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FINITE INTEGRATION BY PARTS (SERIES FOR $\Sigma u_x v_x$) PART II

By G. J. LIDSTONE, LL.D., etc.

1. The following additional notes are in sequence to those printed in *J.I.A.* Vol. LXIV, pp. 160–4. It will be shown by numerical illustrations how the formulae there obtained, and others based thereon, may be practically applied when u_x is a polynomial and v_x an arbitrary set of quantities. In these illustrations we shall take $u_x = x^3$ or $(x+3)^3$, and make use of the following table of the function and its differences:

x	$x+3$	u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$
−3	0	0			
			1		6
−2	1	1		6	
			7		6
−1	2	8		12	
			19		6
0	3	27		18	
			37		6
1	4	64	[49]	24	[6]
		[94½]	61	[27]	6
2	5	125		30	
			91		6
3	6	216		36	
			127		6
4	7	343		42	
			169		6
5	8	512		48	
			217		6
6	9	729			

Figures in [ ] are mean values; those in black type show the sets of values used in the Examples.

2. If the indefinite integral, represented by the expressions (3) and (7) in the former text, be denoted by S_x, the required definite sum $\sum_{x=0}^{n-1}(u_x v_x)$ is $S_n - S_0$. We can make either S_n or S_0 vanish—so reducing the definite sum to a single expression instead of the difference between two expressions—by so arranging the summations of v_x as to give zero values to the particular values of

Σv, $\Sigma^2 v$, ... appearing in S_n or S_0. This we are entitled to do provided we preserve the fundamental relations

$$\Delta\Sigma^n v_x = \Sigma^{n-1} v_x,$$

whence
$$\Sigma^n v_{x+1} = \Sigma^n v_x + \Sigma^{n-1} v_x,$$

and use the latter to build up the whole table of sums, beginning with the zero values.

3. Since there are two expressions for S_x, and in *each* we may make *either* S_n or S_0 take the value 0, there are in all four forms: two of these involve v's and their differences exterior to the "difference-triangle" of the v's actually involved in Σuv. We shall exemplify these four forms by applying them to the following simple case in which $n=4$; the values of u and the Δ's are taken from the table already given, as there shown by the heavy type.

x	0	1	2	3	
v_x	1	3	7	5	
$u_x=(x+3)^3$	27	64	125	216	
$u_x v_x$	27	192	875	1080	Sum = 2174

Form (3). $S_x = u_{x-1}\Sigma v_x - \Delta u_{x-2}\Sigma^2 v_x + \Delta^2 u_{x-3}\Sigma^3 v_x - \dots$ involving backward differences.

(*a*) To use u_{n-1} (the last value involved) and its backward differences (which fall inside the difference-triangle) we make

$$0 = \Sigma v_x = \Sigma^2 v_x = \Sigma^3 v_x \dots \quad \text{for } x=0.$$

The result is

$$\sum_{x=0}^{n-1} u_x v_x = S_n = u_{n-1}\Sigma v_n - \Delta u_{n-2}\Sigma^2 v_n + \Delta^2 u_{n-3}\Sigma^3 v_n - \dots. \quad (3a)$$

	x	v_x	Σv_x	$\Sigma^2 v_x$	$\Sigma^3 v_x$	$\Sigma^4 v_x$	Calculation
	0	1	0	0	0	0	$216\times 16 = +3456$
	1	3	1	0	0	0	$-91\times 16 = -1456$
	2	7	4	1	0	0	$+30\times 6 = +180$
$n-1$	3	5	11	5	1	0	$-6\times 1 = -6$
n	4		16	16	6	1	2174
							$=\Sigma uv$

The sums are taken downwards and are here stepped-down a line at a time so as to fall on the proper line of x. The sums are all positive; the negative signs in the calculation come from the formula.

(*b*) To use u_{-1} (the value preceding the first involved in the sum) and its backward differences (which fall outside the difference-triangle) we make

$$0 = \Sigma v_x = \Sigma^2 v_x = \Sigma^3 v_x \dots \quad \text{for } x=n.$$

The result is

$$\sum_{x=0}^{n-1} u_x v_x = -S_0 = u_{-1} \times -\Sigma v_0 + \Delta u_{-2}\, \Sigma^2 v_0 + \Delta^2 u_{-3} \times -\Sigma^3 v_0 + \dots \qquad \dots\dots(3b)$$

	x	v_x	$-\Sigma v_x$	$\Sigma^2 v_x$	$-\Sigma^3 v_x$	$\Sigma^4 v_x$	Calculation
	0	1	16	48	102	183	$8 \times 16 = 128$
	1	3	15	32	54	81	$7 \times 48 = 336$
	2	7	12	17	22	27	$6 \times 102 = 612$
$n-1$	3	5	5	5	5	5	$6 \times 183 = 1098$
n	4		0	0	0	0	2174
							$= \Sigma uv$

The sums are taken upwards to the top line and are alternately negative and positive. Thus the products will become positive.

[If the table be continued downwards by the same rules it will be found that (*whatever the values of* $v_{n+1}, v_{n+2}, \dots$) there are two consecutive 0's in the column of Σ^2, three in that of Σ^3, and so on. Thus there is a wedge of zeros at the foot of the table, corresponding to the wedge at the top of the table in the previous example. This property will be used later (para. 12).]

Form (7). $S_x = u_x \Sigma v_x - \Delta u_x \Sigma^2 v_{x+1} + \Delta^2 u_x \Sigma^3 v_{x+2} - \dots$

involving forward differences.

(*a*) To use u_n (the value succeeding the last in the sum) and its forward differences (which fall outside the difference-triangle) we make

$$0 = \Sigma v_x = \Sigma^2 v_{x+1} = \Sigma^3 v_{x+2} \dots \quad \text{for } x = 0.$$

The result is

$$\sum_{x=0}^{n-1} u_x v_x = S_n = u_n \Sigma v_n - \Delta u_n \Sigma^2 v_{n+1} + \Delta^2 u_n \Sigma^3 v_{n+2} - \dots . \qquad (7a)$$

	x	v_x	Σv_x	$\Sigma^2 v_x$	$\Sigma^3 v_x$	$\Sigma^4 v_x$	Calculation
	0	1	0	0	0	0	$343 \times 16 = +5488$
	1	3	1	0	0	0	$-169 \times 32 = -5408$
	2	7	4	1	0	0	$48 \times 54 = +2592$
$n-1$	3	5	11	5	1	0	$-6 \times 83 = -\ 498$
n	4		16	16	6	1	2174
	5			32	22	7	$= \Sigma uv$
	6				54	29	
	7					83	

The sums are taken downwards and are here stepped-down a line at a time so as to fall on the proper line of x. In practice they would be made to begin on the line of $x = 0$. The sums are all positive; the negative signs in the calculation come from the formula.

(*b*) To use u_0 and its forward differences (which fall inside the difference-triangle) we make

$$0 = \Sigma v_x = \Sigma^2 v_{x+1} = \Sigma^3 v_{x+2} \ldots \quad \text{for } x = n.$$

The result is

$$\sum_{x=0}^{n-1} u_x v_x = -S_0 = u_0 \times -\Sigma v_0 + \Delta u_0 \, \Sigma^2 v_1 - \Delta^2 u_0 \, \Sigma^3 v_2 + \ldots \quad (7b)$$

	x	v_x	$-\Sigma v_x$	$\Sigma^2 v_x$	$-\Sigma^3 v_x$	$\Sigma^4 v_x$	Calculation
	0	1	16				$27 \times 16 = 432$
	1	3	15	32			$37 \times 32 = 1184$
	2	7	12	17	22		$24 \times 22 = 528$
$n-1$	3	5	5	5	5	5	$6 \times 5 = 30$
n	4		0	0	0	0	2174
							$= \Sigma uv$

The sums are taken upwards and are here stepped-down a line at a time so as to fall on the proper line of x. In practice they might be made to begin on the line of $x = 0$. The sums are alternately negative and positive. Thus the products in the formula all become positive.

[If the table be continued downwards there will be a wedge of zeros at the foot, whatever the values of $v_{n+1}, v_{n+2}, \ldots$, as in the case of the table for formula (3 *b*). See note at the foot of that table, para. 3.]

4. While it is convenient to illustrate the four variant forms separately it will be seen that they are closely related. The downward sums of (3 *a*) are those of (7 *a*) re-aligned; and the upward sums of (7 *b*) are those of (3 *b*) re-aligned. Also if (3 *a*) be applied to the *u*'s and *v*'s written in reverse order we shall get exactly the figures of (7 *b*) in reverse order; and similarly with (3 *b*) and (7 *a*).

5. If the summation is divided into two parts, say $\sum_{0}^{a-1}$ and $\sum_{a}^{n-1}$ (where *a* is any interior* point), we may use

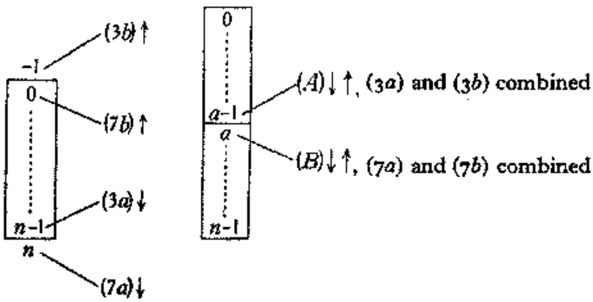
(A) Formula (3 *a*) for $\sum_{0}^{a-1}$, i.e. the top half, and formula (3 *b*) for $\sum_{a}^{n-1}$, i.e. the bottom half; both formulae involving the same multipliers, viz. u_{a-1} and its backward differences; or

(B) Formula (7 *a*) for the top half and (7 *b*) for the bottom half, both involving u_a and its forward differences.

* We use the word "interior" rather than "central" because *a*, the point of division, may be *any* point between the extremes, i.e. it is not necessarily at the precise centre of the range included in the summation. Hence it is immaterial whether *n* is even or odd.

By thus dividing the range into two parts we shall have much smaller figures to deal with, especially if the number of terms is large, in which case the higher sums increase very rapidly.

6. If we use a line slanting $\genfrac{}{}{0pt}{}{\text{upwards}}{\text{downwards}}$ to represent values of u and its $\genfrac{}{}{0pt}{}{\text{backward}}{\text{forward}}$ differences, the formulae may be represented diagrammatically thus:



The arrows show the directions of the summations involved

7. For an example of the use of (A) and (B) we shall take the data in Table IV of Elderton's book, *Frequency Curves*... [ed. 1, 2 or 3, p. 21], calling the data $v_0, \ldots v_9$ and forming $\Sigma\,(u_x . x^3)$. Plan (A) is that used by Elderton, Table IV (A), and otherwise proved by him. See table on the next page.

VARIANT FORMULAE

8. In the first paper, starting from the following fundamental relations

$$\Sigma\,(u_x v_x) = u_{x-1}\,\Sigma v_x - \Sigma\,(\Delta u_{x-1}\,\Sigma v_x), \qquad \ldots\ldots(5)$$

$$= u_x\,\Sigma v_x - \Sigma\,(\Delta u_x\,\Sigma v_{x+1}), \qquad \ldots\ldots(8)$$

we found the two standard series (3) and (7) by expanding the last term of (5) by a repetition of (5), and the last term of (8) by a repetition of (8), respectively. But we may equally well expand the last term of (5) by (8), or the last term of (8) by (5). Thus at any stage we have a choice of two next terms, and so in a formula of n terms there are in all 2^n different forms. We shall show that these can be formed by simple rules. On examining the process of expansion it will be seen that if we reach a new term by the use

Example referred to in para. 7

Data				Plan A				Plan B			
x	v_x	x^3	$v_x x^3$	Σv_x	$\Sigma^2 v_x$	$\Sigma^3 v_x$	$\Sigma^4 v_x$	Σv_x	$\Sigma^2 v_x$	$\Sigma^3 v_x$	$\Sigma^4 v_x$
0	29	0	0								
1	23	1	23	29				29			
2	81	8	648	52	29			52	29		
3	151	27	4077	133	81	29		133	81	29	
4	192	64	12288	284	214	110	29	{284 **716**	214	110	29
5	239	125	29875	{476 **524**	{498 **978**	{324 **1648**	{139 **2587**	**524**	{498 **978**	324 {822	139 463
6	157	216	33912	**285**	**454**	**670**	**939**	**285**	**454**	670	{1285
7	93	343	31899	**128**	**169**	**216**	**269**	**128**	**169**	**216**	**269**
8	29	512	14848	**35**	**41**	**47**	**53**	**35**	**41**	**47**	**53**
9	6	729	4374	**6**	**6**	**6**	**6**	**6**	**6**	**6**	**6**
	1000		131944								

Plan A — Calculation

$524+476=1000$	$\times 64=64000$
$978-498=480$	$\times 37=17760$
$1648+324=1972$	$\times 18=35496$
$2587-139=2448$	$\times 6=14688$
	131944

Plan B — Calculation

$716+284=1000$	$\times 64=64000$
$978-498=480$	$\times 61=29280$
$670+822=1492$	$\times 30=44760$
$269-1285=-1016$	$\times 6=-6096$
	131944

Notes. Sums from the top downwards are printed in ordinary type, and are alternately + and −. Sums from the bottom upwards are printed in black type and are all +. The sums are printed on the lines of x to which they theoretically belong, with the result that the final downward and upward sums are brought together, as shown by the signs {, but in practice a different arrangement might be adopted. In Plan A the *top* sums are truncated; in Plan B the *lower* sums are truncated. The multipliers, viz. u and its appropriate differences, are taken from the table in the first paragraph. The point a is 5 in Plan A, 6 in Plan B.

of (5) this will involve reducing by 1 the previous suffix of u, and in passing to the *next* term the suffix of v is unchanged, whichever formula we use. But if we reach a new term by the use of (8) this will involve no change of the suffix of u, but in passing to the *next* term the suffix of v must be reduced by 1, whichever formula we use. Hence we are led to the following.

GENERAL RULES

(1) The form of the series is

$$\Sigma u_x v_x = u\Sigma v - \Delta u\ \Sigma^2 v + \Delta^2 u\ \Sigma^3 v - \ldots,$$

where the suffixes of u and v remain to be inserted.

(2) The first term is either $u_x \Sigma v_x$ or $u_{x-1}\Sigma v_x$. In order to make the Rules general we shall consider that the first term is preceded by an imaginary term of zero value, which may be called the 0th term, with suffixes x and x.

(3) In passing from any term (including the 0th) to the next the suffix of u may remain unchanged *or* be reduced by 1.

(4) If at any step [say from the nth term to the $(n+1)$th, where n may be zero] the suffix of u is $\begin{matrix}\text{unchanged}\\ \text{reduced by 1}\end{matrix}$, then at the *next* step [i.e. from the $(n+1)$th term to the $(n+2)$th] the suffix of v is $\begin{matrix}\text{increased by 1}\\ \text{unchanged}\end{matrix}$.

9. The writer has communicated these Rules to Mr D. C. Fraser, M.A., F.I.A., with the suggestion that it must be possible to derive them from his elegant Hexagon Diagram as extended to Σuv (see his important paper, *T.F.A.* Vol. XV, pp. 163–9). It is understood that he will prove this in a supplementary Note, *ibid.* Vol. XVII.

10. The Rules lead immediately to the standard series (3) and (7), and to extensions. Take as an example the following case:

Term	0th	1st	2nd	3rd	4th	5th	...
Suffix of u	x	x	$x-1$	$x-1$	$x-2$	$x-2$	...
Suffix of v	x	x	$[x+1]$	$[x+1]$	$[x+2]$	$[x+2]$	...

Here we first write down the unbracketed values. Starting with the 0th term the differences of the suffixes of u are $0, -1; 0, -1; \ldots$. Hence by the Rules, starting with the 1st term, the differences in the suffixes of v are $+1, 0; +1, 0; \ldots$, and thus we build up the values in [ ]. We are thus led to the formula

$$\Sigma u_x v_x = u_x \Sigma v_x - \Delta u_{x-1}\Sigma^2 v_{x+1} + \Delta^2 u_{x-1}\Sigma^3 v_{x+1} - \Delta^3 u_{x-2}\Sigma^4 v_{x+2}\ \ldots, \qquad \ldots\ldots\text{(A)}$$

where the sequence of the Δ's corresponds to the Gauss Backward Interpolation formula beginning with u_x. In the same way we find

$$\Sigma u_x v_x = u_x \Sigma v_x - \Delta u_x \Sigma^2 v_{x+1} + \Delta^2 u_{x-1} \Sigma^3 v_{x+2} - \Delta^3 u_{x-1} \Sigma^4 v_{x+2} \ldots, \quad \ldots\ldots(\mathrm{B})$$

which corresponds to the Gauss Forward Formula, beginning with u_x. Similarly

$$\Sigma u_x v_x = u_{x-1} \Sigma v_x - \Delta u_{x-1} \Sigma^2 v_x + \Delta^2 u_{x-2} \Sigma^3 v_{x+1} - \Delta^3 u_{x-2} \Sigma^4 v_{x+1} \ldots, \quad \ldots\ldots(\mathrm{C})$$

corresponding to the Gauss Forward Formula beginning with u_{x-1}.

11. Taking the mean of (A) and (B) we get the following formula involving central differences or mean differences as in Stirling's Interpolation Formula:

$$\Sigma u_x v_x = u_x \Sigma v_x - \tfrac{1}{2}\left(\Delta u_x + \Delta u_{x-1}\right) \Sigma^2 v_{x+1} + \Delta^2 u_{x-1}\, \tfrac{1}{2}\left(\Sigma^3 v_{x+1} + \Sigma^3 v_{x+2}\right) - \ldots, \quad \ldots\ldots(\mathrm{D})$$

and taking the mean of (A) and (C) we get

$$\Sigma u_x v_x = \tfrac{1}{2}\left(u_x + u_{x-1}\right) \Sigma v_x - \Delta u_{x-1}\, \tfrac{1}{2}\left(\Sigma^2 v_x + \Sigma^2 v_{x+1}\right) + \tfrac{1}{2}\left(\Delta^2 u_{x-1} + \Delta^2 u_{x-2}\right) \Sigma^3 v_{x+1} + \ldots, \quad \ldots\ldots(\mathrm{E})$$

involving central differences or mean differences as in Bessel's Interpolation Formula. These summation formulae have been given, and otherwise proved, in various places (see G. F. Hardy's *Lectures*... [1909], Note E, pp. 125–8, etc.).

12. Like the standard series (3) and (7) already discussed, the formulae given in this section all give the indefinite sum S_x, and the definite sum $\sum_{x=0}^{n-1} u_x v_x$ is equal to $S_n - S_0$. In applying the formulae either S_n or S_0 is made to vanish by suitable arrangement of the summations round an interior point. No fresh conventions are required to secure this result, for it will be found that the wedges of zeros [which, as previously pointed out, exist at the top or bottom of the columns of sums in (3*a*), (3*b*), (7*a*) and (7*b*)] coalesce in such a way that zero values are obtained also for the differences and mean differences that are required to vanish in applying the central-difference formulae (D) and (E).

13. As an illustration of the central-difference formulae (D) and (E) we shall apply them to our preceding example, para. 7.

The bulk of the summation work will be the same, but for clearness we shall show it in full. In these cases it is usual and convenient to begin registering the sums on the top line.

x	v_x	Σ	Σ^2	Σ^3	Σ^4
0	29	29	29	29	29
1	23	52	81	110	139
2	81	133	214	324	463
				[573]	[874]
3	151	284	498	822	1285
			[736]		
4	192	476	974		
5	239	524	978	1648	
			[716]	[1159]	
6	157	285	454	670	939
					[604]
7	93	128	169	216	269
8	29	35	41	47	53
9	6	6	6	6	6
	1000				

The appropriate values of u and its central differences or mean differences are taken as before from the table in para. 1.

(D) Calculation

$524+476=1000$	$\times 64$	64000
$978-498=480$	$\times 49$	23520
$1159+573=1732$	$\times 24$	41568
$939-463=476$	$\times 6$	2856
		131944

(E) Calculation

$524+476=1000$	$\times 94\frac{1}{2}$	94500
$716-736=-20$	$\times 61$	-1220
$670+822=1492$	$\times 27$	40284
$604-874=-270$	$\times 6$	-1620
		131944

14. When the number of terms involved is at all considerable, there is an undoubted advantage in the use of one or other of the formulae referred to some interior point. In general it would not appear that those just illustrated, involving central differences, have much if any advantage over the formulae (A) and (B) illustrated in para. 7. There are, however, some cases in which the central-difference formulae are greatly to be preferred: viz. when we can so choose the interior point a that alternate differences or mean differences vanish. For examples reference may be made to G. F. Hardy's *Lectures*, Note E, pp. 125–8; and to A. W. Joseph's paper, *J.I.A.* Vol. LXV, p. 285, where examples quoted from a paper of Aitken's are given.

INTERCONNEXION OF FORMULAE

15. The plan of these notes is to deduce all the summation formulae directly from the fundamental relations

$$\Delta u_x v_x = u_{x+1}\Delta v_x + v_x \Delta u_x,$$
$$\Delta \Sigma^t v_x = \Sigma^{t-1} v_x,$$

without using interpolation formulae, or formulae expressing $\Sigma^t v_x$ in terms of the v's involved, or even binomial coefficients. It will illustrate the very close connexion between differencing and summing, and the consistent interlocking of different parts of the general theory, if we show that these classes of formulae can themselves be *deduced from* the summation formulae, instead of serving as their basis.

16. In formula ($3b$) put for u_x

$$(x+t-1)_{(t-1)} \equiv (x+t-1)(x+t-2)\ldots(x+1)/(t-1)!.$$

Then u_x vanishes for $x=-1, -2, \ldots, -(t-1)$, and so all the differences involved in the formula vanish until we reach $\Delta^{t-1}u_{-t}=1$. Thus the right-hand side reduces to a single term, and reversing the sides we get

$$(-)^t \Sigma^t v_0 = \sum_{x=0}^{n-1}(x+t-1)_{(t-1)} v_x.$$

Taking α instead of 0 as the origin this becomes

$$\begin{aligned}(-)^t \Sigma^t v_\alpha &= \sum_{x=0}^{n-1}(x+t-1)_{(t-1)} v_{\alpha+x}\\ &= \sum_{x=\alpha}^{\omega}(x-\alpha+t-1)_{(t-1)} v_x \qquad (\omega\equiv\alpha+n-1)\\ &= (t-1)_{(t-1)} v_\alpha + t_{(t-1)} v_{\alpha+1} + \ldots + (\omega-\alpha+t-1)_{(t-1)} v_\omega,\end{aligned}$$

where all the factorials are of order $(t-1)$. Since

$$(x-\alpha+t-1)_{(t-1)} = (x-\alpha+t-1)_{(x-\alpha)}$$

the expression may alternatively be written

$$\begin{aligned}(-)^t \Sigma^t v_\alpha &= \sum_{x=\alpha}^{\omega}(x-\alpha+t-1)_{(x-\alpha)} v_x\\ &= v_\alpha + t_{(1)} v_{\alpha+1} + (t+1)_{(2)} v_{\alpha+2} + \ldots + (\omega-\alpha+t-1)_{(\omega-\alpha)} v_\omega.\end{aligned}$$

These expressions give the tth summation taken *upwards* from the line of ω to the line of α with the convention that sums of all orders vanish on the line $\omega+1$. This may be conveniently represented by the suggestive symbol $\overset{\alpha}{\underset{\omega}{\uparrow}}{}^{t}\, v_x$, which is specially useful when v is constant so that a suffix is inapplicable. Putting $v=1$,

$$(-)^t \overset{\alpha}{\underset{\omega}{\uparrow}}{}^{t}\, 1 = \sum_{x=\alpha}^{\omega} (x-\alpha+t-1)_{(t-1)}$$

$$= [(x-\alpha+t-1)_{(t)}]_{\alpha}^{\omega+1}$$

$$=(\omega-\alpha+t)_{(t)},$$

since the factorial in [ ] vanishes at the lower limit.

17. In $(7b)$ put Δu_x for u_x and $v=1$ and we get

$$u_n-u_0=\sum_{x=0}^{n-1}\Delta u_x$$

$$=\Delta u_0 \times - \overset{0}{\underset{n-1}{\uparrow}}\, 1 + \Delta^2 u_0 \overset{1}{\underset{n-1}{\uparrow}}{}^{2}\, 1 - \Delta^3 u_0 \overset{2}{\underset{n-1}{\uparrow}}{}^{3}\, 1 + \dots .$$

Using the last formula in the preceding paragraph, this becomes

$$u_n-u_0=n_{(1)}\Delta u_0+n_{(2)}\Delta^2 u_0+n_{(3)}\Delta^3 u_0+\dots,$$

which is equivalent to the advancing-difference interpolation formula.

18. In the same way we may obtain expressions for *descending* sums and for the backward-difference interpolation formula. In $(3a)$ put

$$u_x=(x+t-1-n)_{(t-1)}.$$

Then u_x vanishes for $x=n-1,\ n-2,\ \dots,\ n-t+1$, and so all the differences involved in the formula vanish until we reach $\Delta^{t-1}u_{n-t}=1$. Thus the right-hand side reduces to a single term, and reversing the sides we get

$$(-)^{t-1}\,\Sigma^t v_n=\sum_{x=0}^{n-1}(x+t-1-n)_{(t-1)}v_x$$

$$=\sum_{x=0}^{n-1}(-)^{t-1}\,(n-x-1)_{(t-1)}v_x,$$

or, excluding zero terms from the summation on the right and cancelling $(-)^{t-1}$ which appears on both sides,

$$\begin{aligned}\Sigma^t v_n &= \sum_{x=0}^{n-t} (n-x-1)_{(t-1)} v_x \\ &= (n-1)_{(t-1)} v_0 + (n-2)_{(t-1)} v_1 + \ldots + t v_{n-t-1} + v_{n-t},\end{aligned}$$

which involves $n-t+1$ terms, from v_0 to v_{n-t}. Inspection of the scheme (3*a*) in para. 1 shows that the formula gives the *downward* sum standing on the line of n (which is arbitrary) with the convention that sums of all orders are zero on the line of 0. This sum may be conveniently represented by the symbol $\underset{n-t}{\overset{0}{\downarrow}}{}^{t}\, v_x$. Making $v=1$ we get, in this notation,

$$\begin{aligned}\underset{n-t}{\overset{0}{\downarrow}}{}^{t}\, 1 &= \sum_{x=0}^{n-t} (n-x-1)_{(t-1)} \\ &= -\left[(n-x)_{(t)}\right]_0^{n-t+1} \\ &= n_{(t)},\end{aligned}$$

since the factorial in [ ] vanishes at the upper limit. Changing the origin from 0 to α, and writing ω for $n-t+\alpha$, this becomes

$$\underset{\omega}{\overset{\alpha}{\downarrow}}{}^{t}\, 1 = (\omega-\alpha+t)_{(t)} = (-)^t \underset{\omega}{\overset{\alpha}{\uparrow}}{}^{t}\, 1.$$

19. In (3*a*) put Δu_x for u_x, and $v=1$, and we get

$$\begin{aligned}u_n - u_0 &= \sum_{x=0}^{n-1} \Delta u_x \\ &= \Delta u_{n-1} \times \underset{n-1}{\overset{0}{\downarrow}}\, 1 - \Delta^2 u_{n-2} \times \underset{n-2}{\overset{0}{\downarrow}}{}^{2}\, 1 + \Delta^3 u_{n-3} \times \underset{n-3}{\overset{0}{\downarrow}}{}^{3}\, 1 - \ldots \\ &= n_{(1)} \Delta u_{n-1} - n_{(2)} \Delta^2 u_{n-2} + n_{(3)} \Delta^3 u_{n-3} - \ldots,\end{aligned}$$

which is equivalent to the backward-difference interpolation formula.

20. In the same way the Gauss interpolation formulae and the central-difference formulae, and expressions for the corresponding

sums, may be derived from the summation formulae given in paras. 10–11. The close inter-connexion of the formulae is thus strikingly illustrated.

LIMITS OF ERROR

21. If, as assumed in our numerical illustrations, u_x is a polynomial of degree t, whose differences vanish after $\Delta^t u$, the summation formulae are accurate if taken to the term involving that difference. But if u_x is not a polynomial the results are approximate only, and it is important to find limits for the error involved. Steffensen (*Interpolation* [1927], p. 101, § 116) has shown how to determine remainder-terms of the form

$$R = u_\xi^{(t+1)} \Sigma^{t+2} v,$$

where the term $u_\xi^{(t+1)}$ represents the $(t+1)$th differential coefficient of u_x for $x=\xi$, and ξ is a value of x between the extreme values involved. But this formula is not always suitable, because

(1) A further sum of v is involved: this may be laborious, and if available might perhaps be better utilized to bring in another term in the summation formula.

(2) Since ξ is unknown we can only say that $u_\xi^{(t+1)}$ falls between its greatest and least values in the range, and it may not be easy either to evaluate $u_\xi^{(t+1)}$ or to determine its limiting values. In particular in a function of unknown form [such as a_x, A_x, q_x, ..., in a table not based on a mathematical formula] the differential coefficients cannot be accurately evaluated.

22. It is therefore practically useful to base limits of error on the *differences* of u, which are always known. Omitting suffixes, which depend on the particular formula in use, let the summation formula be

$$\Sigma uv = u\Sigma v - \Delta u \Sigma^2 v + \ldots + (-)^t \Delta^t u \Sigma^{t+1} v. \quad \ldots\ldots(\text{F})$$

On reference to Part I of the paper (*loc. cit.* pp. 160–3), it will be seen that the last term in this formula is the first term in the expansion of a remainder-term $(-)^t \Sigma (\Delta^t u\, \Sigma^t v)$. If the greatest and least values of $\Delta^t u$, over the range of the sum $\Sigma^t v$, be denoted by $\Delta^t_{\max}$ and $\Delta^t_{\min}$, the accurate value of the last term in (F) lies between

$$(-)^t \Sigma (\Delta^t_{\max} \Sigma^t v) = (-)^t \Delta^t_{\max} \Sigma^{t+1} v$$

and

$$(-)^t \Sigma (\Delta^t_{\min} \Sigma^t v) = (-)^t \Delta^t_{\min} \Sigma^{t+1} v,$$

from which we get at once the limits of the error involved in the use of (F). Assuming that $\Delta^t u$ is not changing rapidly these limits depend chiefly on $\Sigma^{t+1} v$, and they may therefore be considerably contracted by using one of the formulae of paras. 5–11, in which upward and downward summations are made from some *interior* point, such as the point a in para. 5; for in this case the sums of higher orders are greatly reduced in comparison with those arising when the summations are made from a *terminal* point. Moreover, if we end with a term involving a difference of even order, we take the *difference* instead of the *total* of the upward and downward sums; and if the interior point a is so selected that the upward and downward sums of order $(t+1)$ are not greatly different the limits of error are comparatively small. To this end if v_x is generally $\begin{matrix}\text{increasing}\\\text{decreasing}\end{matrix}$ the point a should be $\begin{matrix}\text{below}\\\text{above}\end{matrix}$ the centre of the range. It will be seen that the "interior" or "point-a" formulae not only involve smaller figures but are also likely to be more accurate than the terminal formulae, if u_x is not a simple polynomial. Note, however, that for the point-a formulae the limits of error must usually be found separately for the upper and lower sections; and if we end with a term involving upward and downward sums of different signs, $\Delta^t_{\max}$ in one section must be combined with $\Delta^t_{\min}$ in the other section, and *vice versa*.

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23. *Note.* Throughout our illustrations, as stated in the small print notes, the sums (except in para. 13) have been registered on the line of x corresponding to the subscript of Σ^t in the appropriate formula. This has been done in order to show clearly the correspondence between the working and the formula. Further, in summing long series by machine, the sums will necessarily fall on the lines and not between them. But for some purposes it is convenient to space the sums so that $\Sigma^t u_x$ falls on the line of $x-\frac{1}{2}t$, as in the examples given on p. 103. [Cf. Steffensen, *loc. cit.* pp. 94 and 98; Fraser, *loc. cit.* pp. 143, 146, 173.] By this arrangement it is seen that, reading the table from right to left, the figures to the left of any column are its successive differences, spaced in the usual way.

Ex. (3 *a*). *Downward sums*

x	u_x	Σu	$\Sigma^2 u$	$\Sigma^3 u$
−2				
−1				
		0		
0	1		0	
		1		0
1	3		1	
		4		1
2	7		5	
		11		6
3	5		16	
		16		22
4			32	
				54
5				
6				

Ex. (3 *b*). *Upward sums*

u_x	Σu	$\Sigma^2 u$	$\Sigma^3 u$
			−102
		48	
	−16		−54
1		32	
	−15		−22
3		17	
	−12		−5
7		5	
	−5		0
5		0	
	0		