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THE GILT-EDGED MARKET REFORMULATED

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1. BACKGROUND

1.1. In this note it is argued that models of the gilt-edged market which are based on yield curves are unnecessarily restrictive and should not be expected to give a satisfactory statistical 'fit' in current conditions. The new model which is formulated relates market prices directly to the life and coupon without diverting into the computation of redemption yields. Indeed, it is suggested that the yield calculation destroys the inherent simplicity of the underlying equations —which follow from a simple assumption concerning the return from different portfolios. The method avoids the inconsistency inherent in the conventional analysis of discounting future investment proceeds at a uniform rate of interest when the yield curve itself implies that interest rates will vary in the future.

1.2. It is shown that the price of a stock can be represented as the present value of the net interest income and redemption proceeds. However, unlike conventional methods, a varying force of interest[1] is employed, being the same for all stocks at any point of time. The market prices themselves define the varying rates of interest and income tax and contain an implicit forecast (on the basis of consistency) of all future prices. It must be stressed that these price projections should not be thought of as a real forecast any more than the implicit assumptions of a yield calculation forecast that the stock can be bought or sold on the yield basis throughout its life.

1.3. As formulated by Marshall[2] and Pepper,[3] the yield curve connects the gross redemption yields of stocks with different redemption dates or volatilities. Burman and White[4] have formulated the adjustments required to incorporate the effect of coupon on the curves using an expectation hypothesis, and have introduced the notion of par yield curves. Hamilton[5] gives an excellent summary of the methods used by stockbrokers on a daily basis. The papers of Grant,[6] Brew,[7] and Pepper and Salkin[8] contain further theoretical extensions. Some of the ideas in Clarkson's work[9] have been incorporated in further work by Burman *et al.*[10], [11]

1.4. If there is an exact relationship between yield and volatility, the definition of the latter immediately implies a relationship between yield and price alone (see Appendix). This position is quite untenable except as a crude approximation. If the yield curve is formulated in terms of life, this is a perfectly valid mathematical model (see § 2.6) but difficulties arise as soon as an attempt is made to reconcile the two quite separate sets of assumptions. Redemption yields are defined as the annual income that can be derived from a stock if a simple model is used to describe future levels of interest rates. This definition cannot be reconciled with the interest rate models implied by the observed yield curve. These difficulties suggest that it might be more fruitful to seek a simpler relationship connecting the price of a stock with its coupon and life directly, without the diversion of yields based on assumptions known to be false. Whether the model is required for the purpose of anomaly switching exercises, or to confirm the terms of a new issue, it is the relationship between the price, coupon and life that is required.

2. THE MODEL

2.1. Some simplifying assumptions are required in the initial formulation. Consider the market to consist of a large number of stocks with fixed redemption dates and repayment at £100. For each date there are several stocks with different coupons and to avoid the problem of accrued interest, stocks are assumed to pay interest daily. All prices are therefore 'clean'.

2.2. Consider a group of stocks with the same redemption date. By taking several different portfolios of such stocks, some will give the same interest income and redemption proceeds as others whatever uniform rate of taxation is applied to the income of the portfolios. Consider for example the three stocks

(a) Gas 3% 1990/5
(b) Funding 6% 1993
(c) Treasury 9% 1994

Ignoring for the time being the differences in redemption dates, the same income and capital will result from the purchase of £50 nominal of (a) and £50 of (c) as that of £100 of (b). This is quite independent of considerations of taxation. Given that investors follow an active policy of switching into or out of any stocks, and therefore anomalies are corrected quickly, we should expect the value of all portfolios giving the same proceeds to be the same. In this example the price of (b) should be midway between that of (a) and (c). The linearity of this relationship is demonstrated by plotting the 'cleaned' prices of stocks against their coupons in Figure 1.

2.3. This argument is now formulated mathematically. Suppose that the value that investors place on £100 of capital at the redemption date of a group of stocks with the same date is V. The value of a unit annuity per annum payable until redemption is A. Then for a stock with coupon g its price P will be

$$P = gA + V \tag{2.3.1}$$

The factors A and V could be cast into the usual form involving compound interest factors, but this is quite unnecessary. At no time is it assumed that the same uniform yield basis occurs in both A and V. It is shown in the Appendix that the equation can always be written

$$P = g(1-t_n)\,a_{\overline{n}|}^{i} + 100\,v_i^n \tag{2.3.2}$$

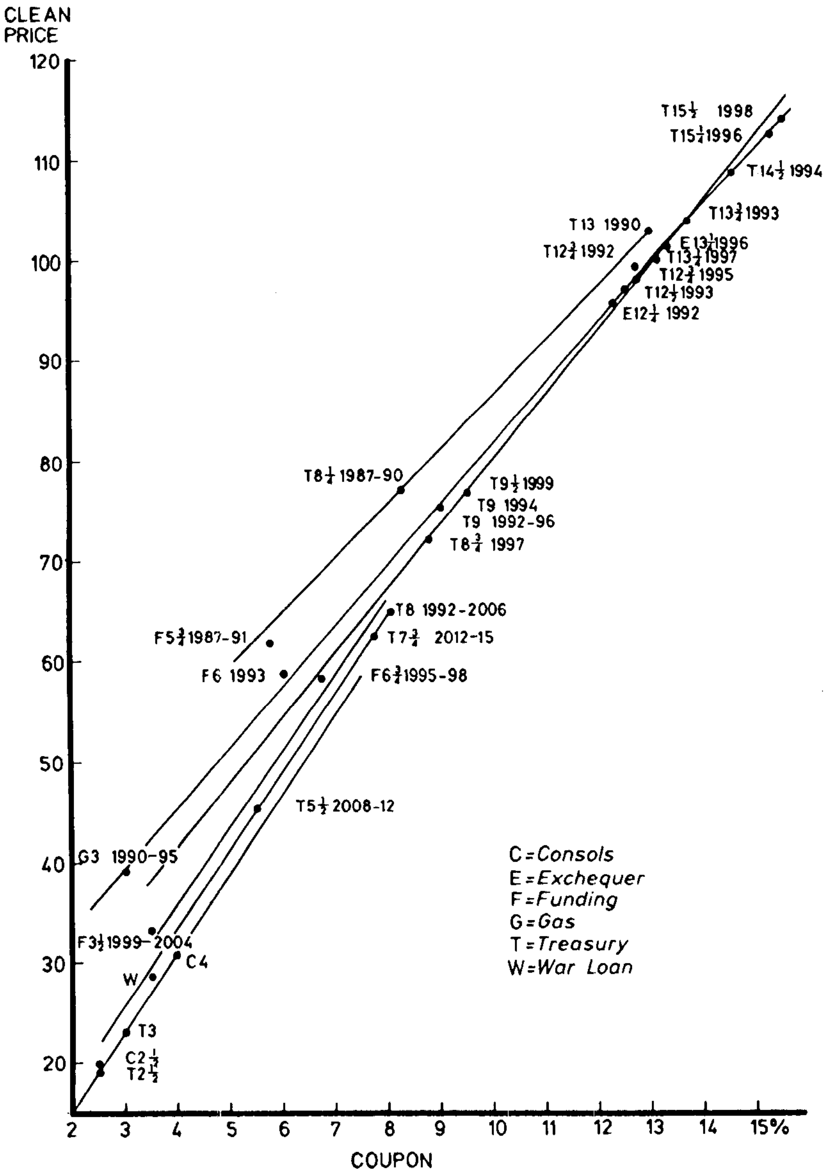

FIG. 1: Situation on 25 March 1977

for some tax rate t_n and interest rate i, the a and v functions having the usual meanings.

2.4. In other words, stocks with the same redemption date must have the same net redemption yield for some rate of tax. In the example of Figure 1 for the 1993–95 stocks the corresponding net redemption yield is 10·9% at a taxation rate of 16%. Alternatively we can say that investors put a value of 6·0 on a unit annuity payable until 1994 and a value of 22 on the redemption proceeds.

2.5. Now, consideration must be given to whether equation (2.3.1) holds for all stocks with the same redemption dates but with varying coupons which make certain stocks attractive to different classes of taxpayers. The validity of the equation clearly depends on an active market in those stocks for which the equation is to hold with all classes of investors switching between the various stocks. Now for taxed investors who deal with coupons between say 3% and $7\frac{1}{2}$% market prices should satisfy

$$P = g\,A_{taxed} + V_{taxed} \qquad \text{if } g \leqslant 7\tfrac{1}{2}\% \tag{2.5.1}$$

where A_{taxed} denotes the value to an investor who pays income tax of an annuity of 1 per annum gross and V_{taxed} denotes the corresponding value of £100 at the redemption date. Similarly, for gross funds

$$P = g\,A_{gross} + V_{gross} \qquad \text{if } g \geqslant 5\%, \text{ say.} \tag{2.5.2}$$

Provided that there are sufficient stocks with coupons between say 5% and $7\frac{1}{2}$% dealt in by all classes of investors, since there is only one market price for each stock (except for special ex-dividend situations) A and V must be the same for all investors. Even if there are few stocks in this coupon range at a particular redemption date, arguments involving continuity will show that the relationship still holds if there are stocks with these intermediate coupons and slightly different redemption dates. Clarkson(9) rejects this argument and always expects higher order terms in g to take account of the differing income/capital requirements of investors with different tax positions. Our model can be considered as the limiting case of his family of equations. It is shown in the Appendix that linearity is a necessary condition for equal performance.

2.6. Thus in Figure 1 the slope measures the value of a unit annuity and the intercept the value of £100 of capital repaid at the redemption dates. In general, summing up the model in one equation

$$P(n,g) = gA(n) + V(n) \tag{2.6.1}$$

where $P(n,g)$ is the 'cleaned' price of a stock with a life of n and coupon g. $A(n)$ and $V(n)$ are factors which depend only on n. This simple equation replaces the usual redemption yield/yield curve relationship found by solving

$$P(n,g) = g(1-v^n)/i + 100\,v^n,\ v = 1/(1+i) \tag{2.6.2}$$

$$i = F_1(n) \text{ or } i = F_2(dP/Pdi) \tag{2.6.3}$$

for some functions F_1 or F_2.

2.7. Equation (2.6.2) is a special case of equation (2.6.1) but by forcing the connection between A and V implied by the former, impossible and unnecessary constraints are imposed. However, by considering equation (2.3.2) one should not be surprised to find the yield curve formulation to be satisfactory when the market is dominated by gross investors. Clearly, by solving equation (2.6.2) for the redemption yield i we may be destroying the simplicity and linearity of the model expressed in equation (2.6.1). If the terms of a new issue are required, equation (2.6.1) is very much easier to apply than the usual equations which in any case require further adjustments for low coupons.[4] The remainder of Section 2 is devoted to an analysis on lines analogous to reinvestment rate theories or the equal performance model.

2.8. The function $V(n)$ which capitalizes the value of £100 due n years hence will not in general be of the form $(1+i)^n$ for some fixed rate of interest i. By considering factors such as $V(n)/V(n-1)$ one can estimate the market's implicit assumptions about the one-year return on capital $(n-1)$ years hence. In fact, the differential coefficient of the logarithm of $V(n)$ gives the market's forecasts of short-term interest rates n years hence.

Table 1. *Values of V_n and the implied force of interest at the redemption date of each stock shown*

Stock	$V(n)$	$-d(\log V)/dn$	Stock	$V(n)$	$-d(\log V)/dn$
TRY-9-78	95·798	5·026	T5-86/89	38·932	11·775
TRY10H78	94·718	5·411	TRY13-90	37·863	11·779
EX5-76/8	93·997	5·632	T8Q87/90	36·172	11·786
TRY11H79	92·004	6·052	F5T87/91	33·097	11·801
TRY-3-79	89·294	6·209	TRY12T92	30·286	11·821
E4Q74/79	89·161	6·208	EXQ12Q92	28·103	11·836
TRY10H79	88·630	6·194	TRY12H93	25·642	11·855
E3H76/79	88·402	6·185	FDG-6-93	25·167	11·858
TRCV9-80	86·832	6·078	TRY13T93	24·179	11·866
TRY9H-80	85·796	5·983	TRY14H94	23·877	11·869
T3H77/80	85·368	5·939	TRY-9-94	21·998	11·885
F5Q78/80	85·368	5·939	G3-90/95	20·880	11·895
EXQ13-80	83·189	5·752	TRY12T95	19·608	11·907
TRY11H81	82·515	5·719	T9-92/96	18·872	11·914
T3H79/81	82·081	5·765	TRY15Q96	18·579	11·917
TRY9T-81	81·476	5·773	EX-13Q96	18·499	11·918
EXQ12T81	77·196	6·413	TRY13Q97	17·070	11·931
T8H80/82	77·159	6·422	TRY8T-97	15·899	11·943
TRY-3-82	76·612	6·586	T6T95/98	14·693	11·955
TRY14-82	76·112	6·718	TRY15H98	13·991	11·963
TRY12-83	69·642	8·348	TRY9H-99	13·504	11·968
F5H82/84	65·204	8·935	F3H99/04	6·950	12·059
T8H84/86	54·664	11·343	T8-02/06	5·292	12·099
F6H85/87	50·779	11·657	T5H08/12	2·533	12·247
T7T85/88	46·969	11·738	T7T12/15	1·876	12·338
T3-78/88	44·812	11·754			

2.9. The same argument can be applied to the function $A(n)$. $A(n)-A(t),(t<n)$ represents the present value of a unit annuity payable from time t to time n. For consistency the value of this annuity t years hence should be $100\{A(n)-A(t)\}/V(t)$. Clearly, if the market's implicit assumptions are fulfilled, one can compute the price of every stock any time in the future. For example, the price of an undated stock with coupon g is given by

$$P_{undated}(n=0) = g\,A(\infty) \tag{2.9.1}$$

$$P_{undated}(n=t) = 100\,g\,\{A(\infty)-A(t)\}/V(t) \tag{2.9.2}$$

Example: War Loan $g = 1{\cdot}75\%$, $A(\infty) = 15{\cdot}79$

Table 2. *Values of* A, V *and* P, *the implied price of War Loan at the redemption date of each stock shown.*

Stock	A	V	P	Stock	A	V	P
TRY-9-78	1·007	95·798	27·003	T8Q87/90	10·087	36·172	27·585
TRY10H78	1·215	94·718	26·926	F5T87/91	10·468	33·097	28·132
EX5-76/8	1·354	93·997	26·875	TRY12T92	10·830	30·286	28·650
TRY11H79	1·741	92·004	26·721	EXQ12Q92	11·124	28·103	29·045
TRY-3-79	2·277	89·294	26·481	TRY12H93	11·469	25·642	29·484
E4Q74/79	2·304	89·161	26·468	FDG-6-93	11·537	25·167	29·567
TRY10H79	2·410	88·630	26·416	TRY13T93	11·680	24·179	29·738
E3H76/79	2·456	88·402	26·393	TRY14H94	11·724	23·877	29·790
TRCV9-80	2·777	86·832	26·225	TRY-9-94	12·004	21·998	30·109
TRY9H-80	2·992	85·796	26·103	G3-90/95	12·174	20·880	30·294
T3H77/80	3·081	85·368	26·050	TRY12T95	12·371	19·608	30·501
F5Q78/80	3·081	85·368	26·050	T9-92/96	12·487	18·872	30·618
EXQ13-80	3·543	83·189	25·760	TRY15Q96	12·533	18·579	30·664
TRY11H81	3·686	82·515	25·667	EX-13Q96	12·546	18·499	30·676
T3H79/81	3·777	82·081	25·610	TRY13Q97	12·775	17·070	30·897
TRY9T-81	3·904	81·476	25·528	TRY8T-97	12·966	15·899	31·074
EXQ12T81	4·737	77·196	25·053	T6T95/98	13·165	14·693	31·253
T8H80/82	4·744	77·159	25·050	TRY15H98	13·282	13·991	31·355
TRY-3-82	4·842	76·612	25·005	TRY9H-99	13·364	13·504	31·425
TRY14-82	4·931	76·112	24·966	F3H99/04	14·504	6·950	32·346
TRY12-83	5·957	69·642	24·706	T8-02/06	14·803	5·292	32·601
F5H82/84	6·518	65·204	24·882	T5H08/12	15·308	2·533	33·231
T8H84/86	7·818	54·664	25·517	T7T12/15	15·429	1·876	33·530
F6H85/87	8·309	50·779	25·779	CONSLS-4	15·789	0	
T7T85/88	8·791	46·969	26·075	WAR LOAN	15·789	0	
T3-78/88	9·056	44·812	26·293	TREASY-3	15·789	0	
T5-86/89	9·756	38·932	27·119	CONSLS2H	15·789	0	
TRY13-90	9·883	37·863	27·295	TREASY2H	15·789	0	

2.10. This argument is the reinvestment rate concept in a new guise. That concept suggests that two stocks can be compared by considering at what price the longer-dated one would have to stand at the redemption date of the shorter-dated one in order that the same effective yield is obtained by buying the shorter and switching into the longer at redemption, or by simply holding the longer stock throughout. By comparing different pairs of stocks and using different tax rates, various differing yield pattern scenarios are obtained, even for one stock. These problems are due largely to the different rates of roll-up of excess (or deficient) income. The methods advanced in this paper are very similar in concept but by using the $V(n)$ function and derivatives, a constantly changing roll-up rate is applied to all the stocks and a self-consistent pattern emerges. Each stock has only one forecast price at each date in the future. The general formula for the price of a stock whose life is n and coupon g, t years hence is

$$P(t) = 100[g\{A(n)-A(t)\}+V(n)]/V(t) \tag{2.10.1}$$

The functions have the following properties

$$V(0) = 100,\ A(0) = 0,\ V(\infty) = 0 \tag{2.10.2}$$

It can be verified that

$$P(0) = gA(n)+V(n) \quad \text{as defined}$$

$$P(n) = 100 \quad \text{as it is redeemed at par.}$$

Thus, given a yield list showing A and V for each stock, it would be a very simple exercise to use equation (2.10.1) to estimate the price of any stock at the redemption date of any other. Any resulting inconsistencies would suggest policy switching situations.

2.11. Equations (2.9.1) and (2.9.2) may be combined using the fact that the yield on undated stocks $Y(t)$ at time t is simply

$$Y(t) = V(t)/\{A(\infty)-A(t)\} \tag{2.11.1}$$

But

$$Y(0) = V(0)/A(\infty) \tag{2.11.2}$$

Combining equation (2.10.1) with (2.11.1) and (2.11.2) we can show that for a dated stock whose price is $P(t)$ at time t and coupon g

$$\{P(t)-100g/Y(t)\}V(t) = \{P(0)-100g/Y(0)\}V(0) \tag{2.11.3}$$

where $Y(t)$ is the yield on undated stocks at time t and $P(t)-100g/Y(t)$ is the amount by which the price of a dated stock should exceed that at which it would stand t years hence if it were undated. Henceforth this excess price is denoted by $E(t)$. Equation (2.11.3) may be written as

$$E(t)V(t) = E(0)V(0) \tag{2.11.4}$$

or

$$E(t)/E(0) = V(0)/V(t) \tag{2.11.5}$$

$V(0)/V(t)$ represents the amount of a unit investment after t years where a varying force of interest is used to compound throughout. Since the right hand side of (2.11.5) only involves the date t, the left hand side which is computed for each stock must be independent of the characteristics of such stocks for the model to hold. This suggests that rather than compute price ratios, one should instead use excess-price ratios for monitoring deviations from the model.

2.12. It should be noted that the functions $V(n)/V(n-1)$ and $\{A(n)-A(t)\}/V(t)$ will be the result of a smoothing process and will not, therefore, refer to any particular stocks but rather to a quantification of investors' expectations about the values of interest rates and annuity values in the future. For convenience in the tables we have computed these functions at the redemption date of each stock. The values will fluctuate from day to day in much the same way as does the level and shape of the conventional yield curve.

Fluctuations in the ratios E_1/E_2 for two stocks cannot be related simply to the profit on a switch in the general case. This profit will, in general, be a combination of both short-term anomalies and of any changes in the level and term structure of interest rates. If, however, both stocks have a comparable running yield F which is small compared to the yield on undated stocks Y, a change of $X\%$ in E_1/E_2 will correspond to a profit of approximately

$$X(1-F/Y)\% \tag{2.12.1}$$

2.13. For a dated stock which always remains on a yield basis i the following equation connects the current and future prices

$$\{P(t)-g/i\}v^t = \{P(0)-g/i\}v^0 \tag{2.13.1}$$

This is the analogue of the more general equation (2.11.3). One consequence of equation (2.11.4) is that for consistency the excess price cannot change sign, since V is always positive. Hence, if a dated stock gives the same return as an undated over one or more small time intervals, the difference between the flat yields of the stocks must always have the same sign.

3. ANALYSIS OF DATA

3.1. The methods developed in Section 2 depend on the assumption concerning the continuous payment of interest. A method of 'cleaning' the net accrued interest from the market price is required. Although the method outlined below can be used to find the best value of income tax to be deducted, in the sample data the full value of accrued interest was removed from the market prices.

3.2. The functions A and V are approximated by the envelope of straight lines where the time axis is defined in terms of v^n where $v = 1/(1+j), n$ is the life, and j is a constant. This method introduces a bias in the smoothing process and a second application of the method corrects for this. Accrued interest is treated as follows:

Value of unit semi-annual annuity starting in six months' time, payable for n periods $A(n)$
Corresponding value for $(n-1)$ payments $A(n-1)$
Value of n payments starting immediately $\zeta + A(n-1)$
where ζ is the value of a unit payment payable immediately (representing average net of tax amount).

If the n payments start in $(1-k)$ periods, by interpolation we have value

$$k\{\zeta + A(n-1)\} + (1-k)A(n) = k\zeta + A(n-k) \tag{3.2.1}$$

$k\zeta$ represents the conventional net accrued: for ex-dividend situations k will be negative.

3.3. A least squares method is used to find a set of functions which give the smallest percentage error in price. This is simplest to formulate in terms of fitting price to the functions where a weight inversely proportional to the square of the price is used for each observation.

3.4. A special adjustment has to be made for Consols $2\frac{1}{2}$, having four payments per annum. This can be considered to be the sum of two annuities, one starting half a period (i.e. three months) after the other. It is merely necessary to add in the regression equations one quarter of the half-yearly coupon to the formula for price.

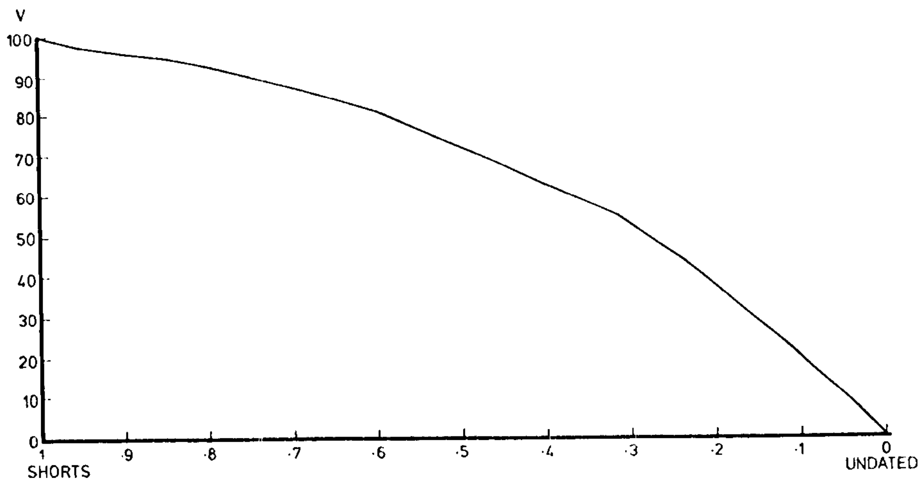

FIG. 2: Values of V (present value of 100) against term on 25 March 1977

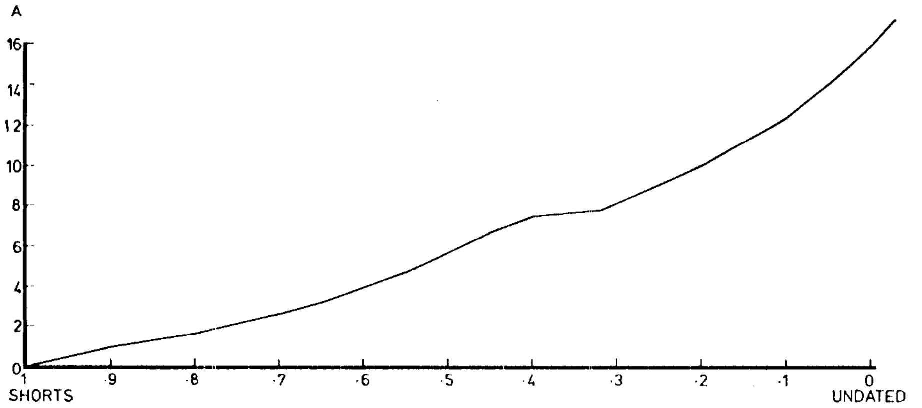

FIG. 3: Values of A (multiplying factor for half-yearly coupon) against term on 25 March 1977

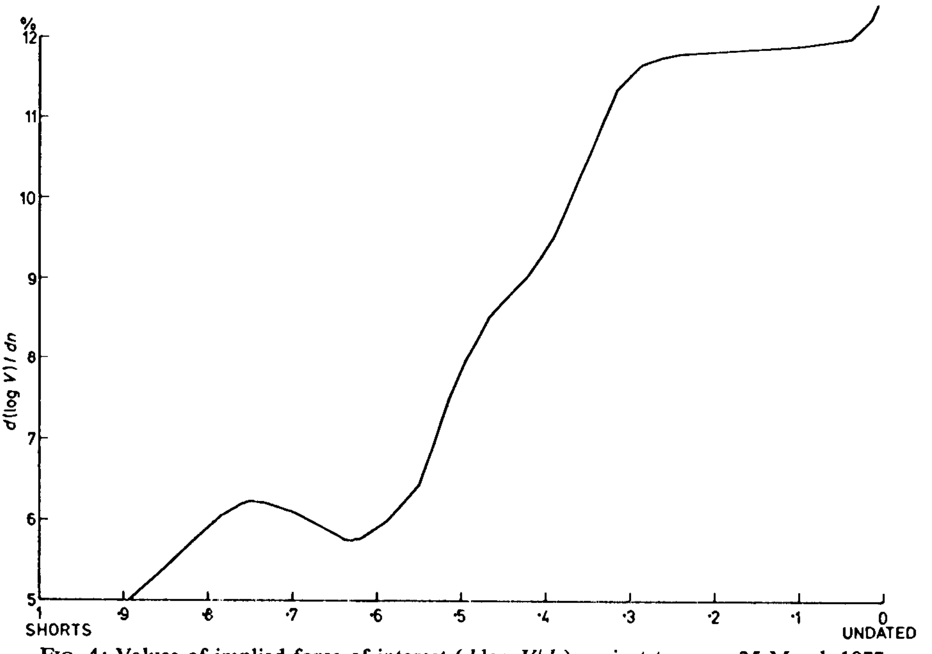

FIG. 4: Values of implied force of interest ($d \log V/dn$) against term on 25 March 1977

4. CONCLUSIONS

4.1. A method has been developed for explaining the price structure of gilt-edged stocks. Unlike yield curve methods, which first compute the yield on the usual assumptions of roll-up rates and then relate this yield to the life, the method admits the market's implied assumptions as to future interest rates. In one operation prices are related to the present value of all the components of interest and capital redeemed by using a unique force of interest defined for all points of time in the future by the market itself.

4.2. The model quantifies investors' expectations (as implied by the structure of prices) of short- and long-term interest rates at all future dates. The same equations may be used to investigate how one's own forecasts might differ from those implied by the model and thereby help to highlight policy moves.

Table 3. *Values of the 'clean' prices and 'fitted' prices on 25 March 1977*

Stock	'Clean' price	'Fitted' price	Stock	'Clean' price	'Fitted' price
TRY-9-78	99·500	100·328	T8Q87/90	77·524	77·781
TRY10H78	101·375	101·097	F5T87/91	61·985	63·193
EX5-76/8	96·500	97·381	TRY12T92	99·695	99·331
TRY11H79	102·250	102·014	EXQ12Q92	95·957	96·240
TRY-3-79	93·250	92·710	TRY12H93	97·466	97·322
E4Q74/79	94·375	94·057	FDG-6-93	58·770	59·778
TRY10H79	100·625	101·285	TRY13T93	104·182	104·479
E3H76/79	92·125	92·701	TRY14H94	108·763	108·877
TRCV9-80	98·750	99·327	TRY-9-94	75·745	76·016
TRY9H-80	98·750	100·006	G3-90/95	39·521	39·141
T3H77/80	90·000	90·760	TRY12T95	98·444	98·475
F5Q78/80	92·125	93·456	T9-92/96	75·280	75·064
EXQ13-80	106·875	106·221	TRY15Q96	112·587	114·146
TRY11H81	103·625	103·712	EX-13Q96	101·011	101·617
T3H79/81	88·250	88·691	TRY13Q97	101·104	101·704
TRY9T-81	99·625	100·506	TRY8T-97	72·579	72·624
EXQ12T81	107·625	107·397	T6T95/98	58·610	59·124
T8H80/82	97·250	97·321	TRY15H98	113·917	116·927
TRY-3-82	86·375	83·874	TRY9H-99	77·100	76·983
TRY14-82	113·000	110·627	F3H99/04	33·165	32·333
TRY12-83	105·605	105·384	T8-02/06	65·153	64·504
F5H82/84	84·900	83·128	T5H08/12	45·589	44·629
T8H84/86	90·934	87·892	T7T12/15	62·684	61·665
F6H85/87	78·088	77·783	CONSLS-4	30·886	31·525
T7T85/88	81·934	81·032	WAR LOAN	28·868	27·578
T3-78/88	58·910	58·396	TREASY-3	23·183	23·631
T5-86/89	62·358	63·321	CONSLS2H	19·610	19·684
TRY13-90	103·400	102·104	TREASY2H	19·271	19·684

Root mean square of percentage price errors = 1·398

4.3. Anomalies may be identified by a consideration of the excess price—the amount by which market price exceeds that of an irredeemable with the same coupon. The logarithm of this excess should be compared with the average of all stocks on a daily basis—or ratios of these excess prices could be used in addition to the normal price ratio charts.

4.4. No assumption concerning the average rate of taxation need be made explicitly. The solution of the model informs us what average rate of income tax has been discounted in the prices.

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APPENDIX

A.1. In § 2.3 we developed the expression

$$P = gA + V \tag{A.1.1}$$

Set

$$V = 100v^n \tag{A.1.2}$$

Where

$$v = 1/(1+i) \tag{A.1.3}$$

(given V we can always solve for i, a net redemption yield).
For any value of A we can substitute

$$A = (1-t_n)(1-v^n)/i \tag{A.1.4}$$

and solve for the 'tax rate' t_n. This demonstrates equation (2.3.2). In general i and t_n will both vary with life n.

A.2. In § 1.4 it was suggested that a yield curve based on volatility can only be an approximation. Suppose that the relationship is

$$i = a/\omega + b \tag{A.2.1}$$

where

$$\omega = -dP/Pdi \quad \text{and } a \text{ and } b \text{ are model parameters*} \tag{A.2.2}$$

Then

$$i = -aPdi/dP + b \tag{A.2.3}$$

and this differential equation is easily solved to give

$$P = Q(b-i)^{-a} \tag{A.2.4}$$

where Q is a constant to be fitted to the data. Given the price, the yield basis follows without reference to the coupon or life. The same arguments apply even when the yield/volatility relationship is more complex than equation (A.2.1) although the solution of the differential equation is more complex.

* The author is indebted to Mr F. P. S. Phillips for suggesting this relationship which has theoretical justification as a first-order approximation.

A.3. In § 1.3 it was stated that linearity is a necessary condition for equal performance. Consider the price of stocks (which pay interest continuously) to be a function of coupon g, life n and time t.

$$P = P(g, n, t) \tag{A.3.1}$$

The value of capital and income after a short time interval Δt is

$$P + \partial P/\partial n \Delta n + \partial P/\partial t \Delta t + g\Delta t \tag{A.3.2}$$

but $\Delta n = -\Delta t$ since life has decreased by the passage of time.
If the performance is the same for all stocks and dependent only on the time t we have

$$\partial P/\partial t - \partial P/\partial n + g = Pq(t) \tag{A.3.3}$$

This Lagrange linear partial differential equation has a general solution

$$PY(t) - g = z(t_0, g)/x(t) \tag{A.3.4}$$

where t_0 is the redemption date of the stock $= (n+t)$.
$Y(t)$ is the yield on irredeemables at time t

$$\left.\begin{aligned} Y(t) &= -x'(t)/x(t) \\ x(t) &= \int \exp\{-b(t)\}\, dt \\ b(t) &= \int q(t)\, dt \end{aligned}\right\} \tag{A.3.5}$$

By setting $t = t_0$ in equation (A.3.4), since stocks are redeemed at par

$$100\, Y(t_0) - g = z(t_0, g)/x(t_0) \tag{A.3.6}$$

back-substitution into equation (A.3.4) demonstrates that the price must be a linear function of g for the equal performance model to remain valid.