S(t)$ to perform better than $B(t)$ we would not price derivatives differently.

2.2. Forward Prices. A forward contract is an agreement made at time t to buy one pound at time T . The forward price $F(t, T)$ is determined at time t so that the contract cannot provide an arbitrage opportunity. Consider two investments. First, look at the forward market, note the price $F(t, T)$, and invest just enough in the dollar bond to accumulate to the forward price at time T . That is, invest $F(t, T)e^{-r(T-t)}$ in dollar bonds. At time T , pay $F(t, T)$ dollars for 1 pound. Therefore the present value at time t of a pound paid at time T is $F(t, T)e^{-r(T-t)}$. On the other hand the present value at t in pounds of 1 pound paid at time T is $e^{-u(T-t)}$, which is worth $e^{-u(T-t)}C(t)$ dollars. The two present values have to be equal or there is an arbitrage. Therefore,

$$F(t, T)e^{-r(T-t)} = e^{-u(T-t)}C(t)$$

and

$$(13) \quad F(t, T) = e^{(r-u)(T-t)}C(t).$$

The relation comes from the arbitrage property. The probability measures, P and Q , are not involved. However, we can see from the dollar risk-neutral representation (7) of the spot exchange rate $C(T)$ that the Q expectation of $C(T)$ is equal to $F(t, T)$:

$$(14) \quad F(t, T) = \mathbb{E}_{Q\$}[C(T)|\mathcal{F}_t].$$

The analog for the forward price of 1 dollar $G(t, T)$ is obtained by interchanging currencies. The result is

$$(15) \quad G(t, T) = e^{(u-r)(T-t)}C(t)^{-1} = F(t, T)^{-1}.$$

Now consider a forward contract opened at $t = 0$ to buy 1 pound at $t = T$. Let $F = F(0, T) = e^{(r-u)T}C_0$. What is the value of the buyer's position at time t ? The payoff function is $f(c) = c - F$ since

at time T the buyer pays the agreed price F and gets a pound worth $C(T) = S(T)/D(T) = e^{-uT}S(T)$. Hence the value at time t is

$$\begin{aligned}
 V(t) &= e^{-r(T-t)}E_Q[f(C(T))|\mathcal{F}_t] \\
 &= e^{-r(T-t)}E_Q[C(T) - F|\mathcal{F}_t] \\
 &= e^{-r(T-t)}(C(t)e^{(r-u)(T-t)} - F) \\
 &= C(t)e^{-u(T-t)} - Fe^{-r(T-t)} \\
 (16) \quad &= (F(t, T) - F)e^{-r(T-t)}
 \end{aligned}$$

The buyer agreed at time 0 to pay F for a pound at time T . If the market price of a pound delivered at time T has increased ($F(t, T) > F$), then the buyer's forward contract has a gain, a positive value. It could be sold for the price given by (16) without creating an arbitrage. On the other hand, if the price declines ($F(t, T) < F$), then the buyer would have to pay another person $-V(t)$ to assume the buyer's role. If the buyer holds the forward contract to maturity, then the gain (or loss) $F(T, T) - F = C(T) - F$ is realized at time T . Now we take a brief look at a closely related contract - currency futures.

2.3. Futures. A futures contract, as far as we are concerned here, differs from a forward contract only in that the contract requires daily *marking-to-market*. The gains or losses relative to the current futures price are realized daily rather than at time T . After a gain or loss is realized the futures contract is reset to the current futures price. Let $F_m(t, T)$ denote the futures market price² in dollars at time t for 1 pound delivered at time T . It turns out that for models with a deterministic dollar bond, the forward and futures prices of pounds are equal. This is a result of no arbitrage pricing. We illustrate this now for a simple futures contract opened at time t , requiring delivery of 1 pound at time T , and requiring marking - to - market at a single time s , $t < s < T$. For $t > s$, the contract is identical to a forward contract and so forward and futures prices are identical after time s . Consider the buyer's position at $t < s$. The buyer and sell consider at time t two future cash flows:

$$\begin{aligned}
 &F_m(s, T) - F_m(t) \text{ at time } s \text{ and} \\
 &F_m(T, T) - F_m(s, T) \text{ at time } T.
 \end{aligned}$$

Since the futures contract essentially becomes a forward contract at time s , then $F_m(s, T) = F(s, t)$ and $F_m(T, T) = F(T, T) = C(T)$.

²The subscript m suggests the mark-to-market distinguishing feature of futures.

With no arbitrage pricing, the futures price is set at time t so that the dollar risk neutral expected value of the future cash flow is zero:

$$\begin{aligned} E_Q [e^{-r(s-t)}(F(s, T) - F_m(t, T)) | \mathcal{F}_t] + \\ E_Q [e^{-r(T-t)}(C(T) - F(s, T)) | \mathcal{F}_t] = 0 \end{aligned}$$

The forward price $F(s, T)$ is set at time s so that $F(s, T) = E_Q[C(T) | \mathcal{F}_s]$ according to (??). Therefore, when the expectations are calculated at time t , $F(s, T)$ and $C(T)$ have the same expected value so the second term is zero. Hence, we find that $F_m(t, T) = E_Q[F(s, T) | \mathcal{F}_t]$. Since forward prices are expected spot prices (relative to Q), then $\{F(t, T) : t \geq 0\}$ is a Q -martingale and, therefore, $E_Q[F(s, T) | \mathcal{F}_t] = F(t, T)$. From these relations we conclude that

$$F_m(t, T) = E_Q[F(s, T) | \mathcal{F}_t] = F(t, T)$$

or in other words the futures contract and the forward contract always set the same prices. The equivalence of forward and futures prices (when the interest rate is deterministic) holds in general, regardless of the number of times the futures contract is marked to market. This is well known [?]. We will no longer distinguish forward and futures currency prices. The foreign exchange market includes both contracts. Futures contracts are normally exchange traded and forward contracts are normally arranged through banks or other financial intermediates.

2.4. European Currency Options. The second example is a European currency option. The valuation formula was derived by Garman and Kohlhagen [13] using different methods. Consider a call option providing the owner the right to buy 1 pound at time T for an exercise price of K . The dollar value of a pound at time T is $C(T)$. Therefore the value of the call right at time T is $f(C(T))$ where

$$f(c) = (c - K)^+ = \begin{cases} c - K & \text{if } c > K, \\ 0 & \text{otherwise.} \end{cases}$$

The market price at time t of the call option is the Q -discounted expected value:

$$\begin{aligned} V(t) &= e^{-r(T-t)} E_Q[f(C(T)) | \mathcal{F}_t] \\ &= e^{-r(T-t)} E_Q[(C(T) - K)^+ | \mathcal{F}_t] \\ &= e^{-r(T-t)} E_Q[(C(T) - K) I_{\{C(T) > K\}} | \mathcal{F}_t] \\ &= e^{-r(T-t)} E_Q[C(T) I_{\{C(T) > K\}} | \mathcal{F}_t] - e^{-r(T-t)} K E_Q[I_{\{C(T) > K\}} | \mathcal{F}_t] \end{aligned}$$

where $I_{\{C(T) > K\}}$ is the indicator random variable with value 1 if $C(T) > K$ (the option is in the money) and 0 otherwise (out of the money).

Now we use (7) the separation theorem on the first expectation:

$$\begin{aligned}
E_Q[C(T)I_{\{C(T)>K\}}|\mathcal{F}_t] &= e^{-uT}E_Q[S(T)I_{\{S(T)>Ke^{uT}\}}|\mathcal{F}_t] \\
&= e^{-uT}S(t)e^{r(T-t)}E_{Q_T}[I_{\{S(T)>Ke^{uT}\}}|\mathcal{F}_t] \\
&= C(t)e^{(r-u)(T-t)}E_{Q_T}[I_{\{C(T)>K\}}|\mathcal{F}_t] \\
&= C(t)e^{(r-u)(T-t)}Q_T(C(T) > K|\mathcal{F}_t)
\end{aligned}$$

Now using (7) again we see that the event

$$C(T) > K | \mathcal{F}_t$$

is Q_T -equivalent to the event

$$\exp(\sigma(\widetilde{W}(T) - \widetilde{W}(t))) > \frac{K}{C(t)} \exp((u - r - \frac{1}{2}\sigma^2)(T - t))$$

which, in turn, is equivalent to

$$\widetilde{W}(T) - \widetilde{W}(t) > \frac{1}{\sigma} \left(\log \left(\frac{K}{C(t)} \right) + (u - r - \frac{1}{2}\sigma^2)(T - t) \right)$$

Because $\widetilde{W}(T) - \widetilde{W}(t)$ is normal with mean 0 and variance $T - t$, we finally obtain an expression we can calculate from tabulated values and the known parameters:

$$E_Q[C(T)I_{\{C(T)>K\}}|\mathcal{F}_t] = C(t)e^{(r-u)(T-t)}(1 - \Phi(z_1))$$

where

$$z_1 = \frac{1}{\sigma\sqrt{T-t}} \left(\log \left(\frac{K}{C(t)} \right) + (u - r - \frac{1}{2}\sigma^2)(T - t) \right)$$

and $\Phi(z)$ is the cumulative density function for the standard normal distribution.

A similar calculation using the Q -measure leads to a formula for the other expectation:

$$E_Q[I_{\{C(T)>K\}}|\mathcal{F}_t] = 1 - \Phi(z_2)$$

where

$$z_2 = \frac{1}{\sigma\sqrt{T-t}} \left(\log \left(\frac{K}{C(t)} \right) + (u - r + \frac{1}{2}\sigma^2)(T - t) \right)$$

Finally we obtain the Black/Scholes formula for the price at time t of a European call option maturing at time T :

$$\begin{aligned} V(t) &= e^{-r(T-t)} E_Q[C(T) I_{\{C(T) > K\}} | \mathcal{F}_t] - e^{-r(T-t)} K E_Q[I_{\{C(T) > K\}} | \mathcal{F}_t] \\ &= e^{-r(T-t)} C(t) e^{(r-u)(T-t)} (1 - \Phi(z_1)) - e^{-r(T-t)} K (1 - \Phi(z_2)) \end{aligned}$$

$$(17) = C(t) e^{-u(T-t)} \Phi(d_1) - e^{-r(T-t)} K \Phi(d_2)$$

$$(18) = (F(t, T) \Phi(d_1) - K \Phi(d_2)) e^{-r(T-t)}$$

where we used $d_i = -z_i$ and the normal distribution property that $1 - \Phi(z) = \Phi(-z)$ for convenience. The first form (17) give the value in terms of the current exchange rate. The second form (18) gives the value in terms of the current (time t) forward price for pounds at time T .

3. INTEREST RATE PARITY

Sometimes the forward price relation (13)

$$F(t, T) = e^{(r-u)(T-t)} C(t)$$

is called *covered interest rate parity*. As we noted earlier it follows from the no arbitrage condition. It does not involve expectations or investor attitudes about risk. If a model assumes no arbitrage, then the covered interest rate parity relation must hold. Confusion about risk and expectations are the basis of an “argument” creating Siegel’s paradox [32, 33, 29, 18]. The “argument” goes like this: Assuming all investors are indifferent between holding either dollars or pounds then equating expected yields at time 0 on two pound investments maturing at time T requires that

$$e^{uT} = E[C(0)e^{rT}/C(T)] = C(0)e^{rT}E[1/C(T)].$$

This is called *uncovered interest rate parity*. By the no arbitrage relation, we have from equation (13) with $t = 0$:

$$e^{uT} = C(0)e^{rT}/F(0, T)$$

By comparison of the two equations we should have $F(0, T) = 1/E[1/C(T)]$. By Jensen’s inequality [4],

$$E[1/C(T)] > 1/E[C(T)]$$

hence

$$F(0, T) = C(0)e^{(r-u)T} = 1/E[1/C(T)] < E[C(T)].$$

This is the paradox – the forward price $F(0, T)$ of the time T exchange rate cannot be equal to the expected value $E[C(T)]$ of the time T spot rate. As Hull [16] explains (perhaps too tersely), there is no

paradox here, but one has to be careful about these expectations. If the expectation $E[C(0)e^{rT}/C(T)]$ is calculated relative to the physical measure, then it is not necessarily equal to e^{uT} . Using (7) and the P -measure we obtain

$$\begin{aligned} E_P[C(0)e^{rT}/C(T)] &= C(0)e^{rT}E_P[C(0)^{-1}\exp(-(\mu - \tfrac{1}{2}\sigma^2)T - \sigma W(T))] \\ &= e^{(r-\mu+\sigma^2)T}. \end{aligned}$$

Only if $r - \mu + \frac{1}{2}\sigma^2$ is equal to u , or

$$\mu = r - u + \tfrac{1}{2}\sigma^2$$

will this be equal to e^{uT} . With this value of μ the P -measure Brownian motion representation (5) of S is

$$\begin{aligned} S(T) &= S(t)\exp((u + \mu)(T - t) + \sigma(W(T) - W(t))) \\ &= S(t)\exp((u + r - u + \tfrac{1}{2}\sigma^2)(T - t) + \sigma(W(T) - W(t))) \\ &= S(t)\exp((r + \tfrac{1}{2}\sigma^2)(T - t) + \sigma(W(T) - W(t))) \end{aligned}$$

This is equivalent to the dollar risk-neutral representation of S . The only way to have the forward price be the same as the expected value of the spot currency rate is to use the $Q\$$ -measure. This measure will not necessarily agree with the physical measure. Frankel [12] describes the Seigel paradox slightly differently. If we use the dollar risk neutral measure, then the forward price of pounds in the expected value of the market price for pounds:

$$\begin{aligned} &E_Q[C(T)|\mathcal{F}_t] \\ &= C(t)\exp((r - u - \tfrac{1}{2}\sigma^2)(T - t))E_Q\left[\sigma(\widetilde{W}(T) - \widetilde{W}(t))\right] \\ (19) \quad &= C(t)\exp((r - u)(T - t)) = F(t, T) \end{aligned}$$

According to Frankel [12, page 193], the paradox is that using the same measure (the dollar risk neutral measure) the forward price of dollars *cannot* be equal to the expected value of the market price for dollars:

$$\begin{aligned} &E_{Q\$}[C(T)^{-1}|\mathcal{F}_t] \\ &= C(t)^{-1}\exp(-(r - u - \tfrac{1}{2}\sigma^2)(T - t))E_{Q\$}\left[-\sigma(\widetilde{W}(T) - \widetilde{W}(t))\right] \\ &= C(t)^{-1}\exp((u - r + \sigma^2)(T - t)) \\ &= F(t, T)^{-1}e^{\sigma^2(T-t)} \\ (20) \quad &> F(t, T)^{-1} = G(t, T) \end{aligned}$$

Of course, relative to the pound risk neutral measure, the forward rate for dollars is equal to the expected value of the market price for dollars.

This follows from merely interchanging the roles of dollars and pounds.

$$E_{Q^L}[C(T)^{-1}|\mathcal{F}_t] = G(t, T)$$

We see that there is no paradox built into the model. The empirical question – does covered interest rate parity hold in a particular currency market is a different issue.

4. INCORPORATING STOCHASTIC INTEREST RATES

In this section we extend the basic formulation of the two-country model to allow for stochastic interest rates. Any reasonable model of currency risk must include local economic factors, such as interest rates, inflation, and equities indices that interact with exchange rates and established empirical facts such as the correlation between interest rate levels and exchange rates need to be allowed for. We will enlarge our model allowing only for interest-rate factors. In order to keep the model simple we assume that local interest rates can be described as one-factor models of the class that is discussed in [7]. The same mathematics can be used to extend the model to the case of multi-factor interest rate models. No significant changes are introduced if equities indices are required in both local economies³. This paper is not about estimation/calibration and we have not addressed this important issue. Information on the estimation of a wide class of term structure models can be found in [23], [7], and [9].

Our point of departure is that interest-rate dynamics have already been identified in each local economy. As is standard in stochastic modelling, there is a “reference measure” for the global economy and this is denoted by $\mathbb{P}$. The reference measure serves to define which events are possible and which are excluded. The short-rate dynamics are denoted by the process $r_t^{(1)}$ and available for trade is the money market account B with price process $B_t^{(1)} = \exp(\int_0^t r_u^{(1)} du)$. We assume that the interest-rate dynamics can be described by an arbitrage-free term-structure model. In the first economy we will let $P^{(1)}(t, T)$ denote the price at time t of a zero coupon bond paying one unit of the currency of economy 1 at time T . In the first economy, bond prices will follow a stochastic differential equation:

$$(21) \quad \frac{dP^{(1)}(t, T)}{P^{(1)}(t, T)} = [r_t^{(1)} + \lambda_t^{(1)} \sigma_t^{(1)}(t, T)] dt + \sigma_t^{(1)}(t, T) dW_t^{(1)},$$

³All that happens is two additional noise terms are needed, the change of measure reflects the extra noise terms, and the notation becomes a bit more involved.

where $\lambda^{(1)}$ is the market price of risk process for the local economy 1. We know from financial economics⁴ that providing local economy 1 is arbitrage-free, the bond prices will follow the equation (21), and this point is discussed in [11] and [24].

The traded assets in local economy 1 are the money market account and the zero coupon bonds of all maturities. The discussion we have just provided for local economy 1 can be repeated for local economy 2 – the only difference being a change in the notation to reflect that we are in economy 2. We shall not repeat this discussion and will merely assume that we also have available the traded assets in local economy 2 which are the money market account and the zero coupon bonds of all maturities. Of course, all assets in each local economy are denominated in units of the local currency. In summary, in each local economy we have the traded assets as indicated in Figure 1.

<u>Economy 1</u>	<u>Economy 2</u>
$B_t^{(1)}$	$B_t^{(2)}$
$P^{(1)}(t, T)$	$P^{(2)}(t, T)$

FIGURE 1. Traded assets in each local economy.

These economies have thus far been considered in isolation. Then exchange rate process brings together the traded assets of each economy and permits us to analyse the global economy. the exchange rate process is denoted by D . One unit of currency in economy 2 is equal to D_t units of currency of economy 1 at time t . Conversely, one unit of currency in economy 1 is equal to $1/D_t$ units of currency of economy 2 at time t . We model the exchange rate process as a geometric Brownian motion⁵ process,

$$D_t = D_0 \exp \left(\left[\mu - \frac{1}{2} \sigma^2 \right] t + \sigma Z_t \right).$$

The correlation structure across global economic factors is allowed for thorough the correlations between the sources of noise in each economy:

⁴The local economy can be described in terms of a short-rate process and a risk-neutral measure but this is not important to us in formulating our general model but this point is illustrated concretely in the subsequent section on Gaussian models.

⁵This has been the usual practice in the finance literature but it is known that this is a questionable practice from the empirical perspective.

$W^{(1)}$, $W^{(2)}$, and Z with

$$\langle W^{(1)}, W^{(2)} \rangle_t = \rho_{1,2} t$$

$$\langle W^{(1)}, Z \rangle_t = \rho_1 t$$

$$\langle W^{(2)}, Z \rangle_t = \rho_2 t.$$

With the exchange rate process defined, we can now bring assets from each economy over for trade in the other economy. Consequently, we can define a global asset market in the units of currency of either local economy. This is indicated in Figure 2.

<u>Economy 1</u>	<u>Economy 2</u>
$B_t^{(1)}$	$D_t^{(-1)} B_t^{(2)}$
$D_t B_t^{(1)}$	$B_t^{(2)}$
$P^{(1)}(t, T)$	$D_t^{(-1)} P^{(1)}(t, T)$
$D_t P^{(2)}(t, T)$	$P^{(2)}(t, T)$

FIGURE 2. Globally traded assets in currency of each economy.

We now seek to determine the change of measure that will value all uncertain cash flows in terms of the currency of economy 1.

Theorem 1. *There is a unique change of measure under which all of the traded assets of economy 1 are discounted martingales. It is given by the change of measure on the correlated factors with*

$$\begin{aligned}\eta_t^{(1)} &= -\lambda_t^{(1)} \\ \eta_t^{(2)} &= -\lambda_t^{(2)} - \sigma \rho_2 \\ \nu_t &= \frac{r_t^{(1)} - r_t^{(2)} - \mu}{\sigma},\end{aligned}$$

and can be expressed through the Radon-Nikodym derivative

$$(22) \quad -\ln \left(\frac{dQ^{(1)}}{dP} \Big|_{\mathcal{F}_t} \right) = \int$$

4.1. A General Property of Arbitrage-Free Foreign Exchange Models. It is proved in [28, page 159] and [31] that in an arbitrage-free model of currency risk “the exchange rate between two countries must be the ratio of their state-price densities”. This is an important result for modelling currency risk. In the modelling approach taken in [28],

which he refers to as “the potential approach to term structure modelling”, the importance of this result is summed up as: “[t]his has the practical advantage that if one has adopted the potential approach to term-structure modelling, then once the term structure has been modelled in two countries, the exchange rate between them is determined, no further Brownian motions are needed.” The result influences the way in which global models of currency risk can be constructed. We first recall the connection between equivalent martingale measures and state price densities.

Definition 1. A state price density is a strictly positive adapted process $\{\rho_t : t \geq 0\}$ such that:

Property (1) for every traded asset $\{S_t : t \geq 0\}$ the process $\{\rho_t S_t : t \geq 0\}$ is a $\mathbb{P}$ -martingale, and

Property (2) $\rho_0 = 1$.

Note that Property (1) applies to all traded assets. An equivalent way to describe this definition is that a state price density is a strictly positive adapted process ρ such that $\rho_t S_t$ is a $\mathbb{P}$ -martingale for every traded asset S . Although we will not prove it⁶, there is a one-to-one correspondence between equivalent martingale measures and state price densities through the relation

$$\rho_t = \exp \left(- \int_0^t r_u du \right) \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t}.$$

We will first state and prove the result in the language of state price densities. Except for changes in notation, the statements in both results are equivalent. However, we have given a different proof of each result. Each of these proofs is helpful for shedding light on the workings of currency risk arguments – which so often are based on change of numéraire. The proof of the first result is based on Rogers [28, Section 6].

Theorem 2 (State Price Density Version). *In a complete and arbitrage-free model of global currency risk, the exchange rate between two countries is equal to the ratio of their respective state-price densities. For our two country model, this means:*

$$(23) \quad \frac{\rho_t^{(2)}}{\rho_t^{(1)}} = \frac{D_t}{D_0}, \quad \text{for all } t > 0,$$

⁶An accessible introduction to these ideas may be found in [5, Chapters 5 and 11].

where $\rho_t^{(k)}$ is the state price density for pricing all globally traded assets in the currency of economy k .

Proof. As we know from general arbitrage-free pricing theory, when the model is complete there is a unique state price density process. This is equivalent to the statement that

- (*) *there exists a unique strictly positive adapted process ρ such that $\rho_0 = 1$ and $\rho_t S_t$ is a $\mathbb{P}$ -martingale for every traded asset S .*

Let S_t denote a generic traded asset that is denominated in the currency of economy 2. Then $D_t S_t$ is a traded asset denominated in the currency of economy 1. Consequently, the processes $\rho_t^{(2)} S_t$ and $\rho_t^{(1)} D_t S_t$ are $\mathbb{P}$ -martingales. We may write this second process as

$$\rho_t^{(1)} D_t S_t = \left(\rho_t^{(1)} \frac{D_t}{D_0} \right) D_0 S_t = \tilde{\rho}_t^{(1)} D_0 S_t,$$

where $\tilde{\rho}_t^{(1)} = \rho_t^{(1)} \frac{D_t}{D_0}.$

Therefore, the process $\tilde{\rho}_t^{(1)} D_0 S_t$ is a $\mathbb{P}$ -martingale and thus the process $\tilde{\rho}_t^{(1)} S_t$ is also a $\mathbb{P}$ -martingale. Therefore, the processes $\rho_t^{(2)} S_t$ and $\tilde{\rho}_t^{(1)} S_t$ are both $\mathbb{P}$ -martingales and the processes $\rho_t^{(2)}$ and $\tilde{\rho}_t^{(1)}$ are both state price densities. Therefore, by (*) we find that $\rho_t^{(2)} = \tilde{\rho}_t^{(1)}$. $\square$

Theorem 3. [*Equivalent Martingale Measure Version*] *In a complete and arbitrage-free model of global currency risk, the exchange rate between two countries is equal to the ratio of their respective state-price densities. For our two country model, this means:*

$$(24) \quad \left. \frac{\frac{1}{B_t^{(2)}} dQ^{(2)}/d\mathbb{P}}{\frac{1}{B_t^{(1)}} dQ^{(1)}/d\mathbb{P}} \right|_{\mathcal{F}_t} \equiv \left. \frac{B_t^{(1)}}{B_t^{(2)}} \frac{dQ^{(2)}}{dQ^{(1)}} \right|_{\mathcal{F}_t} = \frac{D_t}{D_0}, \quad \text{for all } t > 0,$$

where $Q^{(k)}$ denotes the equivalent martingale measure for global valuation in the currency of economy k , $k \in \{1, 2\}$.

Proof. We shall prove the result in two ways. The first proof depends on the formulation of our model while the second is based solely on arbitrage-free pricing theory.

Change of Measure Approach: We shall prove the result for the restricted two economy model we have developed. Based on that development, we know the form for each of $Q^{(1)}$ and $Q^{(2)}$ and thus we need only check directly that (24) is satisfied.

General Theory Approach: Suppose that the model is arbitrage-free and complete. Therefore,

(**) *there is a unique equivalent martingale measure for pricing global contingent claims in the currency of each local economy⁷.*

Suppose that the trading interval for the economy is $[0, T]$. Let S_t denote a generic traded asset that is denominated in the currency of economy 2. Then $D_t S_t$ is a traded asset denominated in the currency of economy 1. Therefore, the process

$$(25) \quad \frac{1}{B_t^{(2)}} S_t$$

is a $\mathbb{Q}^{(2)}$ -martingale and the process

$$\frac{1}{B_t^{(1)}} D_t S_t = \frac{1}{B_t^{(1)}} D_t B_t^{(2)} \frac{1}{B_t^{(2)}} S_t$$

is a $\mathbb{Q}^{(1)}$ -martingale. Since $D_t B_t^{(2)}$ is a traded asset denominated in the currency of economy 1, the process

$$\frac{1}{B_t^{(1)}} D_t B_t^{(2)}$$

is a $\mathbb{Q}^{(1)}$ -martingale. Therefore,

$$\mathbb{E}^{\mathbb{Q}^{(1)}} \left[\frac{1}{B_t^{(1)}} D_t B_t^{(2)} \right] = D_0, \quad \text{for all } t \geq 0.$$

Consequently,

$$(26) \quad \frac{d\mathbb{R}}{d\mathbb{Q}^{(1)}} \Big|_{\mathcal{F}_T} = \frac{1}{D_0} \frac{1}{B_T^{(1)}} D_T B_T^{(2)}$$

defines an equivalent change of measure on the trading interval. We claim that the process (25) is an $\mathbb{R}$ -martingale. Indeed, for all

values of $t > u$:

$$\begin{aligned}
\mathbb{E}^{\mathbb{R}} \left[\frac{1}{B_t^{(2)}} S_t \middle| \mathcal{F}_u \right] &= \frac{\mathbb{E}^{\mathbb{Q}^{(1)}} \left[\frac{1}{B_t^{(2)}} S_t \frac{1}{D_0} \frac{1}{B_t^{(1)}} D_t B_t^{(2)} \middle| \mathcal{F}_u \right]}{\mathbb{E}^{\mathbb{Q}^{(1)}} \left[\frac{1}{D_0} \frac{1}{B_t^{(1)}} D_T B_T^{(2)} \middle| \mathcal{F}_u \right]} \\
&= \frac{\mathbb{E}^{\mathbb{Q}^{(1)}} \left[\frac{1}{D_0} \frac{1}{B_t^{(1)}} D_t S_t \middle| \mathcal{F}_u \right]}{\frac{1}{D_0} \frac{1}{B_u^{(1)}} D_u B_u^{(2)}} = \frac{\frac{1}{D_0} \frac{1}{B_u^{(1)}} D_u S_u}{\frac{1}{D_0} \frac{1}{B_u^{(1)}} D_u B_u^{(2)}} \\
&= \frac{1}{B_u^{(2)}} S_u.
\end{aligned}$$

where we have applied Bayes rule (Appendix B) in the first equality. Therefore, both $\mathbb{Q}^{(2)}$ and $\mathbb{R}$ are equivalent martingale measures for global valuation in the currency of economy 2. Therefore, by $(**)$ we find that

$$\left. \frac{d\mathbb{Q}^{(2)}}{d\mathbb{P}} \right|_{\mathcal{F}_T} = \left. \frac{d\mathbb{R}}{d\mathbb{Q}^{(1)}} \right|_{\mathcal{F}_T} \left. \frac{d\mathbb{Q}^{(1)}}{d\mathbb{P}} \right|_{\mathcal{F}_T}$$

Substituting from (26) this becomes

$$\left. \frac{d\mathbb{Q}^{(2)}}{d\mathbb{P}} \right|_{\mathcal{F}_T} = \frac{D_t B_t^{(2)}}{D_0 B_t^{(1)}} \left. \frac{d\mathbb{Q}^{(1)}}{d\mathbb{P}} \right|_{\mathcal{F}_T},$$

which is the same as

$$\frac{d\mathbb{Q}^{(2)}/d\mathbb{P}}{d\mathbb{Q}^{(1)}/d\mathbb{P}} = \frac{D_t B_t^{(2)}}{D_0 B_t^{(1)}}.$$

□

5. THE GAUSSIAN MODEL

In this section we describe a concrete case of the model we have developed:

- Gaussian interest rates in each local economy, and
- lognormal exchange rate factor.

The interest rate models could be developed as a multi-factor economy. However, for simplicity we have chosen to restrict the local models to the single-factor case. A Gaussian version of this model from the Heath-Jarrow-Morton perspective may be found in [20].

There are three sources of uncertainty in the global economy and these are modelled by three correlated Brownian motions $W^{(1)}$, $W^{(2)}$, and Z with correlations:

$$\begin{aligned}\langle W^{(1)}, W^{(2)} \rangle_t &= \rho_{1,2} t \\ \langle W^{(1)}, Z \rangle_t &= \rho_1 t \\ \langle W^{(2)}, Z \rangle_t &= \rho_2 t.\end{aligned}$$

The exchange rate is modelled by the geometric Brownian motion process D ,

$$D_t = D_0 \exp \left(\left[\mu - \frac{1}{2} \sigma^2 \right] t + \sigma Z_t \right)$$

The short-rate process for economy 1 is modelled as

$$dr_t^{(1)} = \alpha_1 (\gamma_1 - r_t^{(1)}) dt + \eta_1 dW_t^{(1)}.$$

The market price of risk process for economy 1 is assumed to be equal to a constant which we will denote as λ_1 . This is equivalent to specifying the risk-neutral measure for the local economy 1 through the Radon-Nikodym derivative

$$\left. \frac{d\mathbb{Q}^{(1,loc)}}{d\mathbb{P}} \right|_{\mathcal{F}_T} = \exp \left(-\lambda_1 W_T^{(1)} - \frac{1}{2} \sigma^2 T \right).$$

As is well known, this results in bond prices of the form

$$P^{(1)}(t, T) = \exp \left(-r_t^{(1)} F_1(T-t) - H_1(T-t) \right),$$

where

$$\begin{aligned}D_1 &= \gamma_1 - \lambda_1 \eta_1 / \alpha_1 - \eta_1^2 / (2\alpha_1^2), \\ H_1(s) &= sD_1 - F_1(s)D_1 + F_1(s)^2 \frac{\eta_1^2}{4\alpha_1}, \quad \text{and} \\ F_1(s) &= [1 - e^{-\alpha_1 s}] / \alpha_1.\end{aligned}$$

One may check directly, apply Itô's lemma, that bond prices satisfy the stochastic differential equation

$$\frac{dP^{(1)}(t, T)}{P^{(1)}(t, T)} = [r_t^{(1)} - \lambda_1 \eta_1 F_1(T-t)] dt - \eta_1 F_1(T-t) dW_t^{(1)}.$$

The short-rate process for economy 2 is modelled as

$$dr_t^{(2)} = \alpha_2 (\gamma_2 - r_t^{(2)}) dt + \eta_2 dW_t^{(2)}.$$

The market price of risk process for economy 2 is assumed to be equal to a constant which we will denote as λ_2 . This is equivalent to

specifying the risk-neutral measure for the local economy 2 through the Radon-Nikodym derivative

$$\left. \frac{dQ^{(2,loc)}}{dP} \right|_{\mathcal{F}_T} = \exp \left(-\lambda_2 W_T^{(2)} - \frac{1}{2} \sigma^2 T \right)$$

Identical formulas hold for the bond prices $P^{(2)}(t, T)$ in economy 2.

Recall that in our notation for general model, the probability measure $Q^{(1)}$ is the equivalent martingale measure for valuing all assets in the currency of economy 1. We can make a direct translation from the general results to conclude that this change of measure corresponds to the choice of processes:

$$\begin{aligned} \eta_t^{(1)} &= -\lambda_1 \\ \eta_t^{(2)} &= -\lambda_2 - \sigma \rho_2 \\ \nu_t &= \frac{r_t^{(1)} - r_t^{(2)} - \mu}{\sigma}. \end{aligned}$$

However, this is not the process that defines the Radon-Nikodym derivative for the change of measure $dQ^{(1)}/dP|_{\mathcal{F}_T}$. Similarly, we can describe the operational change of measure for $Q^{(2)}$, the equivalent martingale measure for valuing all assets in the currency of economy 2. As we will not require this measure for the examples in this section we shall leave it to the reader to identify $Q^{(2)}$.

5.1. Forward Contract. One of the simplest foreign exchange contracts is a *forward contract*. Let us price a forward contract for delivery at time T . This means that a unit of the currency of economy 2 is to be exchanged at time T for an amount F in the currency of economy 1. The contract has no value at time 0 and the price F is called the *T-year forward price*. We compute F from the equation:

$$\begin{aligned} 0 &= E^{Q^{(1)}} \left[\frac{1}{B_T^{(1)}} (D_T - F) \right] \\ &= E^{Q^{(1)}} \left[\frac{1}{B_T^{(1)}} D_T \right] - FP^{(1)}(0, T). \end{aligned}$$

To complete the formula we must evaluate the expectation

$$(27) \quad E^{Q^{(1)}} \left[\frac{1}{B_T^{(1)}} D_T \right]$$

As we know from (??), under the probability measure $\mathbb{Q}^{(1)}$ for each fixed $s > 0$ the process

$$(28) \quad \left\{ \frac{1}{B_t^{(1)}} D_t P^{(2)}(t, s) \mid 0 \leq t \leq s \right\}$$

is a martingale. This enables us to quickly evaluate (27) as follows.

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^{(1)}} \left[\frac{1}{B_T^{(1)}} D_T \right] \\ = \mathbb{E}^{\mathbb{Q}^{(1)}} \left[\frac{1}{B_T^{(1)}} D_T P^{(2)}(T, T) \right] \\ = \frac{1}{B_0^{(1)}} D_0 P^{(2)}(0, T) = D_0 P^{(2)}(0, T). \end{aligned}$$

Thus, we find that the forward price satisfies

$$F = \frac{P^{(2)}(0, T)}{P^{(1)}(0, T)} D_0.$$

5.2. Foreign Currency Options. Another important class of contracts are *foreign currency options*. Let us consider a simple European version of such a contract. Under this contract the option holder has the right, but not the obligation, to purchase a unit of the foreign currency at time T for a price K which is fixed at contract initiation. The contingent payoff from this contract at time T is

$$(D_T - K)_+.$$

The price of this option is given by the expectation

$$\mathbb{E}^{\mathbb{Q}^{(1)}} \left[\frac{1}{B_T^{(1)}} (D_T - K)_+ \right].$$

This may be rearranged as

$$(29) \quad P^{(1)}(0, T) \mathbb{E}^{\mathbb{Q}^{(1)}} \left[\frac{\exp \left(- \int_0^T r_u^{(1)} du \right)}{P^{(1)}(0, T)} (D_T - K)_+ \right],$$

which has the advantage that we are in a position to recognise a change of measure in this expression via the Radon-Nikodym derivative

$$(30) \quad \frac{\exp \left(- \int_0^T r_u^{(1)} du \right)}{P^{(1)}(0, T)}.$$

This change of measure is known as the “forward measure” (see [20, chapter 14] and [25]). The mean of D_T under the change of measure is easily evaluated as

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^{(1)}} \left[\frac{\exp \left(- \int_0^T r_u^{(1)} du \right)}{P^{(1)}(0, T)} D_T \right] \\ = \mathbb{E}^{\mathbb{Q}^{(1)}} \left[\frac{1}{P^{(1)}(0, T)} \frac{1}{B_T^{(1)}} D_T P^{(2)}(T, T) \right] \\ = \frac{P^{(2)}(0, T)}{P^{(1)}(0, T)} D_0, \end{aligned}$$

where we have once again used the martingale property expressed in (28). In many cases, when we have Gaussian models, the variance of the logarithm of the random variable the option is written on remains the same under a change of measure because of special properties relating to the joint normal distribution. We can clearly recognise that D_T is a lognormal random variable under $\mathbb{Q}^{(1)}$. Unfortunately, the change of measure given by $d\mathbb{Q}^{(1)}/d\mathbb{P}$ is not normal⁸ and thus general⁹ we must expect

$$\text{Var}^{\mathbb{Q}^{(1)}}(\log D_T) \neq \text{Var}^{\mathbb{P}}(\log D_T).$$

In order to complete our evaluation of (29) we will therefore have to evaluate $\text{Var}^{\mathbb{Q}^{(1)}}(\log D_T)$ directly. Since the change of measure induced by the Radon-Nikodym derivative (30) is normal, we then know that the variance of $\log D_T$ under the full change of measure is the same as $\text{Var}^{\mathbb{Q}^{(1)}}(\log D_T)$. This will then permit us to evaluate the currency option price using the usual formula for the truncated mean of a log-normal random variable.

We know that

$$\log(D_T) = \sigma \tilde{Z}_T - \frac{1}{2} \sigma^2 T + \int_0^T [r_u^{(1)} - r_u^{(2)}] du,$$

where $\tilde{Z}$ is a Brownian motion under the probability measure $\mathbb{Q}^{(1)}$. Therefore,

$$\text{Var}^{\mathbb{Q}^{(1)}}(\log(D_T)) = \text{Var}^{\mathbb{Q}^{(1)}} \left(\sigma \tilde{Z}_T + \int_0^T [r_u^{(1)} - r_u^{(2)}] du \right)$$

⁸This is because the stochastic integral part of $\log(d\mathbb{Q}^{(1)}/d\mathbb{P}|_{\mathcal{F}_T})$ involving dZ_t contains a term of the form $[r_t^{(1)} - r_t^{(2)}]dZ_t$.

⁹See [25, Lemma 6.2] for conditions when the variance is preserved under a change of measure.

Working directly from the stochastic differential for the short-rate process, we find that

$$dr_t^{(2)} = [\alpha_2(\gamma_2 - r_t^{(2)}) - \eta_2(\lambda_2 + \sigma\rho_2)] dt + \eta_2 d\widetilde{W}_t^{(2)},$$

where $\widetilde{W}^{(2)}$ is a Brownian motion under the probability measure $\mathbb{Q}^{(1)}$. Performing a stochastic integration by parts, one can show that

$$\begin{aligned} \int_0^T r_u^{(2)} du &= r_0^{(2)} F_2(T) + (\gamma_2 - \eta_2(\lambda_2 + \sigma\rho_2)/\alpha_2) F_2(T) \\ &\quad + \int_0^T \eta_2 F_2(T-u) d\widetilde{W}_u^{(2)}. \end{aligned}$$

We may do a similar calculation for $r^{(1)}$. Again working directly from the stochastic differential for the short-rate process, we find that

$$dr_t^{(1)} = [\alpha_1(\gamma_2 - r_t^{(1)}) - \eta_1\lambda_1] dt + \eta_1 d\widetilde{W}_t^{(1)},$$

where $\widetilde{W}^{(1)}$ is a Brownian motion under the probability measure $\mathbb{Q}^{(1)}$. One may then check that

$$\begin{aligned} \int_0^T r_u^{(1)} du &= r_0^{(1)} F_1(T) + (\gamma_1 - \eta_1\lambda_1/\alpha_1) F_1(T) \\ &\quad + \int_0^T \eta_1 F_1(T-u) d\widetilde{W}_u^{(1)}. \end{aligned}$$

Consequently,

$$\begin{aligned} \text{Var}^{\mathbb{Q}^{(1)}}(\log(D_T)) &= \text{Var}^{\mathbb{Q}^{(1)}}\left(\sigma \widetilde{Z}_T + \int_0^T \eta_1 F_1(T-u) d\widetilde{W}_u^{(1)} - \int_0^T \eta_2 F_2(T-u) d\widetilde{W}_u^{(2)}\right). \end{aligned}$$

We can then complete the valuation formula using the usual formula for the truncated lognormal expectation.

5.3. A Simple Insurance Example. The most important applications of the general currency model to insurance occur in pricing, reserving, and risk management/hedging. Suppose that a domestic company writes a policy in a foreign country. The foreign policyholder will pay the premiums in the foreign currency and his claims will also be paid in the foreign currency. This will result in a series of net cash flows through time. To illustrate this, let us take a very simple example. Suppose that the claim is paid, if it occurs, at time t and the premium is paid at time 0. Let us denote the amount of the claim, in units of foreign currency, by L and the amount of the premium, also in units of foreign currency, by P . Providing that the policyholders claim

is contingent only on the state variables in the foreign economy, P can be determined through the equation

$$P = E^{Q^{(2, loc)}} \left[\frac{1}{B_t^{(2)}} L \right].$$

Remark. Actually, P can also be determined as an expectation under the global measure $Q^{(2)}$ as

$$P = E^{Q^{(2)}} \left[\frac{1}{B_t^{(2)}} L \right]$$

where this will properly reflect P for arbitrary claim random variables.

The company will want to price the liability in units of the domestic currency, since it is in this currency that they track their profits and reserves. The price of the policy in domestic currency is

$$(31) \quad E^{Q^{(1)}} \left[\frac{1}{B_t^{(1)}} D_t L \right]$$

If the claim does not depend on any of the underlying state variables (i.e. is constant if the risks are treated deterministically), such as would be the case when this product was a one-year term insurance, then (31) is equal to

$$E^{Q^{(1)}} \left[\frac{1}{B_t^{(1)}} D_t \right] L = D_0 P^{(2)}(0, t) L.$$

Of course, this is nothing surprising for the price today in domestic currency of one unit of foreign currency to be paid at time t is clearly equal to $D_0 P^{(2)}(0, t)$ by general reasoning. The interesting case is when the claim does depend on state variables (either local or global) in which case (31) cannot be simplified and must be directly computed. It is in these cases that the full currency model is essential.

5.4. Foreign Returns Hedging. Suppose that a financial intermediary operating in the global economy is concerned that the investment returns of its competitors over some time horizon, say $[0, t^*]$ might outperform what it can earn in its domestic investment market. Let us suppose that both companies are investing in T -year zero coupon bonds. A simple hedge against this contingency is to purchase an option that pays the excess of the domestic dominated returns on the foreign T year bond over the returns on the domestic T year bond.

This contract would pay some notional amount of the contingent payment

$$(32) \quad \left(\frac{D_t^*}{D_0} \frac{P^{(2)}(t^*, T)}{P^{(2)}(0, T)} - \frac{P^{(1)}(t^*, T)}{P^{(1)}(0, T)} \right)_+.$$

We have not chosen this example because it is particularly realistic. The importance of the example is that the contingent claim expressed in (32) depends on all state variables and thus must be valued under the full currency risk model¹⁰. The numerical evaluation of

$$(33) \quad \mathbb{E}^{\mathbb{Q}^{(1)}} \left[\exp \left(- \int_0^{t^*} r_u^{(1)} du \right) \left(\frac{D_t^*}{D_0} \frac{P^{(2)}(t^*, T)}{P^{(2)}(0, T)} - \frac{P^{(1)}(t^*, T)}{P^{(1)}(0, T)} \right)_+ \right]$$

is relatively sophisticated because in general we will not have a closed formula for $P^{(1)}(t^*, T)$ nor $P^{(2)}(t^*, T)$ in terms of the short-rate trajectories $\{r_s^{(1)} : 0 \leq s \leq t^*\}$ or $\{r_s^{(2)} : 0 \leq s \leq t^*\}$. Consequently, for each simulation run to evaluate $\int_0^{t^*} r_u^{(1)} du$ we will have to run several thousand supplementary simulations over $[t^*, T]$ to evaluate $P^{(1)}(t^*, T)$ and $P^{(2)}(t^*, T)$. This problem is typical when running simulations for sophisticated term structure models in which the prices of fixed income securities are not known explicitly in terms of the models state variables.

6. CONCLUSIONS

In this paper we have developed a general model for currency risk. We have approached it from the “classical” perspective of interest-rate modelling based on the modelling of the short-rate process. This has the advantage that the reader can understand the mathematics of exchange rate models without requiring a new mathematical tools. Alternative approaches are available (at least in preliminary form) in the literature. The two prominent alternatives are:

- the “potential approach” of Rogers [28], and
- using a Heath-Jarrow-Morton formulation for the term structure of interest rates.

Approaches based on the Heath-Jarrow-Morton framework are described in [20, chapter 17] and more tersely in [1, section 6.5]. The potential approach was developed in Rogers [28] and is discussed in [3, section 6.2].

The class of processes that we have adopted for the exchange rate dynamics and interest-rate dynamics are too restrictive. It is generally

¹⁰In other words, the claim is not adapted to either of the local information structures $\mathcal{F}^{W^{(1)}}$ or $\mathcal{F}^{W^{(2)}}$.

thought that exchange rates are not well fit by geometric Brownian motion and that other classes of processes, GARCH processes being one example, are needed to adequately capture the empirical characteristics of exchange rate dynamics. Melino and Turnbull [19] model the exchange rate process using a stochastic volatility process and show how to estimate this model using the GMM approach. Broader classes of processes for both exchange rate dynamics and interest-rate dynamics are needed. Better models of the exchange rate dynamics can be constructed using the types of models discussed in [30]. Some examples of the types of processes that could be used for the interest-rate dynamics are studied in [23], [14], [21], and [26].

We have not given an indication of how the model is to be calibrated for practical use. This is an involved and important topic that will be the subject of future research. These models can be estimated using a variety of techniques such as maximum likelihood, GMM estimation, and Bayesian estimation.

APPENDIX A. GIRSANOV'S THEOREM FOR CORRELATED BROWNIAN MOTIONS

The usual version of Girsanov's theorem is stated for standard Brownian motion. For our model we require a version of the result which allows for correlated Brownian motions. However, the general version of Girsanov's theorem for semimartingales, which may be found in [20, page 467], [15, Chapter XII], or [27, Chapter VIII], is too general for our purposes here. The result we discuss is valid for an arbitrary n -dimensional correlated Brownian motion and is useful in a variety of applications to financial modelling. The result is important because we often seek to change measure based on readily identifiable factors in the financial model. If uncertainty is modelled using a correlated Brownian motion, it is common that the change of measure is naturally described in terms of drift adjustments to the components of this process. Such was the case when we computed the change of measure for our general currency risk model.

It is convenient to introduce some notation for handling vector processes. All vectors will be considered as column vectors. If x is a vector, the transpose of x is denoted x^t . Consequently, if Z denotes an n -dimensional stochastic process then $Z^t = (Z_t^{(1)}, Z_t^{(2)}, \dots, Z_t^{(n)})$ where $Z^{(i)}$ denotes the i^{th} component of Z . When we integrate vector processes it is convenient to have a notation that permits us to compactly express certain vector integrals. We will use paranthesis to denote the inner product of two vector processes. For example, if A and B are

two n -dimensional vector processes then

$$(A_t, B_t) = A_t^{(1)} B_t^{(1)} + A_t^{(2)} B_t^{(2)} + \dots + A_t^{(n)} B_t^{(n)}.$$

We can use the same notation to simplify the writing of stochastic integrals. For instance, if M is an n -dimensional semimartingale and an appropriate n -dimensional integrand, denoted by A , we have

$$\int A_t^{(1)} dM_t^{(1)} + \int A_t^{(2)} dM_t^{(2)} + \dots + \int A_t^{(n)} dM_t^{(n)} = \int (A_t, dM_t).$$

Lastly, we employ the obvious convention

$$\int A_t dt = \left(\int A_t^{(1)} dt, \int A_t^{(2)} dt, \dots, \int A_t^{(n)} dt \right).$$

An n -dimensional correlated Brownian motion is an n -dimensional continuous stochastic process with independent increments such that the increment $W_t - W_s$ is normally distributed with mean zero and covariance matrix $(t - s)C$, where C is a symmetric positive definite matrix. It is clear that $C = E[W_1 W_1^T]$, the covariance matrix of W_1 . Elements of the matrix C will be denoted by ρ_{ij} . In the notation of [17], we have $\langle W^{(i)}, W^{(j)} \rangle_t = \rho_{ij} t$. An n -dimensional standard Brownian motion is thus an n -dimensional correlated Brownian motion with C being the identity matrix. Since C is a symmetric positive definite matrix, there exists a lower triangular matrix Σ such that $C = \Sigma \Sigma^t$, the Choleski factorisation of C (see [6] for example). If W is an n -dimensional correlated Brownian motion with covariance matrix $C = \Sigma \Sigma^t$ then it may be checked that $\Sigma^{-1}W$ is a standard Brownian motion which generates the same filtration as W .

Theorem 4. Let W be an n -dimensional correlated Brownian motion with covariance matrix $C = \Sigma \Sigma^t$. Let B denote the n -dimensional standard Brownian motion $B \equiv \Sigma^{-1}W$. Let λ be an n -dimensional adapted process. Then the process

$$W_t - \int_0^t \lambda_u du$$

is an n -dimensional correlated Brownian motion with covariance matrix $C = \Sigma \Sigma^t$ under the equivalent change of measure

$$\begin{aligned} \ln \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_T} \right) &= \int_0^T (\Sigma^{-1} \lambda_u, dB_u) - \frac{1}{2} \int_0^T (\Sigma^{-1} \lambda_u, \Sigma^{-1} \lambda_u) du \\ &= \int_0^T (\Sigma^{-1} \lambda_u, \Sigma^{-1} dW_u) - \frac{1}{2} \int_0^T (\Sigma^{-1} \lambda_u, \Sigma^{-1} \lambda_u) du. \end{aligned}$$

Proof.

$$\begin{aligned} W_t - \int_0^t \lambda_u du &= \Sigma B_t - \Sigma \int_0^t \Sigma^{-1} \lambda_u du \\ &= \Sigma \left(B_t - \int_0^t \Sigma^{-1} \lambda_u du \right) \\ &= \Sigma \tilde{B}_t, \end{aligned}$$

where $\tilde{B}_t$ is a standard Brownian motion under the change of measure

$$\left. \frac{dQ}{dP} \right|_{\mathcal{F}_T} = \exp \left(\int_0^T (\Sigma^{-1} \lambda_u, dB_u) - \frac{1}{2} \int_0^T (\Sigma^{-1} \lambda_u, \Sigma^{-1} \lambda_u) du \right)$$

by the usual form of Girsanov's theorem for Brownian motion [17, page 191]. $\square$

In practice one must often carry out valuation using Monte Carlo simulation or other numerical techniques. Consequently, it is important to know the explicit form of the risk-neutral measure so that the simulation can be programmed and the calculations carried out. If we are working in terms of a correlated Brownian motion W then it is convenient to express the change of measure in terms of this process. The following corollary shows how this may be done.

Corollary 1. *Let W be an n -dimensional correlated Brownian motion with covariance matrix $C = \Sigma \Sigma^t$. Let λ be an n -dimensional adapted process. Define the process ζ by $\zeta_t = C^{-1} \lambda_t$. Then the process*

$$W_t - \int_0^t \lambda_u du$$

is an n -dimensional correlated Brownian motion with covariance matrix $C = \Sigma \Sigma^t$ under the equivalent change of measure

$$\ln \left(\left. \frac{dQ}{dP} \right|_{\mathcal{F}_T} \right) = \int_0^T (\zeta_u, dW_u) - \frac{1}{2} \int_0^T (\zeta_u, C \zeta_u) du.$$

Proof. The proof is nothing more than an application of linear algebra to the integrands. We need only check that

$$\begin{aligned} \int_0^T (\Sigma^{-1} \lambda_u, \Sigma^{-1} dW_u) - \frac{1}{2} \int_0^T (\Sigma^{-1} \lambda_u, \Sigma^{-1} \lambda_u) du \\ = \int_0^T (\zeta_u, dW_u) - \frac{1}{2} \int_0^T (\zeta_u, C \zeta_u) du, \end{aligned}$$

as the rest of the statement of the corollary will follow from the theorem.

$$\begin{aligned}
& \int_0^T (\Sigma^{-1} \lambda_u, \Sigma^{-1} dW_u) \\
&= \int_0^T ((\Sigma^{-1})^t \Sigma^{-1} \lambda_u, dW_u) \\
&= \int_0^T (C^{-1} \lambda_u, dW_u) \\
& \int_0^T (\Sigma^{-1} \lambda_u, \Sigma^{-1} \lambda_u) du \\
&= \int_0^T ((\Sigma^{-1})^t \Sigma^{-1} \lambda_u, \lambda_u) du \\
&= \int_0^T (C^{-1} \lambda_u, \lambda_u) du \\
&= \int_0^T (C^{-1} \lambda_u, C C^{-1} \lambda_u) du
\end{aligned}$$

□

APPENDIX B. BAYES' RULE FOR CHANGE OF MEASURE

We briefly describe Bayes rule as it is used in the proof of Theorem 3.

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