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**FRACTALS
THE PHYSICISTS' WOLF STALKING THE ACTUARIAL FLOCK?**

A DISCUSSION OF THE ROLE OF ACTUARIES IN THE DEVELOPING WORLD OF COMPLEXITY

by

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and

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Dedicated to those practical actuaries who thought they had
escaped from the clutches of seriously tricky maths

Look out it's behind you.

Preface

Perhaps the biggest challenge in writing this paper has been to establish a tone that is both robust and appropriate.

The problem has been borne from a fundamental conflict; how do you address a subject that ought to be considered by the general populous of the profession and hence should be argued on a level of principle, when the very heart of that subject is witnessed by complex modern mathematical techniques, which in their turn require to be considered on a level of detail.

That the paper is termed 'Fractals', however, merely points to the roots of this document and to the tricky maths in its nether regions. Please do not be deterred by it. A diversion into the pages of 'New Scientist' gave rise to an inquisitive journey, and this paper represents the current staging post of that mathematical research. However it has also raised some issues which are more fundamental, issues that are raised in the early pages of the paper and which may warrant consideration by a wider audience.

Hence we have attempted to structure the paper in the manner of a funnel, with the widest concepts and concerns first and the narrowest principles last. We have also adopted what we hope is a fairly light approach to our narrative, for if successful we may yet gain the attention of the wider audience. Our primary hope is that many will read the widest concepts and at best place them into the context of the debate on the future of the profession (or at worst discard them). Our secondary hope is that a few will slide down the funnel and into the dark nether regions of the paper and come to contribute to the serious consideration of the use of complex and chaotic systems in the understanding of risks.

• *The illustration on the front is the "Frothy Basin Attractor"*

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SECTION 1

Themes



The Fractal Fern

“The sciences do not try to explain, they hardly even try to interpret, they mainly make models. By a model is meant a mathematical construct which, with the addition of certain verbal interpretations, describes observed phenomena. The justification of such a mathematical construct is solely and precisely that it is expected to work.”

John Von Neumann

1.1 Themes

This paper, like the title, is somewhat schizophrenic in nature for we have two points that we would like to present.

The first theme is that fractal geometry offers scope for risk analysis and may, in time, lead to actuarial processes based upon it. This is a fairly technical presentation even at this early stage in our research and as such most of the discussion is carried in Sections 3 and 4.

The second theme is that the actuarial profession is not geared up to embrace technical advances efficiently when those advances are rooted in the science of mathematics. This theme is discussed at greatest length in the first two sections.

The two themes are related by the fact that the research into the first concept has uncovered significant advancement in the application of fractal geometry in areas that are closely related to those of actuarial interest by related disciplines. These areas include demography, oncology, epidemiology and complex modelling.

At a time when the profession is looking to take its applied skills into wider areas of risk management, we found it a little disconcerting to discover that the pure mathematics behind those skills was moving. What has been of greater concern is that it has been moving behind the backs of most of the profession for some years now. Is it remotely possible that the development of investment derivatives by an allied discipline is the thin edge of a much larger wedge?

The two themes are expanded further below.

1.1.1 Why might a Fractal be of use to an Actuary?

Fractal geometry burst onto the mathematical scene in the early 1980's following a relatively short gestation period in the preceding 20 years. It brought with it a new glossary of terms and has given a different perspective to a whole range of natural phenomena. Above all it is a strikingly visual form of mathematics and up to a point this can shield the potential for serious application of the relatively simple concepts that lie behind it.

In its purest sense a 'fractal' is a set of points in a space of as many dimensions as you wish to operate in, that have been plotted by the repetitive application of mapping algorithms. The mapping algorithms, however, remain the same for each round of transformation. If those mappings apply scale changes then you end up with a set that either tends to the infinite or to the infinitesimal but one that is 'self similar' at all orders of magnification. There also exist 'natural' fractals which are objects that reveal a certain degree of self similarity on a number of scales and which can be approximated to by the use of the pure or 'mathematical' fractals. In mathematical fractals the parts are copies of the whole transformed; in natural fractals the parts are only reminiscent of the whole.

The geometry is that which applies to this scale relationship and which is, by definition, discontinuous and non-differentiable. Looking at a set of observations, or at a structure, as the end result of the repetitive mapping of a fixed set of algorithms rather than as a set with a 'snap-shot' relationship is what makes fractal geometry different from traditional geometry. This perspective has clearly been facilitated by the power of PCs.

Many risk profiles appear to show the trademarks of natural fractals, particularly if plotted over time.

However, there are only three alternative scenarios for fractals and any other new techniques emerging;

- They may be of no use at all, or at least are of no greater value than simpler measures or other well established tools.
- They will become an additional tool, such as Chi-squared tests, that give an extra piece of information on a set of data and may one day take up a chapter in the statistics text book for student reading.
- They will evolve new analytical processes that may be applied to the relationship combinations that drive long term sensitivities. This will provide the profession with fast, visual, methods of understanding the mechanics that they are managing.

At this juncture you may be tempted to close the paper and escape before you become immersed in a mire of formulae and jargon. If so then we will shortcut the later conclusions for you by saying that in our opinion the application for fractals lies somewhere between the last two scenarios, but it is not debated any further until section 2.

1.1.2 Why might a Fractal be a threat to an Actuary?

The mathematical landscape has appeared relatively static for some time. Einstein shook the world of physics into a new era of discovery, medicine continues to march on and new findings constantly revolutionise chemistry, but good old mathematics has largely bumbled along unencumbered by the arrival of anything too dramatic for many decades.

Speed of application has changed due to ever improving technology, of course, but our children's maths text books are carrying the same subject matter that we studied 30 years ago; albeit with a change in emphasis. Not so physics and chemistry (wasn't a quark a sound that was emitted from a West Country duck?). So the profession upon which much of the mechanics of financial services has been based; the actuarial profession (arguably the one and only commercially based profession with its roots deep in pure mathematics), has been quite stable as a profession with its face firmly towards the commercial world in which it operates.

Commercial operations based on medicine or physics (such as audio-visual and electronic companies) have their technical professions split, with a substantial proportion being based in research processes; so that the industry may become more accurate, faster, cheaper etc etc.

Actuaries have had no need to take this stance. Yet with only a few full-time actuademics (to coin a phrase) the volume of output from that source is huge. If such a small, albeit select, group can push barriers forward from a small research base, what is the industry missing, for such a group cannot have much time left over to explore too many alternatives? This was not a great risk in a relatively static science, of course. Furthermore the industry in which actuaries worked was very stable, as well as being very narrow.

However, the science on which the industry is based (mathematics) is no longer static, nor is our traditional industry (be it insurance, pensions or investment) and the profession is also looking beyond these traditional boundaries towards wider horizons. Commercially, finding a way of doing everything more accurately, faster, cheaper is the big challenge; for the profession to contribute its traditional values effectively in this new and

volatile environment, it will have to embrace these commercial demands.

Commercial developments within our traditional industries and the application of our basic skill sets to non-traditional industries are being well debated and lie beyond the scope of this paper. It is the developments in the world of mathematics that occupy the thoughts of these pages. To consider this the functions of the actuary may arguably be split into two; the application of techniques and the techniques themselves. While mathematics offers some opportunity to widen the scope of the former, the major impact clearly lies with the latter function. This definition of function does not envisage actuaries pushing forward the evolution of mathematics but it implicitly accepts that actuaries should embrace such evolution in the development of their processes and applications.

Fuzzy Logic, Chaos Theory, Fractals, Complexity and Genetic Algorithms are all 'modern' arrivals on the mathematical scene. While interesting in themselves they are not here in their own right. It is their potential to contribute to practical and commercial application that is important. The question is:

**Do these new concepts offer any potential to improve the techniques
(processes) used in actuarial applications ?**

As you may have gathered by now we believe that it is not this question that is important. The most pressing question is different:

**Is the profession in a position to answer the first question, in relation to
any new mathematical concept, quickly ?**

There is some reason to have concern about this question for historically process changes have taken a long time to gain a foothold in actuarial training.

In the late 1970s the actuarial study notes on life office practice had a small paragraph on cash flow projections and their application in product pricing. Yet some offices were already running their control processes on them.

One actuary's anecdote is that when expressing concern to his MD that he didn't have pensions experience and was going to sit his pensions final next, he was advised that if he wanted to pass the pension exam the last place he should be was in a practical pensions environment, picking up all of the modern practices which the examinations didn't reflect.

The Institute and Faculty have put a lot of effort in getting good practice into the training of actuaries recently, but what about the unproven practices? Where do they get tested ?

Should the actuarial profession be up there hunting with the most effective tools, for if not is it possible that another branch of the scientific fraternity will make the connection and do it instead ?

Isn't it worrying that Dr Mandelbrot (he of "fractalicious" mug designs) dedicates space in his theses to showing how a technique already being applied vigorously in particle physics could analyse investment trends?

Fractals may not be a gateway to more efficient processes, but the certain message from this is, as they say in the movie, 'we are not alone' in professional terms. If it isn't fractals then it might be Chaos Theory, Complexity or Neural Networks, and what about Surreal Numbers? Given that this very new branch of mathematics claims to be able to code every type of number (including imaginary numbers) in a combination that is either point up or point down it could revolutionise mathematics in a computerised (ie binary) world.

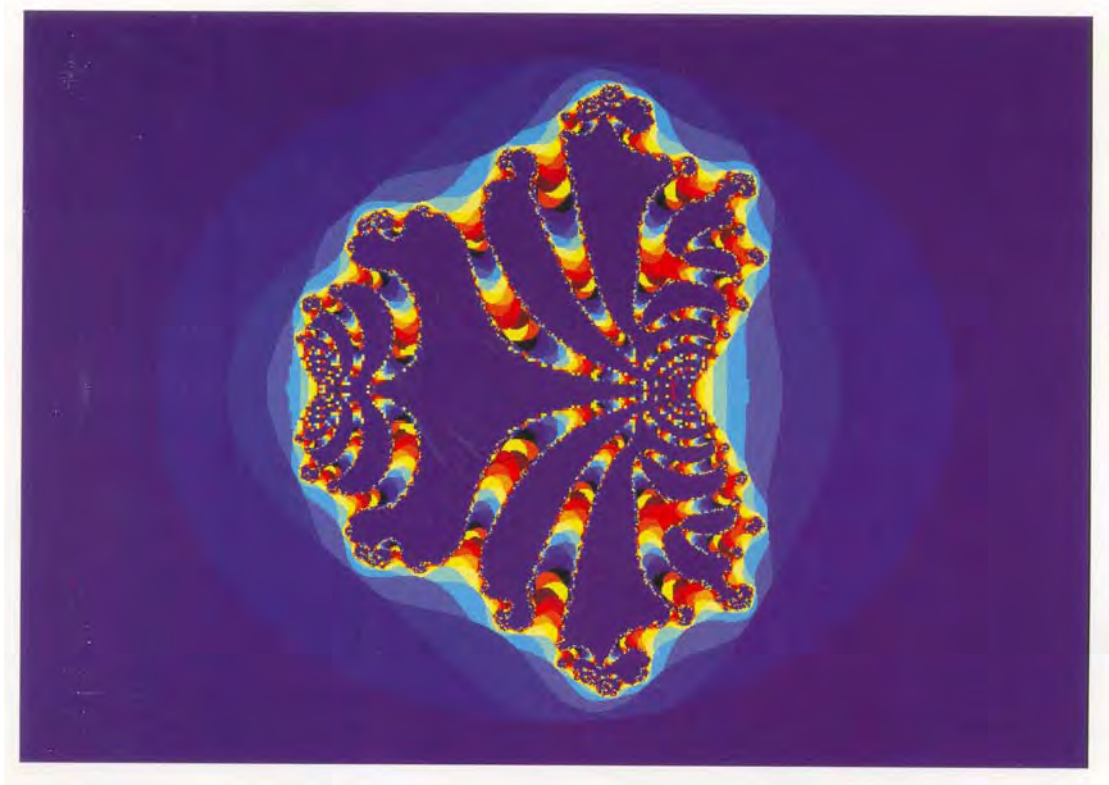
For so long actuaries have played a core role in the practice of life insurance, pensions and investment and latterly general insurance, researching the performance of their company and its assets. How does a profession so heavily rooted into commercial applications find the resource to address effective consideration of any new technique?

Perhaps it should not try to address it in developmental terms, merely act as a watchdog on fiscal prudence in the application of techniques brought forward by others, a role it has arguably taken on in the development of derivatives.

Perhaps the fact that this paper gives no definitive answer, only questions, means that the profession is actually going to have to face up to a need to find a way to escalate the research into our own science as well.

SECTION 2

Complexity, Chaos and Risk



The Spider Fractal

“Clouds are not spheres, mountains are not cones, and lightning does not travel in a straight line.”

B Mandelbrot

2.1 Actuaries and Mathematics - the Premise

No paper entitled 'Fractals' can hope to escape without reference to Benoit Mandelbrot, but his quote, taken at the start of this section, is one of the few references we shall make to him in this wider discussion of the role of actuaries with mathematics in general and modern mathematics in particular. It is an interesting quotation however, not simply because it is rooted in the observations that gave birth to fractal mathematics, but because it touches on the principles of actuarial science as well.

It is an analogy that is worth pursuing, for while clouds are not spheres, mountains are not cones, and lightning does not travel in a straight line, an actuary would base his work on the assumption that this is what they are, with the final shape caused by variations (mostly random) around the norm.

The most obvious example is that of mortality, described often in terms of a formula around which the actual observations might reasonably be random variations, well within acceptable confidence limits.

The most topical example, meanwhile, would be that of stochastic modelling where the results are random variations around a selection of (usually) mathematically 'smooth' distributions, of which more later.

However, this mathematical principle is only one of several underlying most of the application processes used by actuaries in their work. While most definitions of actuaries rightly revolve around the applications of our work, there is, not surprisingly, a commonality of mathematical principles that give rise to the processes adopted for those applications. These principles are the current pretexts upon which actuarial processes are based.

It seems to us to be worth trying to define the current pretexts underlying data analysis. If a set of simple pretexts can be established that give a definition of the function to be played by mathematical processes for actuarial applications, set in terms of principle, it then becomes significantly easier to measure the worth of any new mathematical technique and hence to consider the resources to be applied to the interrogation of its application.

Bear in mind that it is not impossible for a new mathematical concept to change the pretexts (exponents of pure chaos theory may suggest that this is required of the pretext given below), but while we touch on this possibility it is not explored in any depth in this paper.

We believe there are currently five such pretexts for the use of mathematical processes in long-term contingency risk management.

Principle Pretext 1

There is underlying smoothness (homogeneity)

As mentioned above, for many risks it is assumptive that a set of experiences are merely random variations around a common underlying risk pattern. Where such commonality is deemed to exist for a group of risks they are regarded as being 'homogeneous' and it is possible to find, or estimate, the mathematical model that reflects the underlying risk pattern.

For many risks the actuarial profession has established a linear profile, or a series of linear profiles, as the underlying pattern. From these profiles, formulae and processes have been established to allow effective risk management applications to be brought to life. It is not necessary for all of the profiles to be fixed by formulae, however, and decrement tables have long permitted a lack of smoothness to be incorporated. Bear in mind, though, that actuarial text books have always recommended that even where a formula is not ensuring smoothness that the actuary looks for continuity in decrement changes by looking at differences. This is analogous to the cloud being a sphere, a mountain - a cone and lightning a straight line.

Principle Pretext 2

There is random variation around underlying smoothness

In the above scenario the actual differences in experience are assumed to be driven by random variations around the underlying risk profile. In other words the 'experience will differ from that of the past- as expressed by statistics- quite by coincidence, ie without following a definite pattern' (Gerathewohl, 1980).

The actuary is expected to have due regard for the potential risk, not only of adverse random variations in experience but in adverse movements in the underlying risk shape. To achieve this the actuary must err toward conservatism.

It has to be said, of course, that some of the 'randomness' is not really random, it is merely expedient to assume that it is. Some elements of risk are occasionally taken out of this assumed bracket (random) and placed in the grouping that follows (non-random) if the risk is both different enough to no longer warrant being basically homogeneous with the other groups, and that it can be measured as a risk with a different profile.

This concept of randomness has been in existence for some time. However, until recently it has been left to the inherent conservatism of actuaries to defend against adverse random variations, the necessary margin required being largely a matter of subjective judgement. Alternatively (or as well) the use of reinsurers, who have taken the principle a stage further by spreading such variations across wider ranges of risk, have been used in order to permit random variations more opportunity to even themselves out.

It has only been the relatively recent arrival of stochastics that has taken this subject onto a higher and more objective plane. It has been a classic case of new mathematics using higher performance technology and generating new techniques to arrive at a less deterministic approach to ensuring that actuarial pricing and valuation of risk are robust enough to withstand most reasonable spreads of random variation.

Now clouds are fluffy, mountains are rugged and lightning is jagged, all without compromising actuarial principles.



Principle Pretext 3

There can be non-random variation around underlying smoothness (heterogeneity)

If the actual variation is too great as to be explained by random changes, the actuary is required to consider that the underlying risk profile is no longer suitable. The actuary is now faced with heterogeneity.

In recent decades mortality rates have been differentiated by age, sex, and more recently by smoking habits when such heterogeneity was deemed to have gone beyond reasonable variation to the point of posing a threat of selection (by prospective policyholders) against life offices.

In the case of a pension fund there may also be numerous groupings, caused not only by underlying claim differences (eg sex, occupation) but by different benefits (eg ages of retirement) or different types of movement (eg deaths, retirements, job leavers) historically resulting in double or multiple decrement tables.

It is possible that the new risk patterns are related to other risk patterns in the same group, but equally it is not impossible for them to look completely different.

So clouds may yet be domes, mountains may be cubes and lightning may be stairways.

Principle Pretext 4

There is a point of spurious accuracy

The first three pretexts ought to explain all of the risk types within any one group. However, whether processes emerging from them will be used will depend upon the importance of additional accuracy in setting the risk profile. Benjamin and Haycocks (1970) considered the subdivision of mortality in the context of practical homogeneity and summarised the issue very effectively by stating that:

“...in fact one can go on until one is almost forced to the conclusion that each individual should be considered in isolation and group measurement is impossible! Certainly in respect of mortality any population is in practice very heterogeneous. A mortality investigation always raises the question of subdivision. The amount of subdivision must depend on the purpose the investigation is to serve. Thus if a population is mixed and rates are obtained from such a population, these rates can be used only in connection with similarly mixed populations.”

Again, judging where the line is drawn has been largely subjective to date and much guidance has been written concerning where ‘intrinsic roughness’ within observed data (ie non-random) may be ignored in the graduation of a risk. Such consideration is bound to be guided by the amount of data available. However this act remains mathematical if only because that judgement is, at the very least, about which of two mathematical processes to adopt. At best the judgement itself may possibly be assessed mathematically.

A very different tale may yet serve to illustrate what happens, however, if your model is too simple, too assumptive of homogeneity. It is termed the story of the ‘spherical chicken’ (Kaye, 1989) and goes roughly like this:

Fundamentally the spherical chicken syndrome is the extensive study of oversimplified models of real systems. It apparently dates back to late 1960s USA when a physicist was

asked to study the heat generated by a hut full of chickens. The object was to determine how the heating system should react to conditions during the winter months in order to keep the chickens comfortable with the aim of maximising their egg laying. After six months of study the physicist reputedly informed his sponsors that his best endeavours to get a good model established has run aground on the complexity of the surface areas of feathers, legs and wings. 'Not to worry', he comforted them, for he had now developed a computer model that 'works out how much heat is lost from the surface of a spherical chicken.' History records that the outcome arising from the implementation of his recommendations was not a success, but the exact fate of the chickens remains a mystery. The moral of the story is to beware of relying too much on the arguments of 'spurious accuracy' for they will change the nature of the problem. Sometimes you need to insist on the resource to develop the new technology, tools or knowledge, for it may cost less in the long run than the potential mistakes due to the erroneous conclusions from making decisions based on spherical chickens, spherical clouds, conical mountains or linear lightning.

Principle Pretext 5

A time series exists for risk variation

For many risks it is accepted that the risk profile being used will indeed date. Benjamin & Haycocks (1970), again pointed out in relation to mortality analysis that:

"...the actuary traditionally employs the life table as a model but he knows that the past experience from which that table has been derived will never be exactly reproduced in the future. Mortality is itself constantly varying; there are fluctuations about an underlying trend..."

In mortality assumptions used for annuitants, for example, fairly complex processes are adopted to make allowance for a suitable trend of improving mortality into the future.

Allowing for such time series will also require mathematical alteration to the basic risk profiles established by one of the earlier pretexts.

Basically clouds change shape, mountains crumble and lightning moves on.

These are then the five key areas of usage, by actuaries, of mathematics in the investigation of risk. It is against these pretexts that we shall measure the potential employment of some of the modern mathematical processes, notably of fractals. Our conclusions are recorded within this second section and evidenced thereafter.

This section also then considers the principle of ongoing actuarial research into applications and the threats potentially posed to the profession by only considering emerged processes as opposed to emerging processes.

2.2 Actuaries and Mathematical Development - the usage

Having established the conventions for use of mathematical developments it seems worth considering how this manifests itself in the work of actuaries by looking at the trail of involvement on a longitudinal basis. We see this as three broad areas: the application and processes referred to earlier plus underlying mathematics. This section considers these three areas in turn, but in section 2.3 we will concentrate on the latter, and most fundamental, element.

Our reasoning for this approach is that the former two (application and processes) are the almost exclusive domain of actuaries, our fortresses if you will. The method of their development has not only held the profession in good stead for decades, even centuries, but has actually been an asset of incalculable value, involving as it does, practitioners in the interrogation of developmental work. Our concerns lie entirely with the latter area (mathematical bases).

2.2.1 Application

The most visible part of the work is obviously in the application of processes to achieve results. Such applications may be directly or indirectly for 'commercial' application. We take the word commercial in a very loose sense here to mean both for private purposes (notably companies of course) and for public purposes be those of a fiscal nature or of a social nature.

The key areas of application have long been in three main areas: investment, life and pensions, although these have been expanded by the involvement of the profession in non-life risk management in the form of general insurance.

Applications take the form, across all of the areas of:

- Pricing, be that of a product, benefit or service.
- The setting of adequate reserves for future fixed or contingent liabilities, together with appropriate margins.
- Portfolio management, be that of risks or of assets held against risks, aimed at minimising the net exposure to risk.
- The on-going monitoring of the assumptions made in the above areas.

Informally, actuarial skills have been adopted in a wide variety of ancillary areas of analysis for many years, some explicitly involving risk (such as the financial structures of sales operations) and some a long way removed (such as road and rail timetable planning). These areas may yet expand formally, however, given the work of the Future of the Profession Working Party and the Wider Fields Board. This was highlighted most effectively in last year's Presidential Address (Ferguson, 1996) from whence the following plea is taken:

"Let us be, and be seen to be, experts in the practical application of financial modelling and risk analysis in all areas of business. Let us be the preferred qualification for applied statisticians who see benefit in belonging to a professional body."

The one clear identifier of this element of work is that it is used in a practical, often commercial, often multi-discipline environment. Ferguson's quotation stresses that practicality of application and the text from which it was extracted is focused strongly on the application skills of our profession. Clearly the majority of actuaries are employed in this broad area of work.

2.2.2 Processes

However, by definition, applications are based on processes which have been developed by actuaries to provide robust tools and rules that effectively minimise risk.

These processes have usually been applied across numerous areas of application. Decrement tables have been applied to pensions and life areas in the past and cash flow projection techniques have been applied across even wider areas, just as stochastic methods are being applied now. Ultimately the effectiveness of these processes is measured against the success of their application, but also against the fundamental principles of actuarial work now embodied in the Manual of Actuarial Practice. This area is of interest to most actuaries and involves a reasonable amount of academic, as well as practical, development within the profession.

It is embedded within the profession's culture that practitioners are encouraged to contribute to these debates, either by attending papers or by actually writing them; a feature that, while the base science has been relatively stable, has been one of the keystones of the profession's durability and success. With some 60 of 64 Institute Presidents being 'front-line' practitioners, either corporately or in the management of state insurance responsibilities and the remaining four being 'second line' consultants, it is clear to see that pure researchers have to date, played a very limited role in the profession's development at any level.

This said, it is not our purpose to question this particular structure. It has worked well and so long as actuaries continue to stay true to the view that 'every man is a debtor to his profession' there is no reason to believe that the structure will not continue to offer much the most effective manner of process development and interrogation that could be conceived.

2.2.3 Fundamental Mathematics

Behind all of this, however, lie the mathematical bases, be they statistics, numerical analysis, basic algebra or differential calculus.

Until recent years the mathematical landscape has been fairly static. Hence there has been no need to consider how developments may affect the processes or applications. There were none, if you will excuse the sweeping generalisation. These concerns were expressed fully in section 1, but in the context of this section it is worth stressing that this particular domain is not the exclusive domain of actuaries. As new forms of fundamental mathematics appear, therefore, they potentially open up an avenue of access to the applications currently under the stewardship of this profession, should processes be so developed.

It is also true that if the profession is to be successful in taking its skills into wider fields of commercial risk management, it is sure to meet other disciplines entering, or already present in, such fields utilising some of these techniques for modelling purposes. As will be seen in section 3, oncology, the study of cancer, has been investigating chaotic and fractal models for over a decade for example.

As will be seen within the narrow context of fractals, such processes are beginning to emerge in related areas of work. Hence in section 1 we asked two questions:

Do these new concepts offer any potential to improve the techniques (processes) used in actuarial applications ?

A question we aim to give our views on, in specific relation to fractals, but that in general terms begs the ancillary question:

Is the profession in a position to answer the first question, in relation to any new mathematical concept, quickly ?

This could, however, arguably lead to a third question:

Is it the aim of the profession to be involved in the development of such fundamental concepts at all or is it sufficient to aim to interrogate emerging processes (be they from other professional or academic disciplines) and adapt them for application in risk management work?

We believe that these matters need some debate and this section ends with the humble offering of our own views on the subject.



2.3 Actuaries and Fundamental Mathematics - the Options

There is an ever broadening range of developments that could be considered within this section. They have been emerging over the past few decades spurred on by the ever increasing power and, just as important, the accessibility, of computers for use by individuals. In particular some very broad but complicated concepts have been able to be brought to life. An example of this might be the emergence of 'surreal numbers'. Martin Kruskal of Rutgers University addressed the Isaac Newton Institute at Cambridge University in 1995 on this particular subject - a new family of numbers.

The proposition is that surreal numbers can represent, within one numbering system, all natural counting numbers, all negative numbers, all fractions and even irrational numbers. What is more it can represent infinity (and beyond), as well as the infinitesimal (and below). Surreal numbers, just for the record, are represented by arrows, either pointing upwards or pointing downwards (and combinations of both obviously). This simple binary option makes them particularly applicable to computers, whence they can be 'powered up'. At this juncture we shall put this particular topic to bed, but only after pointing out that this numbering system has been pioneered by a team of physicists, albeit that the concept was inspired by the work of mathematician John Conway on game theory.

The point is that technology is not only providing the capacity to explore new mathematical domains, it is, in some branches of science, delivering access to phenomena that themselves ask more questions of mathematics, that are then themselves in need of solution. Telescopes and microscopes and related technology are, in particular, driving astrophysicists and particle physicists respectively in to the search for more accurate and more flexible mathematical bases for their work.

The most advanced of the 'new' areas (although its roots go back over a century to the work of Henri Poincaré) is that of chaos theory, and its sister 'on the edge', complexity. Hence this section will give an overview of chaos theory, complexity and finally fractals in order to consider whether they hold any potential to meet the conventions laid out in section 2.1.

2.3.1 Chaos

"Complete disorder; utter confusion" is how The New Collins Concise English Dictionary defines chaos, a fairly succinct definition that would fit most people's understanding of the term. Of course, disorder is not something that lends itself to mathematics by definition, yet the definition does not in itself say that chaos is meant to be random by nature. Indeed the Longman Dictionary gives one definition of chaos as "the irregular and unpredictable behaviour of a dynamic system which nonetheless has an underlying order" and it is this definition that gives rise to the mathematics of Chaos Theory.

Chaos Theory works with sequences of events that contain surprise and unpredictability and while chance is certainly one source of surprise, it is not an essential ingredient. Such unpredictability, ie failure to be able to predict or forecast results accurately, can actually arise even where the events of the future are exactly determined by the events of the past and there is no chance or random influence upon those results at all. Peak and Frame (1994) refer to this as 'Deterministic Chaos', an interesting phrase given the connotations for the actuarial profession.



The elements that actuaries observe in many of the rates and indices that they measure, referred to earlier, usually show an underlying trend and variation around that trend. This is true of the stock market, of mortality, of sickness rates and indeed it is the existence of this phenomenon that invites actuarial science to be applied. If you regard the trend as a 'signal' and the variation as 'noise' however, you can see that the same phenomenon exists in electronics and in physics in general. Of course some 'signals' carry very little 'noise' and are thus very predictable, others are almost all 'noise'. Here then we can link back to our statements of the use of mathematics in actuarial science. Actuaries are able to apply their processes where there is a strong 'signal' (Principle Pretext 1) and where the 'noise' is safely assumed to be random (Principle Pretext 2).

Chaos assumes that much of this noise is not random at all. The compelling argument is that we don't know all of the rules that govern every influence on the behaviour of the observed systems and hence we are unable to predict events accurately, simply because the model is based on incomplete ideas that thus lack in predictive power (the 'spherical chicken' again). More importantly, if all of the rules really are deterministic there is still room for the intrusion of disarray when many variables are involved, whence the 'randomness' of results is simply the fact that the complexity of the underlying system has succeeded in overwhelming our ability to model the results.

As if to underline yet again the cross discipline interest in these areas it was a meteorologist attempting to model convection in the atmosphere who put some real tangible meaning into this whole process, one Edward Lorenz. His complex model took warm air up and cold air down and predicted, in otherwise stable conditions, very attractive ordered cloud patterns. Applying slight change gave very different results, but none more surprising than when he simply reran his computer projection with ostensibly the same data to check his result, except that he rounded the initial conditions to three decimal places rather than his original six; the final result was "pure chaos".

The implication is that no physical, biological or social state is infinitely precise; there is always some uncertainty or assumption. If the underlying dynamics are what are termed as being in a chaotic state (in the sense referred to above, ie deterministic) then any initial uncertainty will grow so large that long-range prediction becomes impossible. Lorenz referred to this as the 'Butterfly Effect', for in meteorological terms even the most gentle flapping of a butterfly's wings can ultimately give rise to a wide divergence of meteorological conditions.

Hence chaos is irregular output from a deterministic source, whence the future results of a chaotic system are completely governed by its past. It is not chance nor is it randomness, but it is confounding, for in chaotic systems measurement discrepancies compound. However, if you understand the system you can still predict your limits within which results will sit effectively.

The 'Butterfly Effect' could imply that all chaotic systems can produce huge variation from tiny inaccuracies at outset, but if you will pardon the continuing use of the analogy, not all Caribbean hurricanes are caused by butterflies in the Amazonian rainforests. The actual range of effect will depend upon how near the underlying system is to a state of equilibrium (or stability). The more complex a system is, and/or the more variable any of its components are and the less stable it will be, thus the more effect any small inaccuracy will ultimately have.

Up to a point, this is all intuitive stuff. Knowing when a measurement has to be treated with greater accuracy is part and parcel of an actuary's responsibility and requires him or her to have some knowledge of the nature of the risks involved in the system he or she is working with. For many risks, particularly the 'binary' risks such as mortality

(either you are dead or you are not), the basic system is simple although the causes of the claim may not be, of course. However, actuaries are moving into areas where risks become more complex (household insurance can give rise to claims from many sources) and far more susceptible to these chaotic considerations. It is interesting that in general insurance the vast majority of risks are covered for a year at a time; this limits the exposure to the underlying system throwing up a fundamental shift in risk. In chaotic terms the industry has accepted that the premium rating methods are not accurate enough when it comes to predicting longer term time series because of the lack of 'equilibrium' in the underlying risks; whence there are real risks of a wide divergence of results if you allow your initial inaccurate estimates to compound.

As in most things in life, there is always more than one way to overcome any perceived obstacles. Digging deeper and deeper into the working of observed systems in order to model them accurately is only one solution to managing risk systems. The insurance industry, often without the technical guidance of actuaries, has long been adept in finding ways to manage such matters with practical solutions. However, as a large number of Lloyds syndicate members might testify, if those processes go wrong it can be in a big way.

2.3.2 Complexity

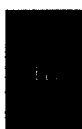
It is debatable as to whether chaos forms part of complexity or complexity forms part of chaos, or indeed whether they are really the same thing. Perhaps the biggest differentiator is only a practical one; a chaotic system is unpredictable, a complex one is predictable. Thus all complex systems are potentially chaotic if inaccuracy is introduced and all chaotic systems are potentially complex if it were possible to remove the inaccuracies.

Complexity is arguably multi-dimensional thinking, but it sits at the very edge of chaos. The link may be illustrated by work published in 1995 by mathematicians from the University of Cincinnati: Ken Myer, Christopher McCord and Quidong Wang who used 18-dimensional geometry to disprove a 70-year-old theory about the dynamics of three bodies moving freely in space, influenced only by each other's gravity. Previous 'vagaries' have now been fixed by the use of multi-dimensional mathematics, chaos that has turned into complexity.

This is true in many other areas of physics. The universe appears to have three visible dimensions, four if you add time, but many observed phenomena are only modelled accurately if you use in excess of 10 dimensions. Without this you gain a model that is a little inaccurate and thus chaotic by nature.

In a practical sense this is a topic of conversation whose major use, we would propose, is to clear a crowded bar at Christmas. However, actuaries are used to working with n -dimensions in the formulae that have long driven the assumptions underlying risk curves.

In risk terms, complexity is the definitive Eldorado that models a risk pattern with absolute predictability. We candidly suggest that it is practically impossible to achieve and will remain so for the foreseeable future. The risks that we model, as well as the risks that we aspire to model, are too complex in terms of the factors that influence them at a detailed level. However they all fit the nature of risks to which the processes may apply. In other words they may be viewed as an underlying trend. Interestingly enough if this were not so there would be no need for an actuarial profession at all.



2.3.3 Derivatives of Chaos and Complexity

Hence we are really looking to see whether the thrust of Chaos Theory, and its way of looking at such complex systems, will yield processes that will manage risk patterns more effectively than those processes already in play. To this end a range of techniques has indeed emerged from complexity (as the ideal mathematical model on the edge of chaos) and a few are listed below, before we concentrate on the key technique, that of fractals.

'Strange Attractors' emerge from certain chaotic systems where the underlying system, even with inaccuracies present, will tend to work within known boundaries. This area is a link back to that of statistical distributions and the world of histograms. Histograms start with the observations and yield statistical mathematics from the results. Strange Attractors provide the mathematical basis that yields the observations from fundamentally chaotic systems.

'Genetic Algorithms' are an interesting side track emerging from the computer sciences. They are 'clever' algorithms that work within defined rules while having the ability to adjust their route until they 'learn' the quickest way to get to the end result. If a complex model is too complex you can attack it with a genetic algorithm that starts with your best guess and beavers away until it improves it.

'Neural Networks' are related to genetic algorithms in that they perform a range of cognitive feats which are developed by discovering their own rules and subsequently applying them. They are processing devices (algorithms or actual hardware) that are loosely modelled after the neuronal structure of the mammalian cerebral cortex but on a much smaller scale.

'Quantum Chaos' is the collision between quantum theory and chaos theory. Quantum mechanics describes the behaviour of matter on an atomic scale and changed the understanding of the actions of matter. By rights, chaos should not exist in quantum systems, the laws of quantum mechanics actually forbid it. Classically chaotic systems, such as the swing of a pendulum (when measured in fine detail) and even Brownian motion, are ultimately made up of atoms which are subject to the laws of quantum mechanics. The interest here lies in the fact that somewhere between pure (simple?) quantum mechanics and chaotic systems there must be a link. The research into quantum chaos may yield a new means of turning what is perceived as chaos into complexity. Maybe not.

You will notice from this that the mathematics involved in this area is regularly throwing up conclusions that had been reached long before from observation alone. It is tempting to say that this complex mathematics is only generating the same processes as already existed. However, we find this very heart-warming, for it gives added confidence in the robustness of those processes. It also gives credibility to the mathematics of chaos and complexity, however, whence any new conclusions cannot easily be discarded. Which brings us neatly round to the subject of fractals.

It is worth mentioning **'Extreme Value Theory'** (EVT) before leaving this section, if only because it too is attracting much interest, appears to be a new concept and looks a candidate for having its roots in chaos. In fact EVT is only related to chaos and complexity by the fact that it is founded on the complete antithesis of the theory behind them. At the heart of EVT is the standard statistical idea that the frequency of random events follows a mathematical rule which leads to, or is, a distribution. EVT says that extreme values also follow their own special family of curves and that it is possible to deduce the shape of these curves from basic probability theory. For the record one proposed benefit of this is that "It can help actuaries predict the likelihood of events that are incredibly rare, but so devastating that they would threaten the survival of their companies" (Matthews, 1996).

2.3.4 Fractals

We have begun each section with an illustration of a fractal.

Mandelbrot (1982) states that: “The most useful fractals involve chance and both their regularities and their irregularities are statistical. Also the shapes tend to be scaling, implying that the degree of their irregularity and/or fragmentation is identical at all scales.”

From this point on we shall ignore the philosophical debate at the heart of chaos theory and complexity as to whether or not, with enough knowledge, everything is predictable. For all practical purposes we shall assume, with due deference to Einstein, that God does indeed play dice. This assumption fits very well with fractals and is embodied in Mandelbrot’s definition given above. This is a critical point and will be elaborated upon later.

The history and the basic mathematics behind fractal geometry is given in Section 3. Its potential to contribute to processes, even directly to applications, is also explored more fully in Section 3 after a generalised introduction in Section 2.4. The purpose here is to relate fractals to chaos and complexity and highlight their particular characteristics that make them of potential interest.

Fractal shapes, of any number of dimensions or form, are generated by applying simple algorithms repetitively. It is possible in theory, but not absolutely necessary in practice, to repeat this through to infinity, to derive that ‘shape’. This resultant shape will thus be similar on whatever scale you are observing it at. Looking at this process longitudinally, rather than simply as a result, has produced a new form of geometry, termed fractal geometry. Those that use it distinguish it from traditional geometry by terming the latter as ‘Euclidean’ geometry for fairly obvious reasons and we shall adopt the same stance from here on.

Because fractal shapes replicate to infinity (or to the infinitesimal) they cannot be properly defined by Euclidean geometry, this can only happen if a boundary exists which cannot be exceeded, whence the definition is not completely accurate. The potential problem is amply demonstrated by consideration of an island, such as Great Britain (explored in more detail later), where the coastline is actually infinite in length, but the area within it has a finite limit to its size. You can define the latter with Euclidean geometry to sufficient accuracy, but not the former. Put simply, fractal geometry will define the coast, and thus the area within it to sufficient accuracy to replicate the shape.

As the name implies, fractals are usually discontinuous, or fractured, distributions. Each point or area of the shape is uniquely defined in terms of what has gone before (by the algorithm) and is not necessarily a continuous development thereof. Hence fractals are correctly a subset of chaos. Equally, because the fractal is accurately described by the algorithm it could be taken as being a subset of complexity, until you realise that you can never actually reach that final definition in Euclidean terms and perhaps it is a pure chaotic system, even though fully defined.

While in this headache creating mode, we would point out that fractals do not have some of the other usual Euclidean properties, such as tangents. A visual representation of a fractal may appear to have a line against which you can set a tangent, but look closer and the line changes shape and the tangent has to move, look closer still and its moved again, and so on.

So here is a mathematical geometry based on models that are discontinuous in nature but that tend to the infinitesimal. The mathematical processes emerging from this geometry offer a new means of measurement that begs investigation. For a profession involved in risks that are discontinuous and moving over time, but that manages those risks by grouping the individual observations, it demands investigation.

2.4 Actuaries and Fractals

The fundamental question is whether fractals offer the scope to be used in any of the Principle Pretexts referred to in Section 2.1. Therefore we shall consider fractals in specific relation to the five pretexts stated there. There is a second consideration that arises from this debate, which is whether fractals could give rise to new Principle Pretexts in risk evaluation. This topic is touched upon in Section 3, but its investigation is outside the scope of this paper.

Principle Pretext 1

There is underlying smoothness (homogeneity)

This is assumptive in a fractal so long as this pretext is assumed to read ‘There is an underlying function’. In actuarial terms, however, most functions used in this context are continuous (ie smooth) and while a fractal is continuous in the sense that the underlying algorithm continuously repeats itself it is usually discontinuous when relating one part of the fractal to another. Implicitly this means that most fractals cannot be differentiated; yet continuity is implicit within the development of most standard actuarial tools.

Homogeneity is certainly in the nature of fractals, indeed there is repetitive homogeneity present by definition. A fractal shape is determined by a formula and is exactly repeatable by using the same formula. Thus it is an homogeneous distribution on a line/on a plane/in space. If that distribution is invariant under displacement and is also invariant under a change of scale the fractal is termed ‘self similar’.

The practical question, though, is whether a fractal shape ‘fits’ a risk pattern effectively, in particular whether it would fit a risk pattern more accurately than existing models would. self similar fractals are very good at estimating and thus modelling otherwise apparently random shapes and distributions. It is possible to get a better ‘fit’ in numerous applications using a fractal distribution rather than using a traditional Euclidean distribution. Whether this falls foul of Principle Pretext 4 is considered below, for, of course it is possible to replicate any finite set of observations exactly with a Euclidean formula.

Principle Pretext 2

There is random variation around underlying smoothness

This is an area that can cause some confusion to those with a passing knowledge of fractals. This is because the best known fractals, the prettiest if you like, are exactly self similar. Fractal geometry does not depend on the assumption that there is no underlying randomness, indeed it is able to sit comfortably with statistical tools that measure randomness around an underlying distribution.

At this juncture we can’t avoid using a Mandelbrot (1980) quotation, for as the prime promoter of this geometry his own emphasis of this point is fairly crucial:

“..the reader may have formed the impression that the notion of fractal is wedded to self similarity. Such is emphatically not the case, but fractal geometry must begin by dealing with the fractal counterparts of straight lines.....call them ‘linear fractals’.”

In other words self similar fractals are the simplest theoretical form, but the 'real' complex fractal forms are not exactly self similar, the parts are only reminiscent of the whole. The central characteristic of a natural fractal is that the object offers (at least approximate) invariance under magnification.

Mandelbrot offers the following definition:

"To be valid, the claim that any given natural phenomenon is fractal must be accompanied by the description of a specific fractal set, to serve as a first approximation model, or at least as a mental picture."

We are therefore talking about a distribution form that merely replaces the core distributions used to meet Principle Pretext 1. The assumptions with regard to Principle Pretext 2 remain unaltered.

Principle Pretext 3

There can be non-random variation around underlying smoothness (heterogeneity)

Following the same argument as used in Principle Pretext 2, it can be seen that if one fractal distribution were to model effectively a section of the data being considered, but were to leave 'noise' around that distribution that was clearly not random, there would be cause to believe that the data was heterogeneous.

Interestingly enough, however, it is quite feasible to generate one fractal algorithm that in itself develops into distinct 'heterogeneous' groupings. It is possible that some heterogeneous groupings in a Euclidean analysis could become homogeneous in a fractal analysis. Happily this is beyond the bounds of this section. Suffice to say that fractals accept this pretext also.

Principle Pretext 4

There is a point of spurious accuracy

It is at this juncture that the whole consideration of this specific field of mathematics comes to a head. It is possible that the whole consideration of fractal geometry fits beyond the point of spurious accuracy for practical actuarial applications. It is quite clear to us that the emerging work has to be tested rigorously against this pretext. It should be quite clear that this has yet to be done thoroughly and that it may yet fail.

At this point in time it appears that fractals may fail this pretext for a number of areas of risk management, in particular for simple or binary risks such as mortality. There are some complex multi line risks, or extreme risks such as catastrophes, however, where fractal distributions offer the chance to plot distributions more accurately or more simply than traditional methods.

In terms of pure analysis, of understanding the drivers of individual risks, fractals do offer a new perspective which gives, or looks likely to give, additional information on the nature of the risks and their interplay. For the purposes of most actuaries involved in practical applications this may not yield anything fundamental, but for professional risk carriers and underwriters it becomes another window onto the nature of those risks.



Principle Pretext 5

A time series exists for risk variation

Last but not least is the pretext offering the greatest scope of all. It is very clear that while risk patterns do indeed vary over time, their intrinsic structure usually remains unaltered, ie it is invariant and implicitly fulfils one of the requirements for the definition of a natural fractal to be fulfilled.

In this area fractals naturally lend themselves to the prediction of developing risk profiles.

Thus fractals appear to be able to fulfil all of the pretexts required and to offer an alternative distribution form to those already used. It therefore means that they could, potentially, be used within existing processes and certainly for existing applications, subject to the provisos of Principle Pretext 4. For example, there is no reason why a fractal distribution could not effectively underpin part or even all of a stochastic model.

A second possibility is that fractals will lead to new processes. It is too early to speak of this with any authority. However, it would be surprising if they did not do so in time if they are able to work as models for the underlying distribution.

The third possibility is that they actually change the applications. One is tempted to say that if this were to be the case then the whole actuarial profession would be redefined as a result. From where we sit at present this seems highly unlikely, for there is nothing to suggest that fractal geometry is anything more than, at best, a new means of measuring distributions. What is more likely, and is thus addressed in the next section, is that expertise in fractals and other forms of chaos based processes will permit different disciplines to approach risk analysis and offer competition to the actuarial profession.

2.5 Summary

We would like to summarise this second section of the paper in two parts in relation to the mathematics and the profession.

2.5.1 Fractals

Fractal geometry appears to us to be the most promising area of the new forms of mathematics for application in the actuarial domain. Of all of the complexity and chaotic derivatives, it has the most proven practical application and, in theory, it lends itself to the structure of the problems that this profession has to deal with. Notably its distributions open up a new range of modellable patterns to the practising actuary and its very processes lend themselves to the projection of risk rate developments over time.

We believe that their greatest application looks likely to be in areas of complex risk patterns, but that the additional accuracy, if any, in some of the simpler risk patterns is unlikely to warrant the additional complexity of process involved.

Section 3 takes this whole debate much further and displays the evidence to support this conclusion. It is not, however, the core conclusion arising from our work.

2.5.2 Actuaries

This is the first paper to be published to the UK profession on this subject. It is most certainly not a definitive work. It is also not the journey's end, more of a staging post.

We are aware, however, that in some European Societies (for example in Germany) and in North America (where in the University of Austin, Texas, for example there is an insurance unit investigating chaotic systems in relation to insurance risks) work on these techniques has been underway for some time.

We are also very well aware, as will be expanded upon in Section 3, that fractals in particular are used in a fairly advanced state by mathematical cousins of ours based in disciplines with more than a tenuous link to our field of expertise.

We have, not surprisingly, found some other pockets of UK actuarial expertise in some of the techniques listed earlier as we have researched some of the matters in Section 3 of this paper. We also sincerely hope that others will make themselves known after the paper is published. However, there is no focal point for this sort of research in the UK.

It is fairly easy to surmise why this is so. As stated at the start of the paper, actuaries train in a commercial environment. Mathematical theory is then frozen for most actuaries while they learn the processes which they will have to apply. Equally those processes were established several years earlier by other actuaries, whose mathematical base had also been frozen some years before that.

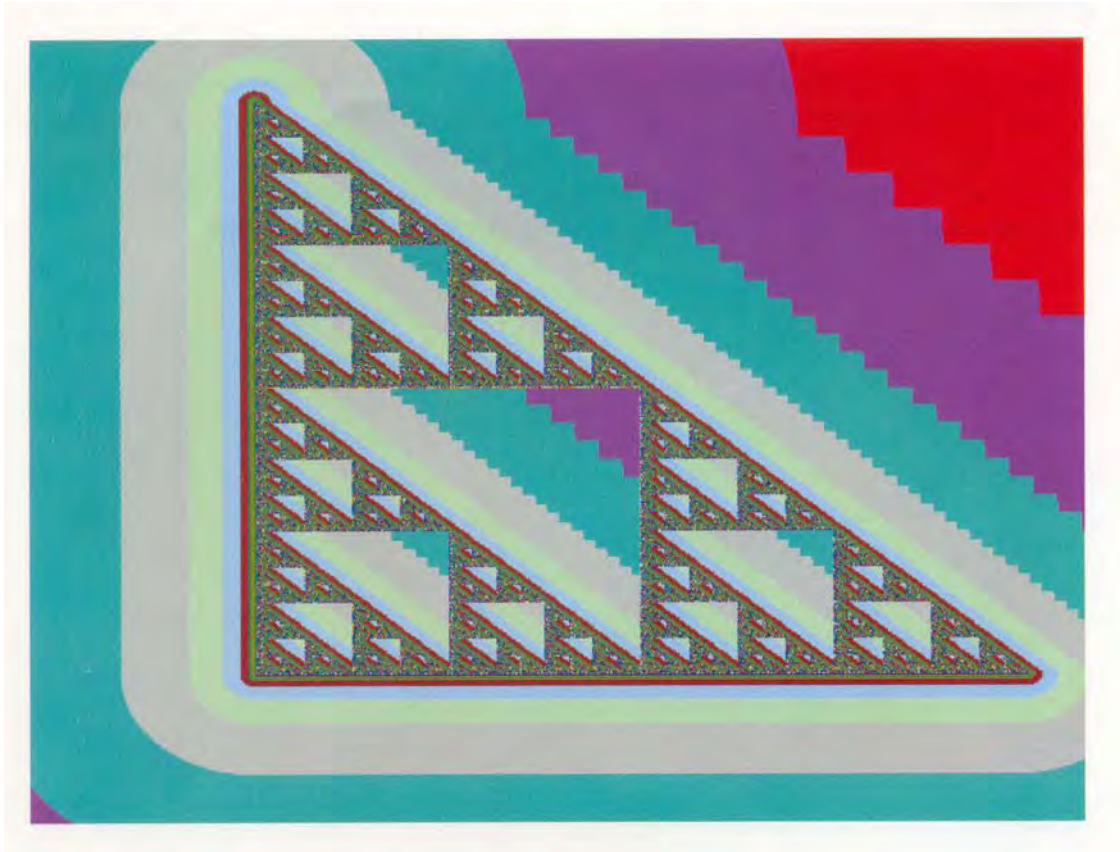
As was expounded in Section 2.1, this organisation has worked very well in maintaining standards of application and in developing processes which are brought for interrogation before the profession, the majority of whom are practitioners (directly or indirectly). This method does not lend itself to robust interrogation of new forms of mathematics, however. They are likely to arrive late, with limited peer group interrogation. Just like this one.

Not surprisingly, the British actuarial profession has one of the highest (if not the highest) standings amongst its international brethren for its practical professionalism and arguably has the same standing above that of professions in neighbouring disciplines. If the mathematical bases move, however, then at the very best, avenues of professional expansion into wider fields will be closed by the emergence of modelling processes from other disciplines. At the very worst those disciplines would offer alternatives to this profession.

It is our view that the profession has to establish vehicles to research the development of mathematical bases for risk analysis, which are rooted in the universities, to gain access to current mathematical research. This must be intertwined with the practical world of commercial application so that mathematical development can have the same rigorous interrogation by commercial actuaries that is now applied to process and application developments. We return to this theme at the end of Section 3.

SECTION 3

Fractals



Sierpinski's Triangle

“Though differentiable functions are the simplest and the easiest to deal with, they are exceptional. Using geometrical language, curves that have no tangents are the rule.”

Jean Perrin 1906

3.1 The History of Fractals

3.1.1 Word Origin

'Fractal' is from the Latin adjective *fractus*. The corresponding Latin verb *frengere* means 'to break: to create irregular fragments'. *Fractus* means both "fragmented" and "irregular". Interestingly enough the word *algebra* is partly constructed from the Arabic word *jabara* which means 'to bind together', the exact opposite of *fractus*.

3.1.2 Early Fractals

"Behold there come seven years of great plenty throughout all the land of Egypt: And there shall arise after them seven years of famine; and all the plenty shall be forgotten in the land of Egypt..." Genesis 41

Arguably knowledge of fractals has been around for a long time. Its hardly surprising that a big discovery was made in Egypt. But we have to move forward from Joseph a few millennia to the twentieth century. H E Hurst was a British hydrologist working on the Nile. The flow of the river depends on rainfall, but that arrives in annual cycles due to the melting snows in the African uplands. His problem was the regulation of the flow of the river to maximise irrigation. What was quite clear to him was that these annual flows did not follow a random walk. Since holding up water by dams is expensive and releasing water too soon may lead to a drought, the model is important. Hurst had a long time series as data - from 622AD to 1469AD.

Hurst took the view that for short periods the Nile flows followed a pure random walk. But then something would happen and this situation would change to another model. The change would be sudden and is called the *joker effect*. This is because in the pre-computer days Hurst used a pack of cards to demonstrate the effects. If the joker appeared the pack was re-shuffled. The results of many reservoirs filling and emptying have profound effects on what might appear a random process. The system would have "memory", changes would be sudden and unpredictable. The more the memory the higher the 'H coefficient'. Mandelbrot named H, the Hurst exponent, in honour of its discoverer. The flow of the River Nile is persistent; the Hurst exponent comes out at 0.91. The river's flow is not very rugged; that is if the flow is high in a year then the flow is likely to be high the next year but by an unpredictable amount.

To a traditionalist the phases could be seen as cycles and an autoregressive model could be attempted. But there is no reason to believe that in the time frames there was a natural cycle. Like weather forecasting, flows can be predicted in the short term but not in the long term.

Hurst published his work in the 1950s before fractals had been developed as a geometric technique. His method using R/s statistics, does not depend on the process being normally distributed.

3.1.3 Mathematical Origin

Fractal geometry was conceived in 1975 and introduced formally by Benoit Mandelbrot in 1977, in a book entitled "Fractals: Form, Chance and Dimensions". The author had been promoting many of the geometric concepts for over 10 years prior to this book, notably his research into 'fractured' distributions, such as the shape of coastlines.

Many of the issues and anomalies thrown up or addressed by Mandelbrot, however, were not new to the mathematical world, but this was the first time that they had been pulled together into one comprehensive theoretical basis.

Louis Fry Richardson had written copious notes on the problems with the measurement of coastlines in the 1920's. Jean Perrin (1906, 1909, and 1913) had published on discontinuous materials and problems in plotting Brownian motion to earn himself the Nobel Prize and push forward the development of probability theory.

Mathematicians had always been aware of the two main features of fractals; irregularity and self similarity, but had never dealt with either effectively. One of the first attempts to approach irregularity was made in 1877 by Cantor, followed by Peano in 1890, and self similarity was considered in the early 18th Century by Leibniz. The struggle to deal with these notions using Euclidean geometry and hence topology (with integer dimensions) was usually fruitless, however. As a result, the need to find a solution was often dismissed as well. Notable French mathematician, Lebesgue (circa 1900), referred to "notions that are new, to be sure, but of which no use can be made after they have been defined" and the whole area was effectively buried in terms of mainstream mathematics for the bulk of the century.

It would certainly be very easy to pick up Lebesgue's quotation in terms of actuarial applications for this field of development today. Whether it is correct is another matter.

3.1.4 The Mandelbrot Set

For the uninitiated, the Mandelbrot Set (see the image at the top of each page) is one of the best known fractal images. It is worth mentioning in isolation for it is likely to be encountered at some point. You can buy the T-Shirt, drink from the mug, or watch a very pretty screen saver based on it. It appears on books, magazines and even some LP covers, it is the nearest thing in mathematics to a pop cult. In many ways the Mandelbrot Set is the prototype fractal, for it is remarkably easy to generate, but is highly complex in structure and the study of its regular irregularities has gone a long way to generate this new form of mathematics. Ironically the Mandelbrot Set has little to do with natural fractals, it is purely a philosophical vehicle that forces questions to be asked about what is simple, what is complex and what is chaotic.

For the record, the Mandelbrot set is a two-dimensional mapping system such that;

$$\begin{aligned}x_{n+1} &= x_n^2 - y_n^2 + a \\ y_{n+1} &= 2.x_n.y_n + b\end{aligned}$$

Shading of the point (a, b) depends on whether either x or y tend to infinity or becomes stationary, as n tends to infinity.

3.1.5 Recent Contributions

Practical developments are covered in a little more depth in Section 3.4, but theoretical movement has been somewhat limited in the past decade other than in the work of applying the geometry to natural phenomena. Kaye (1989) and his team at Laurentian University in particular clearly made a marked contribution to the particle applications of fractals which yielded an expansion of the geometry, but one is drawn to the conclusion that this field is not so much an invention as a discovery, for after some 20 years of intense interrogation by fascinated mathematicians and physicists around the world, the basic geometry promoted by Mandelbrot remains completely intact.

3.2 The Mathematics of Fractals

3.2.1 Introduction

Setting the scope of this section is particularly difficult. For some readers it will verge on the banal: for those who have ignored geometry since school it may be more testing. Space restricts us to continue the funnel analogy outlined in the preface, albeit on a different scale.

For those requiring more detail the bible is Mandelbrot's "The Fractal Geometry of Nature". We have also borrowed heavily from Hastings and Sugihara (1993) in this section.

3.2.2 A Fractal

A fractal is a scale invariant geometric object. By scale invariant we mean self similar or self affine. As such this definition includes most shapes mathematicians deal with: points, straight lines, squares, circles etc. Mandelbrot says that to qualify as a fractal then the object must also have a fractal dimension that exceeds its topological dimension; these objects can be strange. So there are a few concepts to explore.

We define self similarity, then we look at examples of regular fractal objects. This is done by way of some exercises to get across some key concepts. We next look at dimension and then the area of most obvious potential for actuarial applications - the random walk. Finally we give the minimum you need to know about the stable paretian distribution, a distribution used in many fractal processes. No room for proofs, but hopefully enough to get strangers to the subject to the dark nether regions of the paper.

3.2.3 Self Similarity

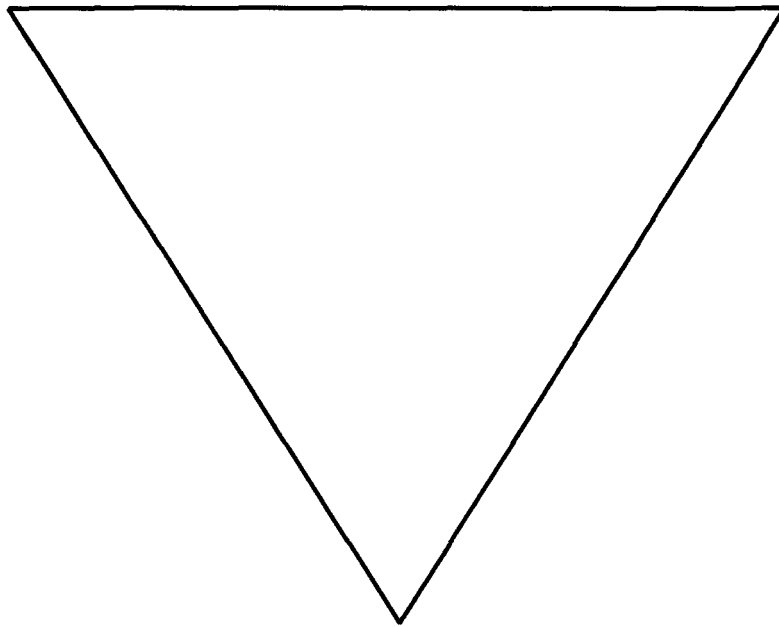
A geometric object is called self similar if it may be written as a union of rescaled copies of itself, with the rescaling uniform in all directions. A geometric object is called self affine if it may be written as a union of rescaled copies of itself, where the rescaling may be dependent on direction. Regular fractals display exact self similarity. Random fractals display a weaker, statistical version of self similarity or more generally self affinity (Hastings and Sugihara). In this paper most examples are self affine, but we have stuck with the term self similar for both types.

3.2.4 Games and Puzzles

This section demonstrates some of the features of regular fractals, before moving onto random fractals. The latter are more relevant to actuarial science.

The first two examples start with a simple triangle. The first shows how complex ideas can come out of simple rules. The second shows that deterministic and fractal processes are not discrete sets. The third example shows how a strange object can come from a very simple equation. In the appendix we set out some programs written in qbasic that you may prefer to use instead of pencil, compass, ruler and paper. So let's start with an equilateral triangle:





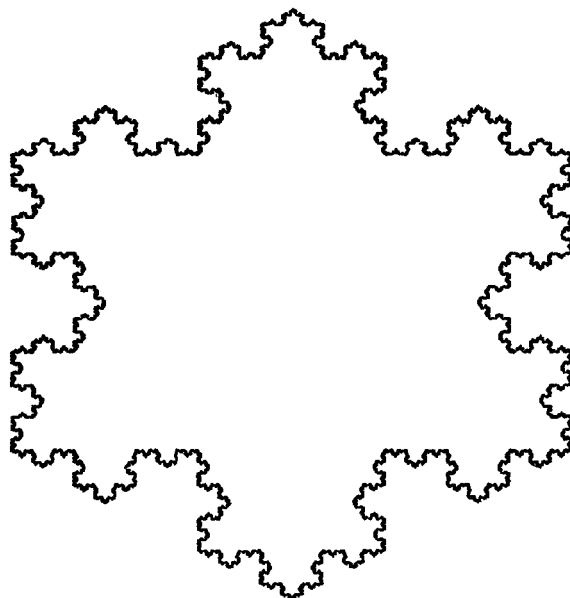
Example 1

Koch's snowflake

In the first example the aim is to draw a snowflake. This concept has come to be known as the Koch Snowflake. The process is as follows:

1. Go to one side of the triangle.
2. Divide the line into three equal lengths.
3. Taking the two mid points as a base construct another equilateral triangle outwards.
4. Move clockwise round the original shape to the next line.
5. When you arrive at the starting point you decide whether to go back to 2 or to go to 6.
6. Admire your object.

The result looks like this:



A variation on this model, and probably a more interesting one, is to throw a coin at the end of step 2 and use the result to decide whether the new triangle faces inward or outwards. The point of this exercise is to demonstrate how self similarity properties can represent complex natural phenomena. Try describing a snowflake using Euclidean geometry!

The simple Koch snowflake has infinite circumference and finite area. On each circuit after the first, the perimeter grows by $4/3$ while the snowflake never passes outside a circle drawn through the original three points. The snowflake demonstrates how we can get a handle on an object with infinite length, and this may prove useful when looking at investment models.

Example 2

Sierpinski's Triangle

Sierpinski's triangle is another complex idea, where the instructions are the most simple way to describe the creation. Start by outlining a new triangle, noting that none of the triangle is coloured yet.

1. Where you have a triangle that is not coloured, draw a triangle with points at the centre of each side of the old triangle.
2. Colour in the new smaller triangle.
3. Go to instruction 1 until you get the drift of the conception.

The result is the frontispiece for this section, albeit not so artistically enhanced. The next game starts yet again with a triangle and this time you need a die as well.

1. Label the points A, B, C.
2. Go to a point anywhere in the triangle and mark it with the pencil.
3. Throw the die.
4. If the die shows 1 or 2 proceed to a point halfway between where you are and point A.
If the die shows 3 or 4 proceed to a point halfway between where you are and point B.
If the die shows 5 or 6 proceed to a point halfway between where you are and point C.
5. Mark the new point and go to 3.

You may prefer to use the program in the appendix. This game is a stochastic process and is an example of an iterative function system. The results are surprising: firstly the points are not spread at random over the triangle, secondly the result is identical to the deterministic approach in the previous game.

This idea of using a random number generator to decide the next plot is very powerful. With a small number of rules the computer will plot convincing looking trees, ferns (see the illustration to section 1) or leaves. It will also plot more standard drawings such as a circle, but Euclidean methods can plot these more easily.



Example 3

The Henon Attractor

Not all attractors are easy to describe. This example developed by Michel Henon comes from astronomy. The system is derived by stretching and shrinking an oval. The equations are:

$$x(t+1) = y(t) + 1 - 1.4x(t)^2$$

$$y(t+1) = 0.3x(t)$$

This maps onto the (x,y) plane as shown on the frontispiece for section 4.

We also include the computer program for this in the appendix. This particular example is of some historic interest as it was one of the first “strange attractors” to be demonstrated. One feature is that if you plot $x(t)$ against $x(t+1)$ you get a similar (but rescaled) object.

The standard method for identifying “strange attractors” is by using this idea. If all $x(t)$ that are close together move to a region close to a point $x(t+1)$, then an attractor is at work. This technique can be extended into n-dimensions by considering a similar consecutive series $\{x(t-n), x(t-n+1), \dots, x(t-1)\}$ as a point in n-space and seeing how close $x(t)$ is for each point. By considering the efficiency gain by adding an extra dimension it is possible to get an idea of the fractal dimension of the attractor. Given the many strange shapes that have been found in non-linear mathematics (the Henon attractor is a relatively straightforward shape!) this is the best clue that can be got from a relatively small data set.

3.2.5 Dimension

The dimension of many objects is already known to us :

- a point - dimension 0
- a line - one dimensional
- a square - two dimensional
- a circle - two dimensional
- a cube - three dimensional.

These are topological dimensions. The objects also have the same fractal dimension, but as we shall see the fractal dimension is not always the same as the topological dimension. And we shall start with an example close to home. The question is: what is the length of the coastline of Britain? Let's put aside all the islands (even though they appear on our map), and use high water mark so as not to get diverted.



Here is the map.



Since it is midwinter and walking round counting out steps is unattractive, we will use dividers on the map to do the same thing. Some commentators call this the coastline of England. Just to show we know better, we choose a unit distance between Edinburgh and Inverness. If you treat the coastline as a polygon with sides of this length the perimeter is about 12.5 units. Now try again with a distance of half the length (from London to Dover as the crow flies). This time we get about 30 units. By changing the scale we get a 20% increase in perimeter (30@.5 against 12.5@1). The point is that at every scale new headlands and inlets will emerge, increasing our estimate. The conjecture is that these new features will emerge, statistically speaking, at the same rate on each reduction in scale. Effectively the perimeter is infinite!

So the coastline of Britain is a natural or random fractal, and we can calculate its fractal dimension.

If the perimeter is $L(x)$ where x is the divider length, then

$$L(x) \sim x^{1-D}$$

Hence we have	$15 = F \cdot 0.5^{1-D}$	where F is a constant
and	$12.5 = F \cdot 1^{1-D}$	
Combining and taking logs	$\log(1.2) = (1-D) \cdot \log(0.5)$	
Hence	$D = 1.26$	

D is defined as the fractal dimension of the coastline of Britain. Many other political boundaries have infinite length, but they can have different fractal dimensions. However not all such boundaries are infinite. Take a look at some American states, such as Wyoming or Colorado: their fractal dimension is one.

Mandelbrot points out that for practical purposes there is a smallest scale that is appropriate, say 20 metres. It is not suggested that the coastline of mainland Britain is the same as the Isle of Wight's; clearly if we know the divider length we can calculate the difference. The key point is that the only way of making sense of the length calculation is by using the fractal dimension.

Although we suggested dividers to measure the map of Britain, the formal mathematical technique uses balls of various radii to cover the coastline. As an aside in our examples in two dimensional ambient space - the sheet of paper - the ball is a circle, but this technique is also done in n-dimensions where n dimensional spheres are used.

In two dimensions we can use shapes other than balls and get the same fractal dimension. For instance, often boxes are the best shape.

We can now return to the Koch snowflake and Sierpinski's triangle and calculate the fractal dimension of their perimeters.

Looking at the Koch snowflake first. Consider the radius of a circle drawn round the initial three points of length unity. If we now use balls radius $1/3$ we need six balls to cover the object, each covers one of the six triangles that make the star shape reached after one circuit of shape building above.

Now consider zooming into the next stage with balls radius $1/9$. 18 of these cover the next layer of triangles, but some of the perimeter of the snowflake remains uncovered. Six more balls are required to cover the inward pointing corners of the star shape - 24 balls in all. Every time we zoom in, by reducing the ball radius by a factor of three we need a fourfold increase in balls to cover the perimeter.

The number of covers N , ball radius s , follows the following rule:

$$\begin{aligned} L(s) &= s. \quad N(s) = \text{constant. } s^{1-d} \\ \text{or} \quad N(s) &= \text{constant. } s^{-d} \end{aligned}$$

$$\begin{aligned} 24 &= \text{constant. } (s/9)^{-d} \\ 6 &= \text{constant. } (s/3)^{-d} \end{aligned}$$

$$\text{Dividing} \quad 4 = (1/3)^{-d}$$

$$\begin{aligned} \text{Taking logs and rearranging} \\ d &= \log 4 / \log 3 = 1.26 \end{aligned}$$

In the case of Sierpinski's triangle it's best to calculate the dimension by using triangles to cover the area. If we start with an equilateral triangle of the same size as the outer edges then trivially one is enough for full coverage. Now consider the cover triangle set with lengths halved. Now we need three to cover the perimeter since none of it is within the inner triangle (the part we coloured in). If we halve the cover triangle again we can cover the object with nine triangles. Hence the number of covers grows by the following rule

$$N(s) \sim s^{-d}$$

$$3 \sim (1/2)^{-d}$$

$$\text{Taking logs and solving} \quad d = \log 3 / \log 2 = 1.58$$

The dimension of the Henon attractor comes out at 1.26, very close to Koch's snowflake.



3.2.6 Random Walks

A random fractal is an object which does not replicate itself exactly over scales. However the various shapes (however defined) appear with the same regularity over all scales. The bedrock of our stochastic processes is the random walk. The random walk is a simple fractal and deserves some particular attention.

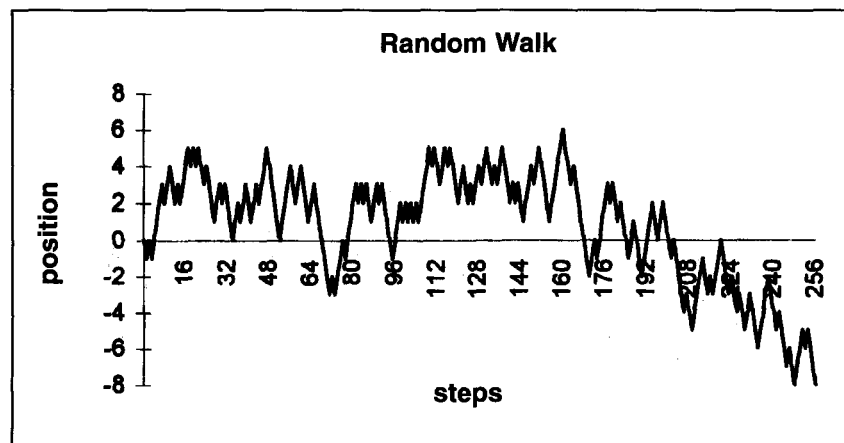
So let's look at the fractal dimension of a random walk. A random walk is the manifestation of Brownian motion; very small steps over infinitesimal time periods can be accumulated to be measured over discrete time steps. Not much different from a coastline! It can be shown that instead of considering different step sizes, as in our tour of Britain, there is an equivalent more practical approach. The approach is as follows:

1. Plot a random walk - ours uses a standard PC random number generator for 256 points:

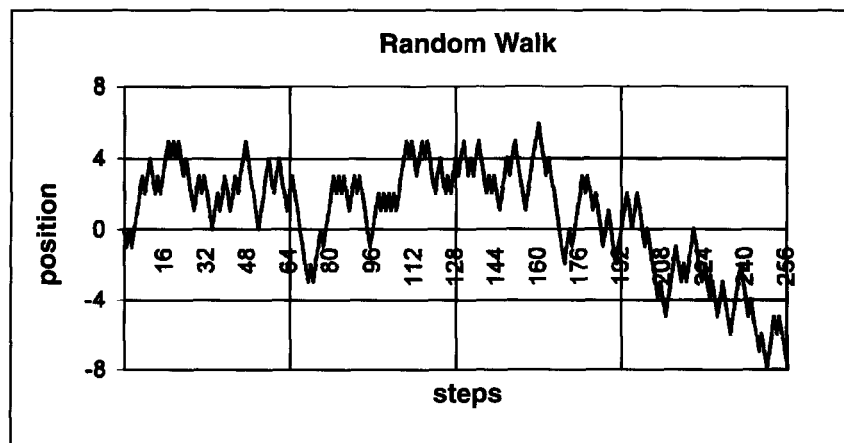
$$R(t+1) = R(t) + 1 \text{ if rand} > 0.5;$$

$$R(t+1) = R(t) - 1 \text{ if rand} < 0.5.$$

$$R(0) = 0$$

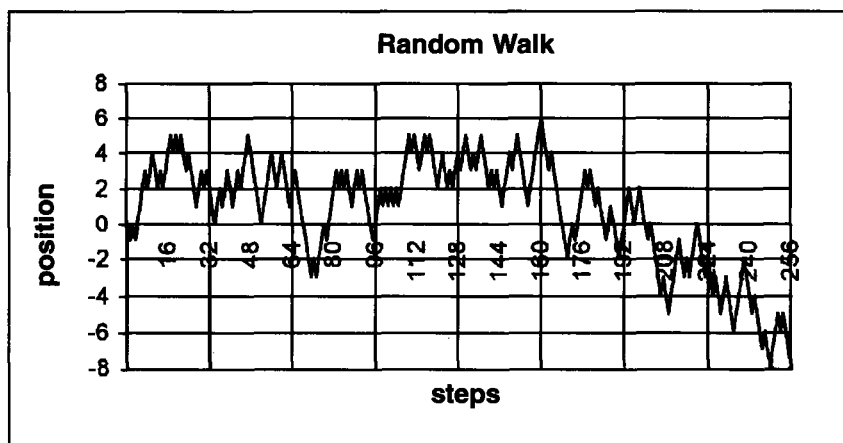


2. Place a grid of 16 boxes over the walk by drawing horizontal and vertical lines breaking the axes into 4 equal segments.



3. Count the number of boxes that the line passes through. We make it 12.

4. Repeat the process outlined in 2, by breaking each of the smaller rectangles into four by doubling the lines drawn through each axis. Count the hits on the new grid - we make it 28.



5. Keep repeating step 4 as long as you like.

The results we got were as follows:

period divided into	hits
4 time steps	12
8	28
16	80
32	212

As the time step is quartered the hits go up approximately 8 fold. (We decided not to find a better fit if only to demonstrate how frustrating it can be to find the fractal dimension in a small data set). This is an example of self similarity and implies that since $8 = 4^{1.5}$, the fractal dimension of a random walk is 1.5. This particular example is untypical as it stays near the x axis more than "expected".

Another way of checking the result is to look at the extremes. If the example is looked at on the scale of one time step the walk passes through the one box available. If we look at the scale of 256 time steps then each step passes through precisely 1 unit that has now been divided into $(256/16=)$ 16 boxes. The random walk on this scale must pass through $256*16 = 4096$ boxes. By this stage the time steps have diminished by 2^8 , whereas the number of boxes has grown to 2^{12} . And $12/8$ brings us back to 1.5. In terms of wiggleness it sits half way between a Euclidean line and a plane. Its dimension is quite close to Sierpinski's triangle.

But a fractal process is not a new idea stealing the clothes of standard statistical models, it is far richer. Since stochastic processes are a core actuarial tool we make no apology for going into a bit more detail. A single variable fractal process can have wiggleness anywhere between the line (dimension 1) and the plane (dimension 2).

3.2.7 Assumptions behind a fractal process

A process $\{y(t)\}$ is Brownian if it meets the following conditions:

For any time step Δt the increments $\Delta y(t) = y(t+\Delta t) - y(t)$ are:

- (i) gaussian
- (ii) of mean 0, and
- (iii) with variance proportional to Δt

Note that on a random walk we assume movement in a straight line of fixed length between each stage. The Brownian process follows a smaller scale random walk over this time frame. A fractal process is a wider family where condition (iii) is replaced by

- (iii') with variance proportional to $(\Delta t)^{2H}$

This relaxation gives rise to a fundamental change in our traditional approach to time series. We normally accept that our data is dirty and we remove the trend, take out cycles such as seasonal effects and eliminate as much “noise” as possible. This is sometimes called “pre-whitening” - called such since white noise occurs in engineering systems when $H = 0.5$. Incidentally, Mandelbrot describes condition (ii) as “vanishing... expectation” - it is a long term expectation.

A fractal process is also stationary so it should be pre-whitened. But whereas in a pure Brownian process successive increments $\Delta y(t)$ and $\Delta y(t+\Delta t)$ are uncorrelated, in all other fractal processes they are correlated. A fractal process is a possible model for some time series which appear to have autocorrelation.

If H is more than 0.5 the process is called “persistent”. If it is less than 0.5 the series is “anti-persistent”.

There is also a class of fractal processes which relax condition (i). The most important one is to allow all stable paretian distributions (see 3.2.10 below). In these processes condition (ii) is replaced by the condition that the distribution is stationary. This possibility is ignored in the next section.

3.2.8 Estimating H, the Hurst Exponent

In 3.1.2 we gave H E Hurst’s contribution to the development of fractals. The Hurst Exponent is a key measure of whether a process is brownian or a more general fractal process. It indicates, loosely speaking, the probability of the joker being drawn from Hurst’s pack of cards. There are several ways that it can be calculated.

The parameter H is simply $(2 - \text{the fractal dimension of the process})$. It measures wiggleness and is used extensively in section 3.5. Some fairly trivial maths $\{1\}$ shows that it is related to the coefficient of correlation by the following formula:

$$2^{2H} = 2 + 2\rho \quad (-0.5 < \rho < 1)$$

This equation gives us the first way of estimating H . Calculate the correlation of $\{\Delta y(t)\}$ over various time steps. If it is fairly constant then $y(t)$ might be a fractal process with

$$H = \log(2 + 2\rho) / \log 4$$

FOOTNOTES

$\{1\}$ treat x_1 and x_2 as two consecutive points. If we double the time step then we observe $(x_1 + x_2)$. Expand $E(x_1 + x_2)^2$ using condition (iii'), noting $\rho = E(x_1 \cdot x_2) / E(x_1)^2$.

There are other ways of estimating H. Here we sketch out some alternatives

1. The variance approach

Calculate the variance of $\Delta y(t)$ over various time steps. Take logs of both series and fit a linear regression line to this data. The slope of the linear regression is $2H$.

2. Local variance

Another approach is to look at the stability of H locally. We calculate the variance on one time step and then we double the step size. Using condition (iii') we find that:

$$H = \{\log E([y(t+2\Delta t) - y(t)]^2) - \log E([y(t+\Delta t) - y(t)]^2)\} / \log 4$$

3. Line Crossing

Looking at the number of times that the random walk crosses the line zero is another traditional approach. In a fractal process it yields the coefficient H since the number of intervals L exceeding a length x, $N(L > x)$ follows the rules:

$$N(L > x) = \text{constant} * x^{-H-1}$$

And again we can carry out a linear regression on the logs.

4. Power Spectrum

Fourier transforms offer some of the most powerful ways of testing fractals. For astute readers it will be no surprise to find a power law at work again, in the power spectrum. A power spectrum of a fractal process follows the rule

$$c(n) = n^{-1-2H}, \text{ where } c(n) \text{ is the amplitude at frequency } n.$$

This rule is particularly useful to transform a stochastic process simulation into one for a fractal process. A Fourier transform obeying this rule can be utilised to get the desired rate of wiggleness for Monte Carlo runs (see Hastings and Sugihara, section 5.6).

5. Growth of range

This approach is considered the most robust. The range $R(t)$ being the difference between the maximum and minimum in that interval can be shown to scale on the following rule:

$$R(t) = \text{constant} * (\Delta t)^H$$

It is the discovery of this rule which started the whole thing off and its history is told in section 3.1.2. Usually the range is rescaled by dividing by the standard deviation. These plots are denoted R/s statistics.

Since this approach is the most used in literature we go into more detail. The data is increasingly subdivided into fewer groups by combining an equal number of consecutive points. To minimise bias it is advisable to use a long series of points where the number of points is highly divisible and only plot R/s for non-overlapping time frames divisible into the total number of points. It can also be shown that this approach is unreliable when n is

small. We recommend ignoring plots where $n < 5$.

Define $\{x(t)\}$ to be a time series of $a = k \cdot n$ time ordered differences $\{y(t+1) - y(t)\}$, k and n being positive integers. This may mean truncating the full series. The idea is to choose a to get as many readings as possible of different scales. The advantages of truncating is that end points discarded in some calculations do not distort results.

Consider a time series, in consecutive order, as $x(1), \dots, x(n)$.

The sample mean is calculated as $m = [x(1) + \dots + x(n)]/n$.

and the sample standard deviation $s = \sqrt{\sum (x(i) - m)^2 / n}$, summing over i .

The next stage is to normalise the data $z(i) = x(i) - m$, for all i .

Now calculate $Y(i) = z(1) + \dots + z(i)$, for all i .

The range, R , is calculated by subtracting the minimum $Y(i)$ from the maximum $Y(i)$.

Finally divide R by s . Take the average of the results over the k observations.

Next recalculate for the next $k' \cdot n' = a$. This is the rescaled range that grows as n increases, and fitting a line to the results gives an estimate of H .

6. Commentary

Finally a word of warning. When we have tried these approaches on a data set we have been surprised at the disparity in answers. They need very long time series. And not long in the traditional statistical sense of more data points. If a yearly series is unsatisfactory moving to an hourly series for the same period does not help. It appears that one needs the long time frame to observe several cycles in order to get good results. One then wonders if the underlying situation might have changed in the intervening period.

3.2.9 Power Laws

The key measure of a fractal is its dimension. It gives us a handle on the scaling characteristics. On regular fractals we have seen that this attribute - although we concentrate on lengths the idea works on mass or area just as well - follows the law

$$N(s) = s^a$$

For random fractals the probability distribution tends asymptotically to:

$$\Pr(x > t) \sim t^{-\alpha}, t > 0$$

This is the hyperbolic function and is often referred to as a power law. Clearly this density function has a discontinuity at $t = 0$, hence it is seen as a geometric interpretation. Note that a power law gives a straight line when plotted on log-log graph paper. One distribution that is geometrically similar is the stable paretian distribution.

3.2.10 Stable Paretian Distributions

These distributions are sometimes called Levy distributions. Both Walter (1995) and Finkelstein (1997) demonstrate some of the properties of this family of distributions. The gaussian distribution is a particularly simple example of such a distribution. The particular interest to risk modellers is the fat tails these distributions have.

For the purposes of this paper the following is important. The distribution has four parameters:

- The characteristic exponent, which lies in the range 0 to 2. The gaussian distribution has an exponent of 2.
- The skewness parameter, which lies between -1 and +1. In most work the parameter is set to zero.
- The scale parameter, a positive number. In the gaussian distribution this parameter is half the variance; it modifies dispersion.
- The location parameter. This parameter does the job we normally associate with the mean. However the mean is infinite for distributions where the characteristic exponent is less than 1. In most situations the mode or median can be used.

Except in three cases (see Finkelstein) the distribution density function cannot be written in a closed form, but the characteristic function can. It is complex and is best understood by reading the preambles provided by Walter and Finkelstein.

If the characteristic exponent is less than 1 the mean of the density function is infinite, but the mode and median are finite. The variance for all distributions except the gaussian is infinite, but it is possible to use the range or quantiles (eg quartiles or deciles). Log stable distributions are sisters to the log normal distribution - $\log x$ is distributed stable paretian. All log-stable distributions apart from the gaussian have infinite mean.

The use of the term "stable" has a specific meaning. If two random variables X, Y have the same density function then the sum $(X+Y)$ transformed to $a(X+Y)+b$, with known a and b , has the same distribution.

It would be possible to use this stability to test whether the law works on underlying stocks constituting a market price index. Mandelbrot has used the fractal property of the tails which tend to follow the fractal law: $P(U>u) = (u/U)^a$. More usually the stability is related to the change in the value of an index over time. The particular fractal process that is being tested is that this stability under addition law works moving from, say, daily to weekly to monthly to annual changes. However Longin (see section 3.4) is looking at the Mandelbrot concept in the tails.



3.3 The Forms of Fractals

“Some fractal sets are curves or surfaces, others are disconnected ‘dusts’, and yet others are so oddly shaped that there are no good terms for them in either the sciences or the arts.”

B Mandelbrot (1982)

The mathematics of fractals is quite a pleasing exercise if lateral mathematics happen to hold a fascination for you. It also happens make use of ever increasing PC power and to give results that can make a mathematician into an artist overnight. Such novelties are almost certain to gain a large following and to give rise to a large volume of pictorially pleasant output “of which no use can be made”, to borrow from Lebesgue’s quote from Section 3.1.3.

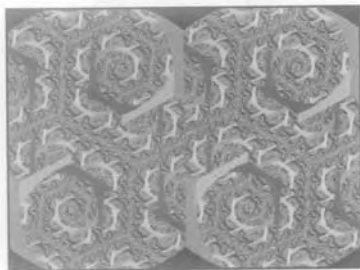
While Lebesgue may be taken as being correct in the narrow sense, there are two very good reasons why it is appropriate to display some of these pure forms at this juncture and to present some basic algorithms which the eager student may wish to attempt in section 4.

Firstly, the potential for the application of fractal geometry to risk management depends upon those risks displaying a naturally fractal shape, be that in their own manifestation or in their relationship with other influences. If fractured, discontinuous, self similarity is present then there ought to be a mathematical fractal model that will approximate to it and thus allow further process work to take place. Displaying some basic fractal ‘shapes’ here is will give the reader some concept of how distributions and relationships that are ‘natural fractals’ might appear. Mandelbrot (1982) stressed the same objective:

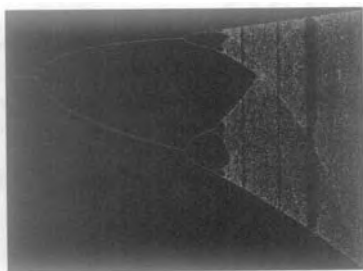
“Showing pretty pictures is not the main purpose in this Essay; they are an essential tool, but only a tool.”

Secondly, it is possible that the very range of images will spark interest amongst some actuaries and actuarial students where words will fail. As we believe that diversity of input is critical to the investigation of this subject it is therefore a justifiable act to carry these images. It is worth stressing that many of the applications referred to in the sections that follow have been motivated by a practitioner ‘seeing’ in pictorial images of mathematical fractals some feature or features, that have been present in his or her real world.

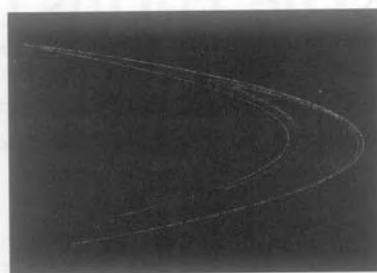
Barnsley:



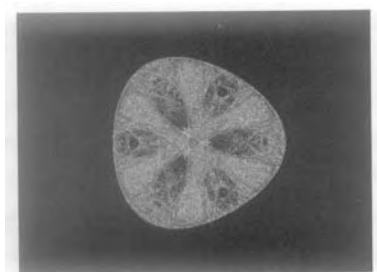
Bifurcation



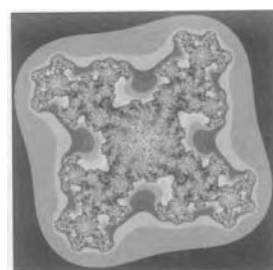
Henon



Icons



Julia4



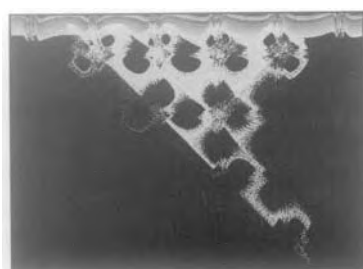
Lorenz



Man-o-war



Popcorn



Crystal



Spiral



Lyapunov



3.4 Fractal Applications

It seemed a worthwhile exercise to run through the areas of application for fractals, before discussing the sorts of applications that seem to justify actuarial investigation.

There are three reasons for this. Firstly, it shows that there is a serious side to some of the new wave of mathematics that is too easily dismissed by calendar and screen saver images, and by the jargon that wraps around the subject. It is possible when faced with a 'fractalicious' fractophilic (see glossary) to dismiss fractals as the mathematical equivalent of Startrek.

Secondly, they indicate the sorts of problem shapes and complexities that lend themselves to fractal analysis, which in turn hint at the areas within the financial and demographic world that might benefit from the application.

Thirdly, they also indicate the sort of areas that might benefit from the actuarial profession's disciplines beyond the financial and demographic world. So, in no particular order, our research indicates the use of fractals in the following fields:

1. Carbonblack Analysis

Carbonblack aggregates have been the centre of attention for a lot of fractal analysis and practical development. Carbonblack is made by burning natural gas with a restricted amount of oxygen, so that carbon is formed in the flame. Around two billion pounds of carbonblack is produced in the US per year, of which around 75% is used by the rubber industry. In the early 1980s some concern was expressed about the safety of carbonblack as used in industry and thus much research was done into the nature of its fine particles. It was found that using fractal geometry gave a great deal of analytical information about the nature of the fine particles not available from traditional analysis.

The key problem with the particles has been that no matter how much you magnify the particles there is always more 'ruggedness' present than was visible at the last, lower, magnification. Given that the visible ruggedness is usually similar at each magnification it would appear that the particles are natural fractals. Thus projecting that ruggedness infinitesimally permits predictions about the particle's reactions to microscopic changes in environment.

This work has been pioneered by Laurentian University, from whence Kaye (1989) has become a leading author on the commercial application of fractals.

2. Powder and Fine particle Metals

A related area has been the understanding of the likely reaction of powder metal particles under different conditions, which is a large practical area of metallurgy research. Observing reactions of particles with different fractal dimensions enables interpolation or extrapolation of the likely optimum shape and dimension of the particles at infinitesimally small sizes for the task in hand (eg paint spraying for maximum shine or strength).

3. Extracting Mineral Grains

It has been shown (Kaye, 1989) that the rock matrix holding mineral grains reacts differently to the form of extraction used according to the fractal dimensions of fine particles of both the ore and the rock. Significant savings in energy can be achieved by applying the right process, and fairly accurate advice can be gained from fractal readings of samples of the minerals to be extracted.

4. Extraterrestrial Geology

Fractals have been used to describe some of the major differences in the structure of lunar rocks as compared to their terrestrial counterparts. Freshly shattered pieces of rock are more rugged than those that emerge from the crushing and grinding of the mining industry, since attrition or erosion will round off any fractal structure in the newly generated fragments. Uneroded freshly shattered rocks exist on the lunar surface and thus some lunar fragments have indeed yielded a natural fractal surface.

5. Respirable Dusts

It is becoming clear that the 'ruggedness' of certain dust particles that may be inhaled is a significant factor in how harmful they are. While their size and chemical composition will determine whether they can get into the lung's alveoli and then react locally to cause damage, the shape seems to determine whether they can effectively 'hook' up and stay at the destination as opposed to simply flying back out from whence they came. This is termed the 'lodgability factor', and it is directly related to the fractal dimension of the particle. Asbestos, nicotine, silica dust (cause of silicosis) and coal dust are all very different chemically, but fractally similar. So what other particles are there in the air with similar fractal dimensions? Do they have a part to play in any diseases as yet not connected to them ?

6. Leaf Structure

Leaves display fractal properties and fractals are being used to measure the distortion in leaf structure caused by biological or chemical attack. For many non-mathematicians it is this sort of similarity which sparks the first interest, for it is relatively easy to generate a leaf shape by means of a fractal algorithm.

This is, of course, the thin end of a very large wedge, for Mandelbrot concentrates a significant amount of time demonstrating the many fractal patterns embodied within nature. Clear trees emerge from algorithms as do rock formations and assorted topological shapes. It should not be surprising then that, given the underlying theory, the elements that make up these formations (particles from rocks and leaves from trees) also display fractal natures.

7. Oncology and Epidemiology

Cancer and epidemic growth models were pioneered in their own right by Eden, but recent work has shown these to display a fundamental fractal structure and can be modelled as such. If fractals can help point the way to answering "How does a disease spread through a randomly heterogeneous material ?" or "How does an epidemic spread through a randomly heterogenous population ?", then the focus it will give to practical research will be of enormous value.

8. City Planning

As long ago as 1984 fractals were being used to describe cartographic features, including the perimeters of cities and their growth. This is, of course, heavily related to the local geography, but more recent studies have looked at the internal fractals of cities' structures to see whether similarities in fractal dimensions are correlated to other features, such as traffic congestion and crime.



9. Geographic Phenomena

An example of the use of fractals here is in the description of island archipelagos, which were observed to have similar properties to fragmentation fractal creations. Using fractal dimensions on real island groups enables computer driven fragmentation models to identify the potential fractal development of such fragmentations and thus give a range of possible histories for the geologists to investigate - and predict the future.

10. Fractal Twins

Did you know that the fractal dimensions of the surface of pyrolytic graphite (carbon) and the surface of a cauliflower are just about identical, as are pictures of them? Also that stress cracks in some metals have the same fractal qualities as the Red Sea (and a picture of the Red Sea from a satellite looks uncannily similar to its fractal twin)? There are a host of these twins, and some triplets too. Not only that but these similarities have opened up new lines of research into whether knowledge on one of the twins will help deepen the understanding of the other in some way; metal cracks are easier to test and gauge responses to than is the Red Sea (which is in a crack in the Earth's crust of course).

As was indicated in 6. above, it is the visual similarity between clear fractal forms (either already demonstrated naturally or those produced deterministically and presented visually) that will spawn consideration of the use of fractals for analysis in otherwise unrelated areas. Indeed it is some of these 'twins' that give rise to clear implications as to their appropriateness for mapping risk patterns.

There are some common themes here. Variability, growth, sudden fragmentation, interrelationships of large numbers of heterogeneous items, correlation and just plain information. But what is not immediately apparent from this list is the fact that the algorithms that drive fractal applications are easily embraced by PC applications. They also enable you to analyse what you have got and to set algorithms running to try to emulate the sort of structures that are facing you.



3.5 Risk Applications

3.5.1 Investment

Background

Mandelbrot has written extensively on time series modelling of markets. Most of his work on market prices pre-date his published developments of fractal geometry. The investment work was in the 1960s as financial economics was being developed. It was well known that the distribution of changes in stock market prices did not exactly follow a normal distribution. There are too many near average changes and more importantly, for risk managers, too many extreme changes. There was no simple way of incorporating this fact in modelling and for most purposes the deviation was small compared with other parameter fitting. The random walk simplification led to the big steps forward in stock market modelling underpinning Black Scholes Option Pricing and the Capital Asset Pricing Model. Mandelbrot's ideas were very difficult to model since the mathematics was intractable. Nevertheless they remain a way of explaining phenomena found widely in stock prices. With the advent of high power PCs we see the actuarial profession widely using Monte Carlo simulations for all kinds of investment risk investigations. Fractal processes are complex, but may have a part to play.

All models are simplifications, and it is possible that for some applications the technical advantages of the pure random walk model outweigh the disadvantages. Shyam Mehta (1996) discusses some of these issues in a recent article in "The Actuary". However as more people have used fractal processes, longer time series have become available and computer power has enabled more analysis, Mandelbrot's ideas have become more viable. This paper is not about investment modelling, but about actuarial applications of fractals. So some specialists might find what follows a gross simplification.

Hurst published his work in the 1950s before fractals had been developed as a geometric technique. His method does not depend on the process being normally distributed. Before he discovered Hurst's work, Mandelbrot had proposed that the stock prices followed a family of curves called Stable Paretian or Levy (after the French mathematician who formulated them). It is important to realise this since often papers only refer to Mandelbrot's 1960s papers. One chapter in "The fractal geometry of nature" published in 1982 is devoted to market prices and this represents a summary. Stable Levy curves will give a better fit to the distribution of stock price changes. The point is that they have four parameters compared with the two in the gaussian distribution. It is one of this family of distributions that Andrew Smith used to investigate a fractal investment model. To quote Smith - "For the fractal model, I have not been able to achieve any significant improvements from dynamic trading" (para 3.6.3). Given the jump process developed by Hurst this is a result we should expect from a fractal model.

Much of Mandelbrot's early work concerned the Levy distributions. In his 1982 work he remained unbowed. The stable Levy model explained the data. This proposal has some disturbing issues. The best fit stable Levy distribution has infinite mean and variance. To Mandelbrot that presents no problem - he suggests that this is little different from "very large". We have already seen the Koch snowflake, where we contended that the perimeter was infinite, but the perimeter of our diagram is, by necessity, finite.

The Fractal Market Hypothesis

Peters (1994) has developed these ideas into a theory which he calls the Fractal Market Hypothesis (FMH). Much financial economics is based on the Efficient Market Hypothesis (EMH). This assumes that current prices reflect all current information to the point where the marginal benefits of acting on information equal the marginal costs incurred by that action. There is an even chance that new information will raise or reduce prices. The EMH becomes a game of chance and the central limit theorem can be utilised.

In contrast, the FMH assumes that the market is made up of a range of investors with different time horizons. Prices are stable when there are investors in the market with a wide range of investment horizons. The investors supply liquidity. Short term traders use technical information, while long term investors use fundamental data. Hence the underlying trend in the market is reflective of changes in expected earnings. Short term trends are more likely the result of crowd behaviour. There is no reason to believe that the length of short term trends is related to the long term economic trend.

Comparing the FMH with the EMH. The FMH suggests that liquidity is important whereas the EMH effectively assumes that it will always be provided. The FMH asserts some particular differences in investors behaviour depending on their time horizon. Under the FMH it is possible for the market to be unstable, when fundamental information loses its value. In such situations the supply of liquidity alters. The FMH says that the market is stable when it has no characteristic time scale or investment horizon. Stable markets are fractal.

Peters shows that the Hurst exponent for the Dow Jones index, based on pre-whitened daily price changes over 102 years is significantly more than the 0.5 implied by the gaussian distribution. This repeated Mandelbrot's findings on cotton prices. If the Hurst exponent is greater than 0.5 then the time series is persistent. In the time series of price changes each data point is correlated with the next. This correlation is seen on a wide range of time scales - hourly, daily, quarterly, annually. It remains in the error terms despite observers using various techniques to modify the model structure.

This persistency has shown up time and time again in investment price series. Usually it is low, somewhere between 0.5 and 0.6. Its low enough for some commentators to ignore, and that's how we get stochastic models based on stationary gaussian distributed random walks.

Peters investigations suggest that liquidity changes and prices jump on average every four years. Above four years investors improve their expected return per unit sample variance. However he finds that for currencies there is no such scaling limit. This may be because his time period is shorter, starting when the Bretton Woods agreement broke up in the late 1960s.

Peters explanation is that the market follows a log stable Levy distribution and hence the mean and variance of changes are infinite.

Economic Implications

Acceptance of a fractal investment model would change a lot of our preconception about investment risk.

- The local sample means and variances would not be a good guides to long term population statistics. Apart from the Normal Distribution, all Levy distributions have infinite variance and some have infinite mean.

- Variance would not be a good guide to volatility.
- We would use different measures of market location and dispersion (median and inter quartile range might replace mean and variance). We would still have tools to control risk. Fractals explain bull and bear markets. They give us no insights into when they end and by how much the change will be. In a mean reversion model the more extreme yields get away from the norm the more likely we are to see a correction in the next time frame and we can estimate and bound that change. In fractal models we lose this benefit.
- A fractal model may give some credence to chartist techniques. Deviations from moving averages over different time periods may pick out attitudes of investors with different time horizons.

Differing schools of thought

As with most new sciences you have to sort out the quacks from the good doctors. Much of the literature is written by fervent disciples of one approach or another. Fractals and chaos have been used successfully in natural sciences and have been proposed in economic problems not least in explaining share price movements. However social sciences have a marked disadvantage against natural sciences: rarely can they carry out controlled experiments. Physicists develop new tools to test theories. Brownian motion explains the movement of pollen in liquids. Chaos explains why apparently random behaviour can be reduced to a simple deterministic model. In natural science stochastic and deterministic approaches are merging.

However economists often look to new techniques to solve perennial problems. Classical economics has no reasonable explanation for the regular variation in prices. Prices should be set so that supply and demand are in equilibrium - a traditional deterministic approach that says nothing about how they are supposed to vary. Clearly there is some stickiness in the market that prevents supply reacting immediately to changes in demand and vice-versa. But it is a great leap forward to suggest that the stickiness leads to a gaussian distribution of price changes.

Financial economics developed as economists discovered stochastic processes. The Efficient Market Hypothesis says that current share prices reflect all public information. Share prices change because new public information becomes available. Future price changes cannot be inferred from past changes.

Recently Kemp (1997) has demonstrated that assumptions can be loosened in derivatives models and the central tenants of the Black Scholes model still hold.

Actuaries have taken on board the idea of random walks for investigations into the risk of ruin over the long term. However the profession has added to its existing deterministic approach. The profession hypothecates, say, that in the long term investments provide a real return; essentially deterministic. Then a random walk is tagged on to explain volatility. Often these are built up into a simultaneous equation model. Our tests are not testing some hypothesis of future market behaviour, but whether our models fit past experience.

This approach has the advantage of being pragmatic. It might be summarised as proposing that the random walk is in the long term pulled back to a price that reflects underlying value. Nothing about why it moves away from fundamental value. The problem we wish to highlight is that we are in a difficult position to assess the possibility of a fractal approach to actuarial analyses. Scientific testing involves forming a hypothesis

and then trying to falsify the hypothesis. It is not sufficient to say that the error term in our models is normally distributed, or becomes an ARCH process or whatever, without saying why.

Kemp (1996) rejects the fractal approach to long term asset/liability modelling quoting work carried out by Longin (1993). Longin found positive first order autocorrelation in his analysis, which is one sign that a fractal process may be at work. Kemp seems to have rejected including autocorrelation in his model since it would allow a dynamic hedger to be able to predict future price movements and this would give any strategy utilising this fact an in-built advantage.

Notwithstanding Kemp's concerns about models involving the Levy distributions, the fractal approach would have several features suitable for this type of work. Indeed this is what Smith found. Longin's analysis showed that the Levy distribution did not fit extreme movements in daily prices. Longin uses another new form of mathematics called Extreme Value Theory (EVT). He has used very similar data to Peters, the S&P index and predecessors from 1885 -1990. By looking at the extreme daily movements in each calendar year, he rejects the idea that stock prices follow a Levy Distribution. He prefers an ARCH approach.

Here is another new technique - EVT - that actuaries should be appraising. The theory is almost an antithesis of chaos theory. It tests for unusual events that can be explained by chance rather than a hidden deterministic model.

As a counterweight to the Longin work are papers by Walter (1995) showing a fractal structure to some French stocks. Walter's analysis is over a relatively short period and only looks at patterns up to a period of 20 days. His estimate of the H exponent is similar to those found in other studies. Longin only inspects the distribution of daily movements and passes no comment on the scaling results found by Peters. Peters proposes a stable levy distribution, but does not fit it, although he indicates the size of the characteristic exponent at about 1.7. The characteristic exponent is of the order of $1/H$.

Finkelstein (1997) has returned to the model proposed by the 1980 working party on maturity guarantees. He fits Levy distributions to the data used by the working party. His model is consistent with a Hurst exponent of about 0.57 on log-dividend yields.

Issues for the profession

The idea of throwing more and more sophisticated statistical techniques at the problems of understanding the stock market is unconvincing. While they may prove that the pure fractal model of Mandelbrot's is not manifest in the data, they seem to ignore the contributions that geometry can make to our understanding of the system. Actuarial investment rules are based on the idea that in the long term there are some underlying economic laws concerning value that should be reflected in stock prices. Economics is complicated and may need non linear explanations. Geometric techniques can help us identify non linear patterns in the data. We should be looking for the attractors hidden inside our not-quite-normally distributed share price random walk.

Such work is beginning. Craighead (1994) has looked at US Treasury Interest Rates. He demonstrates that the rates do not follow a random walk. As Peters has also shown there is a weak persistent bias. He suggests that the market is nearly efficient and at most one could predict the direction of interest rate changes and then only in the long term.

One of the problems with any economic time series is that the market price is affected by outside influences. Larrain (1991) looked at T-Bill rates and used a range of factors to estimate the fundamental value of the stocks. Only then did he look for non-linearities. In an argument similar to Peters he suggests that usually in the short term the market is

more heavily influenced by technical factors. However in the longer term, fundamental value asserts itself. He does point out that at certain times the market suspends belief in fundamental value. Some of the parameters fitted to his model are close to a region where chaos should exist.

Larrain's line of enquiry must be close to giving insights into the various models actuaries are proposing for asset liability work. In the long term our premise is that prices will become relatively close to fundamental value. (We may all have different views on what exactly that might be!) At present most models assume that prices can move away from fundamental value by following a random walk. It is quite possible that at least in some periods, mathematical chaos best describes part of the random variation. Fitting our models to past experience based on an unbiased random walk model is likely to be underestimating the risk to mismatched client's funds.

In actuarial terms an example might be the share dividend yield equation extracted from the Maturity Guarantees Working Party Model (Ford et al 1980). One of the equations, ignoring that it is in log form is:

$$\text{Yield}(t) = 4\% + 0.6 (\text{Yield}(t-1) - 4\%) + \text{random component}.$$

Here is an equation with that underlying actuarial tenant that there is a universal rule that in the long term there is an economic relationship between price and income is constant. It also adds a very fractal idea that whatever its current income/price ratio there is an "attractor" at work. Global determinism, but the final term gives local randomness. This model is an AR(1) model it is a simple autoregressive type or a mean-reversion model.

However the model has some useful simplifications to fit our mathematical techniques. The first technique is to use logarithms. This process keeps all the yields positive and makes the variance relatively stable over time. In this particular model it maintains the log transformation used in the related dividend equation. One disadvantage demonstrated by Finkelstein (1997) is that apart from the gaussian distribution all log-Levy distributions will have infinite mean. Using the yield has advantages in that the measure does not have a trend caused by price inflation.

Another simplification is the use of an autoregressive model. All autoregressive models of finite order are not fractal. In fact if a Hurst exponent is calculated it should show the process looks anti-persistent. Now a fractal model may not be the most suitable, but a good model should show the persistency that has been consistently observed.

This particular problem also exists in Peters work. He fitted an AR(1) model to log-prices and then examined the residuals. He did this to minimise linear dependency and also detrend the series. But it is an arbitrary method to overcome these issues.

The yield equation has been carried forward into "The Wilkie Model" where the random component remains gaussian. The MGWP model has been updated by Finkelstein by fitting a wider Levy distribution as the random component.

The issue is whether our training and other historical baggage is blinding us to new insights. It may be that hidden in our pre-whitening of the data and the use of the gaussian distribution we may be missing something chaotic within that error term.

Lee 1991 suggested that any positive auto correlation model would be inherently unstable. But we know that over all time steps this correlation is in the data. And we know that price movements and the variance in those price movements are more volatile than most models we are using.

3.5.2 Mortality

The bedrock of actuarial science is the life table. The Gompertz and Makeham curves have served the science well. The mortality model is a good example of a deterministic model (the curve) being fitted to a stochastic process (used to calculate the confidence limits of q). As such there is little reason to believe that fractals can add value.

But its not as though new mathematics is not enveloping life tables. MacDonald (1996) has shown how exposure techniques have developed and Renshaw (1991) has shown how a software package - GLIM can be used to graduate experience.

However increasingly premium rates depend on wider factors than age. Werth (1995) recently showed a wide range of statistics that could be used to price mortality. The market for insurance coverages to older aged customers is widening. The rating of people requiring immediate long term care has little to do with their current age.

In some circumstances we will need to resort to Neural Networks, Cluster Analysis or CHAID to rate risk. And in these extremes there might just be a place for fractals. Older lives are at risk in cold weather and to influenza epidemics. Their mortality is not independent of each other and in such situations a fractal process can add value over a pure stochastic process.

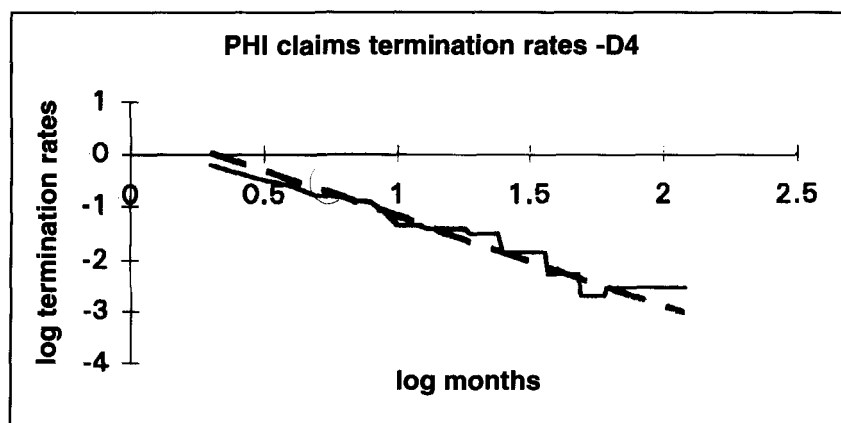
In reinsurance, a fractal model can give insights into stop loss and catastrophe excess loss coverages. In this more risky end of mortality cover it is particularly useful to have alternative models to test the appropriateness of the contingency margins added to traditional techniques based on the generalised poisson distribution.

3.5.3 Morbidity

In CMIR 12 individual permanent health insurance claims relating to 1975-8 were fitted to a multi-state model. This was a big step forward for the professional understanding of such claims. Claims terminations were modelled by fitting a curve with six parameters.

However Clark and Dullaway (1995) have reported on data up to 1990. They show that claims inception and termination rates are unstable. In particular termination rates have tended to fall.

It appears that we need something simpler than a six parameter curve to fit termination rates. The claims termination rates do look fractal in shape. The next diagram shows the fit between their crude deferred four week individual claim termination rates. If the industry accepts that claims termination rates are unstable, then a fractal fit appears a practical approach for monitoring claims rates. Furthermore there is another plausible gain.



There are well rehearsed reasons for why terminations can vary by deferred period, for instance offices with differing underwriting and claims control can be active in different markets. But experience in each deferred period is also influenced by the proportions of claims by cause in that range. In CMIR 8 there is an analysis of claims by cause of illness. From this data we can calculate the exposure to claims for each cause and the recovery rate. Using this information for each deferred period we can calculate the fractal dimension for each cause. The results are set out in the next table.

category		dimension
1	Other Infective	1.11
2	Neoplasms	0.24
3	Endocrine & Metabolic	0.85
4	Mental	0.62
5	Nervous	0.92
6	Circulatory	0.55
7	Acute Respiratory	2.28
8	Chronic Respiratory	0.92
9	Digestive	0.70
10	Genito-Urinary	0.84
11	Musculoskeletal	0.91
12	TRA Injuries	0.52
13	Other injuries	0.70
14	All other	0.92
	All	0.84

Claims reserves could be set by age, the duration of claim from inception of illness and the cause of illness. The fourteen groups above might be too many. However apparently different causes seem to have the same dimension. Those with similar dimension could be grouped together.

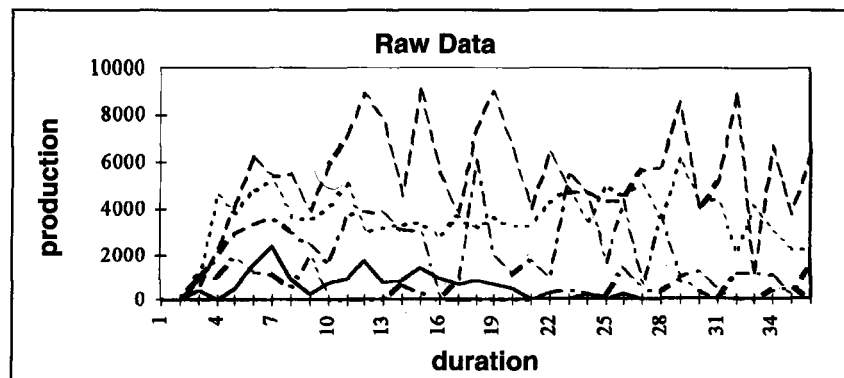
Another advantage is that the calculation of present values using dimensions is quite straightforward.

3.5.4 Life Assurance

Sales Remuneration

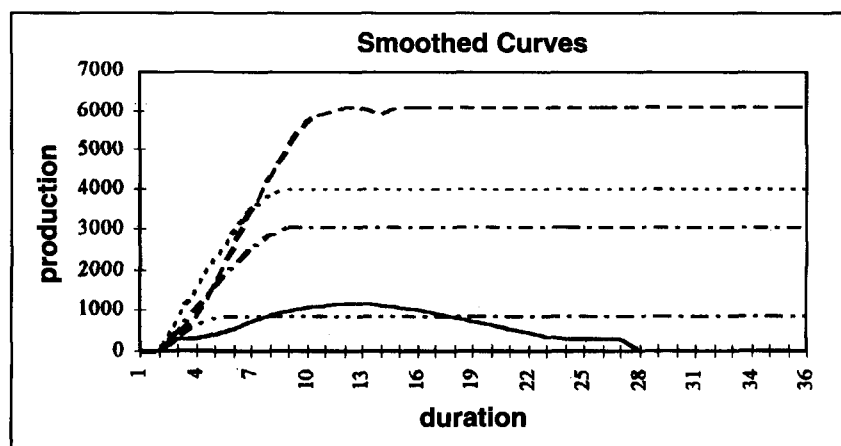
This section outlines our first practical use of fractals. The aim was to cost a new sales consultant structure where a substantial salary was paid for the first time. Previously commission was the main component. This example is simplified to highlight the use of fractals. The aim was to come up with a package of equal to the cost to the company of the existing scheme.

On average the sales force produced 4 cases per month. Production was about £1000 per case, commission would be about £400 per case. In such work it is dangerous to assume all consultants produce the same amount. So the first stage involves gathering information on the sales force, to break the consultants down into reasonably homogeneous groups. In this case we came up with 5 groups with the following career patterns.



source: Jardine Arber

Now we move onto the next actuarial trick and smooth the curves. Given the paucity of the data we assumed a sharp learning curve and then a levelling off. Giving the following results.



source: Jardine Arber

The consultants in training are on a special scale and were treated separately. In costing the package our inclination was to assume a poisson distribution using the long term averages as the best estimate of the mean for each group.

Our solution would then come up with a salary plus bonus which left the cost to the office unchanged. The office had proposed that bonus should be paid if a consultant reached a monthly threshold.

But this approach ignores the culture of the organisation. The whole reason for the change is to alter the culture and this is not reflected in this cost.

The only data that we have available is the “randomness” we have already smoothed out of the curves. Looking back at these curves one can see that in fact they are quite jagged. Carrying out some tests we found that the error term was anti-persistent. A more classical approach may have fitted a Markov process with negative autocorrelation. However this did not appear to fit the facts.

After discussion with our client we decided that the consultants were playing the rules to their advantage. In order to receive an override commission, they had to produce about 5 cases per month. If a consultant was well short there was a tendency to hold back business to increase the chances of override the following month. If they were ahead they would push things through. While this would tend to exhibit mean reverting behaviour, anti-persistence best reflected the decisions that consultants made.

Our conclusions were that the threshold should be based on moving average production and the consultant’s manager should be on a system that countered the tendency. This was achieved by only counting sales from “qualifying” consultants in these packages. We were then able to cost the package by mimicking an anti-persistent fractal process and carrying out the costings. Anti-persistence was assumed to fall slightly.

Now the conclusions may appear to be pure common sense. We agree, the point is that using fractals meant the maths supported the solution.

In passing we note that if a sales force is ranked by production, the probability that a salesman exceeds a level tends to follow the fractal rules. The potential interest is that the dimension is different for each sales force we have looked at.

3.5.5 General Insurance

The whole of the Pacific Rim, including the wealth of Japan and much of California sits earthquake fault lines. They sit on the edge of great moving plates which make up the surface of our planet. The area is uninsurable against earthquakes since there is insufficient worldwide insurance capacity to cover the sums at risk. Furthermore its still not clear how to assess the risk of an earthquake in a particular area.

The mainstream theory is called the seismic gap hypothesis. The proposition is that after a large earthquake the hazard is low since the quake releases most of the local stress. The original hypothesis dates back to the late sixties. There have been modifications since then.

The original hypothesis was tested by seismologists Yan K Kagan and David D Jackson, see Monastersky (1992). They found that the hypothesis simply did not work. Looking back a decade later they found that it was just as likely that a new quake occurred in an area of low risk as there was in a high risk area.

They put forward an alternative theory an equal hazard hypothesis. The analogy is with that of a sand box. Here sand drops on a pile from a funnel at a constant rate. If you observe the surface area of the pile as it grows, the sand does not drop evenly down the sides. Pockets of pressure build up leading to avalanches of different seemingly random sizes. This model is fractal. Any area of any size (short of scales of the order of the entire surface area) is subject to similar pressures.

The battle still rages over the best model. Examination of long records of earthquakes in Japan give unclear results. The problem is that until recently the measuring of the scale of an earthquake was unreliable. What is clear is that the clustering of activity, which is now well observed is important. In the seismic gap model the accumulation of quakes will signal the release of stress and low regional hazard for a period. The equal hazard theory suggests that we will never be able to predict when a cluster will cease.

As we go to press we see that Berkshire Hathaway, the company controlled by the legendary investor Warren Buffet, has received a premium of \$600m to provide large capacity earthquake cover in California over the next four years. Its not clear whether the firm could pay out if there were two "big ones". The risk of the second event is substantially different under the two models.

Now earthquake insurance may not be high on your list of potential new products. But there is a real chance that fractals form the basis of mainstream non life contracts. Hurst's original model came from his work on flood control. Flooding is considered an insurable risk. Furthermore weather systems are known to be chaotic in character. Fractals may well help us model many property risks from storm damage to subsidence.

3.5.6 Catastrophe Risks

Insurance companies need to put a handle on extreme risks, the ones that risk solvency. At the very least it allows them to understand the efficiency of their reinsurance programs. And for reinsurers the comprehension is even more important. In their case it is not possible to price some contracts without this knowledge. In many lines of insurance 20% of events are responsible for 80% of the claims cost. For hurricane insurance this rule is more like 10%-95%. A recent example of this type of risk manifesting itself was in Mortgage Indemnity Insurance. And now it's not just insurers who have these issues. Banks writing long term off the shelf investment derivatives have the potential of many years of profit with a few very big losses.



It is in such areas that the standard stochastic process in isolation is likely to substantially underestimate the risk probabilities. What we have to do is find several models all with their own strengths and weaknesses and use them alongside each other to see if we can get a quantum of the size of risk.

At present the hottest developments are in Extreme Value Theory. If events are ordered in size, then at the highest extremes, the distribution of claims tends to follow one of three distributions: Frechet, Gumbel or Weibull. With a large enough data set it is possible to use these distributions to put a maximum and minimum value on the risk of the “big one”. Longin, who was quoted in the investment section, used this technique. There is a fractal alternative. The fractal approach would look at events over all scales, even relatively small ones, and use this information to extrapolate into the payouts yet to be observed.

It is these sorts of risks that actuaries can advise in wider fields. The Dutch government is prepared to accept a flood surge that will break its sea defences once every ten thousand years. Apparently they have to be 5m high; the highest recorded surge is 4m dating back to 1570. The British Government has commissioned statisticians from Lancaster University to look at sea defences on the eastern coastline. Actuaries would be in a unique position to produce such statistics and put the results in a wider cost benefit perspective. If we are unhappy of moving into untested fields we could practice nearer home with Mortgage Indemnity Insurance.

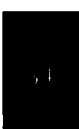
3.6 Fractal Applications to Risk Management

3.6.1 The staging post and the next journey

So we come towards the end of this leg of our journey into the nether regions of Fractals. Its time to see if a “fractuary” is about to usurp our established position as commercial mathematicians.

Along the way we have found more questions that may become the quest for the next stage of our journey.

- Very few natural objects are smooth. Lightning does not travel in straight lines; chickens are not spherical. Is graduation soundly based on some definition of fairness or does it derive from earlier limitations on actuarial mathematics?
- Our tradition has been to use exponentials naturally as a first approximation; bound up in the calculation of the compound interest. Should we be putting more weight on testing whether power laws are better. Until recently the respiratory and circulatory systems of the body were thought to scale exponentially, power laws now look a better explanation.
- Often we use functions that are integratable or differentiable depending on our needs, alternatively we use differences. Fractals are continuous non differentiable functions. Do our technical simplifications limit the value we can add?
- Actuaries have long had the processes to handle jump models. The general insurance industry usually accepts one year premium guarantees and in PHI it is unfashionable to guarantee premiums rates. Are new techniques able to give significantly improved insights into longer term rate guarantees?.
- In the US , Risk Based Capital measures have been introduced in life insurance to show solvency. These appear in part to be based on stable models and gaussian distributions. Our confidence in this approach has been weakened. How do we measure solvency in a world where there is a case for models with very large variance?
- The insurance cycle looks a prime candidate for explanation as a persistent fractal process. The delay in gathering information from unsettled claims has similarities with Egyptian flood cycles. Can we produce a model and does this give insights into claims equalisation reserves (the “dams” so to speak).
- Mortality experience has long been perceived as a stable model with an improving trend. Given the recent experience with AIDS and more historical (for us) plagues and famines, should we consider mortality as a persistent fractal process? How would this effect practical advice on premiums and solvency?



- In many applications the standard random walk will remain the only practical tool. Are there guidelines that can be laid down for when this is safe for a particular application? Is it safe just to use the median or mode instead of the mean?

3.6.2 Fractals, New Mathematics and Actuaries

In this section we have given a brief introduction to fractal mathematics and now it's time to return to the two questions we set in the introduction, viz.:

Do these new concepts offer any potential to improve the techniques (processes) used in actuarial applications ?

and

Is the profession in a position to answer the first question, in relation to any new mathematical concept, quickly ?

We have shown that there are potential applications across a wide range of fields. However with so much unanswered we can only say that the ideas are of some use to the profession and remain on the fence about how much more it will embed in our processes.

On the second question we remain confident that the profession can adapt. The mathematics in these new areas is well within the capabilities of all Fellows. We have no reason to believe that the syllabus will not give new actuaries skills in the new mathematics as the techniques are proven.

There may be a gap in providing the infrastructure for mature actuaries to appraise these new techniques. In this respect we humbly put forward one idea. The BAJ and its forerunners JIA and TFA have a long tradition of publishing discussion on papers of a process or application presented at sessional meetings. While more fundamental papers go through a scrutiny process they are not presented at sessional meetings, and there would be limited demand for such discussions. But there is a place for written contributions on the application and process such contributions could be used especially in a wider field. This may be by way of preamble or responses published in later editions. We cannot expect actuaries to be able to keep abreast of all new developments (the profession now accepts some form of specialisation), but we should expect actuaries to be alert to how new ideas could help their clients. New techniques should be in front of (not behind) the profession. As such we submit this paper as a model.

SECTION 4

The Small Print



The Henon Fractal

“There was a crooked man who walked a crooked mile
who found a crooked sixpence upon a crooked stile, he
bought a crooked cat that caught a crooked mouse and they
all lived together in a crooked little house.”

A Fractal Nursery Rhyme

4.1 Some Fractal Algorithms - and friends

A sample series of mapping algorithms are given below for the would-be fractal student, from whence any number of variants can be developed.

4.1.1 The Newton Map for the Cube Roots of Unity

This is an attractor with three fixed points and if an iterated function system is used starting with random values of x_0 and y_0 they will always tend to one of the three fixed points. To gain an attractive picture simply colour the route in, in one of three colours in accordance with the final fixed point (ie one colour for each of the three points):

$$x_{n+1} = \frac{2x_n}{3} + \frac{x_n^2 - y_n^2}{D}$$

and

$$y_{n+1} = \frac{2y_n}{3} - \frac{2x_n y_n}{D}$$

where

$$D = 3(x_n^2 + y_n^2)$$

Note that any number of dimensions may be used in such iterations.

4.1.2 The Butterfly Effect

This shows how a little initial difference can produce major long term differences, but that can yet be harnessed into algorithmic control;

$$x_{n+1} = s \cdot x_n \cdot (1 - x_n)$$

and

$$y_{n+1} = s \cdot y_n \cdot (1 - y_n)$$

If this is taken such that x_0 and y_0 are not the same (but very similar) the two coordinates diverge when s approaches 4.

However if the average $a_n = (x_n + y_n)/2$ is plotted as a_n against a_{n+1} a distinct relationship shape emerges for different starting points of x_0 and y_0 , demonstrating that the fractal picture is not always in the most obvious relationship.

4.1.3 Variations within an Iterated Function System

Start with an equilateral triangle of unit length sides. Below are three rules;

- 1 Scale the triangle by 0.5.
- 2 Scale the triangle by 0.5 then translate by +0.5 in the x-direction
- 3 Scale the triangle by 0.5 then translate by +0.5 in the y-direction

These rules can in themselves be used to produce a range of fractals. For example:

- a. Apply rule 1 to each original visible triangle.
- b. Apply rule 2 to each original visible triangle.
- c. Apply rule 3 to each original visible triangle.
- d. Combine the resultant triangles into one composite mapping, replacing the original triangles.
- e. Return to a. on all of the new triangles.

or

Start with a shading colour for each new triangle

- a. Apply rule 1 to each original visible triangle.
- b. Apply rule 2 to the triangle just produced.
- c. Apply rule 3 to the triangle just produced.
- d. Change shading colour.
- e. Return to a. for the last triangle.

or

Start with a shading colour for each new triangle

- a. Apply rule 1 to each original visible triangle.
- b. Apply rule 2 to the triangle just produced.
- c. Apply rule 1 to the triangle just produced.
- d. Apply rule 2 to each original visible triangle.
- e. Apply rule 3 to the triangle just produced.
- f. Change shading colour.
- g. Return to a. for the last triangle.

etc

4.1.4 Ford Fractal Froth

Choose two integers, a and b. Draw a circle with radius $1/2b^2$ and centred on the point $(a/b, 1/2b^2)$. Do this for as many combinations of a and b that you like.

The circles never overlap (only touch) and are all tangential to the x-axis. If you do enough circles the result is fractal.

4.1.5 The Carotid-Kundalini function

Plot the curves

$$y = \cos(n.x.\cos(x))$$

where $(-1 < x < 0, n=1,2,3,4\dots)$

Observe the fractal structure with gaps repeated at different size scales and with ever increasing spacing as x reduces in size.



4.2 Computer Programs

Its a long time since we used Basic so apologies to those of you regular programmers who look with horror at the coding. Originally the programs were written to demonstrate fractals to schoolchildren, not to publish.

Sierpinski's Triangle

```
SCREEN 2
CLS
PRINT "Here is the basic triangle for the game."
PRINT "When you want to play the game press any key"
LINE (300, 10)-(10, 180)
LINE (10, 180)-(600, 180)
LINE (600, 180)-(300, 10)
PRINT
INPUT "Press enter to continue"; v$
PRINT "Where do you want to start ?"
INPUT "Input the x coordinate"; x
INPUT "Input the y coordinate"; y
a: INPUT "How many points do you want plotted this time"; plots
IF plots < 1 THEN GOTO b
CLS
FOR i = 1 TO plots
  r = RND(i)
  IF 3 * r < 1 THEN
    x = (x + 300) / 2
    y = (y + 10) / 2
  ELSE
    y = (y + 180) / 2
  IF 3 * r < 2 THEN
    x = (x + 10) / 2
  ELSE x = (x + 600) / 2
  END IF
  END IF
  PSET (x, y)
NEXT i
GOTO a
REM you can end the program here if you wish
b: PRINT "We are now going to look at a small part of the triangle"
PRINT "The part inside the circle"
CIRCLE (230, 160), 85, 6
INPUT "Press enter to continue"; x$
CLS
```



```

PRINT "First we enlarge the circle"
PRINT "You will need more plots this time"
CIRCLE (300, 100), 230
INPUT "Press enter to continue"; x$

c: INPUT "How many points do you want plotted this time"; plots
IF plots < 1 THEN END
CLS
FOR i = 1 TO plots
r = RND(i)
IF 3 * r < 1 THEN
x = (x + 300) / 2
y = (y + 10) / 2
ELSE
y = (y + 180) / 2
IF 3 * r < 2 THEN
x = (x + 10) / 2
ELSE x = (x + 600) / 2
END IF
END IF
IF (x - 230) ^ 2 + (y - 160) ^ 2 < 85 * 85 THEN
PSET ((x - 230) * 230 / 85 + 300, (y - 160) * 230 / 85 + 100)
END IF
NEXT i
GOTO c
END

```

Koch Snowflake

```

REM Koch snowflake
DECLARE SUB a ()
DECLARE SUB b ()
DIM SHARED turns(800) AS STRING, oldturns(800) AS STRING
SCREEN 11, 1
col = 4
scale = 4
pi = 3.1415926
numturns = 3
s = 324 '4*3*3*3*3
direction = 4 * pi / 3
uinit = 240
vinit = 400

```



```

CLS
PRINT "This program draws a snowflake"
PRINT : PRINT : PRINT "Press any key to move onto the next level of detail"
INPUT ""; x$

```

```

FOR i = 1 TO numturns STEP 1
  turns(i) = "R"
NEXT i
CLS

```

```

CALL a
INPUT x$

```

```

FOR stage = 1 TO scale STEP 1
  FOR i = 1 TO numturns STEP 1
    oldturns(i) = turns(i)
  NEXT i
  CALL b
  numturns = numturns * 4
  s = s / 3
  CLS

```

```

CALL a
INPUT x$
NEXT stage

```

```

SUB a
  SHARED s, direction, pi, uinit, vinit, numturns, col
  uold = uinit
  vold = vinit
  FOR i = 1 TO numturns STEP 1
    u = uold + s * COS(direction)
    v = vold + s * SIN(direction)
    LINE (uold, vold)-(u, v), col
    uold = u
    vold = v
  IF turns(i) = "L" THEN
    direction = direction - pi / 3
  ELSE
    direction = direction + 2 * pi / 3
  END IF
  NEXT i
END SUB

```

```

SUB b

```

```

    SHARED numturns
    j = 1
    FOR i = 1 TO numturns STEP 1
        turns(j) = "L"
        j = j + 1
        turns(j) = "R"
        j = j + 1
        turns(j) = "L"
        j = j + 1
        turns(j) = oldturns(i)
        j = j + 1
    NEXT i
END SUB

```

Henon Attractor

```

CLS
SCREEN 2
s1 = 225
f1 = 1.4
s2 = 225
f2 = .35
x = .4
y = .1
FOR i = 1 TO 2000
    x1 = y + 1 - 1.4 * x * x
    y = .3 * x
    IF s1 * (x1 + f1) < 640 AND x1 + f1 > 0 THEN
    IF s2 * (y + f2) < 350 AND y + f2 > 0 THEN
        PSET (s1 * (x1 + f1), s2 * (y + f2))
    END IF
    END IF
    x = x1
NEXT i

```

4.3 Glossary

Actuary	A smooth operator
ARCH process	Autoregressive conditional heteroskedasticity, a nonlinear stochastic process, where the variance is time varying and conditional upon the past variance. see Peters p 306
AR(1) Model	A stochastic process where the next observation is more likely to move towards the mean see Peters p 75
Brownian Motion	see 3.2.7., which follows Hastings and Sugihara. Wilkie (JIA 114(1987) p 60) describes this process as a Weiner Process pointing out that the original description of Brownian Motion was in 3 dimensional space.
Black Scholes Option Pricing	A solution to the problem of pricing options under particular assumptions see Kemp 1997, Appendix A.
Capital Asset Pricing Model	An equilibrium- based asset-pricing model..... The simplest version states that assets are priced according to their relationship to the market portfolio of all risky assets, as determined by the securities' beta. (Peters p 307)
Chaos Theory	The study of nonlinear dynamic systems, see 2.3.1.
CHAID	A method of analysing data by ranking a large input set by the cross correlations with the output set.
Cluster Analysis	A method of dividing a data set into relatively homogeneous groups, by looking at the distance each data point is from the origin, each point with n attributes positioned in n-space.
Extreme Value Theory	The study of the extreme values (the maximum or minimum) of samples of independent random variables, see Longin (1993)
Fourier Transform	The Fourier transform of spatial data describes spatial periodicity.... The Fourier transform of a function f describes f as a sum of multiples of simple periodic functions, sines and cosines, at a fundamental frequency and its harmonics. (Hastings and Sugihara pp62-3)
Fractal	see 3.2.2
Fractal Dimension	see 3.2.5.
Fractal Process	see 3.2.7.
Fractilious	A delicious fractal
Fractophilic	Obsessed by fractals
Fractuary	Akin to an alchemist to chemists, someone who studies discontinuities in mortality rates.

Gaussian	A system or process where the error term follows the gaussian distribution. In other circumstances this distribution would be called the Normal Distribution. We avoided this because one point we wanted to avoid in this paper was the suggestion that this distribution was superior to any other Stable Paretian Distribution
GLIM	A software package used to fit Generalised Linear Models
Hurst exponent	A measure of the bias in fractional brownian motion.
IFS	Iterative Function System see 3.2.4 example 2
Linear	A system using curves which would pass as graduated or smoothed by actuaries; generally continuous and differentiable
Monte Carlo simulations	A series of trials based on a stochastic model
Natural fractal	A random fractal observed in nature
Neural Networks	A model loosely based on the brain, Has the ability to learn and can be said to retain memory.
Power Laws	see 3.2.9.
Random fractal	see 3.2.3.
Self Similarity	see 3.2.3.
Seriously tricky maths	Techniques studied post “O” level (GCSE) mathematics
Stable Levy Distribution	see 3.2.10.
Stable Paretian Distribution	see Stable Levy Distribution
Stochastic Process	A collection of random variables ordered in time. For a single outcome of the process we have only one observation on each random variable and these values evolve in time according to probabilistic laws. (Chatfield p33)
Strange Attractor	Objects of attraction in the phase state of nonlinear dynamic systems that are difficult to define in Euclidean terms. Mandelbrot says that these are all fractals.
Surreal Numbers	see 2.3.

Topological dimension

Traditional concept of dimension viz:
area grows by length squared and
volume by length cubed, as objects are
rescaled.

Wilkie Model

An investment model proposed by AD
Wilkie, see BAJ Vol 1 p777.

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The above are the sources for most of the illustrations in this paper. Please note that there are restrictions on their reproduction.

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