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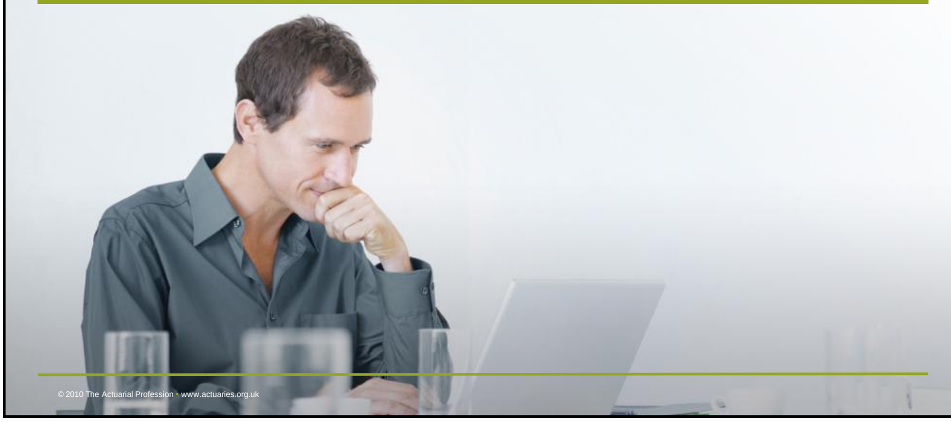
making financial sense of the future

Maximum likelihood and estimation efficiency of the chain ladder
Greg Taylor



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Overview

- Chain ladder loss reserving algorithm
  - Which statistical models generate it by maximum likelihood?
  - What is the bias of their forecasts?
  - How efficient are their forecasts?

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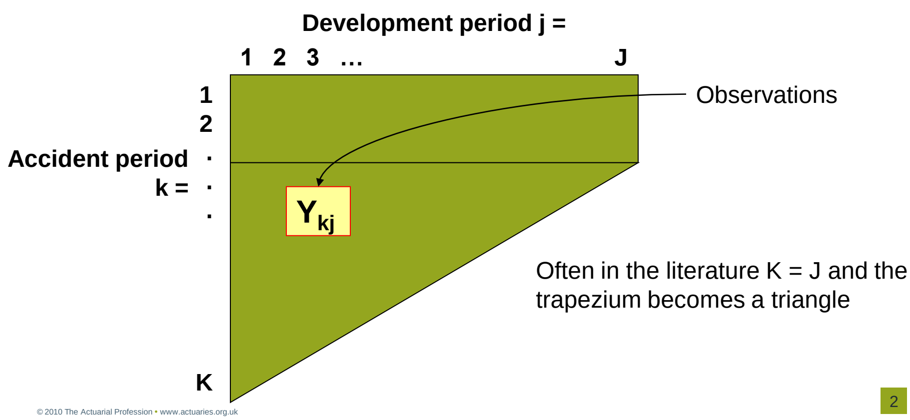
Framework and notation

- Claims reserving trapezium



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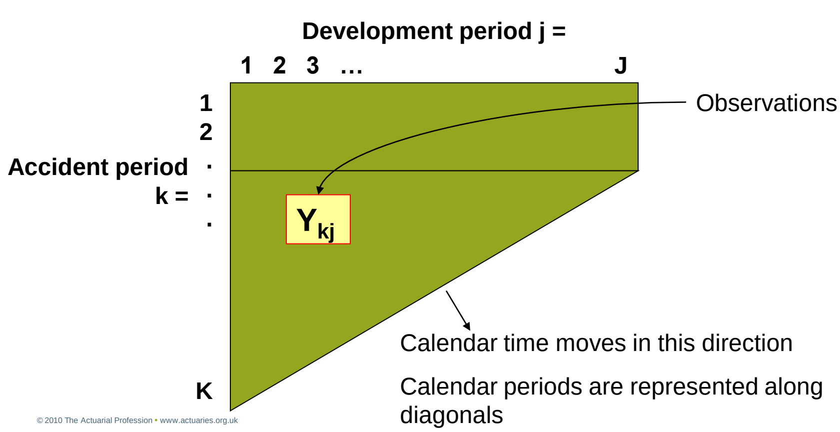
Framework and notation

- Claims reserving trapezium



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Framework and notation

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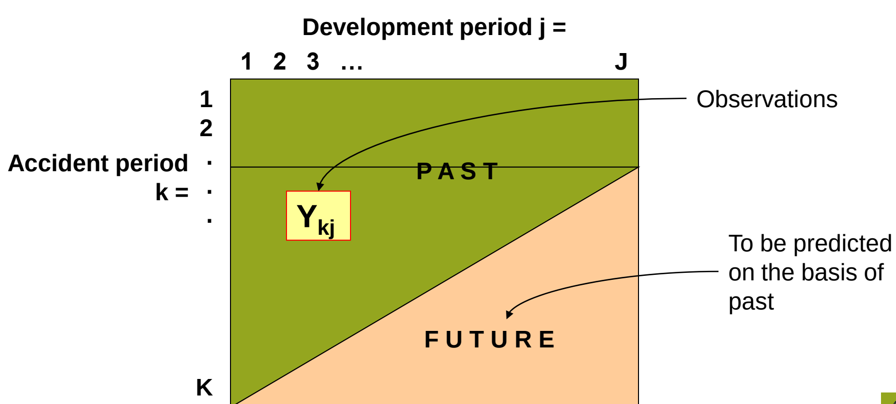

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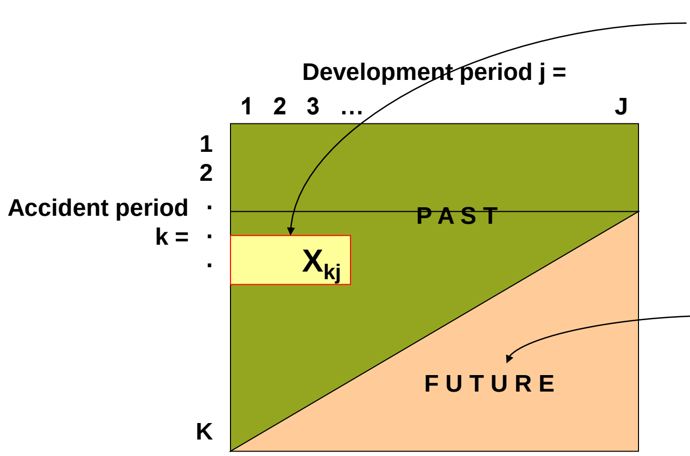
Framework and notation

- Claims reserving trapezium



$$X_{kj} = \sum_{i=1}^{\min J, K-k+1} Y_{ki}$$

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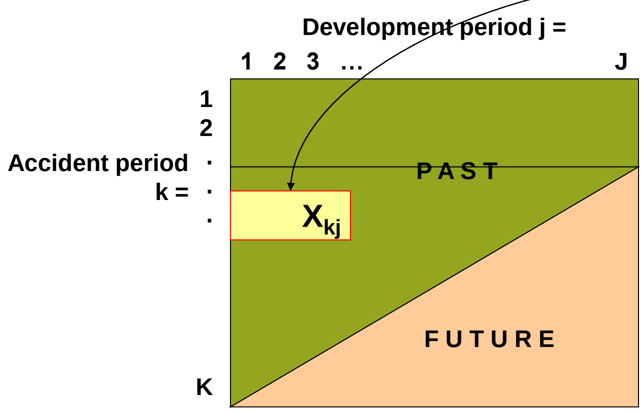
Chain ladder algorithm

- Claims reserving trapezium



Cumulative row sums

$$X_{kj} = \sum_{i=1}^{\min(J, K-k+1)} Y_{ki}$$

Calculate

$$\hat{f}_j = \sum_{k=1}^{K-j} X_{k,j+1} \Big/ \sum_{k=1}^{K-j} X_{kj}$$

Forecast

$$\hat{X}_{kj} = X_{k,K-k+1} \, \hat{f}_{K-k+1} \cdots \hat{f}_{j-1}$$

Last diagonal from the past

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Chain ladder algorithm

- Chain ladder as described is an intuitive algorithm
- But not a statistical model
- Is the algorithm generated by any genuine models?
  - In fact there are two known families of models that do so

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Chain ladder models - recursive

- Mack model (Mack,1993)
  - (M1) Accident periods are stochastically independent, ie $Y_{k_1 j_1}, Y_{k_2 j_2}$ are stochastically independent if $k_1 \neq k_2$.
  - (M2) For each $k = 1, 2, \ldots, K$, the X_{kj} (j varying) form a Markov chain.
  - (M3) For each $k = 1, 2, \ldots, K$ and $j = 1, 2, \ldots, J - 1$, $E\left[X_{k,j+1} \mid X_{kj}\right] = f_j X_{kj}$ for some parameters $f_j > 0$.

    Recursive because each observation depends on predecessor in same row

    $Var\,[X_{k,j+1}|X_{kj}] = \sigma_j^2\, X_{kj}$ for some parameters σ_j.

    Parameters f_j are referred to as **age-to-age factors**

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Chain ladder models – recursive (2)

- Mack model is distribution free
- A distribution is needed to discuss:
  - **Maximum likelihood estimates**
  - **Forecast efficiency**, i.e. forecast error relative to its minimum over all forecasts
- We would like the family of distributions available to be fairly general

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Exponential dispersion family of distributions

- EDF likelihood is

$$\ell(y,\theta,\phi) = [y\theta - b(\theta)]/a(\phi) + c(y,\phi)$$

where
  - θ is a location parameter (the **canonical parameter**)
  - φ is a dispersion parameter (the **scale parameter**)
  - a, b and c are functions with:
    - a continuous
    - b differentiable and one-one
    - c such as to produce a total probability mass of 1
- Properties
  - E[Y] = μ = b′(θ)
    Var[Y] = a(φ) b″(θ) = a(φ) V(μ)
    where V(μ) is called the **variance function**

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Sub-families of the EDF

- **EDF**
  - **Var[Y] = a(φ) V(μ)**
- **Tweedie family**
  - a(φ) = φ
  - V(μ) = μ^p, p≤0 or p≥1

p=0: normal
p=1: (over-dispersed) Poisson
1.5 < p < 2: compound Poisson
p=2: gamma
p=3: inverse Gaussian

- **Over-dispersed Poisson (ODP) family**
  - Tweedie with p=1, i.e. Var[Y] = φμ

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Back to recursive chain ladder models

- Mack model was distribution free
- Equip it with a distribution from the EDF to obtain **EDF Mack model**
  - (M1) Accident periods are stochastically independent, ie $Y_{k_1 j_1}, Y_{k_2 j_2}$ are stochastically independent if $k_1 \neq k_2$.
  - (M2) For each $k = 1, 2, \ldots, K$, the X_{kj} (j varying) form a Markov chain.
  - (M3) For each $k = 1, 2, \ldots, K$ and $j = 1, 2, \ldots, J-1$,

    $E\left[X_{k,j+1} \mid X_{kj}\right] = f_j X_{kj}$ for some parameters $f_j > 0$.

    $Var\,[X_{k,j+1}|X_{kj}] = \sigma_j^2\, X_{kj}$ for some parameters σ_j

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Back to recursive chain ladder models

- Mack model was distribution free
- Equip it with a distribution from the EDF to obtain **EDF Mack model**
  - **(EDFM1)** Accident periods are stochastically independent, ie $Y_{k_1 j_1}, Y_{k_2 j_2}$ are stochastically independent if $k_1 \neq k_2$.
  - (M2) For each $k = 1, 2, \ldots, K$, the X_{kj} (j varying) form a Markov chain.
  - (M3) For each $k = 1, 2, \ldots, K$ and $j = 1, 2, \ldots, J-1$,

    $E\left[X_{k,j+1} \mid X_{kj}\right] = f_j X_{kj}$ for some parameters $f_j > 0$.

    $Var\,[X_{k,j+1}|X_{kj}] = \sigma_j^2\, X_{kj}$ for some parameters σ_j

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Back to recursive chain ladder models

- Mack model was distribution free
- Equip it with a distribution from the EDF to obtain **EDF Mack model**
  - **(EDFM1)** Accident periods are stochastically independent, ie $Y_{k_1 j_1}, Y_{k_2 j_2}$ are stochastically independent if $k_1 \neq k_2$.
  - **(EDFM2)** For each $k = 1, 2, \ldots, K$, the X_{kj} (j varying) form a Markov chain.
  - (M3) For each $k = 1, 2, \ldots, K$ and $j = 1, 2, \ldots, J-1$,

    $E\left[X_{k,j+1} \mid X_{kj}\right] = f_j X_{kj}$ for some parameters $f_j > 0$.

    Var $[X_{k,j+1}|X_{kj}] = \sigma_j^2 X_{kj}$ for some parameters σ_j

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Back to recursive chain ladder models

- Mack model was distribution free
- Equip it with a distribution from the EDF to obtain **EDF Mack model**
  - **(EDFM1)** Accident periods are stochastically independent, ie $Y_{k_1 j_1}, Y_{k_2 j_2}$ are stochastically independent if $k_1 \neq k_2$.
  - **(EDFM2)** For each $k = 1, 2, \ldots, K$, the X_{kj} (j varying) form a Markov chain.
  - **(EDFM3)** For each $k = 1, 2, \ldots, K$ and $j = 1, 2, \ldots, J-1$,

    $E\left[X_{k,j+1} \mid X_{kj}\right] = f_j X_{kj}$ for some parameters $f_j > 0$.

    ~~*Var* $[X_{k,j+1}|X_{kj}] = \sigma_j^2 X_{kj}$ for some parameters σ_j~~

    $$Y_{k,j+1} \mid X_{kj} \sim EDF\ \theta_{kj}, \phi_{kj}; a, b, c$$

    for some functions *a*, *b*, *c* that do not depend on *j* and *k*

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Mack models

EDF Mack model $\supset$ Tweedie Mack model $\supset$ ODP Mack model

EDF Mack model: Just defined

Tweedie Mack model: EDF family restricted to Tweedie

ODP Mack model: Tweedie family restricted to ODP

Distribution-free Mack model: More restrictive variance assumption

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Non-recursive models

- **EDF cross-classified model**

(EDFCC1) The random variables Y_{kj} are stochastically independent

(EDFCC2) For each k = 1, 2, …, K and j = 1, 2, …, J,

(a) $Y_{kj} \sim EDF\ \theta_{kj}, \phi_{kj}; a, b, c$ for some functions a, b, c that do not depend on j and k;

(b) $E\left[Y_{kj}\right] = \alpha_k \beta_j$ for some parameters $\alpha_k, \beta_j > 0$; and

(c) $\sum_{j=1}^{J} \beta_j = 1$.

For EDF Mack, $Y_{k+1,j} \mid X_{kj} \sim$ EDF

For EDF Mack, $E[Y_{k+1,j} \mid X_{kj}] = X_{kj} f_j$

Just removes one degree of freedom from parameters

Non-recursive because each observation independent of predecessors in same row

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Mack and cross-classified models

EDF Mack model ⊃ Tweedie Mack model ⊃ ODP Mack model

Distribution-free Mack model

More restrictive variance assumption

EDF cross-classified model ⊃ Tweedie cross-classified model ⊃ ODP cross-classified model

Just defined

EDF family restricted to Tweedie

Tweedie family restricted to ODP

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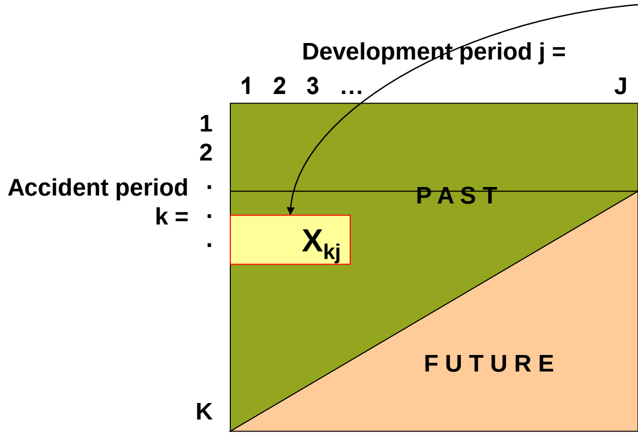
Recall chain ladder algorithm

- Claims reserving trapezium



Cumulative row sums

$$X_{kj} = \sum_{i=1}^{\min(J, K-k+1)} Y_{ki}$$

Calculate

$$\hat{f}_j = \sum_{k=1}^{K-j} X_{k,j+1} \Big/ \sum_{k=1}^{K-j} X_{kj}$$

Forecast

$$\hat{X}_{kj} = X_{k,K-k+1}\ \hat{f}_{K-k+1} \cdots \hat{f}_{j-1}$$

Last diagonal from the past

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Chain ladder algorithm

EDF Mack model $\supset$ Tweedie Mack model $\supset$ ODP Mack model

Distribution-free Mack model

Chain ladder algorithm produces minimum variance estimates of the age-to-age factors among all estimates that are linear combinations of the empirical factors $X_{k,j+1}/X_{kj}$

EDF cross-classified model $\supset$ Tweedie cross-classified model $\supset$ ODP cross-classified model

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Chain ladder algorithm

EDF Mack model $\supset$ Tweedie Mack model $\supset$ ODP Mack model

Distribution-free Mack model

Chain ladder algorithm is maximum likelihood estimator of parameters

EDF cross-classified model $\supset$ Tweedie cross-classified model $\supset$ ODP cross-classified model

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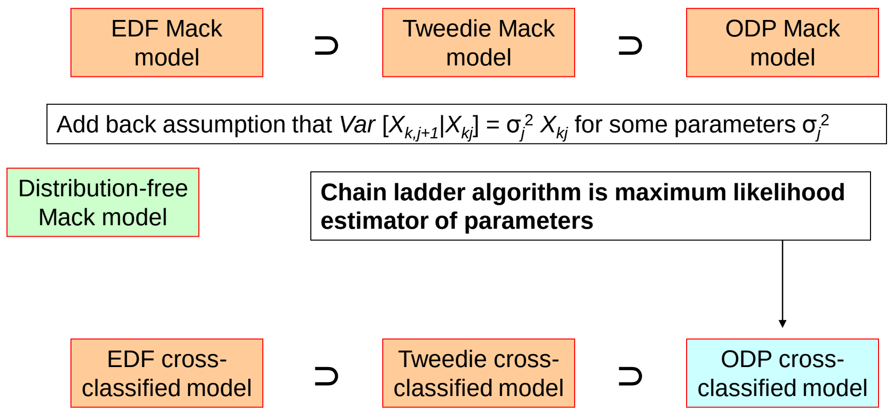
Chain ladder algorithm



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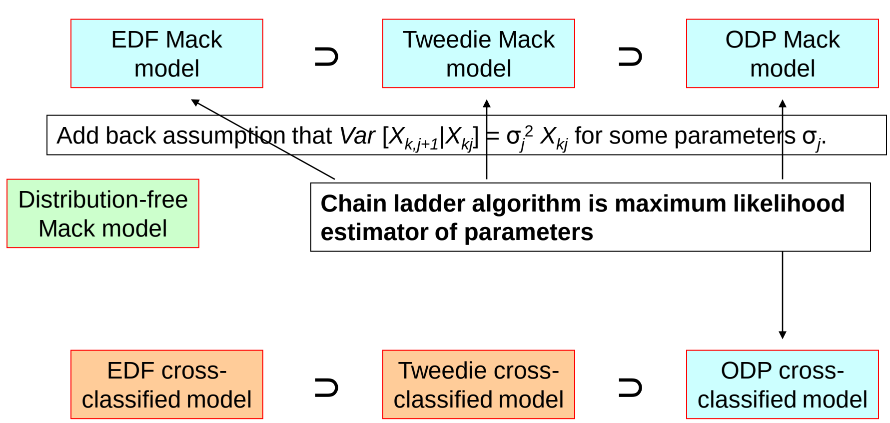
Chain ladder algorithm as MLE



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Chain ladder algorithm as MLE

EDF Mack model ⊃ Tweedie Mack model ⊃ ODP Mack model

Add back assumption that $Var\,[X_{k,j+1}|X_{kj}] = \sigma_j^2\, X_{kj}$ for some parameters σ_j.

Distribution-free Mack model

Chain ladder algorithm is maximum likelihood estimator of parameters

Chain ladder is not MLE

EDF cross-classified model ⊃ Tweedie cross-classified model ⊃ ODP cross-classified model

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Chain ladder bias

EDF Mack model ⊃ Tweedie Mack model ⊃ ODP Mack model

Unbiased estimators of age-to-age factors and loss reserves

Distribution-free Mack model

EDF cross-classified model ⊃ Tweedie cross-classified model ⊃ ODP cross-classified model

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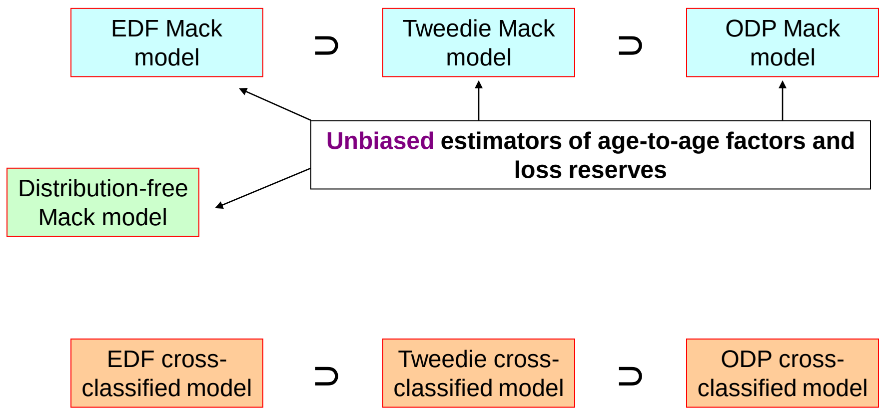
Chain ladder bias



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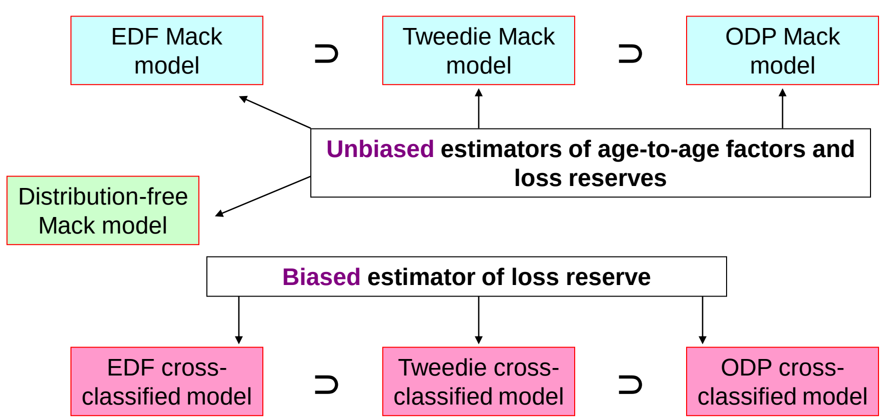
Chain ladder bias



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Chain ladder bias

EDF Mack model ⊃ Tweedie Mack model ⊃ ODP Mack model

Unbiased estimators of age-to-age factors and loss reserves

Distribution-free Mack model

N.B. Same estimators

Biased estimator of loss reserve

EDF cross-classified model ⊃ Tweedie cross-classified model ⊃ ODP cross-classified model

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Chain ladder estimation efficiency

EDF Mack model ⊃ Tweedie Mack model ⊃ ODP Mack model

$Y_{k,j+1} \sim ODP(X_{kj}f_j, \varphi_j)$

Φ_j column dependent only

Distribution-free Mack model

EDF cross-classified model ⊃ Tweedie cross-classified model ⊃ ODP cross-classified model

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Chain ladder estimation efficiency

EDF Mack model ⊃ Tweedie Mack model ⊃ ODP Mack model

$Y_{k,j+1} \sim ODP(X_{kj}f_j,\varphi_j)$

Φ_j column dependent only

Distribution-free Mack model

Minimum variance unbiased estimator of age-to age factors and loss reserve

EDF cross-classified model ⊃ Tweedie cross-classified model ⊃ ODP cross-classified model

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Chain ladder estimation efficiency

EDF Mack model ⊃ Tweedie Mack model ⊃ ODP Mack model

$$Y_{k,j+1} \sim Bin\left(X_{kj}, f_j\right)$$
$$Y_{k,j+1} \sim NegBin\left(X_{kj}, \alpha_j\right) \quad \text{with } \alpha_j / \left(1+\alpha_j\right) = f_j$$

$Y_{k,j+1} \sim ODP(X_{kj}f_j,\varphi_j)$

Φ_j column dependent only

Distribution-free Mack model

Minimum variance unbiased estimator of age-to age factors and loss reserve

EDF cross-classified model ⊃ Tweedie cross-classified model ⊃ ODP cross-classified model

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Chain ladder estimation efficiency

EDF Mack model ⊃ Tweedie Mack model ⊃ ODP Mack model

$$Y_{k,j+1} \sim Bin\ X_{kj}, f_j$$

$$Y_{k,j+1} \sim NegBin\ \ X_{kj}, \alpha_j \quad \text{with } \alpha_j \,/\, 1+\alpha_j = f_j$$

Distribution-free Mack model

Minimum variance unbiased estimator of age-to age factors and loss reserve

$Y_{k,j+1} \sim$ ODP($X_{kj}f_j, \varphi_j$)

Φ_j column dependent only

EDF cross-classified model ⊃ Tweedie cross-classified model ⊃ ODP cross-classified model

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Chain ladder estimation efficiency

EDF Mack model ⊃ Tweedie Mack model ⊃ ODP Mack model

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Distribution-free Mack model

Minimum variance unbiased estimator of age-to age factors and loss reserve

$Y_{k,j+1} \sim$ ODP($X_{kj}f_j, \varphi_j$)

Φ_j column dependent only

$Y_{k,j+1} \sim$ ODP($\alpha_k\beta_j, \varphi_j$)

Φ_j column dependent only

EDF cross-classified model ⊃ Tweedie cross-classified model ⊃ ODP cross-classified model

Bias corrected

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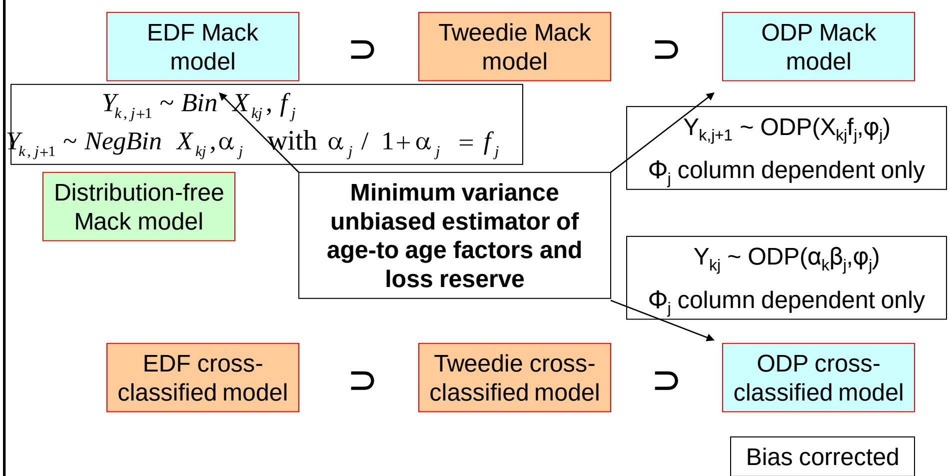
Chain ladder estimation efficiency



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Proof of MVU property

- By means of Lehmann-Scheffé theorem
  - Find sufficient statistics
    - Different for recursive and non-recursive cases
  - Prove completeness of family of admissible distributions

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Summary of results

Model	Chain ladder algorithm is		
	MLE?	Unbiased?	MV?
Recursive:			
Distribution-free	n/a	Y	
EDF	Y*	Y	Binomial, negative binomial or ODP*
Non-recursive:			
ODP	Y	N	Y*
Other	N	N	

* Indicates that some further assumption is required

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Questions or comments?

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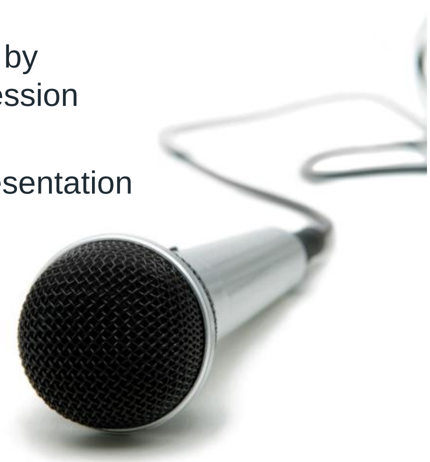

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