EXTREME VALUE TECHNIQUES
PART IV: FIN RE PRICING

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Extreme Value Techniques
Part IV: Fin Re Pricing

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Abstract. We present a state-of-the-art rating methodology for financial reinsurance contracts that is based upon a consistent stochastic model (of the jump diffusion type) for financial market variables (like, e.g., interest rates, foreign currencies, stocks, stock indices, etc.) as well as for (excess-of-loss) reinsurance claims. A lattice-based implementation of this pricing methodology (i.e., a corresponding Fin Re Toolbox) is discussed in some detail and then applied to rate current example Fin Re contracts taken from the Swiss Re New Markets business area.

Keywords. Extreme value techniques (EVT) toolbox, financial/(re)insurance toolbox, Fin Re toolbox, extreme value theory, peaks-over-thresholds model, generalized pareto distribution, reference dataset, optimal self-insured retention (SIR), value proposition (VP) -based client solution, optimal alternative risk transfer solution, securitization structure, financial engineering, Fin Re pricing, Fin Re hedge fund management, Fin Re platform, Fin Re software architecture, Fin Re lattice, lattice manager, dynamic programming procedure, contingent claim price/sensitivity forecast, adaptive update property, loss event contingent cashflow modification, risk-neutral pricing formula.

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Contents.

1. Introduction

Pricing financial reinsurance (Fin Re) contracts involves a proper assessment of

(a) the liabilities (excess-of-loss claims, usually limited or "finite") arising from the reinsurance part of the Fin Re contract;

(b) the financial instruments (fixed income securities, swaps, etc.) "built around" the above liabilities (with cashflows potentially contingent on the loss events on the liability side).

We present here a state-of-the-art pricing/rating methodology for financial reinsurance contracts that is based upon a consistent stochastic model (of the jump diffusion type) for financial market variables (like, e.g., interest rates, foreign currencies, stocks, stock indices. etc.) as well as for (excess-of-loss) reinsurance claims. A lattice-based implementation of this pricing methodology (i.e., a corresponding ***Fin Re Toolbox***) is discussed in some detail and then applied to rate current example Fin Re contracts taken from the Swiss Re New Markets business area.

2. Modelling Excess-of-Loss Claims

List and Zilch [1], Geosits. List and Lohner [2] and List and Lohner [3] describe a consistent set of state-of-the-art techniques and tools for modelling excess-of-loss claims data. These tools are available in the form of a corresponding ***Extreme Value Techniques (EVT) Toolbox*** that runs under Windows 3.1, 95, NT 3.51 and NT 4.0:

EXTREME VALUE TECHNIQUES (EVT)

* ***Claims Data Handling***
* ***Claims Data Analysis***
  - Frequency Statistics
  - Severity Statistics
* ***Excess-of-Loss Claims Modelling***
  - Pareto (PD)
  - Generalized Pareto (GPD)
* ***Advanced Scenario Techniques***
  - Parameter Uncertainty
  - Simulation
* ***Multi-year, Multiline Contract Pricing***
  - Extreme Value Theory (EVT)
  - Increased Limits Factors (ILF)
  - Coverage Futures and Options
* ***Risk-adjusted Capital (RAC)***
  - Calculation
  - Optimization
* ***Value Proposition (VP)***
  - Optimal Coverage Structures
  - Value Quantification

Fig. 1: Extreme Value Techniques (EVT) Toolbox

Extreme Value Techniques have within Swiss Re so far primarily been applied in the development of Swiss Re's recently launched "Beta" program for Oil & Petrochemicals industry high-excess property and casualty layers [that are taken here as an example for more general "catastrophic" non-life (re)insurance exposures]:

"Beta" provides multi-year, high-excess, broad form property and comprehensive general liability coverage with meaningful total limits for Fortune 500 clients in the Oil & Petrochemicals industry ("Beta" is also available in other Fortune 500 segments, its program parameters are industry-specific, however).

Coverage is provided at ***optimal layers*** within prescribed minimum and maximum per occurrence attachment points and per occurrence (i.e., each and every loss: E.E.L., see Fig. 2 below) and aggregate (AGG.) limits, split appropriately between property and casualty. These attachment points and limits are derived from the risk profiles and the needs of the insureds (***Swiss Re's Value Proposition*** for the Oil & Petrochemicals industry).

The aggregate limits provide "Beta" ***base coverage*** for one year and over three years. Simply stated, if the base coverage is not pierced by a loss, then its full, substantial limits (USD 200M property and 100M casualty) stay in force over the entire three year "Beta" policy term.
Insureds might be concerned they would have no (or only a reduced) coverage if losses were to pierce the base coverage. Therefore, "Beta" includes a ***provision to reinstate*** all or a portion of the base coverage that is exhausted.

Lastly, the "Beta" design includes an ***option*** at the inception of the base coverage ***to extend*** its initial three year high-excess insurance coverage (i.e., the property and casualty base coverage and the provision for a single reinstatement of the base coverage) for an additional three year policy term at a predetermined price.

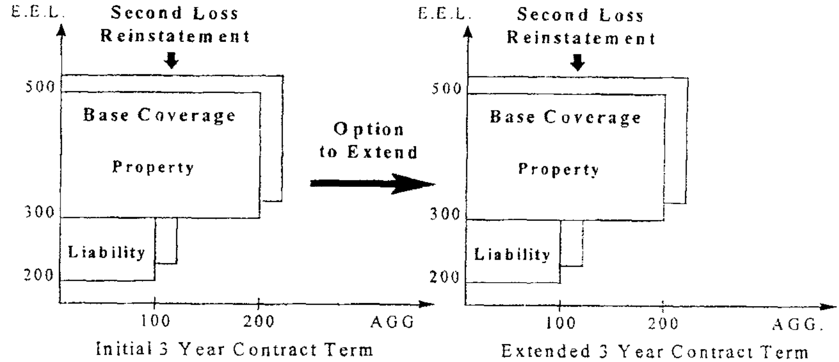

Fig. 2: The "Beta" Insurance Coverage for the Oil & Petrochemicals Industry

From Swiss Re's risk management point of view, optimal layers for "Beta" property and casualty excess coverages are defined as follows:

No annual loss should pierce the chosen property or casualty excess layer more frequently than once every four years (based both on the historical and scenario annual aggregate loss distributions). This translates into a 75% confidence that annual aggregate losses for a given layer of "Beta" coverage will equal zero.[1]

The ***risk quantification process*** leading to the above optimal "Beta" layers for multi-year (i.e., three years) high-excess property and casualty Oil & Petrochemicals industry insurance coverage in principle follows standard actuarial tradition - however with some new elements:

The "Beta" implementation team (consisting of Swiss Re and ETH Zurich[2] personnel) has developed and implemented a consistent and stable (with respect to small perturbations in the input data) actuarial modelling approach for "Beta" high-excess property and casualty layers (see Fig. 3 below). This new methodology is based on ***Extreme Value Theory*** (***Peaks-Over-Thresholds Model***[3]) and fits a ***generalized Pareto distribution***[4] to the ***exceedances of a data-***

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[1] This optimality criterion is mainly derived from Swiss Re's perception (based upon an extensive Oil & Petrochemicals industry analysis) of a ***"Beta"*** or ***"catastrophic" event***. In the case of "Beta" programs with combined single limits/deductibles, lower percentiles and thus shorter contract maturities may be preferable from a marketing point of view.

[2] The ETH Zurich "Beta" implementation team was lead by Prof. Dr. Hans Bühlmann, Prof. Dr. Paul Embrechts (Extreme Value Theory) and Prof. Dr. Freddy Delbaen ("Beta" Options).

[3] It has to be noted that ***claims histories are usually incomplete***, i.e., only losses in excess of a so-called ***displacement*** δ are reported. Let therefore (X_i) be an i.i.d. sequence of ground-up losses, (Y_i) be the associated loss amounts in the "Beta" layer $D \le x \le D + L$ and $Z = \sum_{i=1}^{N} Y_i$ the corresponding aggregate loss. Similarly, let $(\tilde{X}_i)$, $\tilde{X}_i = X_i 1_{X_i > \delta}$, be the losses greater than the displacement δ and $\tilde{Z} = \sum_{i=1}^{\tilde{N}} \tilde{Y}_i$ $\tilde{N} = \sum_{i=1}^{N} 1_{X_i > \delta}$, the corresponding "Beta" aggregate loss amount. Some elementary considerations then show

specific threshold. Once the frequency and severity distribution parameters are determined, ***per claim loss layers are selected and aggregate distributions both within the selected layers and excess of those layers up to the maximum potential individual loss (MPL) in the Oil & Petrochemicals industry (e.g., USD 3 billion for property and USD 4 billion for casualty) calculated***. This procedure is repeated for sequential layers (usually chosen at the discretion of the underwriter to approximate the anticipated "Beta" program structures reflecting the needs of the insureds or the entire Oil & Petrochemicals industry), thus mapping out the "Beta" ***risk potential***. The resulting probabilistic (excess-of-loss) profiles ("Beta" ***risk landscapes*** or ***risk maps***, see Fig. 4 below) can also be used for the securitization[5] of "Beta" portfolio components (see further below).

that $F_Z \equiv F_{\hat{Z}}$ holds for the aggregate loss distributions, provided that $\delta < D$. The ***Peaks-Over-Thresholds Model (Pickands-Balkema-de Haan Theorem)*** on the other hand says that the ***exceedances of a high threshold*** $t < D$ are approximately $G_{\xi,t,\sigma}(x)$ distributed, where $G_{\xi,t,\sigma}(x)$ is the ***generalized Pareto distribution*** with ***shape*** ξ, ***location*** $t \equiv \mu$ and ***scale*** $\sigma > 0$. The threshold $t < D$ is chosen in such a way that in a neighbourhood of t the MLE-estimate of ξ (and therefore the "Beta" premium) remains reasonably stable (see Fig. 3).

[4] The ***generalized Pareto distribution (GPD)*** is defined by

$$G_{\xi,\mu,\sigma}(x) = \begin{cases} 1-\left(1+\xi\dfrac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}} & \xi \neq 0 \\ 1-e^{-\frac{x-\mu}{\sigma}} & \xi = 0 \end{cases}$$

where $x \geq \mu$ for $\xi \geq 0$ and $\mu \leq x \leq \mu - \dfrac{\sigma}{\xi}$ for $\xi < 0$. Compare this with the ***ordinary Pareto distribution (PD)***:

$$F_{\theta,a}(x) = 1-\left(\frac{a}{x}\right)^{\theta}, \quad x > a.$$

[5] From an actuarial standpoint, securitization is a modern capital markets alternative for traditional retrocession agreements (see also Davis and List [4]).

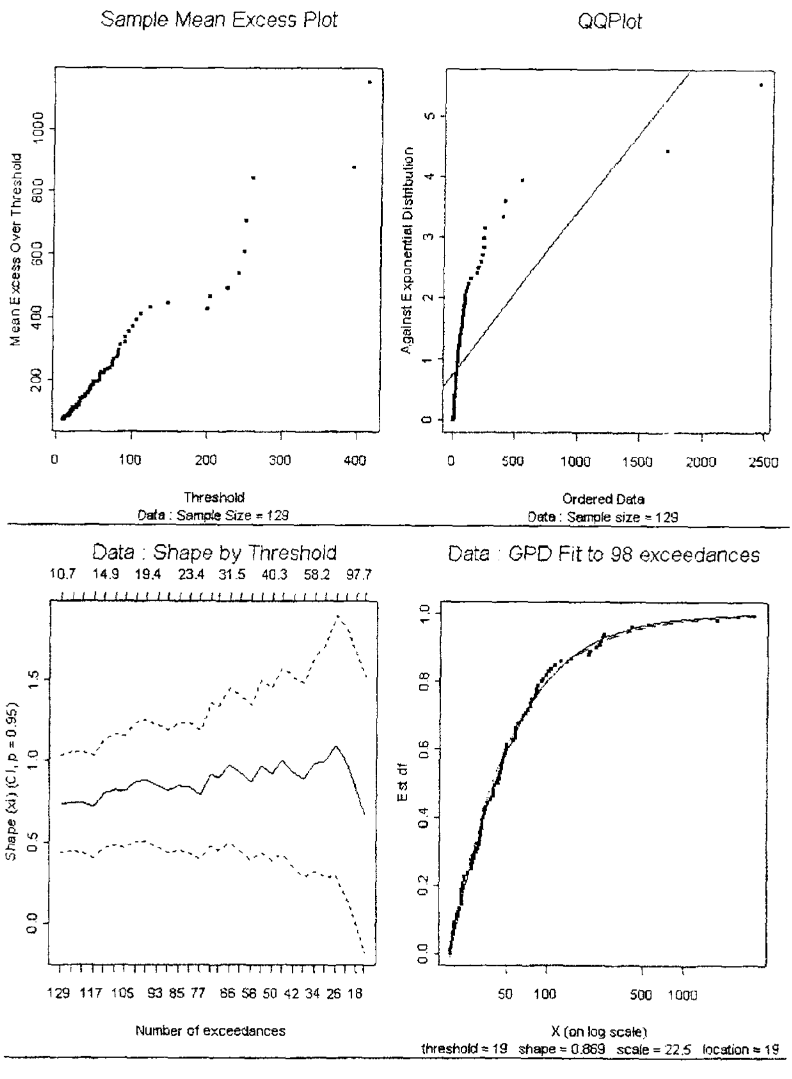


Fig. 3a: Oil & Petrochemicals Industry Severity Parameters (Property)
Solid Line: GPD, Dotted Line: PD

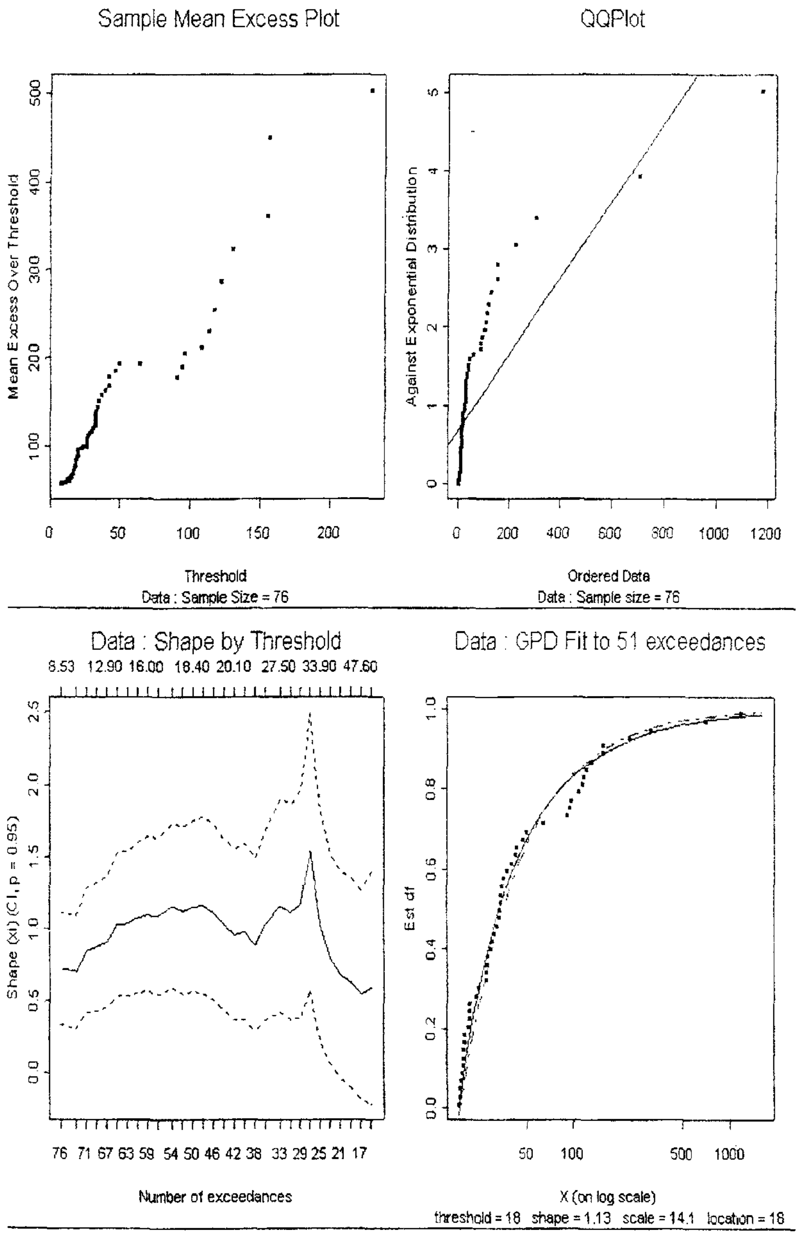
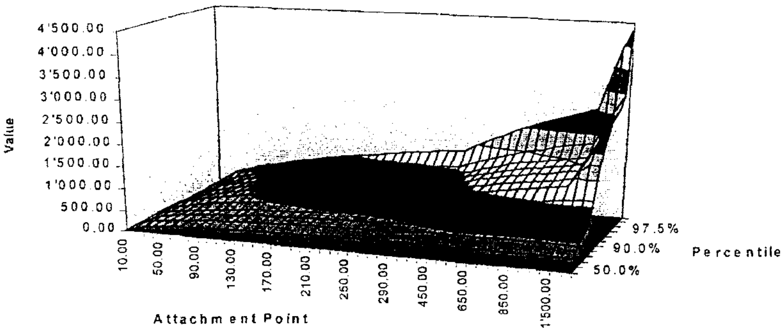


Fig. 3b: Oil & Petrochemicals Industry Severity Parameters (Casualty)
Solid Line: GPD, Dotted Line: PD

Distribution Below Attachment Point



Distribution Above Attachment Point

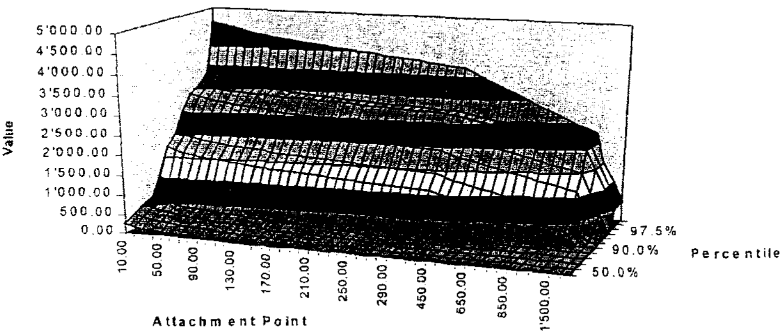
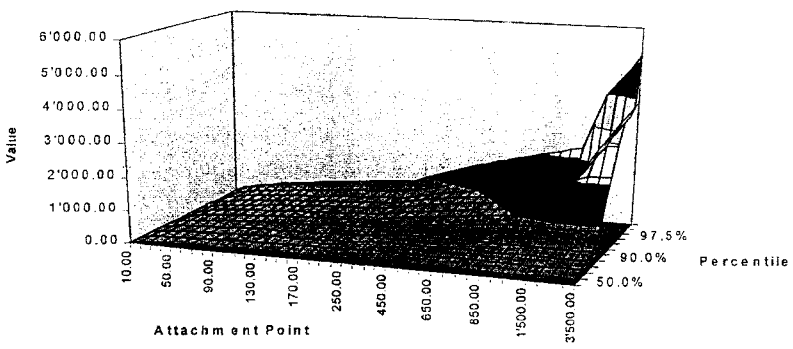


Fig. 4a: Oil & Petrochemicals Industry Risk Landscape[6] (Property)

[6] The ***minimum layer width*** can be determined as follows: Consider the ***80***th ***percentile*** in the risk map containing the one year aggregate loss distributions ***below*** the attachment points 10M, 20M, .., 100M, .., etc. (keeping in mind that this percentile indicates the ***expected maximum loss in the fourth year***) and start with the "Beta" attachment point of 300M, i.e., an expected one year aggregate loss of about 535M. Moving to the upper "Beta" E.E.L. coverage point of 500M (= 300M "Beta" attachment point + 200M "Beta" limit), we have an expected annual aggregate loss of about 630M. ***This means that the expected one year aggregate loss in the envisaged "Beta" property layer is about 95M (= 630M - 535M) or, in other words, the "Beta" property coverage (without reinstatement) absorbes two such expected losses on an E.E.L. and a 3 Y AGG. basis.*** This was according to an extensive analysis (carried out during the "Beta" product engineering process) of the risk preferences in the Oil & Petrochemicals industry Fortune 500 segment considered to be sufficient for catastrophic events causing property damage. ***Similarly, on the casualty side, it transpired that a "Beta" layer width of 100M was considered sufficient; the expected one year aggregate loss in the envisaged "Beta" casualty layer (i.e., 100M xs 200M) being 59M (= 371M - 312M).***

Distribution Below Attachment Point



Distribution Above Attachment Point

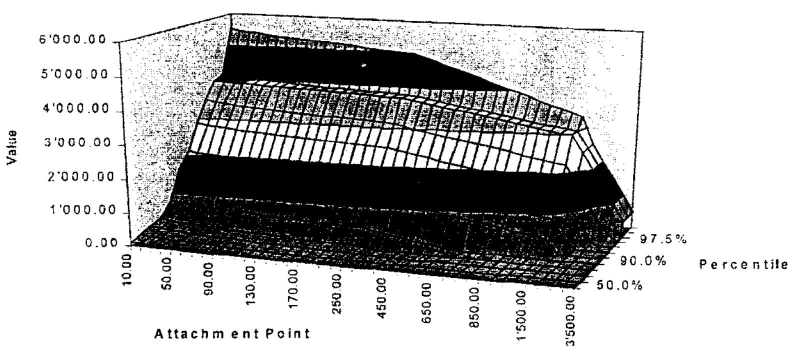


Fig. 4b: Oil & Petrochemicals Industry Risk Landscape[7] (Casualty)

[7] The determination of standard layers (i.e., ***optimal SIRs and limits***) for "Beta" alternative risk transfer solutions in the Oil & Petrochemicals industry (a similar approach is used in the other "Beta" target industries) is very important for the quantification of Swiss Re's Value Proposition for corporate clients in the Fortune 500 group of companies. The Value Proposition argument itself would be as follows: (1) Optimal layers for "Beta" coverages are characterized by ***efficiency and cost transparency***, a ***high degree of structural flexibility to optimally fit clients' asset liability management (ALM) needs*** (see also Davis and List [4]), ***significant capacity for property and casualty, long-term stability (Swiss Re capacity)*** and ***high financial security (AAA capital base)***. (2) "Beta" is a genuine alternative risk transfer product that may also include ***sophisticated financial markets components*** (***balance sheet protection***, see also Davis and List [4]) and a new element in the comprehensive range of Swiss Re's (re)insurance coverages and related services for Fortune 500 companies. Note that the "Beta" program also allows for property and casualty layers different from the standard layers (see List and Zilch [1] and Geosits, List and Lohner [2]).

The ***optimal "Beta" attachment points (=SIRs)*** for the Oil & Petrochemicals industry are:

A. Basic Scenario[8].

Basic Scenario (5% Property, 10% Casualty)		
Property		
BP	Opt. Attachment Point	300.00
EAP	Opt. Attachment Point	350.00
Onshore		
BP	Opt. Attachment Point	250.00
EAP	Opt. Attachment Point	290.00
Offshore		
BP	Opt. Attachment Point	90.00
EAP	Opt. Attachment Point	110.00
Casualty		
BP	Opt. Attachment Point	250.00
EAP	Opt. Attachment Point	300.00

Basic Scenario (5% Property, 10% Liability)		
Fire & Explosion		
BP	Opt. Attachment Point	550.00
EAP	Opt. Attachment Point	600.00
Marine		
BP	Opt. Attachment Point	300.00
EAP	Opt. Attachment Point	350.00
Tanker Pollution		
BP	Opt. Attachment Point	300.00
EAP	Opt. Attachment Point	400.00

Basic Scenario (5% Property, 10% Liability)		
Property Damage		
BP	Opt. Attachment Point	650.00
EAP	Opt. Attachment Point	700.00
Business Interruption		
BP	Opt. Attachment Point	650.00
EAP	Opt. Attachment Point	750.00
Property Damage and Business Interruption		
BP	Opt. Attachment Point	1500.00
EAP	Opt. Attachment Point	1500.00
Offshore		
BP	Opt. Attachment Point	230.00
EAP	Opt. Attachment Point	270.00
General Liability		
BP	Opt. Attachment Point	300.00
EAP	Opt. Attachment Point	450.00
Product Liability		
BP	Opt. Attachment Point	60.00
EAP	Opt. Attachment Point	80.00
Employer's Liability		
BP	Opt. Attachment Point	10.00
EAP	Opt. Attachment Point	10.00
Automobile Liability		
BP	Opt. Attachment Point	10.00
EAP	Opt. Attachment Point	10.00
Marine Liability		
BP	Opt. Attachment Point	40.00
EAP	Opt. Attachment Point	50.00
All Liability Claims		
BP	Opt. Attachment Point	450.00
EAP	Opt. Attachment Point	500.00

[8] The time periods 1997 to 1999 and 2000 to 2002 are called ***"Beta" base period (BP)*** and ***"Beta" extended agreement period (EAP)***, respectively (see List and Zilch [1]). Three different ***reference datasets*** characterizing the Oil & Petrochemicals industry are analyzed (see List and Lohner [3]).

B. Adjustment Scenario[9].

Adjustment Scenario (10% Property, 20% Casualty)		
Property		
BP	Opt. Attachment Point	600.00
EAP	Opt. Attachment Point	800.00
Onshore		
BP	Opt. Attachment Point	500.00
EAP	Opt. Attachment Point	700.00
Offshore		
BP	Opt. Attachment Point	180.00
EAP	Opt. Attachment Point	240.00
Casualty		
BP	Opt. Attachment Point	550.00
EAP	Opt. Attachment Point	850.00

Adjustment Scenario (10% Property, 20% Liability)		
Fire & Explosion		
BP	Opt. Attachment Point	1000.00
EAP	Opt. Attachment Point	1500.00
Marine		
BP	Opt. Attachment Point	600.00
EAP	Opt. Attachment Point	800.00
Tanker Pollution		
BP	Opt. Attachment Point	900.00
EAP	Opt. Attachment Point	1500.00

Adjustment Scenario (10% Property, 20% Liability)		
Property Damage		
BP	Opt. Attachment Point	1500.00
EAP	Opt. Attachment Point	1500.00
Business Interruption		
BP	Opt. Attachment Point	1500.00
EAP	Opt. Attachment Point	2000.00
Property Damage and Business Interruption		
BP	Opt. Attachment Point	2500.00
EAP	Opt. Attachment Point	> 2500.00
Offshore		
BP	Opt. Attachment Point	450.00
EAP	Opt. Attachment Point	600.00
General Liability		
BP	Opt. Attachment Point	700.00
EAP	Opt. Attachment Point	1500.00
Product Liability		
BP	Opt. Attachment Point	120.00
EAP	Opt. Attachment Point	250.00
Employer's Liability		
BP	Opt. Attachment Point	20.00
EAP	Opt. Attachment Point	30.00
Automobile Liability		
BP	Opt. Attachment Point	20.00
EAP	Opt. Attachment Point	30.00
Marine Liability		
BP	Opt. Attachment Point	80.00
EAP	Opt. Attachment Point	130.00
All Liability Claims		
BP	Opt. Attachment Point	1500.00
EAP	Opt. Attachment Point	2000.00

[9] To make this presentation simple, we only consider the ***basic scenario*** and an ***adjustment scenario*** (see List and Zilch [1] for more details on the ***general classes of "Beta" threat scenarios*** identified).

Goodness-of-fit tests (e.g., the LEV comparison test, see List and Lohner [3]) show that ***Extreme Value Techniques are applicable basically from ground-up to the maximum potential loss***. Similar results hold for the other "Beta" target industries in the Fortune 500 segment of Swiss Re's corporate clients.

3. Modelling Interest Rates

Modern ***Value Proposition (VP) -based client solutions for Fortune 500 companies*** often require sophisticated financial engineering, too. Davis and List [4, 5, 6, 7, 8] and Davis and Bühlmann, Bochiccio, Junod and List [9, 10, 11] present the corresponding stochastic models and applications (for excess-of-loss claims on the liability side and interest rates, foreign currencies, stocks and stock indices, etc. on the asset side). Moreover, a sophisticated ***financial/(re)insurance toolbox*** for the design of such alternative risk transfer solutions is outlined: EVT handels the liability side while an extended form of the Rubinstein implied tree model is used for the asset side (with asset cashflows potentially contingent on loss events on the liability side) of such transactions. This toolbox again runs under Windows 3.1, 95, NT 3.51 and NT 4.0:

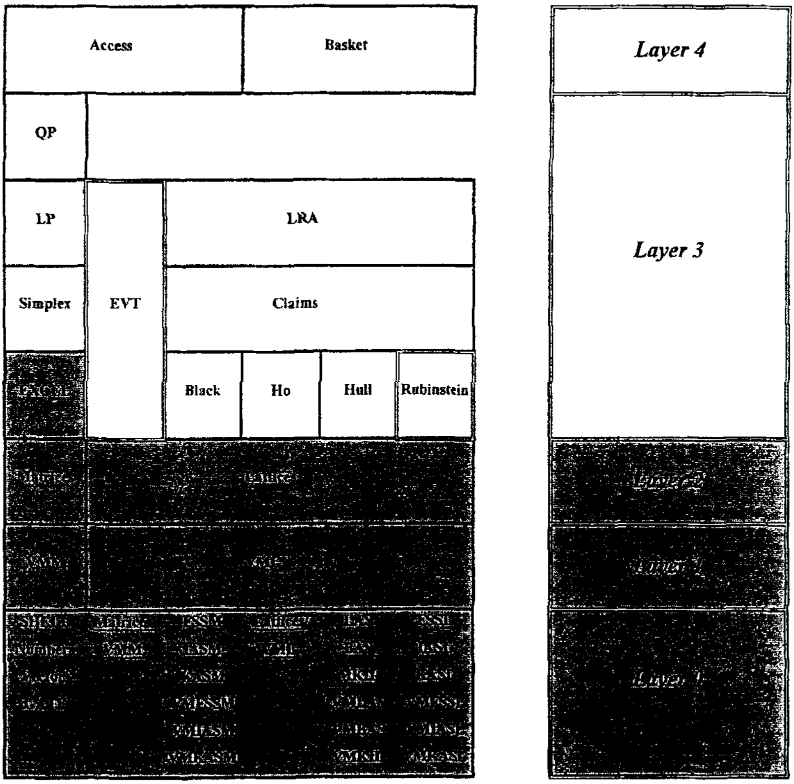


Fig. 5: Financial/(Re)insurance Toolbox (see Davis and List [6])

Lattices and ***matrices*** are the main information processing structures used in corporate and investment banking and (re)insurance applications. These structures tend to be quite large and have to be accessed and updated many times to obtain the results needed in quantitative financial/(re)insurance decision making. The PC is widely used as a convenient low-cost financial services platform in modern banking/(re)insurance. Its main limitations are the small 64 KB data segment size, the typically insufficient RAM size and the usually rather slow and limited harddisk. One of the main objectives in the design of our ***generic PC-based software environment for Fin Re pricing applications (Fin Re Toolbox)*** was consequently to overcome these architectural limitations and to allow networked PCs to process very large financial information structures as efficiently as possible. A direct node access capability and a fast direct data access capability are the two key features which we built into the lattice manager (Lattices) and the virtual memory manager (VML) to achieve this goal. Given the time/state coordinates (i, j) of a lattice node, its address in virtual memory (VML) is looked up in a lattice access structure (LAS) with a binary search algorithm and the node is then directly accessed with one physical memory (RAM, disk or network) operation. Dynamic programming procedures that operate on the lattice are considerably speeded up with the help of a bounds access structure (BAS) which stores the consecutive upper and lower lattice bounds over time. These acceleration structures themselves run on corresponding VML-kernels (VMLAS and VMBAS). Given the address in virtual memory (VML) of a data element (lattice node), its address in physical memory (RAM, disk or network drive) is looked up in an area access structure (IASL) and an address access structure (KASL) which both again use the services of a corresponding VML-kernel (VMIASL and VMKASL). The data element is then as mentioned above directly accessed with one physical memory (context or cache) operation. A similar concept was used to implement large matrices (Matrices, VMM, etc.). With the above two design ideas the processing of large financial information structures on a PC network is always almost at the speed of RAM although the data may actually be stored on disk or even on a (remote) network drive.

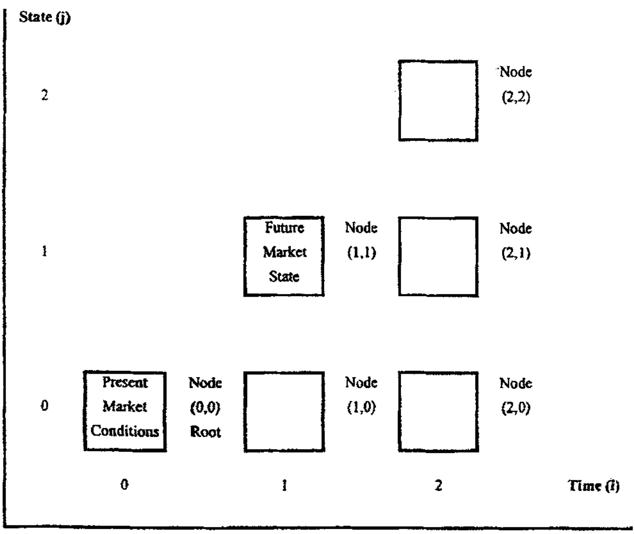


Fig. 6: Fin Re Lattice

The ***Fin Re Toolbox (which is derived from the above more general financial/(re)insurance toolbox)*** determines the current price and the current sensitivities (derivatives risk parameters) of a contingent claim (note that fixed income securities are interest rate contingent claims) as well as their future evolution over the claim's entire lifetime by using a dynamic programming procedure that operates on the underlying (binomial) lattice. Each node in this lattice represents a potential securities market state at a given future time and the root describes the current market conditions that are relevant in a Fin Re pricing context.

Development of a simple Fin Re pricing strategy involving American payer's swaptions (APSW) for example is based on the following lattice structure (see Fig. 7 below). DF0 - DF4 are the discount functions prevailing in the interest rate scenarios under study, the underlying variable is a swap (SWP), P0 - P4 are the prices of the securities under the above mentioned interest rate conditions, alpha and beta are model sensitivities with respect to potential estimation errors in the relevant model parameters (we use the extended Ho & Lee interest rate model, see below) and D0 and D1 (delta), gamma and theta are the contingent claim sensitivities (defined as rates of change of the contingent claim value with respect to instantaneous changes in the underlying initial term structure of interest rates and conditionally expected rates of change of the contingent claim value with respect to changes in time) in the given interest rate scenarios.

With this information about the future dynamics of the underlying securities market variables corresponding (consistent within an arbitrage pricing theory framework) contingent claim ***price and sensitivity (derivatives risk exposure) forecasts***, i.e., expected values

$$E[x_i] = \sum_{j=0}^{i} \pi_{ij} x_{ij} \qquad (3.1)$$

and standard deviations

$$\begin{aligned} D[x_i] &= \sqrt{V[x_i]} = \sqrt{E\left[(x_i - E[x_i])^2\right]} = \sqrt{E[x_i^2] - E[x_i]^2} \\ &= \sqrt{\sum_{j=0}^{i} \pi_{ij} x_{ij}^2 - \left(\sum_{j=0}^{i} \pi_{ij} x_{ij}\right)^2} \end{aligned} \qquad (3.2)$$

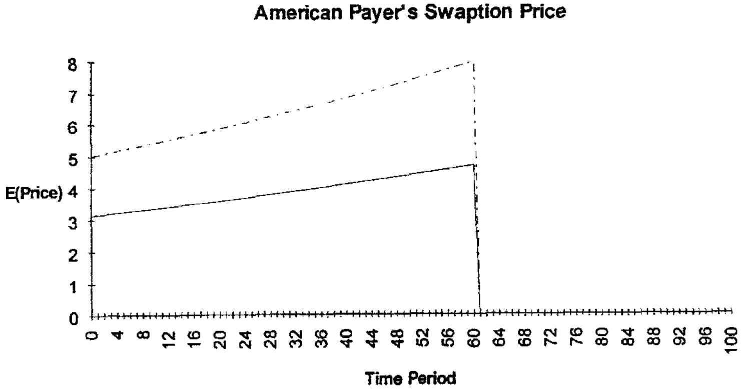
[π_{ij} is the time/state probability[10] associated with node (i, j) and $x_{i0},..,x_{ij},..,x_{ii}$ are the time i realizations of the stochastic process (x_i) denoting the discrete price or derivatives risk parameter dynamics over time], are possible (see Fig. 8 below). These forecasts (which have an ***adaptive update property***) can then be used as an effective quantitative guideline in (conventional) every-day hedging decisions as well as in the design and implementation of longer-term Fin Re portfolio management strategies. Furthermore, this data can be stored in a relational database system and on demand be consolidated into appropriate risk management reports for a book of business, a desk, a department and the entire company (Swiss Re). The simple portfolio management component (Basket) of the PC-based Fin Re platform supports

10 In a Fin Re pricing context, the financial time/state probabilities (risk-neutral) are usually modified to take excess-of-loss probabilities (risk-averse) on the liability side into account. Lattices are very convenient for such applications, as they can store the necessary information on the associated Girsanov transformation of probability measure in each node (see Davis and List [4, 5, 6, 7, 8] and the literature mentioned there for more details).

these tasks on the user interface level. It also contains all the necessary functionality for P&L accounting, derivatives risk management and hedging strategy evaluation.

Nodes							
i-Coordinate		0	1	1	2	2	2
j-Coordinate		0	0	1	0	1	2
PARENTS		0	0	0	0	0	1
		0	0	0	0	1	0
CHILDREN		0	0	1	0	1	2
		1	1	2	1	2	3
LABELS	1	0.993345	0.993358	0.993359	0.993337	0.993338	0.993338
Term	2	0.986747	0.98674	0.986741	0.986703	0.986703	0.986703
Structures	3	0.980173	0.98015	0.98015	0.980115	0.980116	0.980117
DF0	4	0.973627	0.973606	0.973607	0.97357	0.97357	0.973571
	5	0.967127	0.967104	0.967105	0.967113	0.967114	0.967115
DF1	1	0.993345	0.993358	0.993359	0.993337	0.993337	0.993338
	2	0.986747	0.98674	0.986741	0.986702	0.986703	0.986703
	3	0.980173	0.980149	0.98015	0.980115	0.980116	0.980116
	4	0.973627	0.973606	0.973607	0.97357	0.97357	0.973571
	5	0.967127	0.967104	0.967105	0.967113	0.967114	0.967115
DF2	1	0.993345	0.993359	0.993359	0.993337	0.993338	0.993338
	2	0.986747	0.98674	0.98674	0.986703	0.986703	0.986703
	3	0.980173	0.98015	0.98015	0.980116	0.980116	0.980116
	4	0.973627	0.973606	0.973607	0.97357	0.97357	0.973571
	5	0.967127	0.967104	0.967105	0.967113	0.967114	0.967114
DF3	1	0.992359	0.992373	0.992373	0.992352	0.992352	0.992352
	2	0.98479	0.984783	0.984783	0.984745	0.984746	0.984746
	3	0.977258	0.977234	0.977235	0.9772	0.977201	0.977202
	4	0.969768	0.969747	0.969748	0.969711	0.969712	0.969713
	5	0.962338	0.962315	0.962316	0.962324	0.962325	0.962326
DF4	1	0.991375	0.991389	0.991389	0.991368	0.991368	0.991368
	2	0.982838	0.982831	0.982832	0.982794	0.982794	0.982794
	3	0.974354	0.974331	0.974332	0.974297	0.974298	0.974298
	4	0.965928	0.965908	0.965908	0.965871	0.965872	0.965873
	5	0.957577	0.957555	0.957556	0.957563	0.957564	0.957565
Underlying	Price, P0	17.41971	17.53579	17.53705	17.65239	17.65366	17.65493
Variable	P1	17.42034	17.5363	17.53757	17.65278	17.65406	17.65533
SWP	P2	17.41811	17.5345	17.53513	17.65141	17.65205	17.65268
	P3	11.4745	11.56228	11.56342	11.65057	11.65172	11.65287
	P4	6.128498	6.181301	6.182331	6.234473	6.23551	6.236547
	Alpha	0.006382	0.005159	0.005159	0.003926	0.003926	0.003926
	Beta	-15956.1	-12897.3	-19228.7	-9814.39	-16170.3	-22526.4
	Delta, D0	-420.39	-422.391	-422.4	-424.393	-424.402	-424.41
	D1	-378.019	-380.492	-380.5	-382.976	-382.984	-382.992
	Gamma	2996.068	2962.654	2962.71	2928.625	2928.68	2928.734
	Theta	0.116712	0.117243	0.117248	0.118401	0.118406	0.118411
Contingent	Price, P0	3.131769	3.15296	3.152543	3.174251	3.17383	3.17341
Claim	P1	3.131515	3.152746	3.152329	3.174078	3.173657	3.173237
APSW	P2	3.132403	3.153494	3.153286	3.174683	3.174473	3.174263
	P3	5.020499	5.059333	5.058981	5.098397	5.098041	5.097685
	P4	6.674396	6.732898	6.732027	6.79182	6.79094	6.79006
	Alpha	-0.00254	-0.00214	-0.00214	-0.00173	-0.00173	-0.00173
	Beta	6342.29	5342.43	7427.137	4325.679	6427.445	8529.314
	Delta, D0	133.5534	134.801	134.8055	136.0577	136.0622	136.0668
	D1	116.9481	118.3389	118.3022	119.743	119.706	119.669
	Gamma	-1174.17	-1164.04	-1166.96	-1153.62	-1156.56	-1159.5
	Theta	0.020983	0.02108	0.021077	0.021291	0.021287	0.021284

Fig. 7: APSW Lattice Nodes



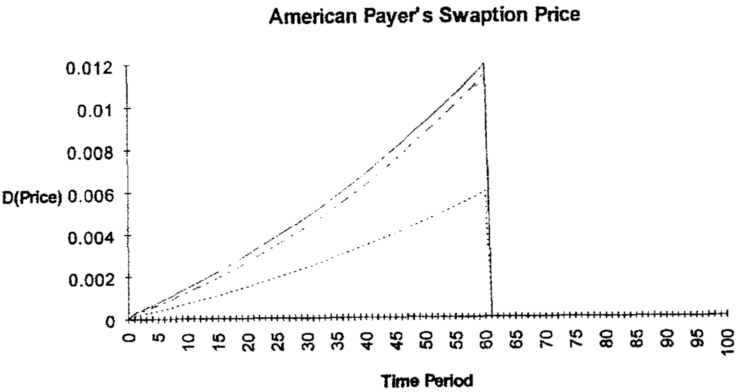
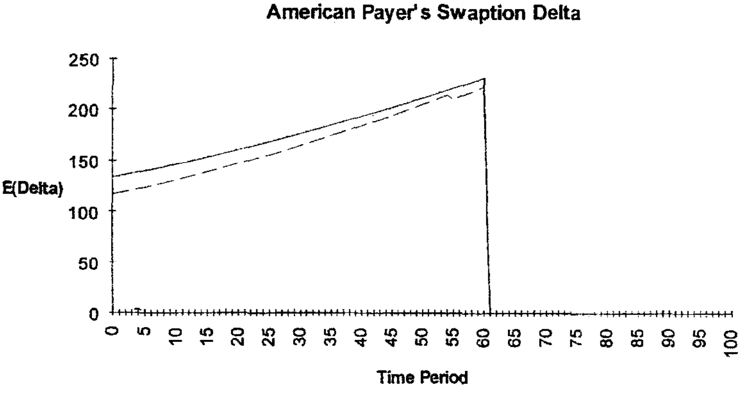


Fig. 8a: Contingent Claim Prices



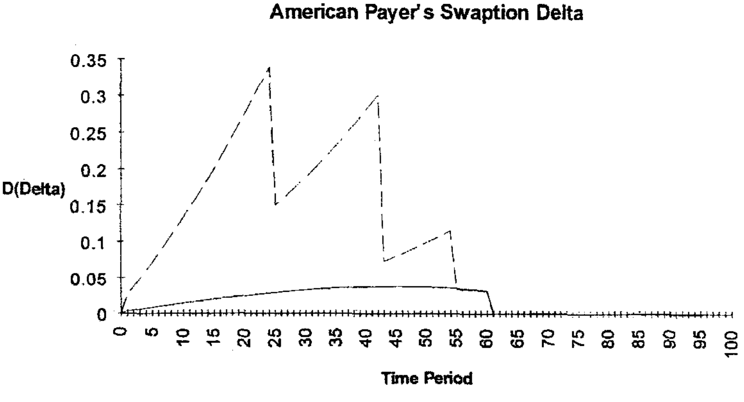
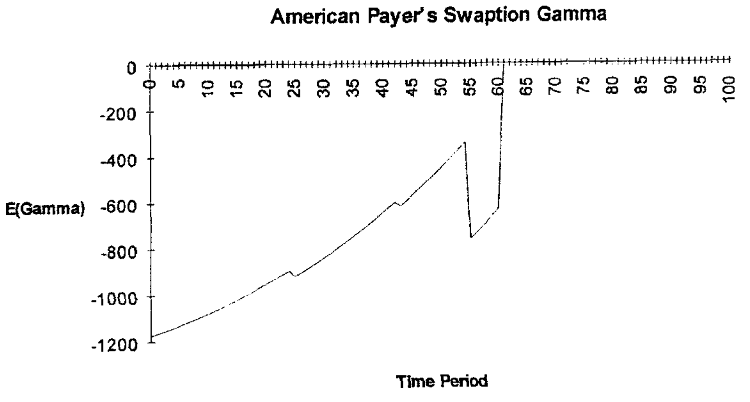


Fig. 8b: Instantaneous Investment Risk



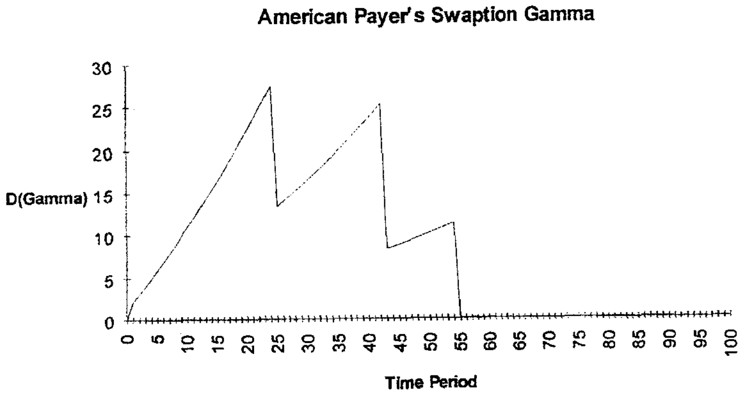
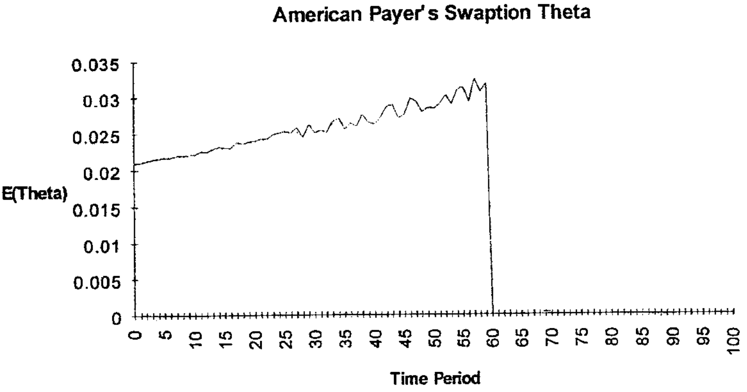


Fig. 8c: Future Risk Dynamics



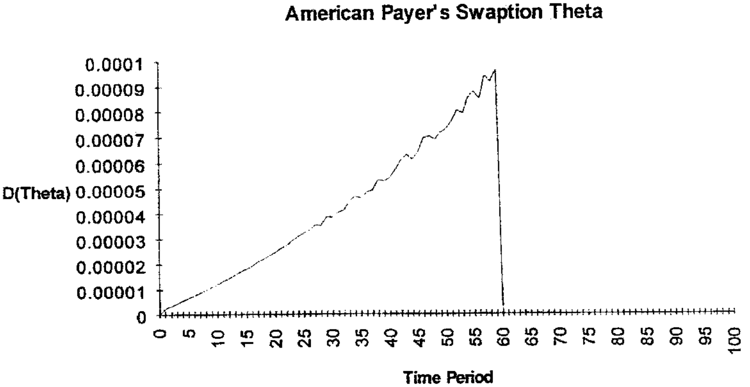


Fig. 8d: Value Appreciation Dynamics

Note that ***derivatives portfolio stress testing as recommended by the G30 banking industry best practices standard*** can be carried out simultaneously with the initial pricing / Fin Re portfolio selection[11] process. The Fin Re Toolbox described here even goes one important step further: the securities and derivatives portfolio determined is optimal in all stress scenarios. This statement means in quantitative terms (see Davis and List [5, 6, 7, 8] for details): If $x_n(t)$ is a lattice approximation of the continuous-time securities market dynamics $x(t)$, $0 \leq t \leq T$, $F_n(t,x)$ is a corresponding discrete-time futures contract value function, furthermore $c_n(t,x)$ and $p_n(t,x)$ are corresponding discrete-time European call and put value functions and moreover $C_n(t,x)$ and $P_n(t,x)$ are corresponding discrete-time American call and put value functions and $x_C^n(t)$ and $x_P^n(t)$ the associated discrete-time optimal exercise boundaries, then we have uniform convergence

$$\lim_{\substack{(\tau,\xi)\to(t,x)\\ n\to\infty}} F_n(\tau,\xi) = F(t,x)$$

$$\lim_{\substack{(\tau,\xi)\to(t,x)\\ n\to\infty}} c_n(\tau,\xi) = c(t,x) \qquad \lim_{\substack{(\tau,\xi)\to(t,x)\\ n\to\infty}} p_n(\tau,\xi) = p(t,x) \qquad (3.3)$$

$$\lim_{\substack{(\tau,\xi)\to(t,x)\\ n\to\infty}} C_n(\tau,\xi) = C(t,x) \qquad \lim_{\substack{(\tau,\xi)\to(t,x)\\ n\to\infty}} P_n(\tau,\xi) = P(t,x)$$

on compacts in $[0,T]\times[0,\infty)$ as well as uniform convergence

$$\lim_{n\to\infty} x_C^n(t) = x_C(t) \qquad \lim_{n\to\infty} x_P^n(t) = x_P(t) \qquad (3.4)$$

on $[0,T]$. If in these numerical approximations the respective discrete-time contingent claim sensitivities are defined according to the finite difference method, then the above uniform convergence results on compacts also hold for the sensitivities. Futhermore, we have uniform convergence

$$\lim_{\substack{(\tau,\xi)\to(t,x)\\ n\to\infty}} n_n(\tau,\xi) = n(t,x) \qquad \lim_{\substack{(\tau,\xi)\to(t,x)\\ n\to\infty}} V_{n_n}^n(\tau,\xi) = V_n(t,x)$$

$$\lim_{\substack{(\tau,\xi)\to(t,x)\\ n\to\infty}} \frac{E_{\tau\xi}^{\{F_i^n\}}[\Delta R_{n_n}^n]}{\Delta\tau} = \frac{E_{tx}^{\{F_i\}}[dR_n]}{dt} \qquad \lim_{\substack{(\tau,\xi)\to(t,x)\\ n\to\infty}} \frac{V_{\tau\xi}^{\{F_i^n\}}[\Delta R_{n_n}^n]}{\Delta\tau} = \frac{V_{tx}^{\{F_i\}}[dR_n]}{dt} \qquad (3.5)$$

on compacts in $[0,T]\times[0,\infty)$ for the optimal solutions of the two Fin Re portfolio selection programs

$$\max_{n(t,x)} V_n(t,x)$$

RA: AC:

$$|n(t,x)^T\delta(t,x)| \leq \delta_0 \text{ (delta)} \qquad V_n(t,x) \geq V_0 \text{ (value)} \qquad (3.6a)$$

$$|n(t,x)^T\gamma(t,x)| \leq \gamma_0 \text{ (gamma)} \qquad A(t,x)n(t,x) + b(t,x) \leq 0$$

$$n(t,x)^T\vartheta(t,x) \geq \vartheta_0 \text{ (theta)} \qquad C(t,x)n(t,x) + d(t,x) = 0$$

---

[11] Fin Re portfolios (to offset potential future liabilities under the reinsurance part of the contracts) are typically chosen by using more or less ***heuristic modifications of classical hedging strategies***. It can be shown, however, that a ***limited risk arbitrage (LRA) approach*** to Fin Re portfolio selection would perform better than hedging strategies (see Davis and List [4] and the corresponding literature mentioned there for details). ***One important reason why LRA techniques are very well suited for this kind of application lies in the fact that they achieve an overall allocation of the asset/liability risks involved that meets set targets at a reasonable price whereas the otherwise commonly applied hedging techniques (for the financial part of a Fin Re program) often unnecessarily avoid financial risks at an unacceptably high price while the (potentially dominating) risk exposure on the liability side remains high.***

(variance of return minimization) and

$$\max_{n(t,x)} n(t,x)^T \lambda(t,x)$$

$$\begin{array}{ll} \text{RA:} & \\ V_n(t,x) \geq V_0 \text{ (value)} & \text{AC:} \\ |n(t,x)^T \delta(t,x)| \leq \delta_0 \text{ (delta)} & A(t,x)n(t,x) + b(t,x) \leq 0 \\ |n(t,x)^T \gamma(t,x)| \leq \gamma_0 \text{ (gamma)} & C(t,x)n(t,x) + d(t,x) = 0 \\ n(t,x)^T \vartheta(t,x) \geq \vartheta_0 \text{ (theta)} & \end{array} \quad (3.6b)$$

(expected return maximization) and their discrete-time approximations

$$\max_{n_n(t,x)} V^n_{n_n}(t,x)$$

$$\begin{array}{ll} \text{RA:} & \text{AC:} \\ |n_n(t,x)^T \delta_n(t,x)| \leq \delta_0 \text{ (delta)} & V^n_{n_n}(t,x) \geq V_0 \text{ (value)} \\ |n_n(t,x)^T \gamma_n(t,x)| \leq \gamma_0 \text{ (gamma)} & A(t,x)n_n(t,x) + b(t,x) \leq 0 \\ n_n(t,x)^T \vartheta_n(t,x) \geq \vartheta_0 \text{ (theta)} & C(t,x)n_n(t,x) + d(t,x) = 0 \end{array} \quad (3.7a)$$

and

$$\max_{n_n(t,x)} n_n(t,x)^T \lambda_n(t,x)$$

$$\begin{array}{ll} \text{RA:} & \\ V^n_{n_n}(t,x) \geq V_0 \text{ (value)} & \text{AC:} \\ |n_n(t,x)^T \delta_n(t,x)| \leq \delta_0 \text{ (delta)} & A(t,x)n_n(t,x) + b(t,x) \leq 0 \\ |n_n(t,x)^T \gamma_n(t,x)| \leq \gamma_0 \text{ (gamma)} & C(t,x)n_n(t,x) + d(t,x) = 0 \\ n_n(t,x)^T \vartheta_n(t,x) \geq \vartheta_0 \text{ (theta)} & \end{array} \quad (3.7b)$$

(where the constraints on the portfolio value are necessary to ensure locally uniform convergence of the conditional moments of the portfolio return)[12].

The Ho & Lee Interest Rate Model. The simple Fin Re pricing application presented at the beginning of this section (no liability contingent cashflows and excess-of-loss probabilities for simplicity of presentation, yet) operates within the ***Ho & Lee interest rate model***

$$dr = \varphi(t)dt + \sigma dz \qquad \varphi(t) = \frac{\partial f}{\partial t}(0,t) + \sigma^2 t \qquad (3.8)$$

(see Ho and Lee [12] and Davis and List [5, 6, 7, 8] for details) [where $f(t,T)$ is the instantaneous forward rate at time t for an investment at time T]. In discrete-time we have then: (1) Security markets clear at time points $0,1,2,..,i,..,H$ (where H is the given investment horizon) which are separated into regular intervals (model time periods). For each of these time points i the initial discount factor $P(i)$ (relative to the time origin 0) is known. Furthermore, at each time point $i = 1,2,..,FH$ [where $FH < H$ is the relevant forward horizon] there are $i+1$ possible future discount functions $P_{ij}(k)$, $j = 0,1,..,i$ and $k = 0,1,..,MH$ [where $MH = H - FH$ is the associated maturity horizon]. (2) The evolution

[12] Note that the Fin Re portfolio selection programs outlined above are of the LRA type (see Davis and List [5, 6, 7, 8] for details). Module LRA of the Fin Re Toolbox (taken from the more general financial/(re)insurance toolbox, see Fig. 5 above) can solve such stochastic (time/state contingent) linear programs very efficiently.

of the term structure of interest rates over the investment period $[0,H]$ is modelled by a recombining binomial lattice with root

$$P_{00}(k) = P(k) \qquad (3.9a)$$

and branching process

$$P_{ij}(k) \begin{cases} \xrightarrow{\ p\ } P_{i+1j+1}(k) & P_{i+1j+1}(k) = h_u(k)\dfrac{P_{ij}(k+1)}{P_{ij}(1)} \\ \xrightarrow{\ 1-p\ } P_{i+1j}(k) & P_{i+1j}(k) = h_d(k)\dfrac{P_{ij}(k+1)}{P_{ij}(1)} \end{cases} \qquad (3.9b)$$

where

$$h_u(k) = \frac{1}{p+(1-p)d^k} \text{ and } h_d(k) = \frac{d^k}{p+(1-p)d^k} \qquad (3.10)$$

are the corresponding upward and downward perturbation functions, the model probability is p and the model delta is d, $0 \le p,d \le 1$. With the length Δt of the model time periods we have then

$$r = -\frac{1}{\Delta t}\log(P(1)) \text{ and } \sigma \approx \sqrt{\frac{p(1-p)\log(d)^2}{r^2 \Delta t^3}} \qquad (3.11)$$

for the current short-term interest rate and the term structure volatility. (3) The ***risk-neutral pricing formula***[13] is in this context

$$v_{ij} = \max\left[L(i,j), \min[P_{ij}(1)\begin{bmatrix} p[v_{i+1j+1} + X(i+1,j+1)] \\ +(1-p)[v_{i+1j} + X(i+1,j)] \end{bmatrix}, U(i,j)]\right] \quad \begin{matrix} 0 \le i \le T-1 \\ 0 \le j \le i \end{matrix} \qquad (3.12)$$
$$v_{Tj} = F(j) \quad 0 \le j \le T$$

where $T \le H$ is the contingent claim maturity. (4) With the forward rates

$$r_{ij}(k) = -\frac{1}{k\Delta t}\log(P_{ij}(k)) \qquad (3.13)$$

we define the ***derivatives risk parameters (contingent claim sensitivities)*** as follows

$$\delta_{ij} = p\frac{v_{i+1j+1} - v_{ij}}{r_{i+1j+1}(1) - r_{ij}(1)} + (1-p)\frac{v_{i+1j} - v_{ij}}{r_{i+1j}(1) - r_{ij}(1)}$$
$$\gamma_{ij} = p\frac{\delta_{i+1j+1} - \delta_{ij}}{r_{i+1j+1}(1) - r_{ij}(1)} + (1-p)\frac{\delta_{i+1j} - \delta_{ij}}{r_{i+1j}(1) - r_{ij}(1)} \qquad (3.14)$$
$$\vartheta_{ij} = p\frac{v_{i+1j+1} - v_{ij}}{\Delta t} + (1-p)\frac{v_{i+1j} - v_{ij}}{\Delta t}$$

(conditionally expected rates of change of the option value with respect to the underlying short-term interest rate and time). (5) The time/state probabilities associated with the Ho & Lee Fin Re lattice are

$$\pi_{ij} = \binom{i}{j} p^j (1-p)^{i-j} \qquad (3.15)$$

and consequently the contingent claim price and sensitivity forecasts

[13] X (intertemporal cashflows) and F (terminal condition) characterize the contingent claim. $L \le v \le U$ are boundary conditions for its price process (see Davis and List [5, 6, 7, 8] for details).

$$\mu_i^v = \sum_{j=0}^{i} \pi_{ij} v_{ij} \quad \sigma_i^v = \sqrt{\sum_{j=0}^{i} \pi_{ij} {v_{ij}}^2 - {\mu_i^v}^2} \quad \text{(price)}$$

$$\mu_i^\delta = \sum_{j=0}^{i} \pi_{ij} \delta_{ij} \quad \sigma_i^\delta = \sqrt{\sum_{j=0}^{i} \pi_{ij} {\delta_{ij}}^2 - {\mu_i^\delta}^2} \quad \text{(delta)} \qquad (3.16a)$$

$$\mu_i^\gamma = \sum_{j=0}^{i} \pi_{ij} \gamma_{ij} \quad \sigma_i^\gamma = \sqrt{\sum_{j=0}^{i} \pi_{ij} {\gamma_{ij}}^2 - {\mu_i^\gamma}^2} \quad \text{(gamma)}$$

$$\mu_i^\vartheta = \sum_{j=0}^{i} \pi_{ij} \vartheta_{ij} \quad \sigma_i^\vartheta = \sqrt{\sum_{j=0}^{i} \pi_{ij} {\vartheta_{ij}}^2 - {\mu_i^\vartheta}^2} \quad \text{(theta)} \qquad (3.16b)$$

(and similar for higher order moments of the corresponding distributions). (6) ***Structured options portfolio optimization (for Fin Re hedging purposes)*** involves the solution of either one of the following (limited risk arbitrage, LRA) programs

$$\max_{n_{ij}} V_{ij}^n \quad \left[V_{ij}^n = {n_{ij}}^T v_{ij}\right] \qquad\qquad \max_{n_{ij}} {n_{ij}}^T \vartheta_{ij}$$

$$\begin{array}{llll} \text{RA:} & & \text{RA:} & \\ \left|{n_{ij}}^T \delta_{ij}\right| \le \delta_0 \text{ (delta)} & \text{AC:} & V_{ij}^n \ge V_0 \text{ (value)} & \text{AC:} \\ \left|{n_{ij}}^T \gamma_{ij}\right| \le \gamma_0 \text{ (gamma)} & A_{ij} n_{ij} + b_{ij} \le 0 & \left|{n_{ij}}^T \delta_{ij}\right| \le \delta_0 \text{ (delta)} & A_{ij} n_{ij} + b_{ij} \le 0 \\ {n_{ij}}^T \vartheta_{ij} \ge \vartheta_0 \text{ (theta)} & C_{ij} n_{ij} + d_{ij} = 0 & \left|{n_{ij}}^T \gamma_{ij}\right| \le \gamma_0 \text{ (gamma)} & C_{ij} n_{ij} + d_{ij} = 0 \end{array} \qquad (3.17a)$$

in a general (state dependent linear optimization) or

$$\max_{\bar{n}_i} \overline{V}_i^n \quad \left[\overline{V}_i^n = {\bar{n}_i}^T \mu_i^v\right] \qquad\qquad \max_{\bar{n}_i} {\bar{n}_i}^T \mu_i^\vartheta$$

$$\begin{array}{llll} \text{RA:} & & \text{RA:} & \\ \left|{\bar{n}_i}^T \mu_i^\delta\right| \le \delta_0 \text{ (delta)} & \text{AC:} & \overline{V}_i^n \ge V_0 \text{ (value)} & \text{AC:} \\ \left|{\bar{n}_i}^T \mu_i^\gamma\right| \le \gamma_0 \text{ (gamma)} & A_i \bar{n}_i + b_i \le 0 & \left|{\bar{n}_i}^T \mu_i^\delta\right| \le \delta_0 \text{ (delta)} & A_i \bar{n}_i + b_i \le 0 \\ {\bar{n}_i}^T \mu_i^\vartheta \ge \vartheta_0 \text{ (theta)} & C_i \bar{n}_i + d_i = 0 & \left|{\bar{n}_i}^T \mu_i^\gamma\right| \le \gamma_0 \text{ (gamma)} & C_i \bar{n}_i + d_i = 0 \end{array} \qquad (3.17b)$$

in a simplified (linear optimization based on forecast expectations) context. Note that the optimal positions (over the whole lifetime of the portfolio) are known before the portfolio is actually set up. This allows Fin Re portfolio managers to use hedging strategies that minimize holding and transaction costs. (7) The parameters of the above outlined Ho & Lee interest rate model are p (model probability), d (model delta) and $R(1), R(2), .., R(i), .., R(H)$ (initial term structure of simple, annualized interest rates). We have then

$$r(i) = \log(1 + R(i)) \text{ and } P(i) = e^{-r(i)i\Delta t} = \frac{1}{(1 + R(i)\Delta t)^i} \qquad (3.18)$$

for the corresponding continuously compounded interest rates and discount factors. In addition to these parameters we now also consider the quantities

$$\Delta p \text{ (probability increment), } \Delta d \text{ (delta increment) and} \qquad (3.19a)$$

$$\Delta R(1), \Delta R(2), .., \Delta R(i), .., \Delta R(H) \text{ (interest rate increments)} \qquad (3.19b)$$

and construct a recombining binomial lattice for the term structures

$$P_{ij}^{(p,d,R)}(k),\ P_{ij}^{(p+\Delta p,d,R)}(k),\ P_{ij}^{(p,d+\Delta d,R)}(k),\ P_{ij}^{(p,d,R+\Delta R)}(k) \text{ and } P_{ij}^{(p,d,R+2\Delta R)}(k). \qquad (3.20)$$

For an interest rate contingent claim we then calculate the corresponding scenario dependent prices

$$V_{ij}^{(p,d,R)},\ V_{ij}^{(p+\Delta p,d,R)},\ V_{ij}^{(p,d+\Delta d,R)},\ V_{ij}^{(p,d,R+\Delta R)} \text{ and } V_{ij}^{(p,d,R+2\Delta R)} \quad (3.21a)$$

and sensitivities

$$\alpha_{ij}^{(p,d,R)} = \frac{V_{ij}^{(p+\Delta p,d,R)} - V_{ij}^{(p,d,R)}}{\Delta p} \qquad \beta_{ij}^{(p,d,R)} = \frac{V_{ij}^{(p,d+\Delta d,R)} - V_{ij}^{(p,d,R)}}{\Delta d} \quad (3.21b)$$

$$\delta_{ij}^{(p,d,R)} = \frac{V_{ij}^{(p,d,R+\Delta R)} - V_{ij}^{(p,d,R)}}{\|\Delta R\|}$$

$$\delta_{ij}^{(p,d,R+\Delta R)} = \frac{V_{ij}^{(p,d,R+2\Delta R)} - V_{ij}^{(p,d,R+\Delta R)}}{\|\Delta R\|} \qquad \gamma_{ij}^{(p,d,R)} = \frac{\delta_{ij}^{(p,d,R+\Delta R)} - \delta_{ij}^{(p,d,R)}}{\|\Delta R\|} \quad (3.21c)$$

$$\vartheta_{ij}^{(p,d,R)} = p\frac{V_{i+1j+1}^{(p,d,R)} - V_{ij}^{(p,d,R)}}{\Delta t} + (1-p)\frac{V_{i+1j}^{(p,d,R)} - V_{ij}^{(p,d,R)}}{\Delta t} \quad (3.21d)$$

Note that the probability and delta exposure

$$\alpha = \frac{\partial v}{\partial p} \text{ and } \beta = \frac{\partial v}{\partial d} \quad (3.22)$$

of a contingent claim in the Ho & Lee interest rate model can be written in the form

$$\alpha = \upsilon\frac{\partial\sigma}{\partial p} + \ldots \qquad \beta = \upsilon\frac{\partial\sigma}{\partial d} + \ldots \quad (3.23a)$$

where

$$\frac{\partial\sigma}{\partial p} \approx \frac{(1-2p)\log(d)^2}{2\sigma r^2 \Delta t^3} \qquad \frac{\partial\sigma}{\partial d} \approx \frac{p(1-p)\log(d)}{d\sigma r^2 \Delta t^3} \quad (3.23b)$$

holds. This extension of the original Ho & Lee interest rate model can still be easily calculated (simultaneous scenario analyses) and is very suitable for Fin Re pricing and portfolio management (hedge portfolios) under varying securities market scenarios.

Fixed Income Securities (Bonds). Fixed income securities play a major role in Fin Re pricing applications: institutional investors such as investment trusts, pension funds and (life) insurance companies invest large funds in order to satisfy future liabilities resulting from the various contractual obligations entered into with their clients. Bonds have cashflow characteristics that make them very attractive investments for these purposes: by monitoring credit risk and call risk and adequately diversifying a bond portfolio by type of issuer, an investor can expect its promised cashflows with a high degree of certainty. The sources of return from investing in a bond are its coupon payments, the interest on these payments and potential capital gains over the investment horizon. Holding aside credit risk and embedded options, there are therefore three components to evaluating the attractiveness of a bond: yield, duration and convexity. If the bond's price is P and its cashflows are $c_1,..,c_t,..,c_T$ (where T is the maturity period), then its ***yield to maturity (YTM)*** is defined by the equation

$$P = \sum_{t=1}^{T} \frac{c_t}{(1+y)^t}. \quad (3.24)$$

Given an investment horizon H, a realized end price P_H of the bond and a set of reinvestment rates $r_2,..,r_t,..,r_H$ the bond's ***realized compound yield (RCY)*** is defined by the equation

$$P(1+Y)^H = P_H + c_H + \sum_{t=1}^{H-1} c_t(1+r_{t+1})..(1+r_H). \quad (3.25)$$

Portfolio managers typically use these simple yield measures as a basis for undertaking ***bond swaps*** in order to enhance the performance of their bond portfolio over some investment period. There are five basic types of bond swaps: ***pure yield pickup swaps***, ***substitution swaps***, ***interest rate anticipation swaps***, ***intermarket spread swaps*** and ***tax swaps***. A rate anticipation swap involves the portfolio manager's expectations about future interest rate movements and the idea is to position the bond portfolio on the basis of its interest rate sensitivity (duration) to take advantage of anticipated shifts in market interest rates: if rates are expected to fall, the portfolio's duration is increased; if rates are expected to rise, high duration bonds in the portfolio are swapped for lower duration bonds in the market. The ***Macaulay duration*** of a bond is

$$D = \frac{\sum_{t=1}^{T} \frac{tc_t}{(1+y)^t}}{P} \qquad (3.26)$$

and the relationship

$$\frac{dP}{P} = -D\frac{dy}{1+y} \qquad (3.27)$$

shows that Macaulay duration is indeed a measure of its first order interest rate exposure. This equation also explains the above mentioned simple bond portfolio optimization strategy. ***Convexity***, the bond's second order interest rate sensitivity, is

$$C = \frac{1}{2P}\frac{d^2P}{dy^2} = \frac{\sum_{t=1}^{T} \frac{(t+1)tc_t}{(1+y)^t}}{2P(1+y)^2} \qquad (3.28)$$

and the relationship

$$\frac{\Delta P}{P} \approx -D\frac{\Delta y}{1+y} + C\Delta y^2 \qquad (3.29)$$

shows that a high convexity bond outperforms a bond with the same yield and duration characteristics but lower convexity in all conceivable interest rate scenarios. Generally, therefore, high convexity bonds offer lower yields, that is, the market prices convexity. This yield discount can be substantial in times of high anticipated interest rate volatility. The ***modified duration***

$$\tilde{D} = D/(1+y) \qquad (3.30)$$

and the convexity C of a bond portfolio are value-weighted averages of the respective component quantities, i.e.,

$$\frac{1}{x_1P_1 + x_2P_2}\frac{d(x_1P_1 + x_2P_2)}{dy} = \frac{x_1P_1(\frac{1}{P_1}\frac{dP_1}{dy}) + x_2P_2(\frac{1}{P_2}\frac{dP_2}{dy})}{x_1P_1 + x_2P_2}. \qquad (3.31)$$

This fact is the basis for another class of simple bond portfolio optimization strategies the objective of which is to improve portfolio performance (RCY) while keeping interest rate exposure (duration and convexity) at the same level. These so called ***duration-equivalent portfolio swaps*** typically replace a bond currently held (***bullet***) with a synthetic security (***barbell***) consisting of two bonds in amounts chosen according to the equation system

$$\frac{x_1P_1\tilde{D}_1 + x_2P_2\tilde{D}_2}{x_1P_1 + x_2P_2} = \tilde{D} \qquad \frac{x_1P_1C_1 + x_2P_2C_2}{x_1P_1 + x_2P_2} = C \qquad (3.32)$$

and work well under parallel shifts of the current term structure of interest rates. Duration adjustments of bond portfolios are usually carried out by using bond futures contracts instead of the bonds themselves. The ***futures price*** [where $S < T$ is the contract maturity period] is related to the bond's ***spot price*** via the equation

$$\frac{F}{(1+y)^S} = P - \sum_{t=1}^{S} \frac{c_t}{(1+y)^t} \qquad (3.33)$$

from which the contract's modified duration and convexity can immediately be calculated, i.e.,

$$\tilde{D}_F = -\frac{1}{F}\frac{dF}{dy} = \frac{\tilde{D}_P - \dfrac{\sum_{t=1}^{S} \frac{tc_t}{(1+y)^t}}{P(1+y)}}{1 - \dfrac{\sum_{t=1}^{S} \frac{c_t}{(1+y)^t}}{P}} - \frac{S}{1+y} \qquad (3.34)$$

$$C_F = \frac{1}{2F}\frac{d^2F}{dy^2} = \frac{C_P - \dfrac{\sum_{t=1}^{S} \frac{(t+1)tc_t}{(1+y)^t}}{2P(1+y)^2}}{1 - \dfrac{\sum_{t=1}^{S} \frac{c_t}{(1+y)^t}}{P}} + \frac{(S-1)S}{2(1+y)^2}. \qquad (3.35)$$

The futures position is then chosen according to the equation system

$$\frac{xF\tilde{D}_F + V_B\tilde{D}_B}{xF + V_B} = \tilde{D} \qquad \frac{xFC_F + V_BC_B}{xF + V_B} = C \qquad (3.36)$$

where V_B, $\tilde{D}_B$ and C_B are the bond portfolio value, modified duration and convexity. The same technique can also be used by managers of mixed asset portfolios (bonds and stocks) to change their optimal asset allocation on a duration-equivalent basis. The bond futures position is in this case chosen according to the equation system

$$\frac{xF\tilde{D}_F + V_I\tilde{D}_I}{V_T} = \tilde{D}_T = \tilde{D}_I \qquad \frac{xFC_F + V_IC_I}{V_T} = C_T \qquad (3.37)$$

where V_I, V_T, $\tilde{D}_I$, $\tilde{D}_T$, C_I and C_T are the initial and target bond portfolio values, modified durations and convexities. Unlike the simple swap strategies described so far, ***structured bond portfolio management strategies*** do not rely on expectations of interest rate movements or changes in yield spread relationships. Instead, the objective is to design a portfolio that will achieve the performance of some predetermined benchmark (***indexing***) or finance a single future liability (***immunization***) or an entire future liability stream (***cashflow matching***). If $P_1^a,..,P_j^a,..,P_N^a$ and $P_1^b,..,P_j^b,..,P_N^b$ are the asked and the bid prices, respectively, of the bonds currently available in a specific bond market and if $c_1^j,..,c_t^j,..,c_H^j$ [where $H \geq T_j$ is the relevant investment horizon] are the (adjusted) cashflows of bond j, $1 \leq j \leq N$, during the investment period under consideration, then the ***general structured bond portfolio management problem*** can be stated in the form

$$\max_{x_j^a, x_j^b} \sum_{j=1}^{N} x_j^b P_j^b - x_j^a P_j^a \qquad (3.38)$$

$$u_1 \leq \sum_{j=1}^{N} [x_j^a - x_j^b] c_1^j \leq v_1$$

$$u_2 \leq \sum_{j=1}^{N} [x_j^a - x_j^b][(1+r_2)c_1^j + c_2^j] \leq v_2$$

$$u_3 \le \sum_{j=1}^{N} [x_j^a - x_j^b][(1+r_3)[(1+r_2)c_1^j + c_2^j] + c_3^j] \le v_3$$

....

$$u_H \le \sum_{j=1}^{N} [x_j^a - x_j^b][(1+r_H)[(1+r_{H-1})[..] + c_{H-1}^j] + c_H^j] \le v_H$$

(see Ronn [14], Ehrhardt [15] and Davis and List [5, 6, 7, 8] for details) where $r_2,..,r_t,..,r_H$ are the consecutive implied one period forward rates in the market and the respective long and short bond portfolio positions satisfy $0 \le x_j^a \le a_j$ and $0 \le x_j^b \le b_j$, $1 \le j \le N$. If $R_1,..,R_t,..,R_H$ is the term structure of simple interest rates per trading period, then the above forward rates satisfy

$$1+r_2 = \frac{(1+R_2)^2}{1+R_1}, .., 1+r_H = \frac{(1+R_H)^H}{(1+R_{H-1})^{H-1}} \qquad (3.39)$$

This linear program has two interpretations: (a) from a bond arbitrage point of view the objective is to maximize the current market value of the portfolio (by exploiting relative mispricing - e.g., as a result of different tax brackets - of bonds in the market) while at the same time constraining the risk exposure of the arbitrage transactions (in terms of their implications on the future portfolio cashflows) to values within a specified tolerance band (u_t, v_t); (b) from a ***term structure estimation*** point of view by solving the ***associated dual problem***

$$\min_{d_t \ge 0, y_j \ge 0} \sum_{j=1}^{2N} y_j \qquad (3.40)$$

$$P_j^b \le \sum_{t=1}^{H} d_t c_t^j + y_j, \; 1 \le j \le N$$

$$\sum_{t=1}^{H} d_t c_t^j - y_{N+j} \le P_j^a, \; 1 \le j \le N$$

$$d_0 = 1, \; 0 \le d_t - (1+\rho) d_{t+1}, \; 0 \le t < H$$

a corresponding (tax-specific) term structure $d_0, d_1,.., d_t,..., d_H$ of discount factors and associated simple interest rates per trading period

$$R_1 = \frac{1}{d_1} - 1, \; R_2 = \sqrt{\frac{1}{d_2}} - 1, .., \; R_H = \sqrt[H]{\frac{1}{d_H}} - 1 \qquad (3.41)$$

that is consistent with a given exogenous (minimal) one period reinvestment rate ρ and prices all bonds in the market within their respective bid/offer spreads (P_j^b, P_j^a) can be obtained.

The Hull & White Class of Interest Rate Models. The ***Hull & White class of (one-factor) interest rate models*** has the generic representation

$$dr(t) = m(\theta(t), \phi(t), r(t), t)dt + s(r(t), t)dz(t) \qquad (3.42a)$$

(see Hull & White [13] and Davis and List [5, 6, 7, 8] for details) and is rich enough to model a wide variety of different interest rate scenarios occurring in practical corporate and investment banking as well as financial (re)insurance applications. Here, we are especially interested in the model

$$dr(t) = [\theta(t) - \phi(t) r(t)]dt + \sigma r(t)^{\beta} dz(t) \qquad (3.42b)$$

of the ***extended Cox, Ingersoll and Ross (CIR) type*** [belonging to the Hull & White class of interest rate models just as the above-mentioned Ho & Lee model

$$dr = \varphi(t)dt + \sigma dz \qquad \varphi(t) = \frac{\partial f}{\partial t}(0,t) + \sigma^2 t \text{].}$$

The generic class of Hull & White models has a very efficient (recombining) trinomial lattice implementation (see Hull and White [13] and Davis and List [6] for details) and is able to fit

(a) the ***current term structure of short-term interest rates*** [specification of the function $\theta(t)$ which is deterministic and varies with time t];
(b) the ***current volatility structure of short-term interest rates*** [specification of the function $\phi(t)$ which is also deterministic and varies with time t].

The risk-neutral pricing formula and the contingent claim sensitivities (derivatives risk parameters) are defined as in the Ho & Lee model above - with the obvious modifications to take the trinomial structure of the underlying lattice implementation into account. Note that the term structure estimation approach (3.40) above can be used to provide customized (i.e., taking client-specific tax-brackets, corporate debt structures, etc. into account) estimates of the initial term structure of short-term interest rates for the Hull & White (and the Ho & Lee) model. Current volatilities-are usually estimated from historical (discount) bond yield data.

4. Modelling Exchange Rates / Stocks / Stock Indices

The Black & Scholes Model. Fin Re pricing in a Black & Scholes securities market setting is driven by the following analytics (see Black and Scholes [16], Black [17] and Davis and List [5, 6, 7, 8] for details):

A. Bond and Stock Options. The ***Black & Scholes equation*** for bond and stock options is

$$\frac{\partial v}{\partial t} + rx\frac{\partial v}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 v}{\partial x^2} - rv = 0 \qquad (4.1)$$

(note that coupons or dividends during the lifetime of the option have to be discounted and subtracted from the current bond or stock price). With

$$d_1(t) = \frac{\log(\frac{x(t)}{X}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \qquad d_2(t) = d_1(t) - \sigma\sqrt{T-t} \qquad (4.2)$$

this linear partial differential equation can in the case of futures contracts and European options be integrated by using a ***risk-neutral valuation argument***. The results are as follows.

Futures (4.3):

$$F(t) = x(t)e^{r(T-t)}$$
$$\delta(t) = e^{r(T-t)}$$
$$\gamma(t) = 0$$
$$\vartheta(t) = -x(t)re^{r(T-t)}$$
$$\upsilon(t) = 0$$
$$\rho(t) = x(t)(T-t)e^{r(T-t)}$$

Call Option (4.4):

$$c(t) = x(t)N(d_1(t)) - Xe^{-r(T-t)}N(d_2(t))$$
$$\delta(t) = N(d_1(t))$$
$$\gamma(t) = \frac{N'(d_1(t))}{x(t)\sigma\sqrt{T-t}}$$
$$\vartheta(t) = -\frac{x(t)\sigma}{2\sqrt{T-t}}N'(d_1(t)) - rXe^{-r(T-t)}N(d_2(t))$$
$$\upsilon(t) = x(t)\sqrt{T-t}N'(d_1(t))$$
$$\rho(t) = (T-t)Xe^{-r(T-t)}N(d_2(t))$$

Put Option (4.5):

$$p(t) = Xe^{-r(T-t)}N(-d_2(t)) - x(t)N(-d_1(t))$$
$$\delta(t) = N(d_1(t)) - 1$$
$$\gamma(t) = \frac{N'(d_1(t))}{x(t)\sigma\sqrt{T-t}}$$
$$\vartheta(t) = -\frac{x(t)\sigma}{2\sqrt{T-t}}N'(d_1(t)) + rXe^{-r(T-t)}N(-d_2(t))$$
$$\upsilon(t) = x(t)\sqrt{T-t}N'(d_1(t))$$
$$\rho(t) = -(T-t)Xe^{-r(T-t)}N(-d_2(t))$$

In the case of American bond and stock options, the above Black & Scholes equation can be solved with numerical techniques, i.e., finite difference methods and lattice approaches. A discretization with the ***implicit finite difference operators*** leads to the (tridiagonal) system

$$a_j v_{ij-1} + b_j v_{ij} + c_j v_{ij+1} = v_{i+1j} \qquad i = 0,..,m-1 \qquad j = 1,..,n-1 \tag{4.6a}$$

of linear equations with (state dependent) coefficients

$$a_j = \frac{j\Delta t}{2}(r - \sigma^2 j) \quad b_j = 1 + \Delta t(r + \sigma^2 j^2) \quad c_j = -\frac{j\Delta t}{2}(r + \sigma^2 j) \tag{4.6b}$$

that can easily be solved backwards in time by using the boundary conditions and early exercise criteria which characterize the given contingent claim v. A discretization with the ***explicit finite difference operators*** substantially simplifies these calculations. The corresponding linear equation system is in this case

$$v_{ij} = a_j v_{i+1,j-1} + b_j v_{i+1j} + c_j v_{i+1j+1} \qquad i = 0,..,m-1 \qquad j = 1,..,n-1 \tag{4.7a}$$

and the (state dependent) coefficients are

$$a_j = \frac{j\Delta t}{2(1+r\Delta t)}(-r + \sigma^2 j) \quad b_j = \frac{1}{1+r\Delta t}(1 - \sigma^2 j^2 \Delta t) \quad c_j = \frac{j\Delta t}{2(1+r\Delta t)}(r + \sigma^2 j) \tag{4.7b}$$

[where the local consistency conditions

$$\sigma^2 \geq r \quad \Delta t \leq \frac{1}{\sigma^2(n-1)^2} \tag{4.8}$$

(see Davis and List [4, 5, 6, 7, 8] and the literature mentioned there) have to be satisfied].

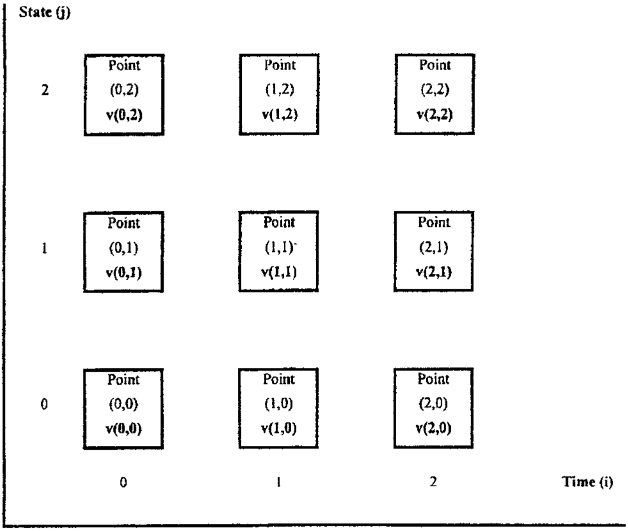


Fig. 9a: Finite Difference Method

The parameters

$$p,\ u \quad \begin{array}{c}\text{risk-averse}\\ \text{state evolution}\end{array} \qquad \tilde{p},\ \tilde{u} \quad \begin{array}{c}\text{risk-neutral}\\ \text{state evolution}\end{array} \tag{4.9}$$

of a ***lattice approximation*** of the bond and stock price dynamics

$$dx = \mu x dt + \sigma x dz = r x dt + \sigma x d\tilde{z}$$

$$E_{tx}[x(t+\Delta t)] = x e^{\mu \Delta t} \qquad V_{tx}[x(t+\Delta t)] = x^2 e^{2\mu\Delta t}\left[e^{\sigma^2 \Delta t} - 1\right]$$

$$\text{(risk-averse state evolution)} \tag{4.10}$$

$$\tilde{E}_{tx}[x(t+\Delta t)] = x e^{r \Delta t} \qquad \tilde{V}_{tx}[x(t+\Delta t)] = x^2 e^{2r\Delta t}\left[e^{\sigma^2 \Delta t} - 1\right]$$

$$\text{(risk-neutral state evolution)}$$

in a risk-averse and in a risk-neutral financial economy are

$$u = \frac{c + \sqrt{c^2 - 4a^2}}{2a} \qquad p = \frac{a - d}{u - d}$$
$$\tilde{u} = \frac{\tilde{c} + \sqrt{\tilde{c}^2 - 4\tilde{a}^2}}{2\tilde{a}} \qquad \tilde{p} = \frac{\tilde{a} - \tilde{d}}{\tilde{u} - \tilde{d}} \tag{4.11}$$

where

$$a = e^{\mu\Delta t} \qquad b^2 = e^{2\mu\Delta t}\left[e^{\sigma^2\Delta t} - 1\right] \qquad c = a^2 + b^2 + 1$$
$$\tilde{a} = e^{r\Delta t} \qquad \tilde{b}^2 = e^{2r\Delta t}\left[e^{\sigma^2\Delta t} - 1\right] \qquad \tilde{c} = \tilde{a}^2 + \tilde{b}^2 + 1 \tag{4.12}$$

holds[14].

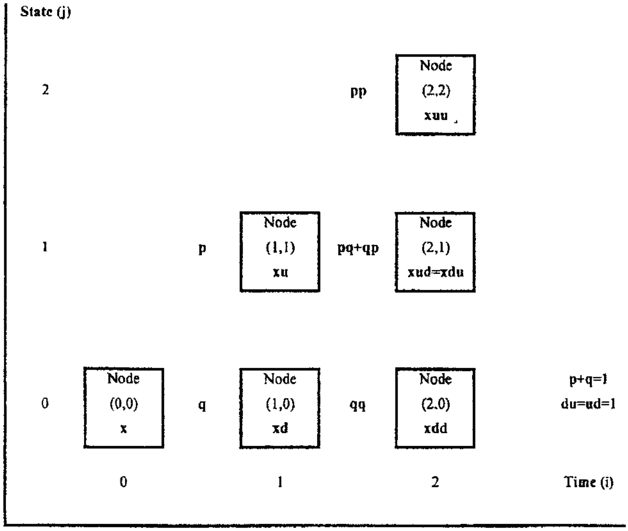


Fig. 9b: Lattice Approach

B. Index Options. The Black & Scholes equation for index options is

$$\frac{\partial v}{\partial t}+(r-y)x\frac{\partial v}{\partial x}+\frac{\sigma^2 x^2}{2}\frac{\partial^2 v}{\partial x^2}-rv=0 \qquad (4.13)$$

where y is the associated constant, continuously compounded, annualized dividend yield [if dividends were reinvested, then the index would grow from value $x(t)$ at time t to value $x(T)e^{y(T-t)}$ at time T]. With

$$d_1(t)=\frac{\log(\frac{x(t)}{X})+(r-y+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad d_2(t)=d_1(t)-\sigma\sqrt{T-t} \qquad (4.14)$$

this linear partial differential equation can in the case of futures contracts and European options be integrated by using a risk-neutral valuation argument. The results are as follows.

Futures (4.15):

$$F(t)=x(t)e^{(r-y)(T-t)}$$
$$\delta(t)=e^{(r-y)(T-t)}$$
$$\gamma(t)=0$$
$$\vartheta(t)=-x(t)(r-y)e^{(r-y)(T-t)}$$
$$\upsilon(t)=0$$

[14] For the risk-neutral pricing formula and the definition of the contingent claim sensitivities (derivatives risk parameters), see further below (Rubinstein model = generalization of the Black & Scholes model).

$$\rho_r(t) = x(t)(T-t)e^{(r-y)(T-t)}$$
$$\rho_y(t) = -x(t)(T-t)e^{(r-y)(T-t)}$$

Call Option (4.16):

$$c(t) = x(t)e^{-y(T-t)}N(d_1(t)) - Xe^{-r(T-t)}N(d_2(t))$$
$$\delta(t) = e^{-y(T-t)}N(d_1(t))$$
$$\gamma(t) = \frac{e^{-y(T-t)}N'(d_1(t))}{x(t)\sigma\sqrt{T-t}}$$
$$\vartheta(t) = -\frac{x(t)\sigma}{2\sqrt{T-t}}e^{-y(T-t)}N'(d_1(t)) + yx(t)e^{-y(T-t)}N(d_1(t)) - rXe^{-r(T-t)}N(d_2(t))$$
$$\upsilon(t) = \sqrt{T-t}\,x(t)e^{-y(T-t)}N'(d_1(t))$$
$$\rho_r(t) = (T-t)Xe^{-r(T-t)}N(d_2(t))$$
$$\rho_y(t) = -(T-t)x(t)e^{-y(T-t)}N(d_1(t))$$

Put Option (4.17):

$$p(t) = Xe^{-r(T-t)}N(-d_2(t)) - x(t)e^{-y(T-t)}N(-d_1(t))$$
$$\delta(t) = e^{-y(T-t)}\left[N(d_1(t)) - 1\right]$$
$$\gamma(t) = \frac{e^{-y(T-t)}N'(d_1(t))}{x(t)\sigma\sqrt{T-t}}$$
$$\vartheta(t) = -\frac{x(t)\sigma}{2\sqrt{T-t}}e^{-y(T-t)}N'(d_1(t)) - yx(t)e^{-y(T-t)}N(-d_1(t)) + rXe^{-r(T-t)}N(-d_2(t))$$
$$\upsilon(t) = \sqrt{T-t}\,x(t)e^{-y(T-t)}N'(d_1(t))$$
$$\rho_r(t) = -(T-t)Xe^{-r(T-t)}N(-d_2(t))$$
$$\rho_y(t) = (T-t)x(t)e^{-y(T-t)}N(-d_1(t))$$

In the case of American index options the above Black & Scholes equation can be solved with numerical techniques, i.e., finite difference methods and lattice approaches. A discretization with the implicit finite difference operators leads to the (tridiagonal) system

$$\begin{gathered} a_j v_{ij-1} + b_j v_{ij} + c_j v_{ij+1} = v_{i+1j} \\ i = 0,..,m-1 \qquad j = 1,..,n-1 \end{gathered} \tag{4.18a}$$

of linear equations with (state dependent) coefficients

$$a_j = \frac{j\Delta t}{2}(r - y - \sigma^2 j) \quad b_j = 1 + \Delta t(r + \sigma^2 j^2) \quad c_j = -\frac{j\Delta t}{2}(r - y + \sigma^2 j) \tag{4.18b}$$

that can easily be solved backwards in time by using the boundary conditions and early exercise criteria which characterize the given contingent claim v. A discretization with the explicit instead of the implicit finite difference operators substantially simplifies these calculations. The corresponding linear equation system is in this case

$$\begin{gathered} v_{ij} = a_j v_{i+1,j-1} + b_j v_{i+1j} + c_j v_{i+1j+1} \\ i = 0,..,m-1 \qquad j = 1,..,n-1 \end{gathered} \tag{4.19a}$$

and the (state dependent) coefficients are

$$a_j = \frac{j\Delta t}{2(1+r\Delta t)}(y - r + \sigma^2 j) \quad b_j = \frac{1}{1+r\Delta t}(1 - \sigma^2 j^2 \Delta t) \quad c_j = \frac{j\Delta t}{2(1+r\Delta t)}(r - y + \sigma^2 j) \tag{4.19b}$$

[where the local consistency conditions

$$\sigma^2 \geq |r-y| \quad \Delta t \leq \frac{1}{\sigma^2(n-1)^2} \qquad (4.20)$$

have to be satisfied]. The parameters

$$p,\ u \ \ \begin{matrix}\text{risk-averse}\\ \text{state evolution}\end{matrix} \qquad \tilde{p},\ \tilde{u} \ \ \begin{matrix}\text{risk-neutral}\\ \text{state evolution}\end{matrix} \qquad (4.21)$$

of a lattice structure approximating the index dynamics

$$dx = \mu x dt + \sigma x dz = (r-y)x dt + \sigma x d\tilde{z}$$

$$E_{tx}[x(t+\Delta t)] = xe^{\mu\Delta t} \qquad V_{tx}[x(t+\Delta t)] = x^2 e^{2\mu\Delta t}\left[e^{\sigma^2\Delta t}-1\right]$$
$$\text{(risk-averse state evolution)} \qquad (4.22)$$
$$\tilde{E}_{tx}[x(t+\Delta t)] = xe^{(r-y)\Delta t} \qquad \tilde{V}_{tx}[x(t+\Delta t)] = x^2 e^{2(r-y)\Delta t}\left[e^{\sigma^2\Delta t}-1\right]$$
$$\text{(risk-neutral state evolution)}$$

in a risk-averse and in a risk-neutral financial economy are

$$u = \frac{c+\sqrt{c^2-4a^2}}{2a} \qquad p = \frac{a-d}{u-d}$$
$$\tilde{u} = \frac{\tilde{c}+\sqrt{\tilde{c}^2-4\tilde{a}^2}}{2\tilde{a}} \qquad \tilde{p} = \frac{\tilde{a}-\tilde{d}}{\tilde{u}-\tilde{d}} \qquad (4.23)$$

where

$$a = e^{\mu\Delta t} \qquad b^2 = e^{2\mu\Delta t}\left[e^{\sigma^2\Delta t}-1\right] \qquad c = a^2+b^2+1$$
$$\tilde{a} = e^{(r-y)\Delta t} \qquad \tilde{b}^2 = e^{2(r-y)\Delta t}\left[e^{\sigma^2\Delta t}-1\right] \qquad \tilde{c} = \tilde{a}^2+\tilde{b}^2+1 \qquad (4.24)$$

holds.

C. Currency Options. The Black & Scholes equation for currency options is

$$\frac{\partial v}{\partial t} + (r-r_f)x\frac{\partial v}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 v}{\partial x^2} - rv = 0 \qquad (4.25)$$

where r_f is the associated constant, continuously compounded, annualized, risk-free foreign interest rate. With

$$d_1(t) = \frac{\log(\frac{x(t)}{X}) + (r-r_f+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \qquad d_2(t) = d_1(t) - \sigma\sqrt{T-t} \qquad (4.26)$$

this linear partial differential equation can in the case of futures contracts and European options be integrated by using a risk-neutral valuation argument. The results are as follows.

Futures (4.27):

$$F(t) = x(t)e^{(r-r_f)(T-t)}$$
$$\delta(t) = e^{(r-r_f)(T-t)}$$
$$\gamma(t) = 0$$
$$\vartheta(t) = -x(t)(r-r_f)e^{(r-r_f)(T-t)}$$
$$\upsilon(t) = 0$$
$$\rho_r(t) = x(t)(T-t)e^{(r-r_f)(T-t)}$$
$$\rho_{r_f}(t) = -x(t)(T-t)e^{(r-r_f)(T-t)}$$

Call Option (4.28):

$$c(t) = x(t)e^{-r_f(T-t)}N(d_1(t)) - Xe^{-r(T-t)}N(d_2(t))$$
$$\delta(t) = e^{-r_f(T-t)}N(d_1(t))$$
$$\gamma(t) = \frac{e^{-r_f(T-t)}N'(d_1(t))}{x(t)\sigma\sqrt{T-t}}$$
$$\vartheta(t) = -\frac{x(t)\sigma}{2\sqrt{T-t}}e^{-r_f(T-t)}N'(d_1(t)) + r_f x(t)e^{-r_f(T-t)}N(d_1(t)) - rXe^{-r(T-t)}N(d_2(t))$$
$$\upsilon(t) = \sqrt{T-t}\,x(t)e^{-r_f(T-t)}N'(d_1(t))$$
$$\rho_r(t) = (T-t)Xe^{-r(T-t)}N(d_2(t))$$
$$\rho_{r_f}(t) = -(T-t)x(t)e^{-r_f(T-t)}N(d_1(t))$$

Put Option (4.29):

$$p(t) = Xe^{-r(T-t)}N(-d_2(t)) - x(t)e^{-r_f(T-t)}N(-d_1(t))$$
$$\delta(t) = e^{-r_f(T-t)}\left[N(d_1(t)) - 1\right]$$
$$\gamma(t) = \frac{e^{-r_f(T-t)}N'(d_1(t))}{x(t)\sigma\sqrt{T-t}}$$
$$\vartheta(t) = -\frac{x(t)\sigma}{2\sqrt{T-t}}e^{-r_f(T-t)}N'(d_1(t)) - r_f x(t)e^{-r_f(T-t)}N(-d_1(t)) + rXe^{-r(T-t)}N(-d_2(t))$$
$$\upsilon(t) = \sqrt{T-t}\,x(t)e^{-r_f(T-t)}N'(d_1(t))$$
$$\rho_r(t) = -(T-t)Xe^{-r(T-t)}N(-d_2(t))$$
$$\rho_{r_f}(t) = (T-t)x(t)e^{-r_f(T-t)}N(-d_1(t))$$

In the case of American currency options the above Black & Scholes equation can be solved with numerical techniques, i.e., finite difference methods and lattice approaches. A discretization with the implicit finite difference operators leads to the (tridiagonal) system

$$a_j v_{ij-1} + b_j v_{ij} + c_j v_{ij+1} = v_{i+1j} \qquad i = 0,..,m-1 \qquad j = 1,..,n-1 \qquad (4.30a)$$

of linear equations with (state dependent) coefficients

$$a_j = \frac{j\Delta t}{2}(r - r_f - \sigma^2 j) \quad b_j = 1 + \Delta t(r + \sigma^2 j^2) \quad c_j = -\frac{j\Delta t}{2}(r - r_f + \sigma^2 j) \qquad (4.30b)$$

that can easily be solved backwards in time by using the boundary conditions and early exercise criteria which characterize the given contingent claim v. A discretization with the explicit instead of the implicit finite difference operators substantially simplifies these calculations. The corresponding linear equation system is in this case

$$v_{ij} = a_j v_{i+1,j-1} + b_j v_{i+1j} + c_j v_{i+1j+1} \qquad i = 0,..,m-1 \qquad j = 1,..,n-1 \qquad (4.31a)$$

and the (state dependent) coefficients are

$$a_j = \frac{j\Delta t}{2(1+r\Delta t)}(r_f - r + \sigma^2 j) \quad b_j = \frac{1}{1+r\Delta t}(1 - \sigma^2 j^2 \Delta t) \quad c_j = \frac{j\Delta t}{2(1+r\Delta t)}(r - r_f + \sigma^2 j) \qquad (4.31b)$$

[where the local consistency conditions

$$\sigma^2 \geq |r - r_f| \quad \Delta t \leq \frac{1}{\sigma^2 (n-1)^2} \qquad (4.32)$$

have to be satisfied]. The parameters

$$p,\, u \;\; \begin{matrix}\text{risk - averse}\\ \text{state evolution}\end{matrix} \qquad \tilde{p},\, \tilde{u} \;\; \begin{matrix}\text{risk - neutral}\\ \text{state evolution}\end{matrix} \qquad (4.33)$$

of a lattice structure approximating the foreign exchange rate dynamics

$$dx = \mu x dt + \sigma x dz = (r - r_f) x dt + \sigma x d\tilde{z}$$

$$E_{tx}[x(t+\Delta t)] = x e^{\mu\Delta t} \qquad V_{tx}[x(t+\Delta t)] = x^2 e^{2\mu\Delta t}\left[e^{\sigma^2\Delta t} - 1\right]$$

$$\text{(risk - averse state evolution)} \qquad (4.34)$$

$$\tilde{E}_{tx}[x(t+\Delta t)] = x e^{(r-r_f)\Delta t} \qquad \tilde{V}_{tx}[x(t+\Delta t)] = x^2 e^{2(r-r_f)\Delta t}\left[e^{\sigma^2\Delta t} - 1\right]$$

$$\text{(risk - neutral state evolution)}$$

in a risk-averse and in a risk-neutral financial economy are

$$u = \frac{c + \sqrt{c^2 - 4a^2}}{2a} \qquad p = \frac{a-d}{u-d}$$
$$\tilde{u} = \frac{\tilde{c} + \sqrt{\tilde{c}^2 - 4\tilde{a}^2}}{2\tilde{a}} \qquad \tilde{p} = \frac{\tilde{a}-\tilde{d}}{\tilde{u}-\tilde{d}} \qquad (4.35)$$

where

$$a = e^{\mu\Delta t} \qquad b^2 = e^{2\mu\Delta t}\left[e^{\sigma^2\Delta t} - 1\right] \qquad c = a^2 + b^2 + 1$$
$$\tilde{a} = e^{(r-r_f)\Delta t} \qquad \tilde{b}^2 = e^{2(r-r_f)\Delta t}\left[e^{\sigma^2\Delta t} - 1\right] \qquad \tilde{c} = \tilde{a}^2 + \tilde{b}^2 + 1 \qquad (4.36)$$

holds.

D. Futures Options. The Black & Scholes equation for futures options is

$$\frac{\partial v}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 v}{\partial x^2} - rv = 0 \qquad (4.37)$$

where we assume a relationship $F(t) = S(t)e^{\alpha(T-t)}$ with constant coefficient α between the futures price $F(t)$ $[= x(t)]$ and the spot price $S(t)$. With

$$d_1(t) = \frac{\log(\frac{x(t)}{X}) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \qquad d_2(t) = d_1(t) - \sigma\sqrt{T-t} \qquad (4.38)$$

this linear partial differential equation can in the case of European options be integrated by using a risk-neutral valuation argument.

Call Option (4.39):

$$c(t) = e^{-r(T-t)}\left[x(t)N(d_1(t)) - XN(d_2(t))\right]$$
$$\delta(t) = e^{-r(T-t)}N(d_1(t))$$
$$\gamma(t) = \frac{e^{-r(T-t)}N'(d_1(t))}{x(t)\sigma\sqrt{T-t}}$$
$$\vartheta(t) = -\frac{x(t)\sigma}{2\sqrt{T-t}}e^{-r(T-t)}N'(d_1(t)) + rx(t)e^{-r(T-t)}N(d_1(t)) - rXe^{-r(T-t)}N(d_2(t))$$

$$\upsilon(t) = \sqrt{T-t}\,x(t)e^{-r(T-t)}N'(d_1(t))$$
$$\rho(t) = (T-t)Xe^{-r(T-t)}N(d_2(t))$$

Put Option (4.40):

$$p(t) = e^{-r(T-t)}\left[XN(-d_2(t)) - x(t)N(-d_1(t))\right]$$
$$\delta(t) = e^{-r(T-t)}\left[N(d_1(t)) - 1\right]$$
$$\gamma(t) = \frac{e^{-r(T-t)}N'(d_1(t))}{x(t)\sigma\sqrt{T-t}}$$
$$\vartheta(t) = -\frac{x(t)\sigma}{2\sqrt{T-t}}e^{-r(T-t)}N'(d_1(t)) - rx(t)e^{-r(T-t)}N(-d_1(t)) + rXe^{-r(T-t)}N(-d_2(t))$$
$$\upsilon(t) = \sqrt{T-t}\,x(t)e^{-r(T-t)}N'(d_1(t))$$
$$\rho(t) = -(T-t)Xe^{-r(T-t)}N(-d_2(t))$$

In the case of American futures options the above Black & Scholes equation can be solved with numerical techniques, i.e., finite difference methods and lattice approaches. A discretization with the implicit finite difference operators leads to the (tridiagonal) system

$$a_j v_{ij-1} + b_j v_{ij} + c_j v_{ij+1} = v_{i+1j} \qquad i = 0,..,m-1 \qquad j = 1,..,n-1 \qquad (4.41a)$$

of linear equations with (state dependent) coefficients

$$a_j = -0.5j^2\sigma^2\Delta t \quad b_j = 1 + \Delta t(r + \sigma^2 j^2) \quad c_j = -0.5j^2\sigma^2\Delta t \qquad (4.41b)$$

that can easily be solved backwards in time by using the boundary conditions and early exercise criteria which characterize the given contingent claim v. A discretization with the explicit instead of the implicit finite difference operators substantially simplifies these calculations. The corresponding linear equation system is in this case

$$v_{ij} = a_j v_{i+1,j-1} + b_j v_{i+1j} + c_j v_{i+1j+1} \qquad i = 0,..,m-1 \qquad j = 1,..,n-1 \qquad (4.42a)$$

and the (state dependent) coefficients are

$$a_j = \frac{j^2\sigma^2\Delta t}{2(1+r\Delta t)} \quad b_j = \frac{1}{1+r\Delta t}(1-\sigma^2 j^2\Delta t) \quad c_j = \frac{j^2\sigma^2\Delta t}{2(1+r\Delta t)} \qquad (4.42b)$$

[where the local consistency condition

$$\Delta t \le 1/[\sigma^2(n-1)^2] \qquad (4.43)$$

has to be satisfied]. The parameters

$$p,\ u \ \ \begin{matrix}\text{risk-averse}\\ \text{state evolution}\end{matrix} \qquad \tilde{p},\ \tilde{u} \ \ \begin{matrix}\text{risk-neutral}\\ \text{state evolution}\end{matrix} \qquad (4.44)$$

of a lattice structure approximating the futures price dynamics

$$dx = \mu x dt + \sigma x dz = \sigma x d\tilde{z}$$

$$E_{tx}[x(t+\Delta t)] = xe^{\mu\Delta t} \qquad V_{tx}[x(t+\Delta t)] = x^2e^{2\mu\Delta t}\left[e^{\sigma^2\Delta t} - 1\right]$$
$$\text{(risk-averse state evolution)} \qquad (4.45)$$
$$\tilde{E}_{tx}[x(t+\Delta t)] = x \qquad \tilde{V}_{tx}[x(t+\Delta t)] = x^2\left[e^{\sigma^2\Delta t} - 1\right]$$
$$\text{(risk-neutral state evolution)}$$

in a risk-averse and in a risk-neutral financial economy are

$$u = \frac{c + \sqrt{c^2 - 4a^2}}{2a} \qquad p = \frac{a - d}{u - d}$$
$$\widetilde{u} = \frac{\widetilde{c} + \sqrt{\widetilde{c}^2 - 4}}{2} \qquad \widetilde{p} = \frac{1 - \widetilde{d}}{\widetilde{u} - \widetilde{d}} \tag{4.46}$$

where

$$a = e^{\mu\Delta t} \qquad b^2 = e^{2\mu\Delta t}\left[e^{\sigma^2\Delta t} - 1\right] \qquad c = a^2 + b^2 + 1$$
$$\widetilde{b}^2 = \left[e^{\sigma^2\Delta t} - 1\right] \qquad \widetilde{c} = \widetilde{b}^2 + 2 \tag{4.47}$$

holds.

The Rubinstein Model. The above simple but quite important Black & Scholes model can immediately be extended to have the following properties (see Rubinstein [18] and Davis and List [4, 5, 6, 7, 8] for details): (1) The market variable follows a ***general diffusion process***

$$\begin{aligned} dx &= x[\mu(t,x)dt + \sigma(t,x)dz] \\ &= x[(r(t,x) - y(t,x))dt + \sigma(t,x)d\widetilde{z}] \end{aligned} \tag{4.48}$$

which evolves in discrete-time on a recombining binomial lattice structure with parameters

$$u(t,x) \quad d(t,x) \quad p(t,x) \quad q(t,x) \tag{4.49a}$$

such that

$$\begin{gathered} u(t,x)d(t+\Delta t, xu(t,x)) = d(t,x)u(t+\Delta t, xd(t,x)) \\ p(t,x)q(t+\Delta t, xu(t,x)) = q(t,x)p(t+\Delta t, xd(t,x)) \\ p(t,x) + q(t,x) = 1 \\ \mu(t,x) \approx \frac{p(t,x)u(t,x) + q(t,x)d(t,x) - 1}{\Delta t} \\ \sigma(t\ x) \approx \sqrt{\frac{p(t,x)u(t,x)^2 + q(t,x)d(t,x)^2 - [p(t,x)u(t,x) + q(t,x)d(t,x)]^2}{\Delta t}} \end{gathered} \tag{4.49b}$$

(risk-averse state evolution) and parameters

$$\widetilde{u}(t,x) \quad \widetilde{d}(t,x) \quad \widetilde{p}(t,x) \quad \widetilde{q}(t,x) \tag{4.50a}$$

such that

$$\begin{gathered} \widetilde{u}(t,x)\widetilde{d}(t+\Delta t, x\widetilde{u}(t,x)) = \widetilde{d}(t,x)\widetilde{u}(t+\Delta t, x\widetilde{d}(t,x)) \\ \widetilde{p}(t,x)\widetilde{q}(t+\Delta t, x\widetilde{u}(t,x)) = \widetilde{q}(t,x)\widetilde{p}(t+\Delta t, x\widetilde{d}(t,x)) \\ \widetilde{p}(t,x) + \widetilde{q}(t,x) = 1 \\ r(t,x) - y(t,x) \approx \frac{\widetilde{p}(t,x)\widetilde{u}(t,x) + \widetilde{q}(t,x)\widetilde{d}(t,x) - 1}{\Delta t} \\ \sigma(t,x) \approx \sqrt{\frac{\widetilde{p}(t,x)\widetilde{u}(t,x)^2 + \widetilde{q}(t,x)\widetilde{d}(t,x)^2 - [\widetilde{p}(t,x)\widetilde{u}(t,x) + \widetilde{q}(t,x)\widetilde{d}(t,x)]^2}{\Delta t}} \end{gathered} \tag{4.50b}$$

(risk-neutral state evolution). (2) Denoting with $n(t,x)$ the number of paths ending in node (t,x) and with $\pi(t,x)$ and $\widetilde{\pi}(t,x)$ the associated risk-averse and risk-neutral time/state probabilities we have

$$\begin{gathered} \pi(t,x) = n(t,x)\left[\frac{\pi(t+\Delta t, xu(t,x))}{n(t+\Delta t, xu(t,x))} + \frac{\pi(t+\Delta t, xd(t,x))}{n(t+\Delta t, xd(t,x))}\right] \\ p(t,x) = \frac{n(t,x)}{n(t+\Delta t, xu(t,x))}\,\frac{\pi(t+\Delta t, xu(t,x))}{\pi(t,x)} \end{gathered} \tag{4.51a}$$

(risk-averse state evolution) and

$$\begin{aligned}\tilde{\pi}(t,x) &= n(t,x)\left[\frac{\tilde{\pi}(t+\Delta t, xu(t,x))}{n(t+\Delta t, xu(t,x))} + \frac{\tilde{\pi}(t+\Delta t, xd(t,x))}{n(t+\Delta t, xd(t,x))}\right] \\ \tilde{p}(t,x) &= \frac{n(t,x)}{n(t+\Delta t, xu(t,x))}\frac{\tilde{\pi}(t+\Delta t, xu(t,x))}{\tilde{\pi}(t,x)}\end{aligned} \quad (4.51b)$$

(risk-neutral state evolution). Note that

$$n(t,x) = n_{ij} = \binom{i}{j} = \frac{i-j+1}{j}\binom{i}{j-1} = \frac{i-j+1}{j} n_{ij-1} \quad [n_{i0} = 1] \quad (4.52)$$

holds and therefore the one step transition probabilities

$$\begin{aligned} &p(t,x) = p_{ij} \quad q(t,x) = q_{ij} \quad \text{(risk-averse state evolution)} \\ &\tilde{p}(t,x) = \tilde{p}_{ij} \quad \tilde{q}(t,x) = \tilde{q}_{ij} \quad \text{(risk-neutral state evolution)}\end{aligned} \quad (4.53)$$

and the time/state probabilities

$$\begin{aligned} &\pi(t,x) = \pi_{ij} \quad \text{(risk-averse state evolution)} \\ &\tilde{\pi}(t,x) = \tilde{\pi}_{ij} \quad \text{(risk-neutral state evolution)}\end{aligned} \quad (4.54)$$

can be determined from a corresponding terminal probability distribution

$$\begin{aligned} &\pi(T,x) = \pi_{mj} \quad \text{(risk-averse state evolution)} \\ &\tilde{\pi}(T,x) = \tilde{\pi}_{mj} \quad \text{(risk-neutral state evolution)}\end{aligned} \quad (4.55)$$

by solving equations (4.51a) and (4.51b) backwards in time from $i = m-1$ to $i = 0$. Simultaneously solving equations (4.49b) and (4.50b) at each node also leads to the remaining lattice parameters

$$\begin{aligned} &u(t,x) = u_{ij} \quad d(t,x) = d_{ij} \quad \text{(risk-averse state evolution)} \\ &\tilde{u}(t,x) = \tilde{u}_{ij} \quad \tilde{d}(t,x) = \tilde{d}_{ij} \quad \text{(risk-neutral state evolution).}\end{aligned} \quad (4.56)$$

Specifically, we have

$$\begin{aligned} u(t,x) &= 1 + \mu(t,x)\Delta t + \sigma(t,x)\sqrt{\frac{q(t,x)}{p(t,x)}\Delta t} \\ d(t,x) &= 1 + \mu(t,x)\Delta t - \sigma(t,x)\sqrt{\frac{p(t,x)}{q(t,x)}\Delta t}\end{aligned} \quad (4.57a)$$

(risk-averse state evolution) and

$$\begin{aligned} \tilde{u}(t,x) &= 1 + [r(t,x) - y(t,x)]\Delta t + \sigma(t,x)\sqrt{\frac{\tilde{q}(t,x)}{\tilde{p}(t,x)}\Delta t} \\ \tilde{d}(t,x) &= 1 + [r(t,x) - y(t,x)]\Delta t - \sigma(t,x)\sqrt{\frac{\tilde{p}(t,x)}{\tilde{q}(t,x)}\Delta t}\end{aligned} \quad (4.57b)$$

(risk-neutral state evolution). (3) If $U(R) = \log(R)$ is the representative utility of the return R of a risky investment in equilibrium, then the myopic optimization program

$$\begin{aligned} &\max_{\tilde{u}(t,x),\tilde{d}(t,x)} p(t,x)U(\tilde{u}(t,x)) + [1 - p(t,x)]U(\tilde{d}(t,x)) \\ &\tilde{p}(t,x)\tilde{u}(t,x) + [1 - \tilde{p}(t,x)]\tilde{d}(t,x) = 1 + [r(t,x) - y(t,x)]\Delta t\end{aligned} \quad (4.58)$$

allows us (KKT first order conditions, see Davis and List [4] and the literature mentioned there for details) to relate risk-neutral and risk-averse one step transition probabilities, i.e., we have

$$p(t,x) = \tilde{p}(t,x)\frac{\tilde{u}(t,x)}{1 + [r(t,x) - y(t,x)]\Delta t} \quad (4.59)$$

(4) The risk-neutral pricing formula[15] is in this context

$$v_{ij} = \max\left[L(i,j), \min[e^{-r_{ij}\Delta t}\begin{bmatrix} \tilde{p}_{ij}[v_{i+1j+1} + X(i+1,j+1)] \\ +\tilde{q}_{ij}[v_{i+1j} + X(i+1,j)] \end{bmatrix}, U(i,j)] \right] \quad \begin{matrix} 0 \le i \le m-1 \\ 0 \le j \le i \end{matrix} \qquad (4.60)$$

$$v_{mj} = F(j) \quad 0 \le j \le m$$

[note that for standard American options

$$L(i,j) = \begin{cases} \max[x_{ij} - X, 0] & \text{(call)} \\ \max[X - x_{ij}, 0] & \text{(put)} \end{cases} \qquad (4.61)$$

is the corresponding intrinsic value] and the contingent claim sensitivities (derivatives risk parameters) are

$$\begin{aligned} \vartheta(t,x) &\approx p(t,x)\frac{v(t+\Delta t, x+\Delta x) - v(t,x)}{\Delta t} + q(t,x)\frac{v(t+\Delta t, x-\Delta x) - v(t,x)}{\Delta t} \\ &= E_{tx}[\frac{\Delta v}{\Delta t}] \end{aligned} \qquad (4.62a)$$

$$\begin{aligned} \delta(t,x) &\approx p(t,x)\frac{v(t+\Delta t, x+\Delta x) - v(t,x)}{(x+\Delta x) - x} + q(t,x)\frac{v(t+\Delta t, x-\Delta x) - v(t,x)}{(x-\Delta x) - x} \\ &= E_{tx}[\frac{\Delta v}{\Delta x}] \end{aligned} \qquad (4.62b)$$

$$\begin{aligned} \gamma(t,x) &\approx p(t,x)\frac{\delta(t+\Delta t, x+\Delta x) - \delta(t,x)}{(x+\Delta x) - x} + q(t,x)\frac{\delta(t+\Delta t, x-\Delta x) - \delta(t,x)}{(x-\Delta x) - x} \\ &= E_{tx}[\frac{\Delta \delta}{\Delta x}] \end{aligned} \qquad (4.62c)$$

(conditionally expected rates of change) and

$$\begin{aligned} \delta(t,x) &\approx \frac{v(t+\Delta t, x+\Delta x) - v(t+\Delta t, x-\Delta x)}{(x+\Delta x) - (x-\Delta x)} \\ \gamma(t,x) &\approx \frac{\delta(t+\Delta t, x+\Delta x) - \delta(t+\Delta t, x-\Delta x)}{(x+\Delta x) - (x-\Delta x)} \end{aligned} \qquad (4.63)$$

(finite difference method) where

$$x + \Delta x = xu(t,x) \qquad x - \Delta x = xd(t,x) \qquad (4.64)$$

holds. Theta can be defined via the relationship

$$\vartheta(t,x) = \lambda(t,x) - x\mu(t,x)\delta(t,x) - \frac{x^2\sigma(t,x)^2}{2}\gamma(t,x) \qquad (4.65)$$

and the approximation

$$\begin{aligned} \lambda(t,x) &\approx p(t,x)\frac{v(t+\Delta t, x+\Delta x) - v(t,x)}{\Delta t} + q(t,x)\frac{v(t+\Delta t, x-\Delta x) - v(t,x)}{\Delta t} \\ &= E_{tx}[\frac{\Delta v}{\Delta t}] \end{aligned} \qquad (4.66)$$

[$xu(t,x)d(t+\Delta t, xu(t,x)) \neq x$ in general]. (5) A risk-neutral terminal probability distribution $\tilde{\pi}(T,x) = \tilde{\pi}_{mj}$ can be determined by solving the quadratic program

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15 X (intertemporal cashflows) and F (terminal condition) characterize the contingent claim. $L \le v \le U$ are boundary conditions for its price process (see Davis and List [5, 6, 7, 8] for details). In the simpler Black & Scholes model considered above, the discount rate r and the risk-neutral transition probabilities $\tilde{p}$ and $\tilde{q}$, $\tilde{p} + \tilde{q} = 1$, are constant.

$$\min_{\tilde{\pi}_{mj}} \sum_{j=0}^{m} [\tilde{\pi}_{mj} - \tilde{\pi}_{mj}^0]^2 \qquad \left[\sum_{j=0}^{m} \tilde{\pi}_{mj} = 1 \quad \tilde{\pi}_{mj} \geq 0\right]$$
$$x^b \leq e^{-(r_0 - y_0)(T-t)} \sum_{j=0}^{m} \tilde{\pi}_{mj} x_{mj}^0 \leq x^a \quad c_k^b \leq e^{-r_0(T-t)} \sum_{j=0}^{m} \tilde{\pi}_{mj} \max[x_{mj}^0 - X_k, 0] \leq c_k^a \tag{4.67}$$

where

$$dx_0 = (r_0 - y_0)x_0 dt + \sigma_0 x_0 d\tilde{z} \qquad \left[\tilde{u}_0 \quad \tilde{d}_0 \quad \tilde{p}_0 \quad \tilde{q}_0 \quad \tilde{\pi}_{ij}^0\right] \tag{4.68}$$

is a simple (standard) approximation of the market variable dynamics

$$dx = x[(r(t,x) - y(t,x))dt + \sigma(t,x)d\tilde{z}] \tag{4.69}$$

and c_k^b and c_k^a are the bid and ask prices at time t of European call options with maturity T and exercise values X_k. The market variable itself is also assumed to be the price of a traded asset (with x^b and x^a the bid and ask prices, respectively, at time t).

"Catastrophic" Claims Portfolio Securitization. In the final part of this section, we should like to briefly mention another potentially interesting area of application for the limited risk arbitrage (LRA) techniques underlying the Fin Re Toolbox (see Davis and List [4] and the literature mentioned there for details): the securitization of "catastrophic" non-life (re)insurance exposures in the capital markets.

One important reason why LRA techniques are very well suited for this kind of application lies in the fact that they achieve an overall allocation of the asset/liability risks involved that meets set targets at a reasonable price whereas the otherwise commonly applied hedging techniques (for the finance part of a securitization program) often unnecessarily avoid financial risks at an unacceptably high price while the (potentially dominating) risk exposure on the liability side remains high.

Based upon a ***risk management target for "Beta" portfolio excess-of-loss probabilities*** (see Fig. 4 above), a corresponding ***securitization structure*** might then look as follows (in simple terms that could be made more precise with some financial engineering, see Davis and Bühlmann, Bochicchio, Junod and List [9, 10, 11] for details):

1. AAA Swiss Re bond with coupon
   a. r fixed = best financial markets conditions
   b. + x variable = linked to performance of underlying risk portfolio

   and a maturity schedule that is adapted to the coverage structure of the underlying risk portfolio, i.e., in the case of "Beta" a maturity of at least 3 years.
2. For tax reasons, (the fixed part r of) the coupon would (at the investor's discretion) not actually be payed out, i.e., the bond would be of the deep-discount type.
3. In the case of a catastrophic loss in the underlying risk portfolio, the notional principal of the bond would be transformed into a long-term loan (i.e., the investor would not loose any money). With some financial engineering, interest rates at best financial market conditions could be guaranteed to both sides at the outset.
4. In the case where an existing Swiss Re share-holder participates in such a structure, the notional principal of the bond could alternatively be transformed into Swiss Re shares at a fixed price (i.e., Swiss Re would not actually have to pay it back).
5. The limited risk arbitrage (LRA) techniques outlined in the publication series (Davis

and List [4, 5, 6, 7, 8] and Davis and Bühlmann, Bochicchio, Junod and List [9, 10, 11]) could be used to effectively exploit any opportunities for arbitrage profits offered by the global financial markets (disparity in interest rate regimes, exchange rates, etc.).

Note that such a *securitization* program can also be designed according to specific risk management requirements (w.r.t. exposures, capital, cashflows, etc.) of a particular client. Usually, however, securitization (the "back-end" of an ***alternative risk transfer process***) is completely transparent to "Beta" clients, i.e., Swiss Re takes complete care of the allocation of (re)insurance risks in the capital markets.

Returning to LRA (advanced Fin Re pricing and portfolio management) techniques now, we note that the above mentioned financial engineering critically depends upon an efficient model for interest-rates, stocks and foreign currency which is also reflective of the "Beta" excess-of-loss probabilities. The main idea is ***to start with an interest rate model (as interest rates are the most significant factor in the above securitization scheme) and to combine this in a consistent way with a model for stocks / stock indices / currencies (on the same lattice)***:

A. Processes.

(I) $dx(t) = x(t)[\mu dt + \sigma dz(t)]$ for stocks, stock indices and currencies
Reference: J.C. Hull, *Options, Futures and Other Derivative Securities*, Prentice-Hall 1993
Generalization: M. Rubinstein, *Implied Binomial Trees*, Journal of Finance 49, 771 - 818 (1994)

(II) $dr(t) = [\theta(t) - \phi(t)r(t)]dt + \sigma r(t)^{\beta} dz(t)$ for interest rates (volatilities)
Reference: J.C. Hull and A. White, *One-Factor Interest-Rate Models and the Valuation of Interest-Rate Derivative Securities*, Journal of Financial and Quantitative Analysis 28, 235 - 254 (1993)

B. Rubinstein Implied Tree (consistent with Hull & White interest rates).

Stock / Stock Index / Currency Dynamics (as in the Rubinstein model above):

$$\begin{aligned} dx(t) &= x(t)[\mu(t,x(t))dt + \sigma(t,x(t))dz(t)] \\ &= x(t)[(r(t,x(t)) - y(t,x(t)))dt + \sigma(t,x(t))d\tilde{z}(t)] \end{aligned} \qquad (4.70)$$

Interest Rates and Dividend Yields (Ito formula[16]):

16 Let $y = f(t,x)$, $dx(t) = a(t,x(t))dt + b(t,x(t))dz(t)$, $a(t,x) \in R^m$, $b(t,x) \in R^{m \times n}$. Then (*Ito formula*): $dy = [f_t + a^T \nabla_x f + 0.5 tr(bb^T \nabla_x^2 f)]dt + (\nabla_x f)^T b dz$.

$$dr(t) = \left[\frac{\partial r}{\partial t}(t,x(t)) + x(t)\mu(t,x(t))\frac{\partial r}{\partial x}(t,x(t)) + \frac{x(t)^2\sigma(t,x(t))^2}{2}\frac{\partial^2 r}{\partial x^2}(t,x(t))\right]dt + x(t)\sigma(t,x(t))\frac{\partial r}{\partial x}(t,x(t))dz(t) \quad (4.71a)$$

$$dy(t) = \left[\frac{\partial y}{\partial t}(t,x(t)) + x(t)\mu(t,x(t))\frac{\partial y}{\partial x}(t,x(t)) + \frac{x(t)^2\sigma(t,x(t))^2}{2}\frac{\partial^2 y}{\partial x^2}(t,x(t))\right]dt + x(t)\sigma(t,x(t))\frac{\partial y}{\partial x}(t,x(t))dz(t) \quad (4.71b)$$

Hull & White Interest Rates (comparison of respective drift and diffusion terms):

$$\begin{aligned} &\frac{\partial r}{\partial t}(t,x) + x\mu(t,x)\frac{\partial r}{\partial x}(t,x) + \frac{x^2\sigma(t,x)^2}{2}\frac{\partial^2 r}{\partial x^2}(t,x) = \theta(t) - \phi(t)r(t,x) \\ &x\sigma(t,x)\frac{\partial r}{\partial x}(t,x) = \sigma r(t,x)^\beta \end{aligned} \quad (4.72)$$

Simplification $r(t,x) = a + bt + cx + dx^2$:

$$\begin{aligned} &b + x\mu(t,x)(c+2dx) + x^2\sigma(t,x)^2 d = \theta(t) - \phi(t)r(t,x) \\ &x\sigma(t,x)(c+2dx) = \sigma r(t,x)^\beta \end{aligned} \quad (4.73a)$$

$$\begin{aligned} &\mu(t,x) = \frac{\theta(t) - \phi(t)r(t,x) - b}{x(c+2dx)} - \frac{d\sigma^2 r(t,x)^{2\beta}}{x(c+2dx)^3} \\ &\sigma(t,x) = \frac{\sigma r(t,x)^\beta}{x(c+2dx)} \end{aligned} \quad (4.73b)$$

This defines the stock / stock index / currency evolution (consistent with interest rates). The simplification $y(t,x) = e + ft + gx$ then leads to the parameters (initial conditions)

$$a = r_0 - cx_0 - dx_0^2 \quad (4.74a)$$

$$b = \theta_0 - \phi_0 r_0 - \mu_0 x_0(c+2dx_0) - \frac{d\sigma^2 r_0^{2\beta}}{(c+2dx_0)^2} \quad (4.74b)$$

$$c = \frac{\sigma r_0^\beta}{\sigma_0 x_0} - 2dx_0 \quad (4.74c)$$

$$e = y_0 - gx_0 \quad (4.74d)$$

with the remaining model specifications

for $r(t,x)$: $$\frac{1}{2}\frac{\partial^2 r}{\partial x^2}(t,x) = d; \quad (4.75a)$$

for $y(t,x)$: $$\frac{\partial y}{\partial t}(t,x) = f \qquad \frac{\partial y}{\partial x}(t,x) = g. \quad (4.75b)$$

Using this model, the LRA (advanced Fin Re pricing and portfolio management) techniques presented in this paper can be implemented on a notebook computer with reasonable response times for both lattice construction and contingent claim (portfolio) evaluation. The corresponding sophisticated financial/(re)insurance toolbox runs under Windows 3.1, 95, NT 3.51 and NT 4.0 (see Davis and List [6] and Fig. 5 above).

5. A Financial Reinsurance (Fin Re) Toolbox

The Financial Components. In a first step, we focus our attention on ***Fin Re pricing*** (rather than Fin Re hedge portfolio management) and choose simple stochastic models for financial market state evolution:

(a) the ***Ho & Lee*** and ***extended Cox, Ingersoll and Ross (CIR)*** models for short-term interest rates;
(b) the ***Black & Scholes*** model for stocks, stock indices and foreign currencies.

Later on, more sophisticated models like the above described combination of the extended CIR model for interest rates and the Rubinstein model for stocks / stock indices / foreign currencies can be used together with the LRA module for advanced ***Fin Re hedge fund management*** (see Davis and List [4, 5, 6, 7, 8] and Davis and Bühlmann, Bochicchio, Junod and List [9, 10, 11]).

Modelling Loss Event Contingent Claims. In order to keep the implementation of our Fin Re pricing models simple, we make the following ***working assumptions***:

(a) ***loss events arising from the reinsurance part of a Fin Re contract do not affect the financial market state evolution***, i.e., any such effects are limited to the Fin Re contract itself [the future cashflows of which can therefore, conditional on a loss event, be modified in accordance with this event and then valued using the above (unchanged) financial models and risk-neutral valuation techniques];
(b) ***Fin Re contracts are not traded securities*** (yet) and are therefore rated with actuarial techniques, i.e., a loading proportional to the variation of the discounted (loss event contingent) cashflows of the Fin Re contract across all considered loss event scenarios is added to the corresponding expected value.

Our ***Fin Re rating approach*** is then:

(1) Determination of the ***financial market parameters*** relevant in a Fin Re pricing context (i.e., current term structure of interest rates, volatilities of short-term interest rates, exchange rates, stocks and stock indices).

(2) ***Calibration*** of the financial models, i.e., valuation of a characteristic set of benchmark securities and readjustment of the model parameters if necessary (i.e., if the calculated prices are too far off the observed market values).

(3) Determination of the ***occurrence times and severity of excess-of-loss events*** under the reinsurance part of the Fin Re contract (using the above-mentioned EVT Toolbox or some equivalent actuarial approach, see the concrete example below and also Fig. 4 above):

LOSS EVENT SCENARIOS									
Scenario	Probability	Time of Loss				Severity of Loss			
		1	2	3	....	1	2	3	....
1									
2									
3									
4									
5									
6									
7									
8									
9									
10									

(4) Determination of the ***corresponding (loss event contingent) cashflow modifications*** to the financial part of the Fin Re contract (see the concrete example below):

CASHFLOW IMPLICATIONS									
Scenario	Probability	Time of Cashflow				Loss Event Contingent Cashflow Modification			
		1	2	3	....	1	2	3	....
1									
2									
3									
4									
5									
6									
7									
8									
9									
10									

Recall from the above risk-neutral pricing formulas (see also Davis and List [5, 6, 7, 8]) that the financial part of any Fin Re contract can be uniquely characterized by

(a) a ***cashflow function*** $X(i, j)$;

(b) a ***terminal condition*** $F(j)$;

(c) two ***boundary conditions*** $L(i, j) \leq U(i, j)$.

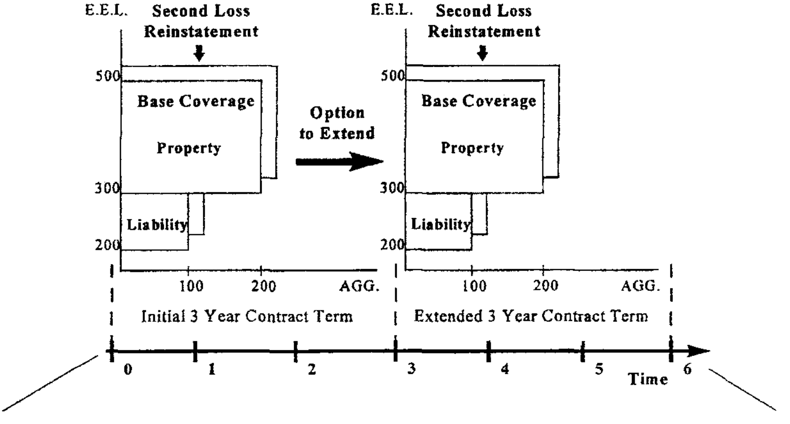
Contingent on a loss event scenario (row in the above tables), these characteristic functions have in a Fin Re pricing context now to be modified (loss event contingent cashflow modifications) before they are applied in the risk-neutral pricing formula. This is a consequence of our first working assumption stated at the beginning of this section.

(5) ***Risk-neutral valuation of the modified (loss event contingent) financial part*** of the Fin Re contract (one valuation per loss event scenario considered). We obtain, conditional on the loss event scenarios considered, a price forecast (stochastic process) and also forecasts for the contingent claim sensitivities (i.e., delta, gamma, theta, etc.).

(6) ***Actuarial rating*** of the resulting (loss event contingent) Fin Re price distribution. The loading is determined in accordance with Swiss Re's Value Proposition (i.e., RAC-based) pricing principle (see List and Zilch [1] and Geosits, List and Lohner [2]). Furthermore, we just take the expectations of the contingent claim sensitivity (derivatives risk parameter) forecasts across all considered loss event scenarios.

6. A Note on Implementation (Example Fin Re Contracts)

In this last section of the paper, we are going to look into the implementation of the above outlined approach to Fin Re pricing in some detail. Specifically, as a first example, we consider a ***6 year "Beta" bond written on the Oil & Petrochemicals industry ("Beta" target) portfolio with both coupon and principal (in USD) at risk.***



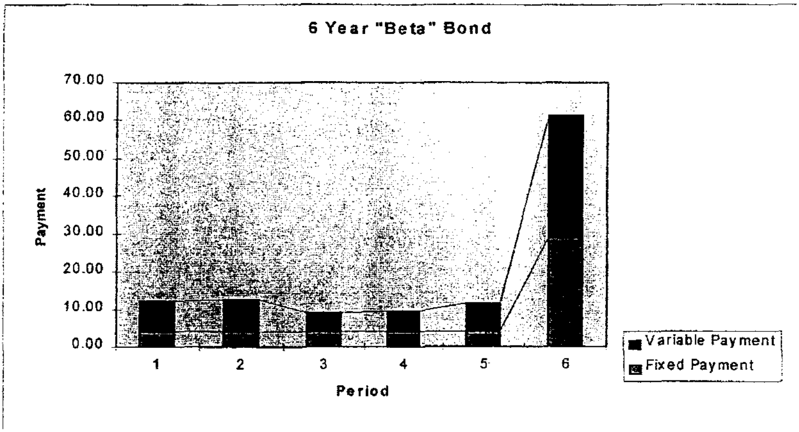


Fig. 10: 6 Year "Beta" USD Bond[17]

6 Year "Beta" Bond	
Notional Principal	100.00
Guaranteed Principal	25.00
Fixed Coupon	4.00%
Variable Coupon	6.00%

Period	Fixed Payment	Variable Payment	Total Payment
1	4.00	8.07	12.07
2	4.00	8.62	12.62
3	4.00	5.09	9.09
4	4.00	5.51	9.51
5	4.00	7.72	11.72
6	29.00	32.21	61.21

Price	100.00
Period	**Payment**
1	12.07
2	12.62
3	9.09
4	9.51
5	11.72
6	61.21
Internal Rate of Return	3.40%

As this (Fin Re / securitization) structure is quite involved, we are going to analyze its key components in several separate steps:

(1) The Oil & Petrochemicals industry "Beta" target portfolio (i.e., ***50 standard coverages USD 200M xs 300M property and USD 100M xs 200M casualty***, see List and Zilch [1] and

[17] The above shown variable cashflows of the bond are of course just one realization of its in general ***stochastic (i.e., "Beta" portfolio loss event contingent) cashflows***. So is the ***internal rate of return (IRR)*** shown below.

Geosits, List and Lohner [2]) has the following *one*, *three* and *six year aggregate loss distributions*:

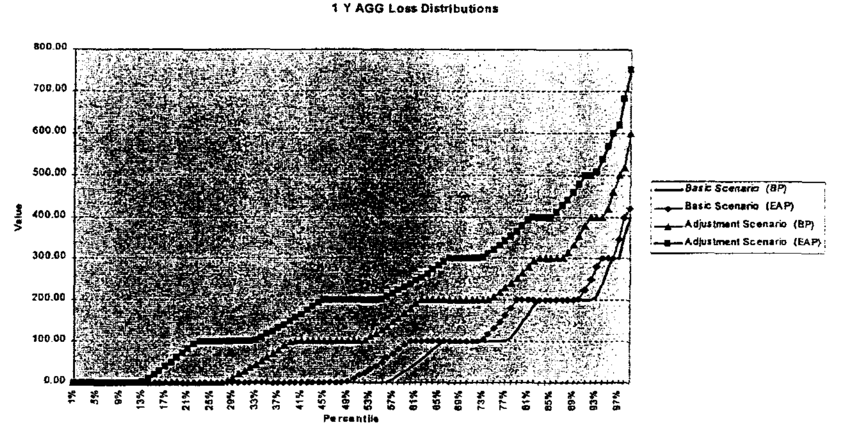


Fig. 11a: 1 Year Aggregate Loss Distributions

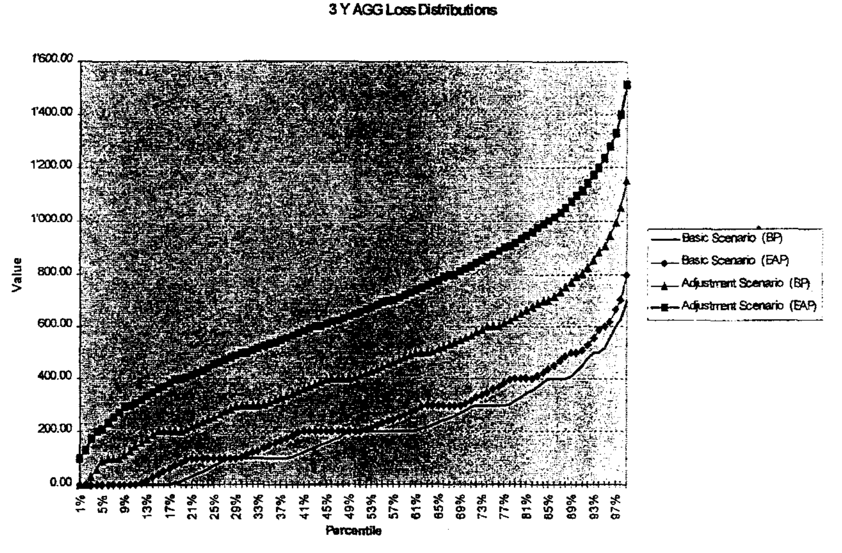


Fig. 11b: 3 Year Aggregate Loss Distributions

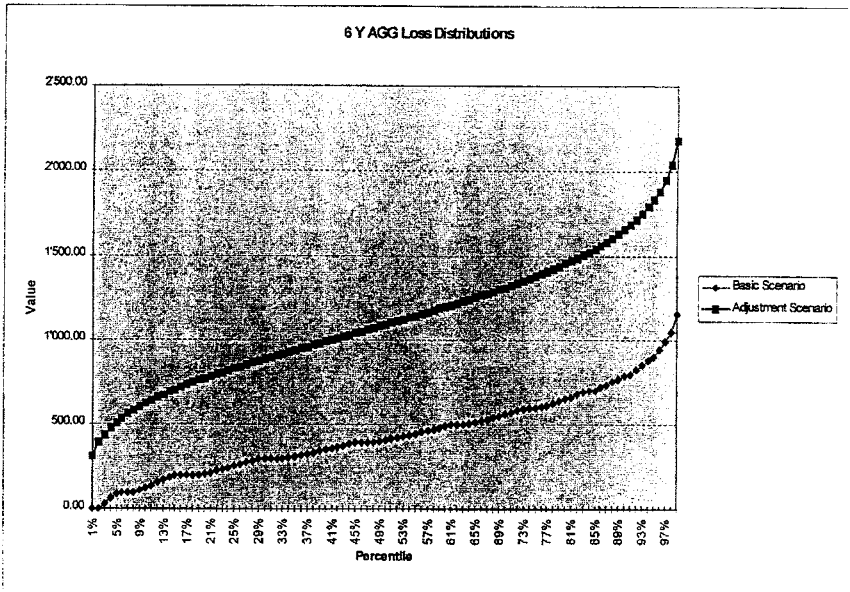


Fig. 11c: 6 Year Aggregate Loss Distributions

(2) The corresponding ***loss event scenarios*** are consequently (in two steps) [18]:

[18] ***The number of loss event scenarios (= size of loss categories) is of course determined by the rules governing their effects on the bond's (variable) cashflows.*** In this application, we use USD 100M steps in the 1 year loss event scenarios, USD 200M steps in the 3 year loss event scenarios and USD 300M steps in the 6 year loss event scenarios. Our Fin Re pricing toolbox can handle any number of loss event scenarios although, in practice, only a few are usually needed.

1 Y Loss Event Scenarios (I)					
Base Period (1997-1999)			Extended Agreement Period (2000-2002)		
Basic Scenario (95% Prob.)			*Basic Scenario (95% Prob.)*		
Size of Loss	Probability of Loss (Cond. on Scen.)	Probability of Loss (Unconditional)	Size of Loss	Probability of Loss (Cond. on Scen.)	Probability of Loss (Unconditional)
0.00	55.00%	52.25%	0.00	48.00%	45.60%
100.00	22.00%	20.90%	100.00	24.00%	22.80%
200.00	15.00%	14.25%	200.00	17.00%	16.15%
300.00	5.00%	4.75%	300.00	7.00%	6.65%
400.00	2.00%	1.90%	400.00	2.00%	1.90%
500.00	1.00%	0.95%	500.00	1.00%	0.95%
600.00	0.00%	0.00%	600.00	1.00%	0.95%
700.00	0.00%	0.00%	700.00	0.00%	0.00%
800.00	0.00%	0.00%	800.00	0.00%	0.00%
900.00	0.00%	0.00%	900.00	0.00%	0.00%
1000.00	0.00%	0.00%	1000.00	0.00%	0.00%
Adjustment Scenario (5% Prob.)			*Adjustment Scenario (5% Prob.)*		
Size of Loss	Probability of Loss (Cond. on Scen.)	Probability of Loss (Unconditional)	Size of Loss	Probability of Loss (Cond. on Scen.)	Probability of Loss (Unconditional)
0.00	27.00%	1.35%	0.00	12.00%	0.60%
100.00	24.00%	1.20%	100.00	20.00%	1.00%
200.00	23.00%	1.15%	200.00	22.00%	1.10%
300.00	13.00%	0.65%	300.00	18.00%	0.90%
400.00	7.00%	0.35%	400.00	13.00%	0.65%
500.00	3.00%	0.15%	500.00	7.00%	0.35%
600.00	2.00%	0.10%	600.00	4.00%	0.20%
700.00	1.00%	0.05%	700.00	2.00%	0.10%
800.00	0.00%	0.00%	800.00	1.00%	0.05%
900.00	0.00%	0.00%	900.00	1.00%	0.05%
1000.00	0.00%	0.00%	1000.00	0.00%	0.00%

1 Y Loss Event Scenarios (II)			
Base Period (1997-1999)		Extended Agreement Period (2000-2002)	
Size of Loss	Probability of Loss (Total)	Size of Loss	Probability of Loss (Total)
0.00	53.60%	0.00	46.20%
100.00	22.10%	100.00	23.80%
200.00	15.40%	200.00	17.25%
300.00	5.40%	300.00	7.55%
400.00	2.25%	400.00	2.55%
500.00	1.10%	500.00	1.30%
600.00	0.10%	600.00	1.15%
700.00	0.05%	700.00	0.10%
800.00	0.00%	800.00	0.05%
900.00	0.00%	900.00	0.05%
1000.00	0.00%	1000.00	0.00%

Fig. 12a: 1 Year Loss Event Scenarios

3 Y Loss Event Scenarios (I)					
Base Period (1997-1999)			Extended Agreement Period (2000-2002)		
Basic Scenario (95% Prob.)			*Basic Scenario (95% Prob.)*		
Size of Loss	Probability of Loss (Cond. on Scen.)	Probability of Loss (Unconditional)	Size of Loss	Probability of Loss (Cond. on Scen.)	Probability of Loss (Unconditional)
0.00	17.00%	16.15%	0.00	11.00%	10.45%
200.00	44.00%	41.80%	200.00	39.00%	37.05%
400.00	27.00%	25.65%	400.00	32.00%	30.40%
600.00	9.00%	8.55%	600.00	13.00%	12.35%
800.00	3.00%	2.85%	800.00	4.00%	3.80%
1000.00	0.00%	0.00%	1000.00	1.00%	0.95%
1200.00	0.00%	0.00%	1200.00	0.00%	0.00%
1400.00	0.00%	0.00%	1400.00	0.00%	0.00%
1600.00	0.00%	0.00%	1600.00	0.00%	0.00%
1800.00	0.00%	0.00%	1800.00	0.00%	0.00%
2000.00	0.00%	0.00%	2000.00	0.00%	0.00%
Adjustment Scenario (5% Prob.)			*Adjustment Scenario (5% Prob.)*		
Size of Loss	Probability of Loss (Cond. on Scen.)	Probability of Loss (Unconditional)	Size of Loss	Probability of Loss (Cond. on Scen.)	Probability of Loss (Unconditional)
0.00	2.00%	0.10%	0.00	0.00%	0.00%
200.00	17.00%	0.85%	200.00	4.00%	0.20%
400.00	30.00%	1.50%	400.00	15.00%	0.75%
600.00	26.00%	1.30%	600.00	24.00%	1.20%
800.00	15.00%	0.75%	800.00	25.00%	1.25%
1000.00	7.00%	0.35%	1000.00	16.00%	0.80%
1200.00	3.00%	0.15%	1200.00	10.00%	0.50%
1400.00	0.00%	0.00%	1400.00	3.00%	0.15%
1600.00	0.00%	0.00%	1600.00	3.00%	0.15%
1800.00	0.00%	0.00%	1800.00	0.00%	0.00%
2000.00	0.00%	0.00%	2000.00	0.00%	0.00%

3 Y Loss Event Scenarios (II)			
Base Period (1997-1999)		Extended Agreement Period (2000-2002)	
Size of Loss	Probability of Loss (Total)	Size of Loss	Probability of Loss (Total)
0.00	16.25%	0.00	10.45%
200.00	42.65%	200.00	37.25%
400.00	27.15%	400.00	31.15%
600.00	9.85%	600.00	13.55%
800.00	3.60%	800.00	5.05%
1000.00	0.35%	1000.00	1.75%
1200.00	0.15%	1200.00	0.50%
1400.00	0.00%	1400.00	0.15%
1600.00	0.00%	1600.00	0.15%
1800.00	0.00%	1800.00	0.00%
2000.00	0.00%	2000.00	0.00%

Fig. 12b: 3 Year Loss Event Scenarios[19]

[19] ***Considering one, three and six year (distributions and) loss event scenarios may be useful because investors might like different maturity investment opportunities in the Oil & Petrochemicals industry "Beta" portfolio.*** We analyze a ***6 year "Beta" bond*** in detail here; a ***1 year (forward) "Beta" bill*** and a ***3 year (forward) "Beta" note*** might however be sensible complementary instruments to consider. ***Because of the independence of loss events resulting from the "Beta" portfolio and financial markets events, "Beta" loss event contingent claims such as the "Beta" bills, notes and bonds mentioned here enhance the risk/return characteristics of institutional investors' asset allocations (e.g., Markowitz portfolio selection: efficient***

6 Y Loss Event Scenarios (I)		
Period (1997-2002)		
Basic Scenario (95% Prob.)		
Size of Loss	Probability of Loss (Cond. on Scen.)	Probability of Loss (Unconditional)
0.00	2.00%	1.90%
300.00	31.00%	29.45%
600.00	42.00%	39.90%
900.00	19.00%	18.05%
1200.00	6.00%	5.70%
1500.00	0.00%	0.00%
1800.00	0.00%	0.00%
2100.00	0.00%	0.00%
2400.00	0.00%	0.00%
2700.00	0.00%	0.00%
3000.00	0.00%	0.00%
Adjustment Scenario (5% Prob.)		
Size of Loss	Probability of Loss (Cond. on Scen.)	Probability of Loss (Unconditional)
0.00	0.00%	0.00%
300.00	0.00%	0.00%
600.00	8.00%	0.40%
900.00	23.00%	1.15%
1200.00	29.00%	1.45%
1500.00	22.00%	1.10%
1800.00	12.00%	0.60%
2100.00	4.00%	0.20%
2400.00	2.00%	0.10%
2700.00	0.00%	0.00%
3000.00	0.00%	0.00%

6 Y Loss Event Scenarios (II)	
Period (1997-2002)	
Size of Loss	Probability of Loss (Total)
0.00	1.90%
300.00	29.45%
600.00	40.30%
900.00	19.20%
1200.00	7.15%
1500.00	1.10%
1800.00	0.60%
2100.00	0.20%
2400.00	0.10%
2700.00	0.00%
3000.00	0.00%

Fig. 12c: 6 Year Loss Event Scenarios[20]

allocation of risk with higher expected returns) and should therefore, from a microeconomic point of view, be useful instruments to add to the financial markets.

[20] In the sequel, we shall focus only on the above 6 year loss event scenarios in our detailed analysis of the Oil & Petrochemicals industry "Beta" bond. The same principles apply however also to the 1 year "Beta" bill and the 3 year "Beta" note. ***Futures and options on "Beta" bills, notes and bonds are "Beta" loss event contingent claims just as the corresponding underlyings and can be analyzed and priced in exactly the same way.***

(3) The *cashflow implications* are now defined as follows:

1 Y Cashflow Implications					
Base Period (1997-1999)			Extended Agreement Period (2000-2002)		
Size of Loss	Coupon	Principal	Size of Loss	Coupon	Principal
0.00	6.00%	100.00	0.00	6.00%	100.00
100.00	5.00%	95.00	100.00	5.00%	95.00
200.00	4.00%	90.00	200.00	4.00%	90.00
300.00	3.00%	80.00	300.00	3.00%	80.00
400.00	2.00%	50.00	400.00	2.00%	50.00
500.00	1.00%	25.00	500.00	1.00%	25.00
600.00	0.00%	25.00	600.00	0.00%	25.00
700.00	0.00%	25.00	700.00	0.00%	25.00
800.00	0.00%	25.00	800.00	0.00%	25.00
900.00	0.00%	25.00	900.00	0.00%	25.00
1000.00	0.00%	25.00	1000.00	0.00%	25.00

3 Y Cashflow Implications					
Base Period (1997-1999)			Extended Agreement Period (2000-2002)		
Size of Loss	Coupon	Principal	Size of Loss	Coupon	Principal
0.00	6.00%	100.00	0.00	6.00%	100.00
200.00	5.00%	95.00	200.00	5.00%	95.00
400.00	4.00%	90.00	400.00	4.00%	90.00
600.00	3.00%	80.00	600.00	3.00%	80.00
800.00	2.00%	50.00	800.00	2.00%	50.00
1000.00	1.00%	25.00	1000.00	1.00%	25.00
1200.00	0.00%	25.00	1200.00	0.00%	25.00
1400.00	0.00%	25.00	1400.00	0.00%	25.00
1600.00	0.00%	25.00	1600.00	0.00%	25.00
1800.00	0.00%	25.00	1800.00	0.00%	25.00
2000.00	0.00%	25.00	2000.00	0.00%	25.00

6 Y Cashflow Implications		
Period (1997-2002)		
Size of Loss	Coupon	Principal
0.00	6.00%	100.00
300.00	5.00%	95.00
600.00	4.00%	90.00
900.00	3.00%	80.00
1200.00	2.00%	50.00
1500.00	1.00%	25.00
1800.00	0.00%	25.00
2100.00	0.00%	25.00
2400.00	0.00%	25.00
2700.00	0.00%	25.00
3000.00	0.00%	25.00

In order to keep matters simple for this presentation, ***we shall interpret the above tables as follows***:

(a) the fixed coupon (i.e., 4%) and the guaranteed principal (i.e., 25.00) are paid on the respective due dates just as in the case of a straight bill, note or bond;

(b) the adjusted variable coupon (i.e., between 0% and 6%) and the non-guaranteed principal (i.e., between 0.00 and 75.00) are paid at maturity, depending on the outcome of the associated aggregate loss (i.e., 1, 3 or 6 year) in the underlying Oil & Petrochemicals industry "Beta" portfolio. Interest rate adjustments for the coupons are made in order to reflect the time value of money.

(4) Loss event contingent ***risk-neutral valuation*** with the ***extended Cox, Ingersoll and Ross (CIR) model***

$$dr(t) = [\theta(t) - \phi(t)r(t)]dt + \sigma r(t)^{\beta} dz(t)$$

[see (3.42b) above] finally yields:

A. USD Term Structure and Term Structure Volatilities / Model Calibration.

To start with, we note the following ***USD yields*** (as of 11 August 1997):

LIBOR		GOVT BONDS		SWAPS	
1D	5.5313%	2Y	5.9640%	2Y	6.3050%
1W	5.5625%	3Y	6.0910%	3Y	6.4250%
1M	5.6563%	5Y	6.2080%	4Y	6.4960%
3M	5.6406%	7Y	6.3400%	5Y	6.5450%
6M	5.7656%	10Y	6.3910%	7Y	6.6550%
1Y	5.9243%	30Y	6.6640%	10Y	6.7650%

The usual ***cubic spline-interpolation***[21] then leads to the corresponding ***USD term structure of interest rates***:

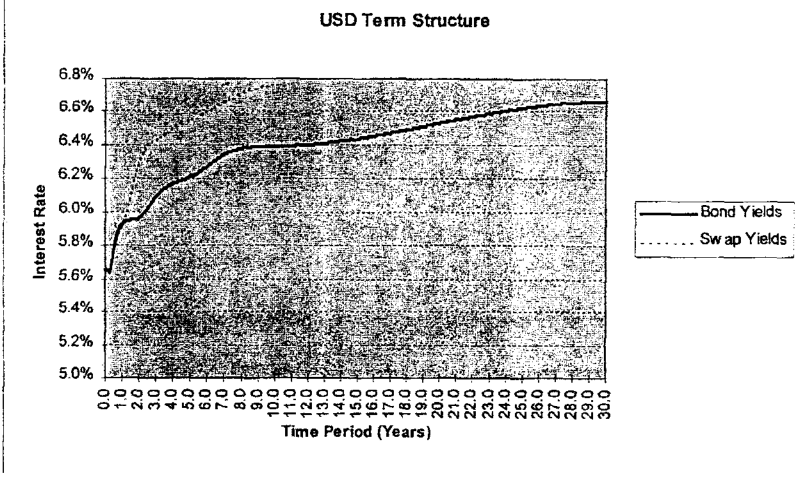


Fig. 13: USD Term Structure of Interest Rates

[21] Note also the ***more sophisticated approaches to term structure estimation outlined in the section on fixed income securities*** (p. 24 - 27).

We choose the bond yield curve for our application and note secondly the associated ***yield volatilities*** (as of 11 August 1997)

GOVT BONDS	
2Y	22.5000%
3Y	24.5000%
5Y	25.0000%
7Y	25.5000%
10Y	24.0000%
30Y	21.0000%

which we translate into a monthly ***USD term structure of interest rate volatilities***

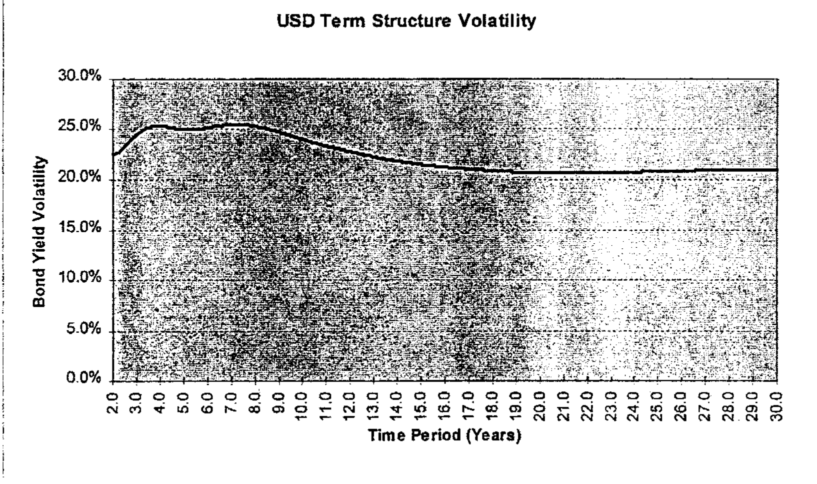


Fig. 14: USD Term Structure of Interest Rate Volatilities

with the same cubic spline-interpolation approach as above. As a final step, we choose $\beta = 0.5$ [22] and $\sigma = 4.25\%$. Note that any ***model calibration*** should

(a) always also incorporate ***future market expectations***[23], not just historically estimated quantities;

(b) include a ***sensitivity analysis*** (i.e., how sensitive are securities prices and risk parameters with respect to changes in the model parameters).

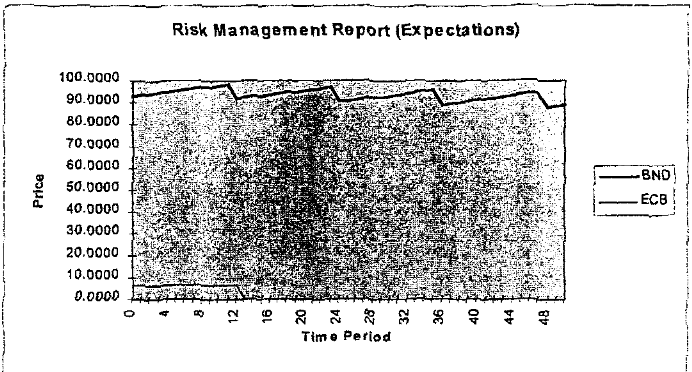
With the above input, the Fin Re Toolbox calculates the 6 Y "Beta" bond's ***price process*** and the corresponding ***risk parameter processes*** (on an expected value basis across loss event

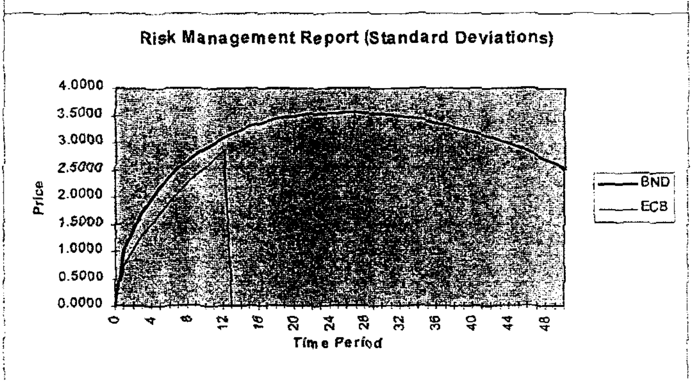
22 $\beta = 0.5$ seems to be a good parameter choice for the US Treasury market also according to various empirical studies looking into the application of the extended CIR model in various bond markets world-wide.

23 We are grateful to Dr. Dellsperger and his team at Credit Suisse Asset Management for helping us out on this rather difficult task.

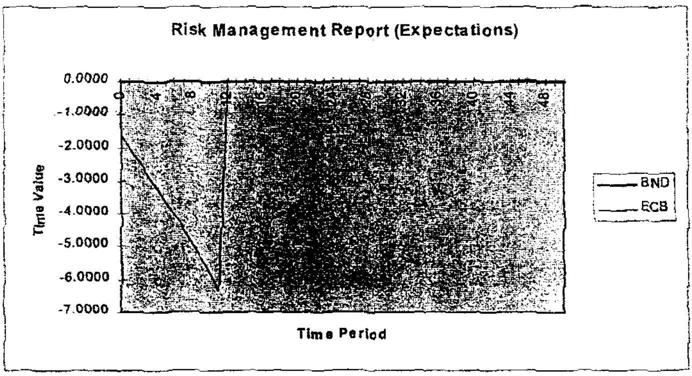
scenarios) as follows [BND = 6 Y "Beta" bond, ECB = 1 Y European call option on the 6 Y "Beta" bond with (loss event contingent) option strike = bond principal].

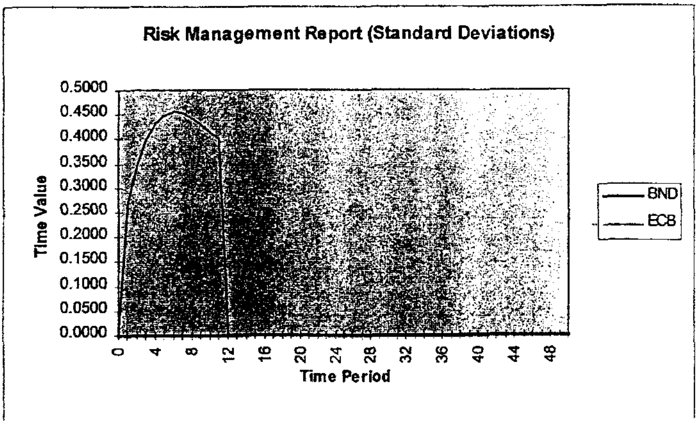
B. Bond Price Process.



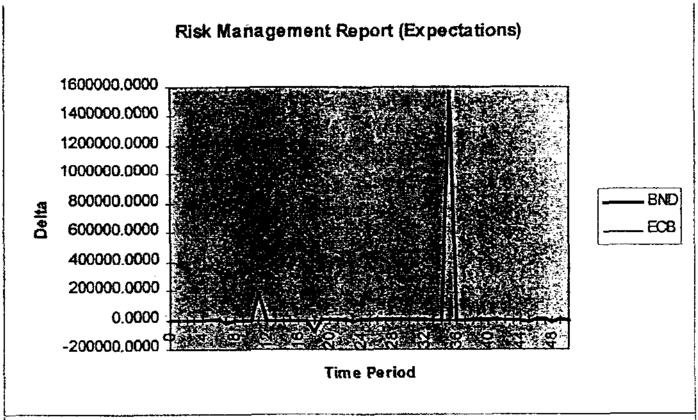


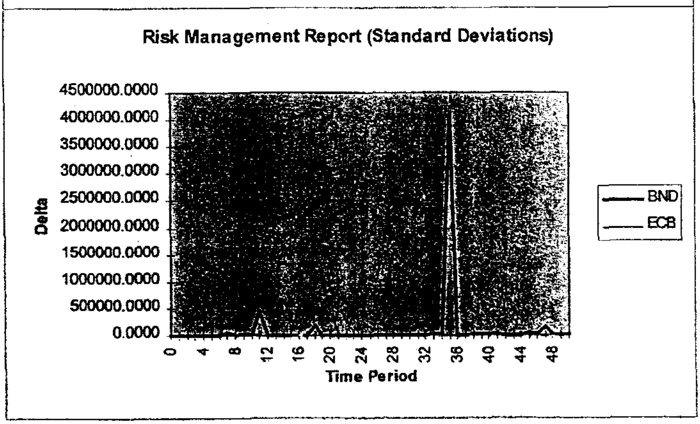
C. Time Value.



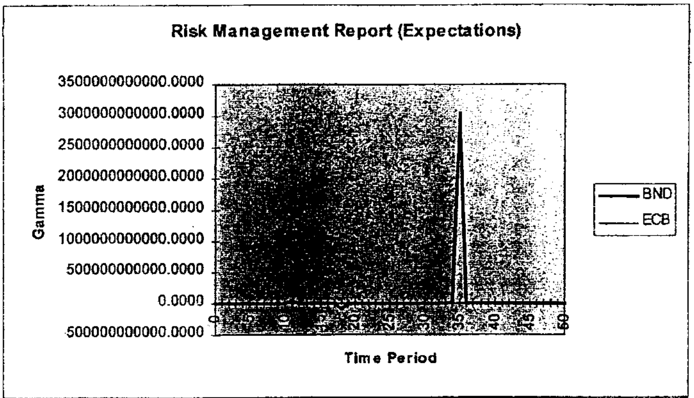

Risk Management Report (Standard Deviations)
Time Value
0.5000
0.4500
0.4000
0.3500
0.3000
0.2500
0.2000
0.1500
0.1000
0.0500
0.0000
0 4 8 12 16 20 24 28 32 36 40 44 48
Time Period
BND
ECB

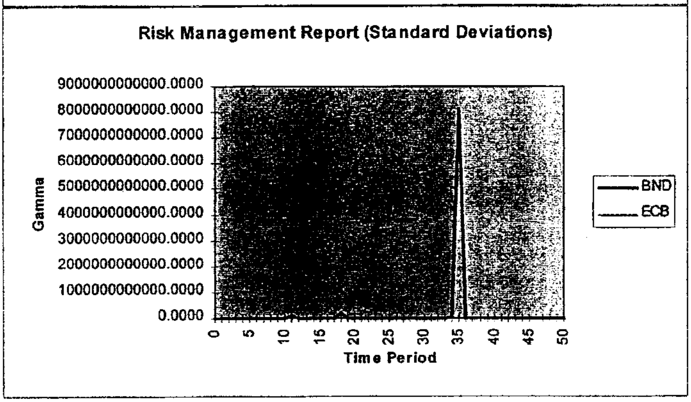
D. Delta.


Risk Management Report (Expectations)
Delta
1600000.0000
1400000.0000
1200000.0000
1000000.0000
800000.0000
600000.0000
400000.0000
200000.0000
0.0000
-200000.0000
Time Period
BND
ECB

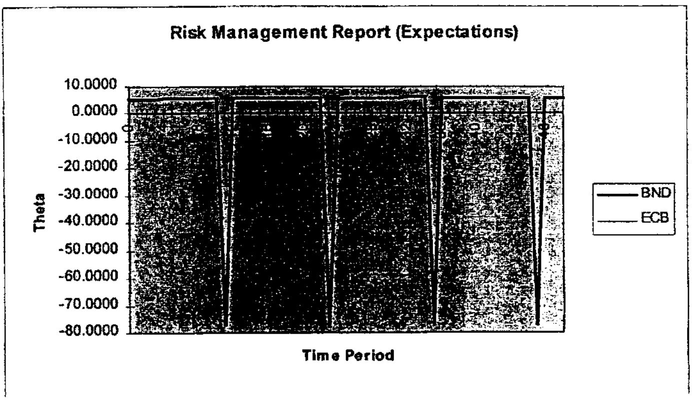

Risk Management Report (Standard Deviations)
Delta
4500000.0000
4000000.0000
3500000.0000
3000000.0000
2500000.0000
2000000.0000
1500000.0000
1000000.0000
500000.0000
0.0000
0 4 8 12 16 20 24 28 32 36 40 44 48
Time Period
BND
ECB

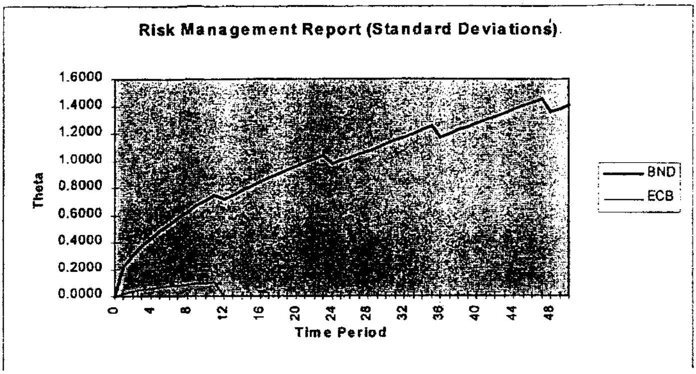
E. Gamma.





F. Theta.





G. Rating.

In order to be consistent with Swiss Re's Value Proposition approach, we choose $k = k_{\text{Oil\&Petrochemicals Industry}} = 1.6324$ (3 Y) (which is equivalent to a RORAC of $r = 6.5\%$ (p.a.) for the Oil & Petrochemicals industry "Beta" target portfolio, see List and Zilch [1] and Geosits, List and Lohner [2]). The Fin Re Toolbox then calculates the ***actuarial prices of the 6 Y "Beta" bond (BND) and the associated 1 Y European and American call and put options (ECB, EPB, ACB, APB) with loss event contingent strikes at the level of the "Beta" bond's principal*** as follows:

Instrument Pricing				
Contingent Claim	Price Expectation	Standard Deviation	[redacted]	Actuarial Price
BND	93.2835	18.4151	1.6324	123.3444
ECB	6.1377	3.3990	1.6324	11.6862
EPB	0.1850	0.3537	1.6324	0.7625
ACB	12.1156	5.1580	1.6324	20.5356
APB	0.1914	0.3871	1.6324	0.8232

As a second example, we now consider the case of an ***excess-of-loss financial reinsurance contract having the attachment point linked to some predefined financial index*** I. The index under consideration can either be an already quoted index such as the Nikkei 225 or the S&P 500 or then a "tailor-made", customized index built up with stocks, bonds or a combination of the two. As an illustration, consider the figure below showing the evolution of the Nikkei 225 stock index over the past twenty-five years:

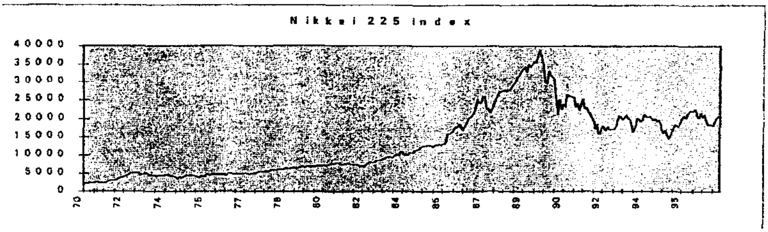


Fig. 15: Nikkei 225 Index Over the Past Twenty-five Years

Let now $A(I)$ be the financially linked attachment point, C the reinsurance cover and L the aggregate loss incurred during the contractually agreed time period. Then the excess-of-loss contract puts the reinsurer under the obligation to pay the policy holder a contingent claim of

$$X(I,L) = \min[\max(L - A(I), 0), C]. \quad (6.1)$$

Verbally, this means that at maturity of the contract the reinsurer pays for the total loss in excess of $A(I)$ with a coverage limit of C. In order to get a basis for the determination of the contract's premium[24], the expected value of the final cashflows given by

$$E = E_{I,L}[X(I,L)] = E[X(I,L)|I,L] \quad (6.2)$$

is needed. Here, the subscripts on E mean that the expectation is taken with respect to the bivariate distribution of both I and L. But as stated in the ***working assumptions*** in section 5 of the paper, ***loss events*** arising from the reinsurance part of the contract ***do not affect the financial market state evolution***. This then implies in our current setting stochastic independence between the index I and the aggregate loss L and thus the expectation operator $E_{I,L}[\cdot]$ "factorizes" as follows:

$$E_{I,L}[\cdot] = E_L[E_I[\cdot|L]]. \quad (6.3)$$

This fact enables us to rewrite the expected value E needed for an actuarial rating of the excess-of-loss contract in the form

$$E = E_L[E_I[X(I,L)|L]]. \quad (6.4)$$

This last formula now tells us that the calculation of the expected part of the contract's premium may be interpreted as a ***financial pricing*** of the loss event contingent cashflow $X(I,L)$. Therefore, at this point, we are again in the same framework as the one outlined in our Fin Re pricing approach in section 5 and in the first example (6 Y "Beta" bond) of this section. We conclude this section by pointing out an interesting ***link between excess-of-loss contracts and option theory***. If we graph the final payoff of the ("dual-trigger") contingent claim $X(I,L)$, we get:

[24] The ***actuarial rating approach*** is as in the above example of the 6 Y "Beta" bond and the associated 1 Y European and American call and put options.

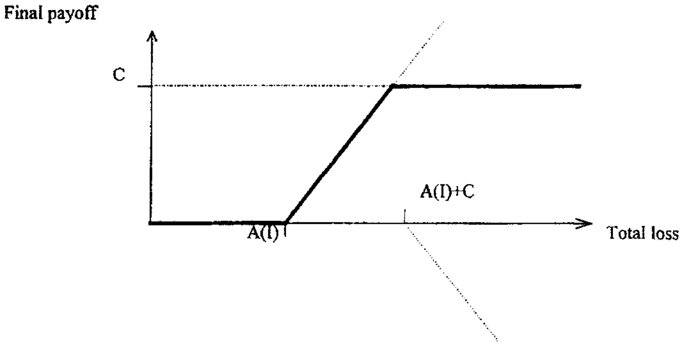


Fig. 16: Excess-of-Loss Contract in Options Theory Framework

This payoff pattern is very interesting because it makes quite obvious the identity

$$\begin{aligned} X(I,L) &= \max(L - A(I),0) - \max(L - [A(I) + C],0) \\ &= \max(L - A(I),0) - \max([L - C] - A(I),0). \end{aligned} \qquad (6.5)$$

From this identity we can conclude that for any fixed level of aggregate loss L the final payoff corresponds to the one of a combined long and short position in a European put option both with the underlying market variable $A(I)$ and strike prices given, respectively, by L and $L - C$. In financial option theory, this strategy corresponds to a ***bull spread with stochastic strike prices***. Keeping in mind that financial point of view, the problem of pricing an excess-of-loss contract can therefore be reformulated as the problem of pricing European put options with stochastic strike prices (see the "Beta" bond options above).

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