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DISCONTINUITY IN THE FORCE OF MORTALITY

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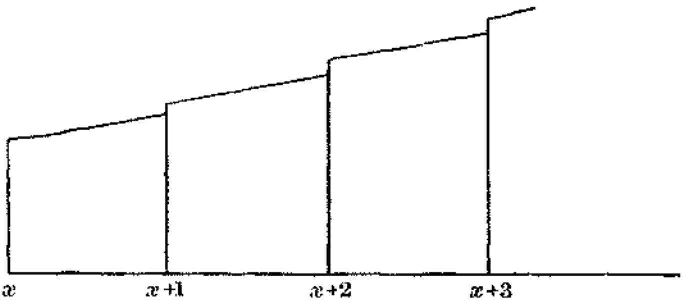
The force of mortality (μ_x) when defined in terms of l_x appears to be a continuous function and is assumed to be so in actuarial practice. There are, however, circumstances in which the force of mortality is discontinuous.

Let us assume that a number of assurances are taken out with annual premiums; then, so long as there is no surrender value, these assurances are withdrawn only because a premium is unpaid. In other words, the withdrawals take place at regular yearly intervals. If those who decide to allow their assurances to lapse are subject to a different mortality from those who continue their assurances, the force of mortality of those remaining assured becomes discontinuous at the end of each year. Thus if the withdrawals are subject to a lighter (or heavier) mortality than the population from which they are withdrawn, the force of mortality of those remaining assured will increase normally during each year and then increase (or decrease) suddenly at the end of each year. Graphically, the force of mortality would take the following form where we assume that withdrawals are subject to lighter mortality:



The extent of the discontinuity will depend on the amount of selection exercised by those who withdraw and by the proportion withdrawing. If, however, we assume a high degree of selection, we cannot accompany it with a large proportion of withdrawals because we should then reach, in extreme cases, impossible rates of mortality.

An imaginary example may help to show the kind of result that might arise in actual experience, and one is given in the table appended to this note.

The experience that would emerge from a mortality investigation of assurances on the books of an assurance company would be of the type indicated in the final section of the table, and, though we have assumed in the part of the table headed "Total population" that we know the rates of mortality if there were no withdrawals, we should not have this information in practice. In the example the force of mortality of the "actual experience" would run from ·01204 at age x to ·01260 at the end of that year of age, but then we should come to the first discontinuity and start again on a new level for $x+1$ and so on.

If the statistics on which we base a mortality table do not imply a continuous function for the force of mortality, it seems illogical to assume that a graduated mortality table found from the rates of mortality will necessarily give a continuous force of mortality. Having found a series of q_x we ordinarily proceed to l_x and then say that $\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx}$. It is often necessary to approximate to this differential coefficient by using l_x, $l_{x\pm 1}$, $l_{x\pm 2}$, etc. If circumstances prevail such as those described in this note, these approximations are theoretically open to criticism.

Another point may be mentioned. Having found q_x^e from the experience we might graduate colog p_x^e by "Makeham's law", and though colog p_x^e would then be of the form $a+bc^x$ we could not say that μ_x takes the same form. In our example, the "total population" is taken from a graduated table which assumes that μ_x is of the form $a+bc^x$, and consequently μ_x^e from x to $x+·999...$ would be correctly represented by $a+bc^x$, where a, b and c have the values implied by the graduation. Now, any three values of colog p_x can be put in the same form so that colog p_x^e could be expressed in that form, but the constants would be different from those implied by the graduation underlying the "Total population". If, from the graduated values of colog p_x^e, we proceeded by the usual methods, we should reach $\mu_x^e = a_1 + b_1 c_1^x$ and should conclude that μ_x^e from x to $x+·999...$ could be expressed as $a_1 + b_1 c_1^x$, whereas we know that within that small range of ages it should be $a+bc^x$.

It is not suggested that the error involved in making the ordinary assumptions is of importance, but I think it may be of interest to call attention to the possibility of discontinuance which might be worth bearing in mind in work on certain special classes of policies, such, for instance, as temporary assurances, especially when there is an option to convert these assurances into a whole life or endowment assurance.

Age	Total population			
	l_x	d_x	q_x	μ_x
x	77918	954	·01224	·01204
$x+1$	76964	986	·01281	·01260
$x+2$	75978	1021	·01345	·01321
etc.				

Age	Withdrawal population			
	New withdrawals	l_x^w	d_x^w	q_x^w
x	—	—	—	—
$x+1$	15393	15393	158	·01025
$x+2$	11397	26632	287	·01076
etc.				

Age	Population remaining, i.e. the population coming into insurance experience			
	E_x	θ_x	q_x^e	μ_x^e
x	77918	954	·01224	·01204
$x+1$	61571	828	·01345	?
$x+2$	49346	734	·01487	?
etc.				

Notes. (1) Withdrawals were taken as 20 % of l_{x+1} at exact age $x+1$ and as 15 % of l_{x+2} at exact age $x+2$.

(2) q_{x+1}^w and q_{x+2}^w were taken as 80 % of q_{x+1} and q_{x+2} respectively.

(3) The experience figures, E_x and θ_x, are found by subtracting the withdrawal population figures l^w and d^w from the total population figures l and d.

(4) $q_{x+r}^e = \theta_{x+r}/E_{x+r}$.

(5) $\mu_x^e = \mu_x =$ ·01204; $\mu_{x+\cdot 999\ldots}^e = \mu_{x+1} =$ ·01260 but $\mu_{x+1}^e > \mu_{x+1}$ and for the reasons indicated in the text no attempt has been made to estimate it.

(6) The figures for total population were taken from H^M Table (Text-Book graduation) for three successive ages but the source is immaterial.