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DRIVING THE PENSION FUND

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ABSTRACT

In an earlier paper the control characteristics of the aggregate method of funding were displayed by way of its response to a spike, step and random variation in the earned rate of interest, together with a simple intuitive method of setting the valuation rate of interest.

The projected unit method is analysed here in the same way.

A further algorithm is developed which aims at driving an opening fund and contribution rate to a desired fund and contribution rate in n years, using the smoothest path of contribution rates.

1. INTRODUCTION

Actuaries have several methods of controlling the funding of pension schemes. There is, however, very little formal comparison between the methods. If it were possible to consider two pension schemes which were identical in all respects except for the method of funding, and we had the complete histories laid before us, what criteria would we use to decide which method of funding had done the better job?

Control engineers have tried to specify certain desired properties of any particular control system in order to judge how well it is behaving according to the desired properties. One approach which they use is to put certain signals into the system and to look at the characteristics of the output signals. Typical input signals are a spike, a step, a ramp, a sine-wave, and a random input.

These names explain themselves. A simple but instructive approach assumes that the system is in a steady state before the input signal; the disturbance to the output signal is then analysed.

Some of the characteristics of the output signal which they watch are:

(1) how long the output signal takes to return to (e.g. within 95% of) its previous level;
(2) when any initial overshoot takes place;
(3) how large the initial overshoot is;
(4) how much the system multiplies the variance of the input.

If the system is a linear system then the effects of different input signals are additive.

2. PENSION FUNDING

In a paper entitled 'An Actuarial Layman looks at Control Theory' (1984)[3] the author examined the characteristics of the 'aggregate' method of funding where the balance between the future liabilities and the fund in hand is spread as a level contribution rate over the working lifetime of the existing members. In this paper we do the same for a method, which is now in common use, known as the 'projected unit' method. In this method the contribution consists of two parts:

(1) the 'future service' contribution rate pays for the cost of benefits which accrue over the following year;
(2) the 'past service' contribution rate spreads any balancing surplus or deficit evenly over a period of years, e.g. 20 years.

3. MODEL FUND FOR THE AGGREGATE METHOD

We simplify considerably. Also the monetary unit is *C*£ (i.e. £ at constant prices), that is allowing for price/salary inflation.

Model Fund
Membership: 1 person at each age 25 to 64: the person aged 65 each year retires and is replaced by someone aged 25.
Mortality and other Decrements: Nil.
Benefit: Lump sum at age 65 of *C*£40.
Contributions: The same absolute amount in *C*£ for each member.

Control System
Funding Method: 'Aggregate funding', i.e. the actuarial valuation each year assumes no new entrants and calculates a single contribution amount p.a. for each member until retirement or until the next valuation.
Funding Basis: The valuation rate of interest is a real rate of interest and is the average of the last m years of the earned real rate of interest on the fund.
Input Signal: Earned real rate of interest each year.
Output Signal: The recommended contribution rate.

Notation
j_t = earned real rate of interest in year t.
F_t = amount of fund at end of year t.
i_t = valuation rate of interest in valuation at end of year t.
c_t = contribution recommended as a result of the valuation at the end of year t, to commence in year $t+1$.

Contributions are paid at the beginning of the year. Retirements and new entrants take place immediately before a valuation and immediately before the next year commences.

We have: $$F_t = (F_{t-1} + 40c_{t-1})(1 + j_t) - 40. \tag{1}$$

At the valuation at the end of year t:

present value of future benefits

$$= 40a_{40}^{(i_t)} \tag{2}$$

present value of future contributions

$$= c_t(40 + 39v_t + 38v_t^2 + \ldots . + v_t^{39}) \tag{3}$$

where $v_t = 1/(1 + i_t)$.

Hence the recommended contribution rate will be

$$c_t = \frac{40a_{40}^{(i_t)} - F_t}{40 + 39v_t + 38v_t^2 + \ldots + v_t^{39}}. \tag{4}$$

Write

$$X_{40} = 40 + 39v + 38v^2 + \ldots + v^{39}$$

$$= \frac{40 - a_{40}}{1 - v}. \tag{5}$$

From (6) we have $$c_t = \frac{40a_{40}^{(i_t)} - F_t}{X_{40}^{(i_t)}} \tag{6}$$

$$F_t = 40a_{40}^{(i_t)} - c_t X_{40}^{(i_t)} \tag{7}$$

and hence $$F_{t-1} = 40a_{40}^{(i_{t-1})} - c_{t-1}X_{40}^{(i_{t-1})}. \tag{8}$$

Substituting from (7) and (8) into (1) and rearranging we have

$$c_t = \frac{40a_{40}^{(i_t)} - (40a_{40}^{(i_{t-1})} - c_{t-1}X_{40}^{(i_{t-1})} + 40c_{t-1})(1 + j_t) + 40}{X_{40}^{(i_t)}}. \tag{9}$$

In a steady state with $j_t = 0$ we shall have $i_t = 0$ and $c_t = 1$.

Present value of future benefits $= 40 \times 40$	$= 1{,}600$
Present value of future contributions $= 40 + 39 + \ldots . + 1$	$= 820$
Balance = Fund	$= 780$
Check: retrospectively, Fund $= 1 + 2 + \ldots + 39$	$= 780$

In a steady state with $j_t = 1\%$, $i_t = 1\%$ we have

$c = \cdot 810$

$F = 727{\cdot}104.$

We note that the new entrant contribution rate is given by

$$40v^{40} = c\ddot{a}_{40} \text{ at } 1\%$$

i.e.

$$c = \cdot 810 \text{ as above.}$$

4. MODEL FUND FOR THE PROJECTED UNIT METHOD

The difference from the aggregate method is that we have the total contribution c_t,

$$c_t = c_t^f + c_t^p$$

where

$$c_t^f = \text{future service contribution, and}$$

$$c_t^p = \text{past service contribution.}$$

Assuming benefit accrues uniformly for the purposes of the past/future service split

$$c_t^f = \frac{v_t + v_t^2 + \ldots + v_t^{40}}{40}.$$

$$= \frac{a_{40}^{(i_t)}}{40}.$$

Spreading past service surplus/deficit over a fixed term of n years

$$c_t^p = \frac{X_{40}^{(i_t)} - 40 - F_t}{40(1 + v_t + v_t^2 + \ldots v_t^{n-1})}$$

$$= \frac{X_{40}^{(i_t)} - 40 - F_t}{40\ddot{a}_n^{(i_t)}}$$

which gives

$$F_t = X_{40}^{(i_t)} - 40 - 40\ddot{a}_n^{(i_t)} c_t^p.$$

The progress of the fund F_t is given by

$$F_t = [F_{t-1} + 40(c_{t-1}^p + c_{t-1}^f)](1 + j_t) - 40.$$

Substituting in this equation for c_{t-1}^f, F_t and F_{t-1} gives the recurrence relationship between c_t^p and c_{t-1}^p

$$c_t^p = \{X_{40}^{(i_t)} - X_{40}^{(i_{t-1})}(1 + j_t) + 40(1 + j_t) - a_{40}^{(i_{t-1})}(1 + j_t) + (1 + j_t)(\ddot{a}_n^{(i_{t-1})} - 1)c_{t-1}^p\}/40\ddot{a}_n^{(i_t)}.$$

As in the earlier paper we suppose that our rule for setting the valuation rate of interest is the intuitive one, to take the average of the rates earned in recent years, i.e.

$$i_t = (j_t + j_{t-1} + \ldots + j_{t-m+1})/m.$$

For a spike input we put

$$j_t = 1\%,\ t = 0$$
$$= 0 \text{ otherwise.}$$

For a step input we put

$$j_t = 1\%,\ t > 0.$$

Taking $n = 20$ as the period for spreading the past service contribution and $m = 5$ as the period of averaging earned interest rates to obtain the valuation rate, for a spike we obtain the output shown in Table 1. Thus the recommended contribution rates in the first year's response to the spike will be

past service	$0 - \cdot 0370 = -\cdot 0370$
future service	$1 - \cdot 0399 = \ \ \cdot 9601$
total contributions	$1 - \cdot 0769 = \ \ \cdot 9231.$

The deviations in Table 1 are plotted in Figure 1. A step change leads to the deviations shown in Table 2 and Figure 2.

It will be noticed that the immediate output for the first year is the same for a step as for a spike, because at that stage we cannot distinguish between them.

The corresponding outputs for the aggregate method for a spike and a step are given in Tables 3 and 4 and Figures 3 and 4.

5. LINEARIZATION

We denote a small change in j_t (from $j_t = 0$) by Δj_t, and similarly we denote Δi_t, Δc_t^p and Δc_t^f.

We also denote

$$\ddot{a}_n(\Delta i_t) = \ddot{a}_0^n + \ddot{a}_1^n \Delta i_t$$
$$a_{40}(\Delta i_t) = a_0^{40} + a_1^{40} \Delta i_t$$
$$X(\Delta i_t) = X_0 + X_1 \Delta i_t$$

where a_0, a_1, X_0 and X_1 are constants.

We have found the recurrence relationship between c_t^p and c_{t-1}^p. We can similarly find the relationship between Δc_t^p and Δc_{t-1}^p resulting from Δj_t as in the earlier paper for the aggregate method.

After much algebra we find

$$\Delta c_t^p = A\Delta i_t + B\Delta i_{t-1} + C\Delta j_t + D\Delta c_{t-1}^p$$

where, recalling $\ddot{a}_0^{40} = 40$ and $\ddot{a}_1^n = n$

$$A = \frac{X_1}{40n}$$

$$B = -\left[\frac{a_1^{40}}{40n} - \frac{X_1}{40n}\right]$$

$$C = -\frac{X_0}{40n}$$

$$D = \frac{n-1}{n}.$$

Using the Z transform we have

$$(1 - Dz^{-1})\Delta c_z^p = (A + B_{z^{-1}})\Delta i_z + C\Delta j_z.$$

We are also assuming

$$\Delta i_t = (\Delta j_t + \Delta j_{t-1} + \ldots \Delta j_{t-m+1})/m$$

i.e.

$$\Delta i_z = \frac{1}{m}\left(\frac{1-z^{-m}}{1-z^{-1}}\right)\Delta j_z$$

whence, after substitution and putting

$$P = \frac{A+B}{1-D}$$

we find

$$\Delta c_z^p = \left\{\frac{P}{m}\left(\frac{1-z^{-m}}{1-z^{-1}}\right) - \left(\frac{PD+B}{m}\right)\left(\frac{1-z^{-m}}{1-Dz^{-1}}\right) + \frac{C}{1-Dz^{-1}}\right\}.\Delta j_z$$

i.e. $\Delta c_z^p = H_z^p \Delta j_z$.

A control engineer would consider adjusting the transfer function H_z^p to alter the characteristics of the control system. In particular he would look at the poles and zeros of the transfer function, treating it as a rational function of the complex variable z.

If Δj_t is a spike then $\Delta j_z = 1$ and the solution becomes

$$\Delta c_t^p = \frac{P}{m} + \left(C - \frac{PD+B}{m}\right)D^t \qquad \text{for } t \leqslant m-1$$

$$= \left(C - \frac{PD+B}{m}\right)D^t + \left(\frac{PD+B}{m}\right)D^{t-m} \qquad \text{for } t \geqslant m$$

and the numerical results are very similar to those produced directly from the original equations.

More simply we have

$$\Delta c_t^f = \frac{a_1^{40}}{40} \Delta i_t.$$

Hence

$$\Delta c_z^f = \frac{a_1^{40}}{40} \Delta i_z$$

which becomes

$$\Delta c_z^f = \frac{a_1^{40}}{40m} \left(\frac{1 - z^{-m}}{1 - z^{-1}} \right) \Delta j_z.$$

For a spike we have the solution

$$\Delta c_t^f = \frac{a_1^{40}}{40m} \qquad \text{for } t \leqslant m - 1$$

$$= 0 \qquad \text{for } t \geqslant m.$$

Corresponding solutions can be found for a step

$$\Delta j_z = \frac{1}{1 - z^{-1}}.$$

6. STATIONARY RANDOM INPUT

To assess the output from a stationary random input Δj_t we find the ratio $\mathrm{var}(\Delta c_t)/\mathrm{var}(\Delta j_t)$.

If we express Δc_z in the form

$$\Delta c_z = (h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots.)\Delta j_z$$

then

$$\Delta c_t = h_0 \Delta j_t + h_1 \Delta j_{t-1} + h_2 \Delta j_{t-2} + \ldots.$$

and if Δj_t, Δj_{t-1}, etc., are independent identical random variables then

$$\mathrm{var}(\Delta c_t) = (h_0{}^2 + h_1{}^2 + h_2{}^2 + \ldots.)\mathrm{var}(\Delta j_t).$$

Furthermore, but putting $\Delta j_z = 1$, a spike, we see that $h_0, h_1, h_2 \ldots.$ form the resulting sequence of output signals.

The solution for Δc_t^p and Δc_t^f for a spike input were given above. We are interested in

$$\Delta c_t = \Delta c_t^p + \Delta c_t^f = h_t.$$

Write

$$J = \frac{a_1^{40}}{40m} + \frac{p}{m}$$

$$K = c - \frac{PD + B}{m}$$

$$L = \left(\frac{PD + B}{m}\right) D^{-m}$$

then
$$h_t = \Delta c_t = J + KD^t \qquad \text{for } t \leqslant m-1$$
$$= (K + L)D^t \qquad \text{for } t \geqslant m$$

whence var $(\Delta c_t)/\text{var}(\Delta j_t) = \Sigma h_t^2$

$$= \sum_{t=0}^{m-1} (J^2 + 2JKD^t + K^2D^{2t}) + \sum_{t=m}^{\infty} (L + K)^2 D^{2t}$$

$$= J^2 m + 2JK\left(\frac{1 - D^m}{1 - D}\right) + K^2\left(\frac{1 - D^{2m}}{1 - D^2}\right) + \frac{(L + K)^2 D^{2m}}{1 - D^2}.$$

From this we obtain SD $(\Delta c_t)/\text{SD}(\Delta j_t)$

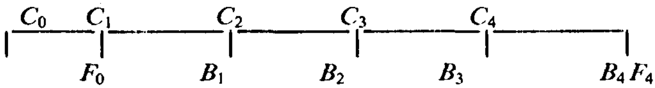
With $n = 20$, for various values of m we have

m	$\text{SD}(\Delta c_t)/\text{SD}(\Delta j_t)$
1	35
2	25
3	20
4	18
5	16
10	12
20	8
30	7

7. DRIVING THE PENSION FUND

Here is an example where the ideas in control theory are used. Although the technique is simple, it is not standard. The example is given in terms of a pension fund, but it has found quite dramatic use for a mutual non-life fund where the problem seemed very difficult using *ad hoc* methods before the following approach was introduced.

Suppose we have a pension fund of amount F_0. Last year's contribution was C_0. Over the next 4 years (say) the outgo on benefits will be B_1, B_2, B_3, B_4. We want the position of the fund at the end of the four years to be F_4 in amount and the contribution amount in year 4 to be C_4.



The rate of interest is j. $J=1+j$.

We might choose C_4 for instance to be the contribution required to bring the fund to a stationary state at the level F_4

$$\text{i.e. } (F+C)J-B=F \text{ whence } C=(B-F_j)/J.$$

On the other hand we might choose C_4 quite differently. F and C together describe 'the state' of the pension fund. We want 'to drive' the state from $F_0\,C_0$ to $F_4\,C_4$. Effectively C_0, F_0, C_4, F_4, B_1, B_2, B_3, B_4 (and j) are all given; we can control the system via C_1, C_2, C_3.

There are various types of control. One is 'bang bang' control, i.e. all or nothing. Another is 'minimum energy' control, i.e. the energy expended in driving from one state to another is minimal. We shall choose the latter and interpret it as aiming to minimize the change in contribution from year to year, i.e. as a maximum smoothness path; our criterion will be to minimize

$$(C_1-C_0)^2+(C_2-C_1)^2+(C_3-C_2)^2+(C_4-C_3)^2. \tag{10}$$

There is an inherent constraint. It is that

$$F_0J^4+C_1J^4+C_2J^3+C_3J^2+C_4J-B_1J^3-B_2J^2-B_3J-B_4=F_4 \tag{11}$$

i.e. accumulated monies must equal F_4.

Re-write (11) in terms of unknowns C_1, C_2 and C_3 and knowns:

$$C_1J^4+C_2J^3+C_3J^2+(F_0J^4+C_4J^4-B_1J^3-B_2J^2-B_3J-B_4-F_4)=0 \tag{12}$$

$$\text{or } C_1J^4+C_2J^3+C_3J^2+K=0 \tag{13}$$

where K=item in ( ).

Hence we want to minimize (10) subject to (13).

This is a standard problem in VIth Form text books on calculus. We introduce the dummy variable as a Lagrange multiplier and minimize the expression

$$H=(13)+\lambda(10)$$

with respect to C_1, C_2, C_3 and λ, i.e we form

$$\frac{\delta H}{\delta C_1}=0 \qquad \frac{\delta H}{\delta C_2}=0 \qquad \frac{\delta H}{\delta C_3}=0$$

and using (13), obtain four equations in four unknowns and solve. λ itself is not of interest.

We obtain

$$\frac{\delta H}{\delta C_1} = J^4 + 2\lambda(C_1 - C_0) - 2\lambda(C_2 - C_1) = 0 \tag{14}$$

$$\frac{\delta H}{\delta C_2} = J^3 + 2\lambda(C_2 - C_1) - 2\lambda(C_3 - C_2) = 0 \tag{15}$$

$$\frac{\delta H}{\delta C_3} = J^2 + 2\lambda(C_3 - C_2) - 2\lambda(C_4 - C_3) = 0 \tag{16}$$

and

$$K + C_1 J^4 + C_2 J^3 + C_3 J^2 = 0. \tag{13}$$

We can rearrange the equations making $\frac{J^2}{2\lambda}$ as the fourth unknown rather than λ.

Put $\frac{J^2}{2\lambda} = L$

$$J^2 L + 2C_1 - C_2 = C_0$$
$$JL - C_1 + 2C_2 - C_3 = 0$$
$$L - C_2 + 2C_3 = C_4$$
$$J^2 C_1 + JC_2 + C_3 = -K/J^2.$$

We obtain 4 simultaneous linear equations in 4 unknowns L_1, C_1, C_2, C_3 from which to obtain C_1, C_2 C_3.

Table 5 shows some numerical values to give the flavour of different situations. The first example is the basic example and shows a steady state. The (real) rate of interest is $j = {\cdot}02$, F = 623, C = 27, and annual outgo is B = 40, hence

$$(F+C)J - B = (623+27)^* \ 1{\cdot}02 - 40 = 623 = F.$$

The following examples start with over-funding, under-funding, high contributions and low contributions in different combinations and move to the steady position with minimum changes in contributions from year to year. The final example, Example 10, moves to a different steady state funding level with the original contribution rate.

More extreme examples can show negative contribution rates in the early years. If that is not acceptable they can be set to zero and the contributions for the remaining years can be left as variables for solution.

There is also no difficulty in extending the time horizon. The total 'energy' required to drive from the opening state to the final state is then less.

There may be an algorithm which eliminates negative values and allows for different time horizons, but a simple trial-and-see approach seems adequate.

It should be noted that if the time horizon is reduced then a further criterion of good design is required. Almost certainly this would be a limitation on the variation allowed within the solution set of contribution rates—corresponding to the limitation on power in an engineering system.

The difficulty we found using *ad hoc* methods was that if we wished to bring the fund up to a higher level and raised the contribution rate to do it quickly, then we were in danger of needing to reduce the contribution rate quite drastically after achieving the required level of, and in order to avoid continuing excessive, funds. The resulting sequence of contribution rates was not acceptable. The effect of different time horizons on the optional contribution path can be shown diagrammatically as in Figure 5.

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Table 1. *Projected Unit Method:* n = *20:* m = *5: spike 1%*

t	j_t	i_t	*cpt* 1 (%)	*cft* 1 (%)	*cptf* 1 (%)
0	·01	·002	−3·70	−3·99	−7·69
1	0	·002	−3·31	−3·99	−7·30
2	0	·002	−2·94	−3·99	−6·93
3	0	·002	−2·59	−3·99	−6·58
4	0	·002	−2·25	−3·99	−6·24
5	0	0	·71	·00	·71
6	0	0	·68	·00	·68
7	0	0	·64	·00	·64
8	0	0	·61	·00	·61
9	0	0	·58	·00	·58
10	0	0	·55	·00	·55
11	0	0	·52	·00	·52
12	0	0	·50	·00	·50
13	0	0	·47	·00	·47
14	0	0	·45	·00	·45
15	0	0	·43	·00	·43
16	0	0	·40	·00	·40
17	0	0	·38	·00	·38
18	0	0	·37	·00	·37
19	0	0	·35	·00	·35
20	0	0	·33	·00	·33
30	0	0	·20	·00	·20
40	0	0	·12	·00	·12
50	0	0	·07	·00	·07
99	0	0	·01	·00	·01

cpt 1 = deviation of past service contribution from 0 p.a.
cft 1 = deviation of future service contribution from 1 p.a.
cpft 1 = *cpt* 1 + *cft* 1, i.e. total deviation from stationary contribution of 1 p.a.

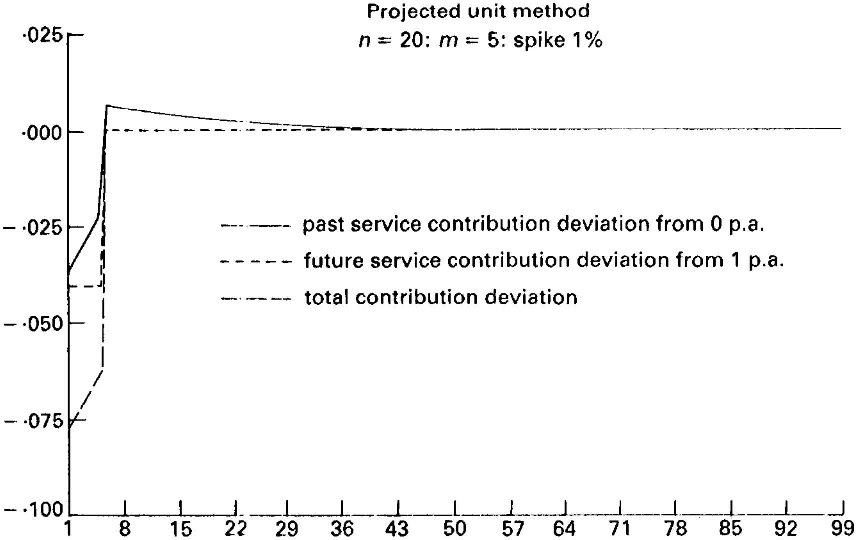
Projected unit method
$n = 20$: $m = 5$: spike 1%
·025
·000
– ·025
– ·050
– ·075
– ·100
1
8
15
22
29
36
43
50
57
64
71
78
85
92
99
past service contribution deviation from 0 p.a.
future service contribution deviation from 1 p.a.
total contribution deviation

Figure 1.

Table 2. *Projected Unit Method:* n = *20:* m = *5: step 1%*

t	j_t	i_t	*cpt* 1 (%)	*cft* 1 (%)	*cptf* 1 (%)
0	·01	·002	−3·70	−3·99	−7·69
1	·01	·004	−7·05	−7·76	−14·81
2	·01	·006	−10·03	−11·33	−21·36
3	·01	·008	−12·67	−14·71	−27·38
4	·01	·01	−14·96	−17·91	−32·87
5	·01	·01	−14·28	−17·91	−32·19
6	·01	·01	−13·63	−17·91	−31·54
7	·01	·01	−13·01	−17·91	−30·92
8	·01	·01	−12·42	−17·91	−30·33
9	·01	·01	−11·86	−17·91	−29·77
10	·01	·01	−11·32	−17·91	−29·23
11	·01	·01	−10·80	−17·91	−28·72
12	·01	·01	−10·31	−17·91	−28·23
13	·01	·01	−9·84	−17·91	−27·76
14	·01	·01	−9·40	−17·91	−27·31
15	·01	·01	−8·97	−17·91	−26·88
16	·01	·01	−8·56	−17·91	−26·48
17	·01	·01	−8·17	−17·91	−26·09
18	·01	·01	−7·80	−17·91	−25·72
19	·01	·01	−7·45	−17·91	−25·36
20	·01	·01	−7·11	−17·91	−25·02
30	·01	·01	−4·47	−17·91	−22·38
40	·01	·01	−2·81	−17·91	−20·72
50	·01	·01	−1·76	−17·91	−19·68
99	·01	·01	−·18	−17·91	−18·09

cpt 1 = deviation of past service contribution from 0 p.a.
cft 1 = deviation of future service contribution from 1 p.a.
cpft 1 = *cpt* 1 + *cft* 1, i.e. total deviation from stationary contribution of 1 p.a.

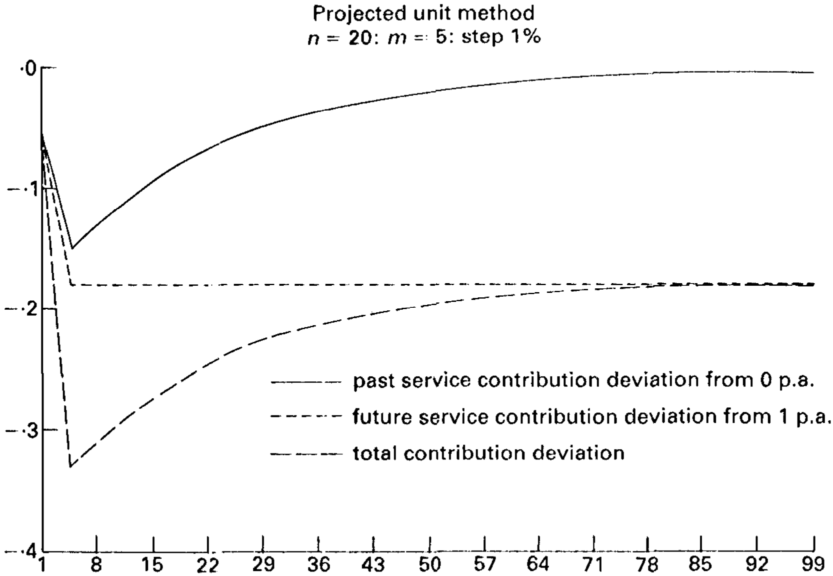
Projected unit method
$n = 20$: $m = 5$: step 1%
·0
—·1
—·2
—·3
—·4
1
8
15
22
29
36
43
50
57
64
71
78
85
92
99
past service contribution deviation from 0 p.a.
future service contribution deviation from 1 p.a.
total contribution deviation

Figure 2.

Table 3. *Aggregate Method:* m = *5: spike 1%*

t	j_t	$ct\,1$ (%)
0	·01	−6·40
1	0	−6·08
2	0	−5·77
3	0	−5·48
4	0	−5·21
5	0	·41
6	0	·39
7	0	·37
8	0	·35
9	0	·34
10	0	·32
11	0	·31
12	0	·29
13	0	·28
14	0	·26
15	0	·25
16	0	·24
17	0	·23
18	0	·21
19	0	·20
20	0	·19
30	0	·12
40	0	·07
50	0	·04
99	0	·00

$ct\,1$ = deviation of contribution from 1 p.a.

Table 4. *Aggregate Method:* m = *5: step 1%*

t	j_t	$ct\,1$(%)
0	·01	−6·40
1	·01	−12·47
2	·01	−18·21
3	·01	−23·60
4	·01	−28·65
5	·01	−28·21
6	·01	−27·79
7	·01	−27·39
8	·01	−27·00
9	·01	−26·63
10	·01	−26·28
11	·01	−25·95
12	·01	−25·63
13	·01	−25·33
14	·01	−25·03
15	·01	−24·76
16	·01	−24·49
17	·01	−24·24
18	·01	−24·00
19	·01	−23·77
20	·01	−23·55
30	·01	−21·84
40	·01	−20·77
50	·01	−20·11
99	·01	−19·10

$ct\,1$ = deviation of contribution from 1 p.a.

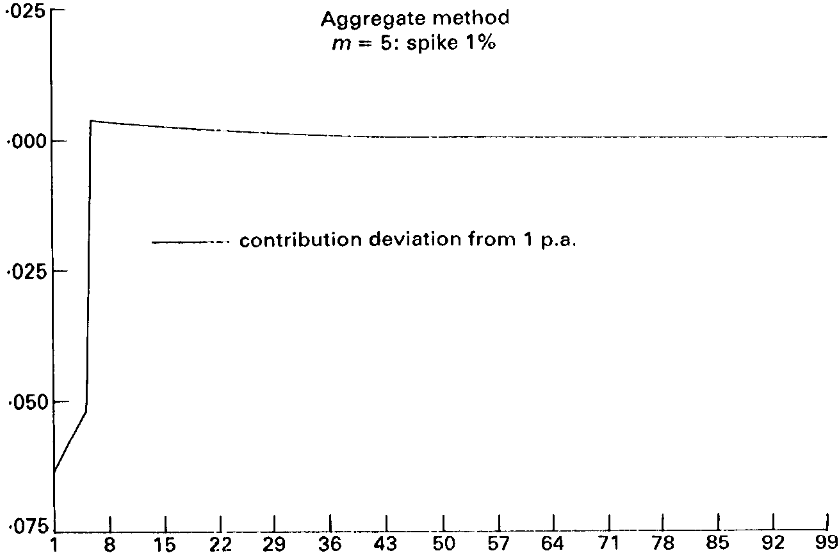

Figure 3.

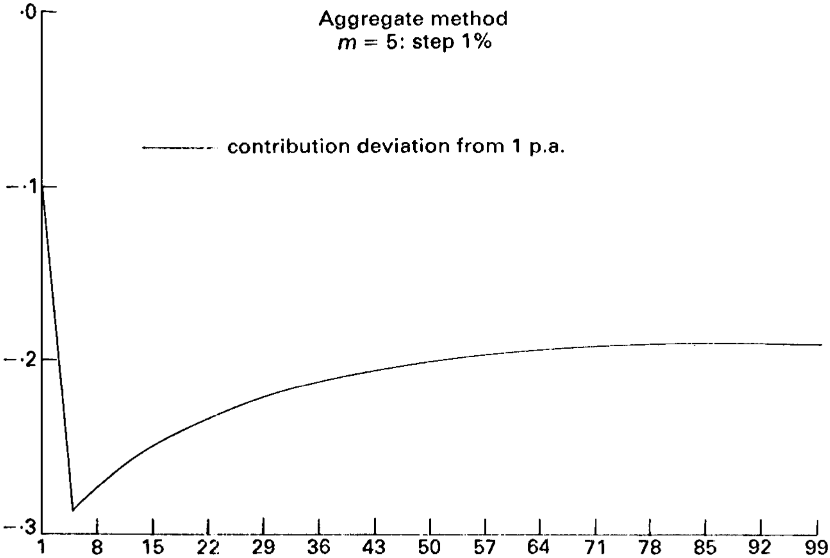

Figure 4.

Table 5

C	C_0		C_1		C_2		C_3		C_4	
F		F_0		F_1		F_2		F_3		F_4
Example 1										
C	27		27		27		27		27	
F		623		623		623		623		623
Example 2										
C	27		46·1		52·3		45·8		27	
F		561		579·2		604·2		623		623
Example 3										
C	32		47·6		51·8		44·8		27	
F		561		580·7		605·2		623		623
Example 4										
C	22		44·6		52·8		46·8		27	
F		561		577·7		603·2		623		623
Example 5										
C	22		25·5		27·5		28·0		27	
F		623		621·5		622·0		623		623
Example 6										
C	32		28·5		26·5		26·0		27	
F		623		624·5		624·0		623		623
Example 7										
C	27		7·9		1·7		8·2		27	
F		685		666·8		641·8		623		623
Example 8										
C	32		9·4		1·2		7·2		27	
F		685		668·2		642·8		623		623
Example 9										
C	22		6·4		2·2		9·2		27	
F		685		665·3		640·8		623		623
Example 10										
C	22		100·5		126·0		99·5		22	
F		623		698·0		800·5		878		878

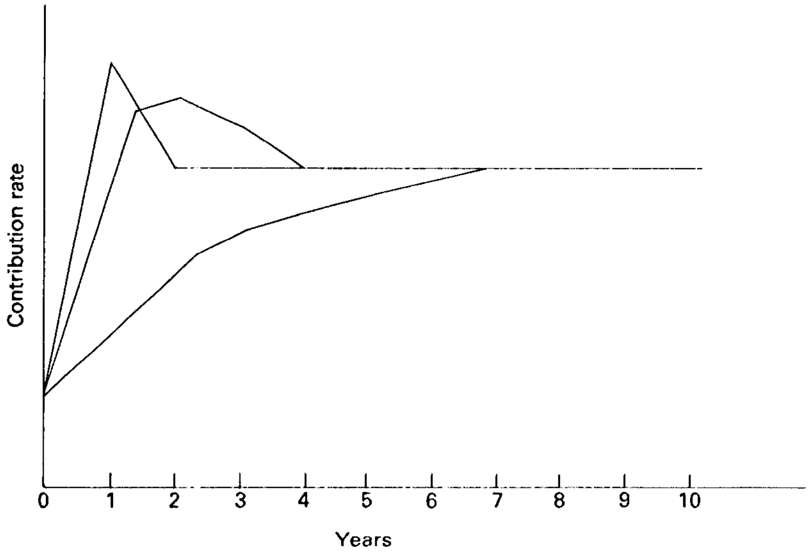

Figure 5.