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THE EFFECT OF A CHANGE IN THE INTEREST RATE ON LIFE ASSURANCE PREMIUMS

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THE recent paper by Dr Hagstroem (*J.I.A.* Vol. LXX, p. 119) directs attention to the very closely related subject of the effect on life assurance premiums of changes in the rate of interest. Some four years ago, in the course of an address before the American Life Convention, Mr V. R. Smith studied the relationship of the interest rate to life assurance premiums from a different angle. This note is an attempt to develop the mathematical theory underlying Mr Smith's method and to present some results on the basis of the A 1924–29 Table. The method suggested may be outlined briefly as follows:

(1) A forecast is made of the invested fund that may be expected in each future year from the policies issued in the current year. The invested fund would vary with class of policy and age at entry and, in annual premium classes, normally would increase to a maximum, perhaps twenty years after issue, and decrease again to zero. It follows the general course of the Reserve Liability multiplied by the probability that the business will be in force.

(2) Consideration is then given to the effect of investing each year the increase in the assets in debentures of a given period, such as ten or twenty years, at market rates of interest, beginning with those available on new investments in the immediate future and varying as the forecast of the future may dictate. For simplification, the effect of one change only from the initial rate is studied. Since the future course of interest rates is problematical this provides as satisfactory a basis for judgment as a more intricate forecast of future developments.

(3) The result can be summarized in a statement of this form: If investments are being made in securities of *n* years' term and new investments can be made in the immediate future at a net rate of 3%, the assumption of $3\frac{1}{2}$% interest in the calculation of the

premium for a whole life policy at age thirty-five will result in an interest profit only if the rate on new investments increases from 3% to 4% before the end of t years. In this form of statement the 4% and the 3% can be transposed with little loss in accuracy.

MATHEMATICAL DEVELOPMENT

In developing the mathematical formulae underlying the method, we begin by calculating a premium rate on certain assumptions as to mortality and expenses which will actually be realized in practice, and on a level interest rate of i. For the sake of simplicity, the surrender value paid to any policyholder has been taken to equal a proportionate share of the accumulated fund, so that no profit or loss arises from surrenders. If we then accumulate a fund, reproducing the forecasts in the premium rate as to expenses, mortality and interest and assuming certain rates of surrender, it is obvious that the fund will reduce ultimately to zero as the last claim or surrender value is paid.

The problem then becomes the effect on this fund of assuming that all other sources of income or outgo remain unchanged, but that the interest rate on new investments during the first t years (resulting from the first t premiums) is i' which is lower (higher) than the level rate i, followed by an interest rate on new investments thereafter i'' which is higher (lower) than the level rate i. If i, i' and i'' are known, t is determined so that the new fund will also reduce to zero as the last claim by death or maturity or surrender value is paid. If i' is less (greater) than i the new fund will always be less (greater) than the fund at the level rate i, equalling the latter at the beginning, when both are equal to the first-year premiums less the first-year expenses, and at the end when both are zero. It is assumed throughout that premiums and expenses are paid at the beginning of the policy year, and claims and surrenders at the end of the policy year.

All investments are made at par in debentures redeemable at par at the end of n years from the date of investment. The coupon rate on these debentures is, of course, the rate, i, i', or i'', which is the market rate current at the time of investment. The purchase of securities at a discount or a premium, would give results equivalent to those obtained by using the method of this paper with debentures of terms slightly longer or shorter than n years.

Assuming then that n is the term of the new investment throughout, that V_k is the invested fund during the $(k+1)$th year at rate i and that U_k is the invested fund during the $(k+1)$th year at rates i' and i'', the growth or decline in V and U from year to year varies only on account of the interest earnings.

(*a*) Considering the relationship between V_k and U_k as k varies from 1 to t during the period when V_k earns rate i and U_k earns rate i', we obtain the equation

$$V_k - U_k = V_{k-1}(1+i) - U_{k-1}(1+i') \qquad \ldots\ldots(1)$$

or

$$U_k = V_k - V_{k-1}(1+i) + U_{k-1}(1+i'). \qquad \ldots\ldots(2)$$

For, if I is the total income from premiums at the end of the kth year and O is the total outgo due to claims, surrenders, expenses, etc., at the same point,

$$V_k = V_{k-1}(1+i) + I - O \qquad \ldots\ldots(3)$$

$$U_k = U_{k-1}(1+i') + I - O. \qquad \ldots\ldots(4)$$

Subtracting (4) from (3) gives equation (1).

By continuously substituting the value of $U_{k-1}(1+i')$ in terms of the V fund in equation (2) an expression is derived for U_t in terms of V_t and earlier V's.

$$U_t = V_t - V_{t-1}(1+i) + V_{t-1}(1+i') - V_{t-2}(1+i')(1+i) + V_{t-2}(1+i')^2 \\ + \ldots + V_1(1+i')^{t-1} - V_0(1+i')^{t-1}(1+i) + U_0(1+i')^t \quad \ldots(5)$$

or since $U_0 = V_0$, collecting like terms of V

$$U_t = V_t + (i'-i)\sum_{k=0}^{k=t-1} V_k(1+i')^{t-k-1}. \qquad \ldots\ldots(6)$$

(*b*) The equations after time t depend on the relationship between n, the term of the investments, and t, the period during which the original rate i' is obtained on the U fund. In the most frequent case n is greater than t and the development which follows deals with this case (leaving the question of investment in perpetuities or in short-term debentures when n is less than t to a later point) and is based upon the funds for a whole life contract, which decrease to zero at the limiting age in the table (ω).

(*b*) (1) Considering the relationship between V_k and U_k as k varies from $(t+1)$ to n we obtain the equation

$$V_k - U_k = V_{k-1}(1+i) - (U_{k-1} - U_{t-1})(1+i'') - U_{t-1}(1+i') \qquad \text{......(7)}$$

or $$U_k = V_k - V_{k-1}(1+i) + U_{k-1}(1+i'') - U_{t-1}(i''-i'). \qquad \text{...(8)}$$

Equation (7) differs from equation (1) only in the fact that a part of U_k, i.e. $(U_{k-1} - U_{t-1})$, is now invested at the second rate of interest i'' while a part, i.e. U_{t-1}, is still invested at the original rate i'. Hence, following the course adopted in obtaining equation (6),

$$U_n = V_n + (i''-i)\sum_{k=t+1}^{k=n-1} V_k(1+i'')^{n-k-1} - (1+i'')^{n-t-1}(1+i)V_t + (1+i'')^{n-t}U_t - U_{t-1}(i''-i')\,s_{\overline{n-t}|}^{(i'')}. \qquad \text{......(9)}$$

(*b*) (2) Considering the relationship between V_k and U_k as k varies from $(n+1)$ to $(n+t)$ we obtain the equation

$$V_k - U_k = V_{k-1}(1+i) - (U_{k-1} - U_{t-1} + U_{k-n-1})(1+i'') - (U_{t-1} - U_{k-n-1})(1+i') \qquad \text{......(10)}$$

or $$U_k = V_k - V_{k-1}(1+i) + U_{k-1}(1+i'') + U_{k-n-1}(i''-i') - U_{t-1}(i''-i'). \qquad \text{......(11)}$$

Equation (10) differs from equation (7) only in the fact that a certain part of the original investments at rate i' in bonds of n years duration has now matured, and the proceeds have been reinvested at rate i''.

Hence, following the course adopted in obtaining equations (6) and (9),

$$U_{n+t} = V_{n+t} + (i''-i)\sum_{k=n+1}^{k=n+t-1} V_k(1+i'')^{n+t-k-1} - (1+i)(1+i'')^{t-1}V_n + (1+i'')^t U_n - U_{t-1}(i''-i')\,s_{\overline{t}|}^{i''} + (i''-i')\sum_{k=0}^{k=t-1} U_k(1+i'')^{t-k-1}. \qquad \text{......(12)}$$

(*b*) (3) Considering the relationship between V_k and U_k as k varies from $(n+t+1)$ to ω we obtain the equation

$$V_k - U_k = V_{k-1}(1+i) - U_{k-1}(1+i'') \qquad \text{......(13)}$$

or $$U_k = V_k - V_{k-1}(1+i) + U_{k-1}(1+i''). \qquad \text{......(14)}$$

All the original investments having now matured, the entire U fund is invested at rate i''.

Hence, following the course adopted in obtaining equations (6), (9) and (12),

$$U_\omega = V_\omega + (i''-i)\sum_{k=n+t+1}^{k=\omega-1} V_k (1+i'')^{\omega-k-1} - (1+i'')^{\omega-n-t-1}(1+i)V_{n+t} + (1+i'')^{\omega-n-t}U_{n+t}. \quad \ldots\ldots(15)$$

(*c*) Substituting in equation (15) the value of U_{n+t} from equation (12); in equation (12) the value of U_n from equation (9); in equation (9) the value of U_t from equation (6); a final equation for U_ω is obtained.

$$\begin{aligned} U_\omega = V_\omega &+ (i''-i)\sum_{k=t}^{k=\omega-1} V_k (1+i'')^{\omega-k-1} \\ &+ (i'-i)(1+i'')^{\omega-t}\sum_{k=0}^{k=t-1} V_k (1+i')^{t-k-1} - (i''-i')(1+i'')^{\omega-t}U_{t-1}\, a_{\overline{n}|}^{i''} \\ &+ (i''-i')(1+i'')^{\omega-n}\sum_{k=0}^{k=t-1} U_k \frac{1}{(1+i'')^{k+1}}. \qquad \ldots\ldots(16)\end{aligned}$$

In the practical application of this formula it is more convenient to substitute for the term

$$(i-i')(1+i'')^{\omega-t}\sum_{k=0}^{k=t-1} V_k (1+i')^{t-k-1} \qquad \text{the equivalent}$$

$$(1+i'')^{\omega-t}\{V_t - U_t\} \qquad \text{(see equation (6)).}$$

In this formula all values except t are known, and as, for the correct value of t, $U_\omega = V_\omega = 0$, t is determined by trial and error. Tables of the functions V_k; $\sum_k^{\omega-1} V_k (1+i'')^{\omega-k-1}$ were prepared for each type of contract for all values of k and of the functions

$$U_k;\ \sum_0^{t-1} V_k \frac{1}{(1+i')^{k+1}};\ \sum_0^{t-1} U_k \frac{1}{(1+i'')^{k+1}}$$

for a period sufficient to cover the usual values of t. The same functions may be used for all values of n.

Variations in the formula occur when n is less than t and when $n=\infty$ (i.e. perpetuities). In the case of endowment assurances the accumulation ceases at maturity. The same method of approach may be used and the formulae are in the Appendix.

When the investments are made in perpetuities (i.e. $n=\infty$) and in respect of endowment assurances, the results are affected seriously by the fact that a loss on sale must be considered since it becomes necessary to realize at the prevailing market rate i'' securities which were purchased to yield i'. The same question may arise to a slight degree in certain other cases, but no correction has been attempted in the formula except for endowment assurances and investment in perpetuities.

ARITHMETICAL RESULTS ON THE A 1924–29 TABLE

Values of t have been calculated on the assumptions that the level rate of interest is $3\frac{1}{2}$%, that the initial rate applicable to investments made during the first t years (i.e. the first t premiums) is 3 % and that the rate for investments and reinvestments thereafter is 4 %.

In order to test the effect of a variation in the term of the investment, the calculations have been made with $n=10$, 20, 30 and ∞.

The effect of the change in interest for different classes of policy is exhibited for whole life assurances, whole life assurances with premiums limited to 20 years and endowment assurances effected at age 35 for terms of 10, 20 and 30 years.

A factor having considerable influence on the result is the withdrawal rate, since with heavy withdrawals the early investments assume a greater significance. Results are tabulated assuming (*a*) that the contracts in force are subject to no cause of decrease excepting death claims, (*b*) that in addition they are subject to withdrawal on the two different scales of rates in Table II (see Appendix A). In the event of withdrawal, as stated before, it is assumed that the accumulated fund on the level rate of interest $3\frac{1}{2}$% (i.e. V_k) will be paid out.

In addition, results are tabulated both for participating and non-participating contracts. The participating contract provides in the premium scale for a simple reversionary bonus addition of £2 % p.a. to whole life assurances and £1. 15. 0 % p.a. to endowment assurances.

The premiums were calculated and the funds accumulated on the basis of A 1924–29 select mortality, and expenses amounting to

£3 % of the sum assured in the first year plus a constant of 4*s.* % of the sum assured and $3\frac{1}{2}$ % of the gross premium in all years including the first.

The number of premiums invested at rate i', taken to the next higher integer, is t. Measured in calendar years from the present if t premiums are invested at rate i' the change in interest takes place between the $(t-1)$th and the tth policy anniversary.

It will be noted that the term of the investments is of very considerable importance. Generally speaking the period t for an investment term of 10 years is at least double the corresponding value for investments in perpetuities. This fact tends to negative the relatively higher rates obtainable on long term investments at the present time, if we may assume that investments can be made at higher rates in the future.

When we compare the results according to class of policy the outstanding feature is the small value of t for the endowment assurances. The question whether a 10-year endowment assurance contract issued at the present time will be a source of profit or loss, as far as the interest element is concerned, is practically decided by the yields obtainable within the next 2 or 3 years. The interest factor is not only of less importance in whole-life and whole-life limited payment assurances (in comparison with mortality) but, in the interest factor itself, the short-term outlook is of relatively less importance.

The lapse rate, while not of such major importance as variations in investment-term and in class of policy, still plays a considerable part in determining the relative importance of the interest on investments in the near and in the distant future.

One rather surprising result is that the with-profit policy is, on the whole, slightly more affected by investment yields in the immediate future than the corresponding without-profit contract. This is due to the fact that the weight of the invested fund is nearer the date of issue for the bonus additions than for the main policy.

CONCLUSION

In the past perhaps too much emphasis has been placed on the interest rate earned on the funds of the company as a whole, and too little emphasis upon the rate obtainable on current investments

Table I. Showing the number of premiums invested at 3 %

Investments made in debentures redeemable in years	Non-participating policies			Participating policies		
	Assuming no withdrawals other than death	Assuming lapses at rate A	Assuming lapses at rate B	Assuming no withdrawals other than death	Assuming lapses at rate A	Assuming lapses at rate B
	Whole-life assurances					
10	22	18	16	22	18	16
20	17	13	11	17	13	11
30	13	10	8	13	9	8
∞	9	7	6	9	6	5
	Whole-life—premiums limited to 20 years					
10	19	16	15	19	16	14
20	14	11	9	14	11	9
30	11	8	7	11	8	7
∞	8	6	5	8	5	4
	10-year endowment assurances					
10	4	3	3	3	3	3
20	2	2	2	2	2	2
30	2	2	2	2	2	2
∞	2	1	1	2	1	1
	20-year endowment assurances					
10	10	9	9	10	9	9
20	6	5	5	6	5	5
30	5	4	4	5	4	4
∞	4	3	3	4	3	3
	30-year endowment assurances					
10	16	14	14	16	14	13
20	11	9	8	11	9	8
30	9	7	6	9	7	6
∞	7	5	4	6	5	4

and the prospect of future movements in this rate. As in the case of mortality, there is no question that the true financial results of issuing contracts on any given basis are determined solely by interest yields obtainable on new investments after their issue. The use of the average rate as a guide when yields have fallen heavily will undoubtedly result in a transfer of funds from old to new policies, unless there is a very rapid return to the so-called normal conditions following the war of 1914-18. How rapid this return must be the table indicates.

APPENDIX A

The rates of termination used in the calculation of the arithmetical results include all withdrawals excepting those occasioned by death or maturity. No analysis of rates of withdrawal was made for this paper and the rates used were taken from *T.A.S.A.*, Vol. XXXIII, p. 387.

Table II. Rate of withdrawal per thousand

End of year	Rate A	Rate B	End of year	Rate A	Rate B
1	100	150	9	27	40·5
2	60	90	10	25	37·5
3	50	75	11	24	36·0
4	44	66	12	23	34·5
5	40	60	13	22	33·0
6	36	54	14	21	31·5
7	32	48	15 and subsequently	20	30·0
8	29	43·5			

APPENDIX B

In deriving formula 16 on p. 384 it was pointed out that variations occur when t is greater than n, and when account is taken of the sale when the interest rate current was i'' of securities purchased when the prevailing rate of interest was i'.

The method of approach in obtaining the formulae applicable to these varying circumstances is precisely the same as that used in the body of the paper and the formulae used in the calculation of the results only are set out.

(1) Whole-life and whole-life limited payment contracts, and

endowment assurances maturing after $(n+t)$ years, t being greater than n but not greater than $2n$,

$$U_\omega = V_\omega + (i''-i)\sum_{k=t}^{k=\omega-1} V_k(1+i'')^{\omega-k-1}$$
$$+(i''-i')(1+i'')^{\omega-n}\sum_{k=t-n}^{k=t-1} U_k\frac{1}{(1+i'')^{k+1}}$$
$$+(i''-i')(1+i'')^{\omega-2n}\sum_{k=0}^{k=t-n-1} U_k\frac{1}{(1+i'')^{k+1}}$$
$$-(i''-i')(U_{t-1}+U_{t-n-1})(1+i'')^{\omega-t}a_{\overline{n}|}^{i''}$$
$$-(1+i'')^{\omega-t}(V_t-U_t).$$

(2) Whole-life and whole-life limited payment contracts and endowment assurances maturing after $n+t$ years, t being greater than $2n$ but not greater than $3n$,

$$U_\omega = V_\omega + (i''-i)\sum_{k=t}^{k=\omega-1} V_k(1+i'')^{\omega-k-1}$$
$$+(i''-i')(1+i'')^{\omega-n}\sum_{k=t-n}^{k=t-1} U_k\frac{1}{(1+i'')^{k+1}}$$
$$+(i''-i')(1+i'')^{\omega-2n}\sum_{k=t-2n}^{k=t-n-1} U_k\frac{1}{(1+i'')^{k+1}}$$
$$+(i''-i')(1+i'')^{\omega-3n}\sum_{k=0}^{k=t-2n-1} U_k\frac{1}{(1+i'')^{k+1}}$$
$$-(i''-i')(U_{t-1}+U_{t-n-1}+U_{t-2n-1})(1+i'')^{\omega-t}a_{\overline{n}|}^{i''}$$
$$-(1+i'')^{\omega-t}(V_t-U_t).$$

(3) All classes of policy, n being equal to ∞,

$$U_\omega = V_\omega + (i''-i)\sum_{k=t}^{k=\omega-1} V_k(1+i'')^{\omega-k-1}$$
$$-(1+i'')^{\omega-t}\left\{(1+i)V_{t-1}-\frac{i'}{i''}(1+i'')U_{t-1}\right\}.$$

This formula follows from the basis that the most convenient practical method of treating depreciation is not to evaluate the loss on sale as the sale of the original investments occurs but to treat the loss in capital values as being met immediately on the change in the interest rate. The results are identical only in the case of investments in perpetuities.

Note. In these first three cases, for endowment assurances ω will equal the term of the policy.

(4) Endowment assurances where the term of the policy m is greater than n, n being greater than t but $(n+t)$ being greater than m,

$$\begin{aligned} U_m = D + V_m + (i'' - i) \sum_{k=t}^{k=m-1} V_k (1+i'')^{m-k-1} \\ + (i'' - i')(1+i'')^{m-n} \sum_{k=0}^{k=m-n-1} U_k \frac{1}{(1+i'')^{k+1}} \\ - (i'' - i') U_{t-1} s_{\overline{m-t}|}^{i''} - (1+i'')^{m-t} (V_t - U_t). \end{aligned}$$

The depreciation, D, is calculated individually for the appropriate values of U_k at the maturity of the policy when the current market rate is i'', by determining the amount of loss entailed in the sale, a number of years before redemption date, of securities bearing interest a coupon rate i'.