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DERIVATION OF A NEW FORMULA FOR THE NUMBER OF MINIMAL LATTICE PATHS FROM (0, 0) TO (km, kn) HAVING JUST t CONTACTS WITH THE LINE $my=nx$ AND HAVING NO POINTS ABOVE THIS LINE; AND A PROOF OF GROSSMAN'S FORMULA FOR THE NUMBER OF PATHS WHICH MAY TOUCH BUT DO NOT RISE ABOVE THIS LINE

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WHITWORTH(1) deals in Chapter V of *Choice and Chance* with the problem of finding the number of minimal lattice paths from $(0,0)$ to (k,k) which do not cross the line $y=x$. By a lattice path is meant a path joining two points with integral coefficients by a line composed of horizontal and vertical steps of unit length. A minimal lattice path from $(0,0)$ to (x,y), say, is a lattice path where the total number of steps is $(x+y)$; in other words, all the steps are onwards. In what follows minimal lattice paths only will be considered, and the words 'minimal lattice' will be omitted.

Although Whitworth deals only with the case where the boundary line (i.e. the line which the path must not cross) is $y=x$, the more general case of a boundary $\alpha y=x$ has been solved provided α is a positive integer(2), (3). The number of paths from $(0, 0)$ to $(\alpha l, l)$ which may touch but never rise above $\alpha y=x$ is $\dfrac{1}{l\alpha+1}\dbinom{l\alpha+l}{l}$.

Grossman(4) announced without proof in 1950 a formula for the number of paths from $(0, 0)$ to (km, kn) which may touch but never rise above the line $my=nx$, where k is a positive integer and m and n are coprime positive integers; thus (km, kn) is any point having positive integral coefficients. Grossman's formula is

$$\Sigma F_1^{k_1}F_2^{k_2}\ldots/k_1!\,k_2!\ldots,$$

where
$$F_j=\frac{1}{j(m+n)}\binom{jm+jn}{jm},$$

the sum extending over all positive integral k_i such that $k_i \geqslant 0$ and $\Sigma ik_i=k$. If $k=1$ this takes the simple form $\dfrac{1}{m+n}\dbinom{m+n}{m}$.

The object of the present note is to supply a proof of Grossman's formula and to extend his result to cover also the problems of enumerating the paths which (i) lie wholly below $my=nx$ and do not touch it between $(0, 0)$ and (km, kn), or (ii) never rise above $my=nx$ but touch it at just t points. The proof of Grossman's formula is believed to be the first, and the extended results are thought to be new.

Figs. 1 and 2 illustrate minimal lattice paths for the case $k=6$, $m=4$, $n=3$.

The first half of this note, including the proof of Grossman's formula, was received early in October 1953—too late for publication in the December number of the *Journal*. The author has taken the opportunity afforded by the delay to pursue his researches further and has added §§ IV and V. Eds. *J.I.A.*

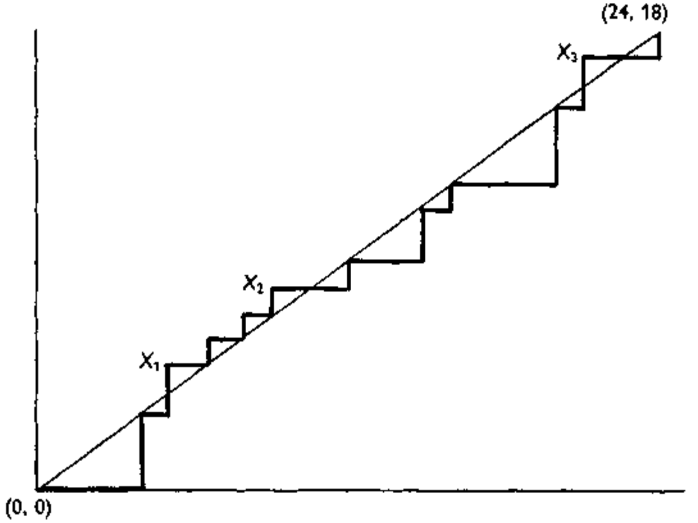

Fig. 1

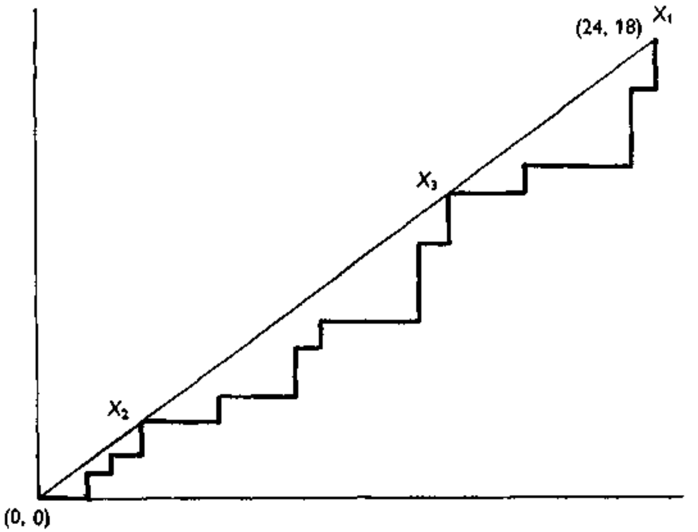

Fig. 2

DEFINITIONS

'Highest point.' A path will be said to have a highest point at a lattice point X on the path if the line of gradient n/m through X cuts the y-axis at a value of y not less than that corresponding to any other lattice point of the path. It is conventional to regard the first point (0, 0) as not belonging and the final point (km, kn) as belonging to the path. The paths in Figs. 1 and 2 have highest points at X_1, X_2 and X_3. In Fig. 2 the highest points lie on the diagonal line $my = nx$.

Such a path which contains no lattice points above the boundary line and has t contacts with this line will be called a 'ϕ-path with t contacts', t in the case of Fig. 2 being 3. The number of such lattice paths will be denoted by $\phi_{k,t}$.

Let $\phi_k = \sum_1^k \phi_{k,t}$, i.e. ϕ_k is the number of paths from (0, 0) to (km, kn) which may touch, but never rise above, the line $my = nx$. A path having this property will be called a ϕ-path.

Let $\psi_k = \phi_{k,1}$, i.e. ψ_k is the number of paths which lie wholly below, and do not touch, the line $my = nx$ except at (km, kn). A path having this property will be called a ψ-path.

'Cyclical permutation.' A cyclical permutation of a lattice path is obtained by removing a section of the path from (0, 0) to (r, s), say, and fitting it at the other end of the path so that (0, 0) falls on (km, kn). The whole path is then moved down and to the left so that (r, s) falls where (0, 0) had been and the other end will now fall where (km, kn) had been. Figs. 1 and 2 illustrate a cyclical permutation.

Two points may be noted: (1) the number of highest points on a path is not affected by a cyclical permutation; (2) any path with t highest points can be transformed into a ϕ-path with t contacts by a cyclical permutation in exactly t ways; each such cyclical permutation corresponds to bringing one of the t highest points to position (km, kn). This is general even if the original path is itself a ϕ-path with t contacts. In this latter case one of the t cyclical permutations is the 'identical' cyclical permutation.

I. Now consider the $(km+kn)\,\phi_{k,t}$ paths formed by permuting cyclically in every possible way (including the identical cyclical permutation) all the ϕ-paths with t contacts. Every path with t highest points will be formed in this way exactly t times. For if not this would mean that a path with t highest points could be transformed by cyclical permutations into a ϕ-path with t contacts in other than t ways. Hence the number of paths with just t highest points is

$$\frac{1}{t}\,k(m+n)\,\phi_{k,t}. \tag{1}$$

Hence the total number of paths from (0, 0) to (km, kn), namely, $\binom{km+kn}{km}$ is the sum of (1) for all values of t, i.e.

$$\binom{km+kn}{km} = \sum_{t=1}^{k} \frac{1}{t}\,k(m+n)\,\phi_{k,t}, \tag{2}$$

i.e.

$$F_k = \sum_{t=1}^{k} \frac{1}{t}\,\phi_{k,t}, \tag{3}$$

where

$$F_k = \frac{1}{k(m+n)}\binom{km+kn}{km}. \tag{4}$$

Now by definition a ϕ-path with t contacts consists of t sections, each section being a ψ-path (when considered in relation to its own starting and ending points). Hence

$$\phi_{k,t} = \Sigma \psi_{a_1}\psi_{a_2}\ldots\psi_{a_t}, \tag{5}$$

the sum extending over all a_i such that $a_i > 0$ and $\sum_{i=1}^{t} a_i = k$. If we write

$$P(x) = \psi_1 x + \psi_2 x^2 + \ldots \text{ ad inf.}, \tag{6}$$

the right-hand side of (5) is the coefficient of x^k in $[P(x)]^t$. Hence the right-hand side of (3) is the coefficient of x^k in $\sum_{t=1}^{k} \frac{1}{t}[P(x)]^t$, or in $\sum_{t=1}^{\infty} \frac{1}{t}[P(x)]^t$, since $[P(x)]^t$ has no term in x^k if $t > k$; i.e. in $-\log[1-P(x)]$ and this is true for all values of k. It follows from (3) that

$$-\log[1-P(x)] \equiv F_1 x + F_2 x^2 + F_3 x^3 + \ldots,$$

or $$P(x) = 1 - e^{-F_1 x - F_2 x^2 - \ldots}. \tag{7}$$

ψ_k, the coefficient of x^k in $P(x)$, is easily deduced from the multinomial theorem; hence ψ_k, the number of paths from (0, 0) to (km, kn) which lie wholly below and do not touch $my = nx$, is given by

$$\psi_k = \Sigma\left[(-1)^{1+\Sigma k_i}\frac{F_1^{k_1}F_2^{k_2}\ldots}{k_1!\,k_2!\ldots}\right], \tag{8}$$

the sum extending over all k_i such that $k_i \geqslant 0$ and $\Sigma i k_i = k$.

II. We have shown that $\phi_{k,t}$ is the coefficient of x^k in $[P(x)]^t$. Using (7) we have thus proved a new theorem, as follows:

'The number of minimal lattice paths from (0, 0) to (km, kn) which have exactly t contacts with the line $my = nx$ (not counting (0, 0)) and which have no lattice points above this line is the coefficient of x^k in

$$(1 - e^{-F_1 x - F_2 x^2 - \ldots})^t, \quad \text{where} \quad F_j = \frac{1}{j(m+n)}\binom{jm+jn}{jm}\text{'}.$$

III. We easily deduce the value of ϕ_k from this general theorem. By definition $\phi_k = \sum_{t=1}^{k}\phi_{k,t}$, and hence ϕ_k is the coefficient of x^k in

$$\sum_{t=1}^{k}(1 - e^{-F_1 x - F_2 x^2 - \ldots})^t,$$

or in $$\sum_{t=1}^{\infty}(1 - e^{-F_1 x - F_2 x^2 - \ldots})^t, \tag{9}$$

since $1 - e^{-F_1 x - F_2 x^2 - \ldots}$ has no term independent of x. Since the sum (9) is merely a geometric progression with common ratio $\{1 - e^{-F_1 x - F_2 x^2 - \ldots}\}$, its value is

$$\frac{1 - e^{-F_1 x - F_2 x^2 - \ldots}}{e^{-F_1 x - F_2 x^2 - \ldots}},$$

i.e. $$e^{F_1 x + F_2 x^2 + \ldots} - 1.$$

Therefore ϕ_k is the coefficient of x^k in $e^{F_1 x + F_2 x^2 + \ldots \text{ad inf.}}$, i.e.

$$\phi_k = \Sigma\frac{F_1^{k_1}F_2^{k_2}\ldots}{k_1!\,k_2!\ldots}, \tag{10}$$

the sum extending over all k_i such that $\Sigma i k_i = k$. (10) is Grossman's formula.

RECURRENCE RELATIONS FOR ϕ_k AND ψ_k

IV. The explicit formulae for ϕ_k and ψ_k are not particularly convenient for arithmetical computation, since they involve in the first place the determination of all the sets of values of k_i. We can, however, exploit the properties of the functions investigated in this note to produce useful recurrence relations for ϕ_k and for ψ_k; these relations have an intrinsic interest.

Since ϕ_k is the coefficient of x^k in $e^{F_1x+F_2x^2+\cdots}-1$, we have

$$\log(1+\phi_1x+\phi_2x^2+\ldots)=F_1x+F_2x^2+\ldots.$$

Differentiate each side of this identity with respect to x, multiply through by $(1+\phi_1x+\phi_2x^2+\ldots)$ and equate coefficients of successive powers of x; then, since by definition $jF_j=\dfrac{1}{m+n}\dbinom{jm+jn}{jm}$, we have

$$(m+n)\,\phi_1=\binom{m+n}{m},$$

$$2(m+n)\,\phi_2=\binom{2m+2n}{2m}+\binom{m+n}{m}\phi_1,$$

$$3(m+n)\,\phi_3=\binom{3m+3n}{3m}+\binom{2m+2n}{2m}\phi_1+\binom{m+n}{m}\phi_2,$$

$$4(m+n)\,\phi_4=\binom{4m+4n}{4m}+\binom{3m+3n}{3m}\phi_1+\binom{2m+2n}{2m}\phi_2+\binom{m+n}{m}\phi_3,$$

and so on,

from which successive values of ϕ_k are easily calculated with the aid of a table of binomial coefficients (e.g. the table by the writer, *J.S.S.* **10**, 65).

Dealing similarly with the relation

$$\log(1-\psi_1x-\psi_2x^2-\ldots)=-F_1x-F_2x^2-\ldots,$$

we obtain for ψ_k,

$$(m+n)\,\psi_1=\binom{m+n}{m},$$

$$2(m+n)\,\psi_2=\binom{2m+2n}{2m}-\binom{m+n}{m}\psi_1,$$

$$3(m+n)\,\psi_3=\binom{3m+3n}{3m}-\binom{2m+2n}{2m}\psi_1-\binom{m+n}{m}\psi_2,$$

$$4(m+n)\,\psi_4=\binom{4m+4n}{4m}-\binom{3m+3n}{3m}\psi_1-\binom{2m+2n}{2m}\psi_2-\binom{m+n}{m}\psi_3,$$

and so on.

These relations can be deduced directly by general reasoning from the geometrical properties of the paths, and this was in fact the method by which the writer first established proofs of the explicit formulae for ϕ_k and ψ_k; this method is, however, rather longer than that given in this note. The similarity between the relations for ϕ_k and those for ψ_k is striking.

As an example of the use of the recurrence relations, we have in the case $m=3$, $n=2$,

$5\phi_1=10$	whence $\phi_1=2$,
$10\phi_2=210+10\phi_1$	whence $\phi_2=23$,
$15\phi_3=5005+210\phi_1+10\phi_2$	whence $\phi_3=377$,
$20\phi_4=125{,}970+5005\phi_1+210\phi_2+10\phi_3$	whence $\phi_4=7229$,
$5\psi_1=10$	whence $\psi_1=2$,
$10\psi_2=210-10\psi_1$	whence $\psi_2=19$.
$15\psi_3=5005-210\psi_1-10\psi_2$	whence $\psi_3=293$,
$20\psi_4=125{,}970-5005\psi_1-210\psi_2-10\psi_3$	whence $\psi_4=5452$.

The explicit formulae are given by (10) and (8), and are

$$\phi_1 = F_1, \qquad \psi_1 = F_1,$$

$$\phi_2 = F_2 + \tfrac{1}{2}F_1^2, \qquad \psi_2 = F_2 - \tfrac{1}{2}F_1^2,$$

$$\phi_3 = F_3 + F_1F_2 + \tfrac{1}{6}F_1^3, \qquad \psi_3 = F_3 - F_1F_2 + \tfrac{1}{6}F_1^3,$$

$$\phi_4 = F_4 + F_1F_3 + \tfrac{1}{2}F_2^2 + \tfrac{1}{2}F_1^2F_2 + \tfrac{1}{24}F_1^4, \quad \psi_4 = F_4 - F_1F_3 - \tfrac{1}{2}F_2^2 + \tfrac{1}{2}F_1^2F_2 - \tfrac{1}{24}F_1^4,$$

and it is easily verified that with $F_1 = 2$, $F_2 = 21$, $F_3 = 1001/3$ and $F_4 = 12{,}597/2$ these give the same values.

It is noteworthy that in the course of our investigation we have proved a very remarkable theorem in number theory, namely, that the sums

$$\Sigma F_1^{k_1}F_2^{k_2}\ldots/k_1!\,k_2!\ldots \quad \text{and} \quad \Sigma(-1)^{1+\Sigma k_i}F_1^{k_1}F_2^{k_2}\ldots/k_1!\,k_2!\ldots$$

yield only integers. Bearing in mind that F_j is not generally an integer and *a fortiori* $F_j^{k_j}/k_j!$ is not an integer, this result is far from being obviously true.

THE CASE $n=1$

V. The problem of finding explicit formulae for ϕ_k and ψ_k when $n=1$ (in which case the boundary line intersects each of the lines $y=1$, $y=2$, $y=3$, etc., at a lattice point) is much simpler than the general problem and, as mentioned at the beginning of this note, has already been solved. It is, however, of interest to investigate this case as follows:

A path from (0, 0) to $(rm, r-1)$, where r is an integer, can cross the boundary line $my=x$ in a horizontal direction only at a lattice point. Let us classify the $\binom{rm+r-1}{r-1}$ paths according to the point at which they last cross the boundary line in a horizontal direction. If for any path this point is (tm, t), the path consists of (i) a section from (0, 0) to $(tm-1, t)$ with no restriction regarding the boundary, (ii) a single horizontal step from $(tm-1, t)$ to (tm, t) and (iii) a path not crossing the boundary from (tm, t) to $(rm, r-1)$. Hence the number of such paths is $\binom{tm+t-1}{t}\phi_{r-t}$, since each of the ϕ_{r-t} paths not crossing the boundary from (tm, t) to (rm, r) ends with a vertical step and is therefore a distinct path from (tm, t) to $(rm, r-1)$. Giving t the values $1, 2, \ldots, (r-1)$ and remembering that there are also ϕ_r paths which do not cross the boundary at all, we have

$$\binom{rm+r-1}{r-1} = \phi_r + \sum_{t=1}^{r-1}\binom{tm+t-1}{t}\phi_{r-t},$$

or

$$\frac{1}{m+1}\binom{rm+r}{r} = \phi_r + \frac{m}{m+1}\sum_{t=1}^{r-1}\binom{tm+t}{t}\phi_{r-t},$$

and since when $n=1$,

$$F_r = \frac{1}{r(m+1)}\binom{rm+r}{r},$$

we have

$$rF_r = \phi_r + m\sum_{t=1}^{r-1} tF_t\phi_{r-t}.$$

Since this is true for all values of r we have identically

$$F_1x+2F_2x^2+3F_3x^3+\ldots=(1+mF_1x+2mF_2x^2+3mF_3x^3+\ldots)(\phi_1x+\phi_2x^2+\ldots),$$

i.e.
$$xF'=(1+mxF')\phi,$$

or
$$xF'=\frac{\phi}{1-m\phi},$$

where
$$F=F_1x+F_2x^2+\ldots,$$

and
$$\phi=\phi_1x+\phi_2x^2+\ldots,$$

and accents denote differentiation with respect to x.

We have also, in the general case, and hence in particular when $n=1$,

$$F=\log(1+\phi),$$

whence
$$xF'=\frac{x\phi'}{1+\phi}.$$

Combining these results,

$$xF'=\frac{\phi+m(x\phi')}{(1-m\phi)+m(1+\phi)}=\frac{\phi+mx\phi'}{m+1}.$$

Equating coefficients of x^r,

$$\frac{1}{m+1}\binom{rm+r}{r}=\frac{\phi_r+rm\phi_r}{m+1},$$

or
$$\phi_r=\frac{1}{rm+1}\binom{rm+r}{r}\quad\text{when } n=1.$$

To find a formula for ψ_r when $n=1$ we recall that in the general case we have shown that

$$1+\phi_1x+\phi_2x^2+\ldots=e^{F_1x+F_2x^2+\ldots}=(1-\psi_1x-\psi_2x^2-\ldots)^{-1},$$

or
$$1+\phi=(1-\psi)^{-1},$$

where
$$\psi=\psi_1x+\psi_2x^2+\ldots,$$

i.e.
$$\phi=\frac{\psi}{1-\psi}.$$

Hence
$$xF'=\frac{\phi}{1-m\phi}=\frac{\psi}{1-(m+1)\psi}.$$

Also since
$$-F=\log(1-\psi),$$

$$xF'=\frac{x\psi'}{1-\psi}.$$

Combining these results

$$xF'=\frac{(m+1)x\psi'-\psi}{(m+1)(1-\psi)-(1-\overline{m+1}\psi)}=\frac{(m+1)x\psi'-\psi}{m}.$$

Equating coefficients of x^r,

$$\frac{1}{m+1}\binom{rm+r}{r}=\frac{r(m+1)\psi_r-\psi_r}{m},$$

giving
$$\psi_r=\frac{1}{rm+r-1}\binom{rm+r-1}{r}.$$

A simple identity satisfied by ϕ is easily deduced from the foregoing argument. Since

$$xF' = (1 + mxF')\phi,$$

and

$$xF' = \frac{x\phi'}{1+\phi},$$

$$\frac{x\phi'}{1+\phi} = \left(1 + \frac{mx\phi'}{1+\phi}\right)\phi,$$

i.e.

$$x\phi' = \phi(1+\phi) + mx\phi\phi'.$$

Adding $x\phi\phi'$ to each side and dividing through by $x\phi(1+\phi)$ we have

$$\frac{\phi'}{\phi} = \frac{1}{x} + \frac{(m+1)\phi'}{1+\phi},$$

i.e.

$$\log\phi = \log x + (m+1)\log(1+\phi),$$

the constant of integration vanishing. Hence

$$\phi = x(1+\phi)^{m+1}.$$

Since $\phi = \dfrac{\psi}{1-\psi}$ we have immediately the corresponding identity for ψ,

$$x = \psi(1-\psi)^m.$$

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