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PENSION FUNDING MODELLING AND STOCHASTIC INVESTMENT RETURNS

by

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ABSTRACT

The paper describes a mathematical model for investigating the behaviour of defined benefit pension schemes in the presence of stochastic investment returns.

Interest is particularly focused on the efficacy of the various control variables at the disposal of the actuary, including the choice of amortization period, the delay in fixing contributions, the frequency of valuation and the choice of funding method. The paper closes with suggestions for topics for further investigation.

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1. INTRODUCTION

Defined benefit pension schemes, which are common in a number of countries including the UK, USA, Canada and the Netherlands, are arrangements for a group of members where the benefits promised in the event of various contingencies are defined by a formula while the contributions (to be paid by the employer and possibly the member) are to be determined by the actuary as part of the regular valuation process.

The fund associated with such a scheme can be regarded as a reservoir into which income from contributions and investment earnings (including the proceeds of sales and maturities and unrealised capital growth) flow and out of which benefit payments on the contingencies of age retirement, disability, death and withdrawal and so on would be made.

The most financially significant benefit is age retirement. For this contingency, the benefit would be in the form of a pension, payable while the member is alive. There may also be a lump sum benefit payable at retirement and an entitlement to a reversionary pension payable to a surviving spouse. The defined benefit formula for the basic annual pension to be member would be normally of the form:

Annual Pension $=$ *K* $\times$ *(number of years of membership)* $\times$ *(earnings averaged over the h years before retirement)*

where K (the accrual rate) and h are specified in the scheme rules. In contrast, the annual contribution formula would be of the form

Annual Contribution $=$ *c* $\times$ *(current pensionable earnings)*

where c is not specified in the scheme rules but is determined by the *"funding method"* used by the actuary at each valuation. The valuations take place at regular intervals and the actuary values the prospective liabilities (i.e. benefit promises) allowing for the value of the future contributions which are expected to be paid at the assumed rate c, and compares this result with the value of the assets currently held in the fund. These calculations (and more detailed analyses) are used to determine c, which is held fixed for the period up to the next valuation.

This financing arrangement depends critically on the presence of regular valuations at which assets, prospective liabilities and future contributions are compared. At each valuation, the actuary is required to make detailed assumptions about the demographic and economic future of the pension scheme. These assumptions may constitute *"best estimates"* of the various parameters but are **not** long term predictions as the actuary will have the opportunity to revise these estimates at the next valuation (say in one year's time) and at subsequent valuations.

The methods and assumptions available to the actuary in these routine valuations are not prescribed in UK, US or Canadian practice. In the event of a surplus or a deficiency being revealed at the next valuation, the contribution rate would be adjusted for the future. The financial status of the scheme would then be reviewed at the next valuation. In the UK, it is common for these valuations to be annual, although legislation requires a valuation to be performed at least every 3½ years.

In recent experience, one of the principal sources of surplus or deficiency has been the rate of investment return on pension scheme assets. In this paper, we focus on the effect of variability in the rate of investment return on the financing of pensions and review the work that has been completed in recent years on the modelling of pension funds to represent the effects of stochastic investment returns.

A mathematical model is used to represent the financial structure of a defined benefit pension scheme, in particular the relationship between the contribution rate in year t, C(t) and the fund level at time t, F(t). The model can be regarded as an extension to that originally proposed by Trowbridge (1952). We focus on the effect of varying investment returns, through the use of different stochastic representations, and consider the behaviour of the moments EF(t), EC(t), Var F(t) and Var C(t) as t varies, and in particular as t tends to infinity. We are also interested in the effect of various control variables at the disposal of the actuary e.g. amortization period (for dealing with valuation surpluses and deficiencies), delay in fixing contributions, frequency of valuation and the choice of funding method.

2. TYPES OF FUNDING METHOD

In common practice, there are a number of different types of pension funding methods used. These can be categorised in a number of ways. Thus, in the UK, the split between accrued benefit methods and projected benefit methods is widely recognised. We shall use a different categorisation based on the mathematical structure of the fundamental equations. Therefore, we shall consider individual and aggregate funding methods.

In pension funding, the normal cost is used to describe the (stable) level of contribution which would apply if all the valuation assumptions made were to be borne out in actual experience. The actuarial liability is used to describe the mathematical reserve held (in a life insurance sense): for some pension funding methods it can be thought of as the difference between the present value of benefit promises expected to be paid out of the pension scheme and the present value of normal contributions expected to be paid to the pension scheme.

With individual funding methods (e.g. Projected Unit Credit and Entry Age Normal), the normal cost (NC) and the actuarial liability (AL) are calculated separately for each member and then summed to give the totals for the population under consideration. With aggregate funding methods (e.g. Aggregate and Attained Age Normal), there may not be explicit determination of a normal cost or actuarial liability; instead the group of members is considered as an entity, ab initio.

Let C(t) and F(t) be the overall contribution and fund level at time t. We consider the case where F(t) is measured in terms of the market value of the underlying assets.

For an individual funding method,

$$C(t) = \sum NC(x,t) + ADJ(t) = NC(t) + ADJ(t) \tag{1}$$

where the summation is taken over all members and where NC(x,t) is the normal cost for a member aged x at time t, NC(t) is the total normal cost at time t and ADJ(t) is an adjustment to the contribution rate at time t represented by the liquidation of the unfunded liability at time t, UL(t). UL(t) is defined by

$$UL(t) = \sum AL(x,t) - F(t) = AL(t) - F(t)$$

where the summation is taken over all members and where AL(x,t) is the actuarial liability for a member aged x at time t and AL(t) is the total actuarial liability, in respect of all members, at time t.

For an aggregate method, the overall contribution is directly related to the difference between the present value of future benefits and the fund. Specifically,

$$C(t) = (PVB(t) - F(t))\ S(t)/PVS(t) \tag{2}$$

where S(t) is the total salaries of active members at time t, PVB(t) is the present value of future benefits (of all members including pensioners) at time t and PVS(t) is the present value of future salaries (of active members) at time t. Here, the difference, PVB(t) - F(t), is spread over the remaining period of membership of current members, effectively by an annuity which allows for expected earnings progression and with expected present value $\dfrac{PVS(t)}{S(t)}$.

This paper considers the behaviour of C(t) and F(t) in the presence of stochastic investment returns of a particular form, to be described below.

3. THE MATHEMATICAL MODEL

At any discrete time t (for integer values t=0, 1, 2 ...) a valuation is carried out to estimate C(t) and F(t), based only on the scheme membership at time t. However, as t changes, we do allow for new entrants to the membership so that the population remains stationary - see assumptions below.

In the mathematical discussion, we make the following assumptions.

1. All actuarial assumptions are consistently borne out by experience, **except** for investment returns.

2. The population is stationary from the start. (We could alternatively assume that the population is growing at a fixed, deterministic rate i.e. that the population is stable in the sense of Keyfitz (1985)).

3. There is no inflation on salaries, and no promotional salary scale. (It would be possible to incorporate a fixed promotional salary scale simply through a change of notation). Inflation on salaries at a deterministic rate is incorporated by considering interest rates that are *"real"* relative to salaries. In parallel we assume that benefits in payment increase at the same rate as salaries. We therefore consider variables to be in real terms. For simplicity, each active member's annual salary is set at 1 unit at entry.

4. The interest rate assumption for valuation purposes is fixed, i_v.

5. The *"real"* interest rate earned on the fund during the period, $(t,t+1)$ is $i(t+1)$. The corresponding *"real"* force of interest is assumed here to be constant over the interval $(t,t+1)$ and is written as $\delta(t+1)$. Thus, $1+i(t+1) = \exp(\delta(t+1))$. $i(t)$ is defined in a manner consistent with the definition of $F(t)$.

6. We define $E[1+i(t)] = E[\exp \delta(t)] = 1+i$. We assume that $i=i_v$ where i_v is the valuation rate of interest. This means that the valuation rate is correct ***"on average"***. This assumption is not essential mathematically but it is in agreement with classical ideas on pension fund valuation.

7. It is assumed that the contribution income and benefit outgo occur at the start of each period (or scheme year).

8. The initial value of the fund (at time zero) is known, i.e. Prob $[F(0)=F_0]=1$ for some F_0.

9. Valuations are carried out at annual intervals (this is relaxed in section 6).

Assumptions 1, 2, 3 and 4, imply that the following parameters are constant with respect to time, t (after rescaling to allow for growth in line with salary inflation):

NC the total normal contribution

AL the total actuarial liability

B the overall benefit outgo (per unit of time)

S the total pensionable payroll

PVB the present value of future benefits (for active members and pensioners)

PVS the present value of future pensionable earnings.

Further, assumptions 1, 2, 4, 7 and 9 imply that the following equation of equilibrium holds:

$AL = (1+i)\ (AL + NC - B)$ or equivalently $B = d.AL+NC$ (3)

where $d=i(1+i)^{-1}$, the compound interest discount rate.

This equation of equilibrium can also be found in the earlier papers of Trowbridge (1952) and Bowers et al (1976).

The paper adopts a discrete time approach. It is possible to use a continuous time formulation, in which case, the mathematical approach is based on stochastic differential equations.

For individual funding methods, there are a number of choices for the ADJ(t) term. The most commonly used are the spread method (UK) and the amortization of losses method (Canada and USA).

Under the spread method, ADJ(t)=k. UL(t) (4)

where k= $(\ddot{a}_{\overline{M}|})^{-1}$ calculated at the valuation rate of interest. So the unfunded liability is spread over M years, where M would be chosen by the actuary. It should be noted that this definition of ADJ(t) uses the same fraction of the unfunded liability regardless of the sign of the latter. So, surpluses and deficiencies would be treated in a comparable manner - this would not always be the case in practice. Typical values of M would be 20-25 years, corresponding approximately to the average remaining period of membership of current members. k is the fraction of UL(t) that makes up ADJ(t) and can be thought of as a penalty rate of interest that is being charged on the unfunded liability, UL(t).

For the amortization of losses method, we introduce the actuarial loss experienced during the intervaluation period (t-1, t), l(t), which is defined as the difference between UL(t) and the value of the unfunded liability if all the actuarial assumptions had been realized during the year (t-1,t). Then ADJ(t) is defined as the total of the intervaluation losses arising during the last m years (i.e. between t-m and t) divided by the expected present value of an annuity for a term of m years calculated at the valuation rate of interest (i.e. spread over an m year period). Thus,

$$ADJ(t) = \frac{\sum_{j=0}^{m-1} l(t-j)}{\ddot{a}_{\overline{m}|}} \quad . \tag{5}$$

As Dufresne (1989) has shown, under these conditions, UL(t) satisfies a recurrence relation

$$UL(t) = (1+i)\,(UL(t+1) - ADJ(t-1)) + l(t)$$

which can be solved to give

$$UL(t) = \sum_{j=o}^{m-1} \lambda_i \, l(t-j) \text{ where } \lambda_j = \frac{\ddot{a}_{\overline{m-j}|}}{\ddot{a}_{\overline{m}|}} \quad . \tag{6}$$

Then $F(t)=AL(t)-UL(t)$ and $C(t)=NC(t)+ADJ(t)$.

Here, m would be chosen by the actuary and would typically lie in the range 5-15 years.

4. STOCHASTIC INVESTMENT RETURNS

4.1 Independent and Identically Distributed i(t)

As a first model, we assume that the earned real rates of investment return, i(t) for $t \geq 1$, are independent and identically distributed random variables, with $i(t) > -1$ with probability 1, and with $Ei(t)=i=i_v$ and $\text{Var } i(t)=\sigma^2 < \infty$.

Dufresne (1988, 1989) has described in detail the properties of

- individual funding methods: spread method for ADJ(t)
- aggregate funding methods

- individual funding methods: amortization of losses method for ADJ(t).

For the spread method,

$$C(t) = NC + k\,(AL - F(t)) \tag{7}$$

and

$$F(t+1) = (1+i(t+1))(F(t)+C(t)-B) \tag{8}$$

Equation (7) includes a negative feedback component, whereby the current status, F(t), is compared with a target (AL) and corrective action is taken to deal with any discrepancy. Then, Dufresne (1988) shows that

$$E\,F(t) = q^t F_0 + r(1-q^t)/(1-q) \tag{9}$$

where $q = (1+i)(1-k)$ and $r = (1+i)(NC+k.AL-B)$,

and

$$\text{Var F(t)} = b \sum_{j=1}^{t} a^{t-j}\,(E\,F(j))^2 \tag{10}$$

where $a = q^2\,(1+b)$ and $b = \sigma^2(1+i)^{-2}$.

Then, $q = \dfrac{\ddot{a}_{\overline{M-1}|}}{\ddot{a}_{\overline{M}|}}$ so if M > 1, 0 < q < 1 and the following limits exist

and
$$\left.\begin{aligned} &\lim_{t\to\infty} E\,F(t) = \frac{r}{1-q} = AL,\ \textit{using}\ (3) \\ &\lim_{t\to\infty} E\,C(t) = NC,\ \textit{using}\ (7) \end{aligned}\right\} \tag{11}$$

If a < 1, then Dufresne (1988) shows that

and
$$\left.\begin{aligned} &\lim_{t\to\infty} Var\ F(t) = \frac{bAL^2}{(1-a)} \\ &\lim_{t\to\infty} Var\ C(t) = \frac{bk^2Al^2}{(1-a)} \end{aligned}\right\} \tag{12}$$

If a ≥ 1, then both of these limiting variances would be infinite. The restriction that a < 1 implicitly places a restriction on the choice of M viz

$$a<1 \text{ is equivalent to } \ddot{a}_{\overline{M}|} < \frac{1}{1-f} \quad \text{where } f = v\sqrt{\frac{1}{1+b}}$$

This is equivalent to $M < M_0 = \frac{1}{\delta} \ln\left[\frac{(1+i)(1+b)^{1/2}-1}{(1+b)^{1/2}-1}\right]$ and provides a restriction on the feasible values of M for convergence. Table 1 provides illustrative values of M_0 for different combinations of i and σ. We note the extent to which M_0 decreases as i and σ each increase.

Dufresne (1988) also considers expressions for the covariances of F and C in the limit and deals separately with the special case $M=1$.

For the aggregate funding methods, equation (8) holds with

$$C(t) = (PVB - F(t))\, S/PVS \tag{7a}$$

so that
$$E\,F(t) = q'^{n}F_0 + r'\,(1-q'^{n})/(1-q') \tag{13}$$

where
$$q' = (1+i)(1-S/PVS),\ r'=(1+i)\,(S.PVB/PVS - B).$$

Then $0<q'<1$ and we note the similarity between equations (9) and (13) and the definitions of q and q'. Indeed, by defining N such that $\ddot{a}_{\overline{N}|} = \frac{PVS}{S}$, we can regard q and q' as being of the same form. Hence similar results to (11) and (12) apply in this case.

For the amortization of losses method, Dufresne (1989) shows that

$$l(t) = (i(t)-i)\,(UL(t-1) - ADJ(t-1) - (1+i)^{-1}AL)$$

and, using (5) and (6), we obtain a difference equation for l(t) viz

$$l(t)=(i(t)-i)\Big(\sum_{j=1}^{m-1}\beta_j\, l(t-j)-(1+i)^{-1}AL\Big) \tag{14}$$

where the coefficients β_i are defined by $\beta_j = \dfrac{a_{\overline{m-j|}}}{\ddot{a}_{\overline{m|}}}$.

$$\left.\begin{array}{lll} \text{Then} & E(l(t)) = 0 \text{ for all } t \geq 1 \\ \text{and} & Cov\ (l(s), l(t+1)) = 0 \text{ for } 1 \leq s \leq t \end{array}\right\} \tag{15}$$

so that the C(t) form an uncorrelated sequence.

If $\sigma^2 \sum_{j=1}^{m-1} \beta_j^2 < 1$, then Dufresne (1989) shows that

$$\left.\begin{array}{l} \lim_{t\to\infty} Var l(t) = \dfrac{\sigma^2(1+i)^{-2}AL^2}{1-\sigma^2\sum_{j=1}^{m-1}\beta_j^2} = x, \ say \\ \lim_{t\to\infty} Var C(t) = \dfrac{mx}{\ddot{a}^2_{\overline{M|}}} \\ \lim_{t\to\infty} Var F(t) = \dfrac{x}{\ddot{a}^2_{\overline{M|}}} \sum_{j=0}^{m-1} (\ddot{a}_{\overline{m-j|}})^2 \end{array}\right\} \tag{16}$$

4.2 Autoregressive Rates of Return

In order to investigate the effects of autoregressive models for the earned real rate of return, we follow the suggestion of Panjer and Bellhouse (1980) and consider the corresponding force of interest and assume that it is constant over the interval of time (t,t+1).

4.2.1 First Order Autoregressive Models

Now it is assumed that the (earned real) force of interest is given by the following stationary (unconditional) autoregressive process in discrete time of order 1 (AR(1)):

$$\delta(t) = \theta + \varphi\,[\delta(t-1) - \theta] + e(t) \tag{17}$$

where e(t) for t=1, 2, ... are independent and identically distributed normal random variables each with mean 0 and variance γ^2.

This model suggests that interest rates earned in any year depend upon interest rates earned in the previous year and some constant level. Box and Jenkins (1976) have shown that, under the model represented by equation (17),

$$E[\delta(t)] = \theta$$

$$Var\,[\delta(t)] = \frac{\gamma^2}{1-\varphi^2} = \upsilon^2 \text{ , say}$$

$$Cov\,[\delta(t),\,\delta(s)] = \frac{\gamma^2}{1-\varphi^2}\,\varphi^{|t-s|} = \gamma(t,s), \quad \text{say.}$$

The condition for this process to be stationary is that $|\varphi| < 1$.

It then follows that $E(\exp \delta(t)) = \exp(\theta + \frac{1}{2}\upsilon^2) = 1+i$ and

$Var(\exp \delta(t)) = \exp(2\theta+\upsilon^2).(\exp(\upsilon^2)-1)$.

We first consider individual funding methods and the spread method for choosing ADJ(t). It is convenient to re-parametrise equation (8) as

$$F(t+1) = (1+i(t+1))\,(QF(t)+R)$$

where $Q=1-k$ and $R=NC-B+k.AL=AL(k-d)$.

Haberman (1992a) then shows that

$$EF(t) = F_0 Q^t c^t e^{z\varphi^t} e^{-z} + \frac{R}{Q}\sum_{s=0}^{t-1} Q^{t-s} c^{t-s} e^{z\varphi^{t-s}} e^{-z} \tag{18}$$

where $z = \upsilon^2\varphi\,(1-\varphi)^{-2}$ and $c = exp\left\{\theta+\frac{1}{2}\upsilon^2\,\left(\frac{1+\varphi}{1-\varphi}\right)\right\}$.

If $Qc<1$ then $\lim\limits_{t\to\infty} E\,F(t)$ exists and the following approximation to the limit is derived

by Haberman (1992a):

$$\lim_{t\to\infty} E\,F(t) \simeq \frac{Rc}{1-Qc}\,e^{-z} \tag{19}$$

For convergence, we require that $Qc<1$. This is equivalent to requiring that

$$M < M_1 = \frac{1}{\delta} \ln\left[\frac{c-1}{vc-1}\right]$$

For given φ, υ and i, there is thus a maximum value of M for which convergence holds. This provides an important restriction on the feasible values of the spread period, M. Table 2 presents values of this maximum feasible M i.e. (M_1) for different combinations of i, φ and υ. Negative values of φ do not lead to any infringement of $Qc<1$ so values have been tabulated only for $\varphi>0$. We note that the extent to which M_1 decreases with increasing i, φ and υ.

Similarly, we can obtain expressions for EC(t) for finite t and in the limit as $t \to \infty$.

It is possible also to consider second moments. Thus, Haberman (1922a) shows that

$$E(F(t)^2) = \frac{2R^2}{Q^2} \sum_{r=1}^{t-1} \sum_{s=0}^{r-1} Q^{t-s} Q^{t-r} \exp((t-s)\theta + (t-r)\theta + \upsilon^2 H(t,r,s))$$

$$+ \frac{R^2}{Q^2} \sum_{s=0}^{t-1} Q^{2(t-s)} \exp(2(t-s)\theta + \upsilon^2 H(t,s,s)) \tag{20}$$

where $$H(t,r,s) = \left[\frac{1+\varphi}{1-\varphi}\right] \frac{(t-s+3(t-r))}{2} - \varphi \frac{(3-2\varphi^{t-r}-2\varphi^{t-s}+\varphi^{r-s})}{(1-\varphi)^2} . \tag{21}$$

Then, $\lim_{t\to\infty} E(F(t)^2)$ exists if $Qc<1$ and $Q^2cw<1$ and Haberman (1922a) obtains the

approximate result:

$$\lim_{t\to\infty} E(F(t)^2) \simeq \frac{e^{-3z}2R^2Qc^2w}{(1-Qc)(1-Q^2cw)} + e^{-4z}\frac{R^2cw}{(1-Q^2cw)} \tag{22}$$

where $w = \exp\left\{\theta+\frac{3}{2}\frac{(1+\varphi)}{(1-\varphi)}\upsilon^2\right\}$.

As noted above, for convergence, as $t\to\infty$, we require $Qc<1$ and $Q^2cw<1$. The requirement that $Q^2cw<1$ is more stringent than $Qc<1$ since $w>c$. This requirement is equivalent to

$$M<M_2 = \frac{1}{\delta}\ln\left\{\frac{\sqrt{cw}-1}{v\sqrt{cw}-1}\right\} .$$

Table 3 presents values of this maximum feasible spread period, $M_2<M_1$, for different combinations of i, φ and υ. Values of $\varphi<-0.3$ do not lead to any infringement of $Q^2cw<1$. We note the extent to which M_2 decreases with increasing i, φ and υ. Since the independent and identically distributed model for investment returns, used in section 4.1, corresponds to $\varphi=0$, we would expect that the maximum feasible values for the spread period for convergence of Var F(t) as $t\to\infty$ to correspond in Tables 1 and 3. Comparison indicates that the values in Table 1 lie approximately midway between the values in Table 3 for $\varphi=\pm0.1$.

For aggregate funding methods similar results can be obtained.

For individual funding methods with the amortization of losses method for choosing ADJ(t), the discussion is complicated because of the presence of non-linear effects in the resulting equations. It is convenient here to model i(t), rather than $\delta(t)$, as a stationary AR(1) process:

$$i(t) = i+\varphi(i(t-1)-i) + e(t) \tag{23}$$

where we repeat that $E(i(t)) = i$, $|\varphi| < 1$ and $\{e(t)\}$ is a sequence of independent and identically distributed normal random variables with mean 0 and variance σ^2.

Gerrard and Haberman (1992) demonstrate how equations (14) and (23) can be combined through the use of generating functions to discuss the behaviour of E(l(t)) for finite t and, in the limit, as $t \to \infty$. Some progress is also made with $E(l(t)^2)$, and hence with Var (l(t)).

4.2.2 Second Order Autoregressive Models

Haberman (1992a) discusses briefly the more complicated case of stationary second order autoregressive models for individual funding methods with the spread method. In this case, equation (17) is replaced by

$$\delta(t) = \theta+\varphi_1(\delta(t-1)-\theta) + \varphi_2(\delta(t-2)-\theta) + e(t) \tag{24}$$

where e(t) for $t=1,2, \ldots$ are independent, identically distributed normal random variables, each with mean 0 and γ^2. In parallel to the results of section 4.2.1, we quote Box and Jenkins (1976) who have shown that, for the above model,

$$E\ (\delta(t)) = \theta$$

$$Var\ (\delta(t)) = \left[\frac{1-\varphi_2}{1+\varphi_2}\right] \left[\frac{\gamma^2}{(1-\varphi_2)^2-\varphi_1^2}\right] = \upsilon^2 \text{ , say}$$

$$Cov(\delta(t),\ \delta(s)) = \upsilon^2\ (\lambda\psi_1^{|t-s|} + (1-\lambda)\ \psi_2^{|t-s|})$$

where $\lambda = \dfrac{\psi_1(1-\psi_2^2)}{(\psi_1-\psi_2)(1+\psi_1\psi_2)}$ and $(1 - \lambda) = \dfrac{\psi_2(\psi_1^2-1)}{(\psi_1-\psi_2)(1+\psi_1\psi_2)}$

and ψ_1^{-1} and ψ_2^{-1} are the solutions of the characteristic equation: $1-\varphi_1 r-\varphi_2 r^2 = 0$.

(If $\varphi_2=0$, these results reduce to the earlier ones for the AR(1) model).

For stationarity, we now require

$$\left.\begin{array}{l} \varphi_1+\varphi_2<+1 \\ \varphi_2-\varphi_1<+1 \\ -1<\varphi_2<+1. \end{array}\right\}$$

It is then possible to construct equations for the moments of F(t) and C(t) in finite time and in the limit, as $t\rightarrow\infty$, which correspond in format to those for the AR(1) case. The full details are not pursued here.

4.2.3 Conditional Autoregressive Models

A disadvantage of the models used in sections 4.2.1 and 4.2.2 is that the uncertainty about $\delta(t)$ is independent of t i.e. $Var[\delta(t)] = \nu^2$, a constant. In reality, we would expect this level of uncertainty to depend on t. A model which allows for this feature would be the conditional AR(1) or AR(2) model considered by Bellhouse and Panjer (1981). In this case, it is assumed that the returns of the past years (and the corresponding forces of interest) are known, as initial conditions. We then believe that the asymptotic results derived in sections 4.2.1 and 4.2.2 would hold also for the conditional processes since, as $t \to \infty$, the initial values δ_0 and δ_{-1} would become increasingly insignificant.

5. OPTIMAL SPREAD PERIOD

In this section we focus on individual funding methods with the spread method, and we shall consider the existence of an *"optimal"* spread period, M. (For the amortization of losses method no such optimal choice of m exists).

In this section, we shall consider the relationship between Var F(t) and Var C(t) as M (or k) varies, with t fixed. Rather than take a particular finite t, we shall consider the limiting variances at $t \to \infty$ and indeed we shall consider these variances relative to the corresponding expectations (i.e. the coefficient of variation). Our consideration of the case where $t \to \infty$ is justified on the grounds that the results are mathematically tractable. We shall now introduce some new notation.

With $a<1$ and $2 \leq M < \infty$ (so $d<k<1$), we define

$$\alpha(k) = \frac{VarF(\infty)}{(EF(\infty))^2} \quad \text{and} \quad \beta(k) = \frac{VarC(\infty)}{(EC(\infty))^2} \tag{25}$$

and we regard α and β as functions of k. We could equivalently regard them as functions of M, given the 1-1 correspondence between k and M. However, it is more convenient to consider α(k) and β(k).

5.1 Independent and Identically Distributed i(t)

For the case of IID i(t), Dufresne (1988) has considered in detail the trade off between Var F(t) and Var C(t) in the limit as $t \to \infty$, as represented by α(k) and β(k), and for finite t under certain conditions. Thus, from (11) and (12), we have that

$$\alpha(k) = \frac{b}{1-y(1-k)^2}$$

$$\text{and } \beta(k) = \frac{AL^2}{NC^2} \cdot \frac{bk^2}{1-y(1-k)^2}$$

where $y=(1+i)^2 (1+b) = E(1+i(t))^2$. Assuming that $y>1$, Dufresne shows that

$$\frac{d}{dk}\alpha(k) < 0$$

$$\text{and} \quad \frac{d}{dk}\,\beta(k)\big|_{k^*} = 0 \quad \text{where } k^* = \; 1-\frac{1}{y} \; .$$

At k=k*, β(k) takes a minimum value. The value of the spread period corresponding to k* will be denoted by M*.

Formally, if y > 1, then both Var F(∞) and Var C(∞) become infinite for some finite M=M$_0$ (when a becomes equal to 1) and there exists a value M* such that

- for M$\leq$M*, Var F(∞) increases and Var C(∞) decreases with M increasing
- for M$\geq$M* both Var F(∞) and Var C(∞) increase with M increasing.

If y=1, Var C(∞)$\rightarrow$0 and Var F(∞)$\rightarrow\infty$ as M$\rightarrow\infty$, although Var F(∞) does stay finite for all M.

If y<1, Var C(∞)$\rightarrow$0 as M$\rightarrow\infty$ and Var F(∞) has a finite limit as M$\rightarrow\infty$.

The particular value of M* is determined by

$$k^* = 1-\frac{1}{y} = \frac{1}{\ddot{a}_{\overline{M}|}}$$

$$\text{i.e. if } i\neq 0 \quad M^* = -\frac{1}{\delta}\,\ln\left[\frac{vy-1}{y-1}\right] \tag{26}$$

$$\text{and if } i=0 \quad M^{*}=1+\frac{1}{\sigma^2} .$$

There is thus a trade off between variability in the fund, represented by α, and variability in the contribution rate, represented by β. This trade-off takes place but only up to $M=M^{*}$. Beyond this point, augmenting M causes both Var F and Var C to increase, With the objective of minimizing variances, any choice of $M>M^{*}$ should be rejected, for clearly some $M<M^{*}$ would reduce both Var F and Var C. If we regard M as being a parameter open to the choice of the actuary, then the optimal choices for M would lie in the region $1 \leq M \leq M^{*}$. Thus, we can describe this region as an ***"optimal"*** region.

Table 4 provides values of M as a function of i and σ (to the nearest integer). In the UK, it is common to choose M to correspond to the average remaining working lifetime of the current membership - with an average age of membership of 40-45 and a normal retirement age of 65 this would correspond to a choice of M in the range 20-25. We see from Table 3, that under particular combinations of i and σ our model indicates that this choice is <u>**not**</u> optimal. If $i=.03$ and $\sigma=.20$ then, for example, smaller values, namely those in the region $1 \leq M \leq 13$, would be more satisfactory.

5.2 <u>Autoregressive Rates of Return</u>

We shall consider here only the case of stationary AR(1) processes as a description for $\delta(t)$. Haberman (1992b) has explored the behaviour of the relative limiting values (as $t \rightarrow \infty$) of Var

F(t) and Var C(t) as functions of M (or equivalently as functions of k), by analysing the properties of the approximate results represented by equations (19) and (22) and the corresponding results for C(t). This mathematical discussion has been supported by numerical investigations.

Haberman (1992a) has based the exploration on the exact equations (18) and (20) and the results are reported here, subject to the constraints that $Qc < 1$ and $Q^2cw < 1$ for convergence, and with $F_0=0$.

The numerical investigation uses the following assumptions:

Population:	English Life Table No. 13 (Males) - Stationary
Entry Age:	30 (only)
Retirement Age:	65
Salary Scale:	Constant
Retirement Benefits:	Level life annuity of 2/3 of salary.

The parameter values used in the calculations are:

i (real):	-.01, +.005, +.01, +.03, +.05
υ	.05, .10, .15, .20, .25
φ	$\pm.1$, $\pm.3$, $\pm.5$, $\pm.7$, $\pm.9$
M	integer values between 1 and 500 (in steps of 1 up to 10, then in steps of 5 up to 100, then in steps of 10 up to 200, then in steps of 50), subject to $Qc<1$ and $Q^2cw<1$ for convergence.

Tables 5 and 6 show values of $\alpha^{1/2}$ and $\beta^{1/2}$, viewed as functions of M, for a range of values of M and φ with i=.01 and υ=.05 held fixed. Values for φ=0.9 are not given because of the failure of the series to converge.

The tables indicate the following general features viz

(i) $\alpha(\varphi,M)$ increases with M (for fixed φ) and with φ (for fixed M)

(ii) $\beta(\varphi,M)$ increases with φ (for fixed M) and decreases with increasing M (for fixed φ) except that for some values of φ (e.g. φ=0.1) there is a minimum at some M^*.

The corresponding values of α and β for different i and υ yield the same general features (details not shown). However, there are some exceptions which we discuss further in the paragraphs below - the exceptions refer to the turning points of $\beta(\varphi,M)$ for fixed φ.

Haberman (1992a) has also investigated numerically the trade-off between Var F(t) and Var C(t) as represented by their limiting values as $t\rightarrow\infty$, i.e. as represented by α and β. When values of α and β are plotted for combinations of i, υ and φ, we find that three distinct patterns emerge, unlike the situation when rates of return are independent, identically distributed random variables (as in section 5.1 and Dufresne (1988) which corresponds approximately to the case φ=0).

The three patterns in terms of profiles of $\beta(M)$ v $\alpha(M)$ are:

TYPE A: the profile has a minimum at M^* so $\underline{1 \leq M \leq M^*}$ is *"optimal"*.

TYPE B: the profile is monotonically decreasing so there is no *"optimal"* region. The choice of M will depend on the characteristics of the scheme sponsor and their particular attitude to the trade-off between variability in F and in C.

TYPE C: the profile is monotonically increasing so $\underline{M=1}$ is *"optimal"*.

When φ takes values -0.9, -0.7, and -0.5, the patterns are of Type B and there is no optimal region.

Table 7 corresponds to φ=-0.3, and shows the classification of α-β profiles and, where appropriate, the optimal regions for M. Tables 8-11 similarly refer to φ=-0.1, 0.1, 0.3 and 0.5. (Tables for φ=0.7 and 0.9 are not reproduced here). Given the set of values of M for which calculations have been performed, the optimal regions for M reported in Tables 7-11 are only approximations. Because we are interested only in general features, no attempts have been made at this stage to estimate more precisely the turning points in the α-β graphs (using, for example, numerical interpolation methods).

From Tables 8 and 9, corresponding to $\varphi=\pm0.1$, we note that the implied optimal values of M are consistent with those shown in Table 4 (from Dufresne (1988)) which would correspond approximately to the case φ=0.

The pattern of optimal M values across Tables 7-11 mirrors that for the IID case. In general, the optimal region decreases as i increases (for fixed υ and φ) and as υ increases (for fixed i and φ). Also, the optimal region decreases as φ increases from -1 to +1 (for fixed i and φ); thus, an increase in the autoregressive parameter φ appears to have a similar effect on the optimal region as an increase in the variance parameter υ.

6. FREQUENCY OF VALUATIONS

A second control variable available to the actuary is the frequency with which valuations are performed. In earlier sections, we have assumed that valuations are annual. Here, we shall consider the case of valuations every 3 years (and then more generally every n years where n is an integer). As noted in section 1, triennial valuations are common in the UK because of legislative and cost considerations. We shall consider only the case of individual funding methods with the spread choice for ADJ(t) and independent, identically distributed i(t). We here introduce j(t) to be the real rate of investment return earned during the t'th (three year) period.

In the triennial case, the equation of equilibrium (3) would become

$$AL=(1+j)(AL+NC'-B') \tag{27}$$

where NC′ and B′ now refer to 3 year rather than 1 year time periods and $(1+j) = (1+i)^3$.

The link between the pairs NC and NC′, B and B′ comes from the following straightforward compound interest relationships:

$$NC'=NC\ \ddot{a}_{\overline{3}|} \quad \text{and} \quad B'=B\ \ddot{a}_{\overline{3}|} \ . \tag{28}$$

Now equation (7) would become

$$C(t) = NC'+k_1\ (AL-F(t)),\ \text{for } t=0,1,2,\ \ldots \tag{29}$$

where $k_1 = 1/\ddot{a}^{j}_{\overline{M/3}|}$ calculated at the real rate of interest j effective over a triennium and corresponding to i effective per year. We note that in equation (29), t is measured in 3 year time units (rather than annual units as in equation (7)).

As noted earlier, $1+j=(1+i)^3$

and so $$k_1 = \frac{1-v^3}{1-v^M} \quad \text{where } v=(1+i)^{-1}.$$

We assume here that the contributions are paid at the start of each triennium. In reality, they would be paid annually; however, this feature introduces complexity into the mathematical formulation. By effectively working in 3 year time units, we avoid such complications.

Haberman (1993a) derives equations that correspond directly with (11) and (12). Thus, if $M>3$,

$$\lim_{t\to\infty} E\ F(t)=AL \quad \text{and} \quad \lim_{t\to\infty} E\ C(t)=NC' \tag{30}$$

And, providing that $y_1(1-k_1)^2<1$ where $y_1=E\ (1+j(t))^2=(1+2i+i^2+\sigma^2)^3 = y^3$,

$$\left.\begin{aligned}\lim_{t\to\infty} Var\ F(t) &= \frac{(Var j(t))AL^2}{(1+j)^2(1-y_1(1-k_1)^2)}\\ \lim_{t\to\infty} Var\ C(t)) &= \frac{(Var j(t))k_1^2AL^2}{(1+j)^2(1-y_1(1-k_1)^2)}.\end{aligned}\right\} \quad (31)$$

We now extend the definitions of α and β in equations (25) so that $\alpha_0(k)$ and $\beta_0(k)$ refer to the annual case and $\alpha_1(k_1)$ and $\beta_1(k_1)$ refer to the triennial case. Then, Haberman (1993a) demonstrates that an optimal spread period, M_1^*, exists for the triennial case providing that $y_1 > 1$. The corresponding value of k_1 is

$$k_1^* = 1-\frac{1}{y_1} = 1-\frac{1}{y^3} \quad (\text{for } i\neq 0).$$

Comparison of the resulting values of M^* (annual case) and M_1^* (triennial case) indicates that

$$M_1^* \simeq M^*+1.$$

Haberman (1993a) also compares the limiting variances in the annual and triennial cases (equations (12) and (31)) and obtains ranges for the spread period for which the variances are increased in the triennial case relative to the annual case. The existence of a spread period M_3 is demonstrated for which, in the triennial case, the relative limiting variances of <u>both</u> F(t) and C(t) are increased for values of M in the range (1, M_3). M_3 and M_1^* are found to be approximately equal, i.e. $M_3 \simeq M_1^* \simeq M^*+1$. This leads to the intuitively reasonable result that, with triennial valuations and a sensible choice of the spread period (i.e. within the optimal range), the limiting variances of both F(t) and C(t) are increased relative to the case where valuations are annual.

Haberman (1993a) also demonstrates how these results may be generalized to apply to valuations avery n years, where n is an integer. Similar results to (30) and (31) can be derived, for example, but expressed in terms of $k_n = \frac{1-v^n}{1-v^M}$, say, $(1+i)^{2n}$ and y^n. Again, the existence of a range of spread periods is described for which, relative to the annual case, the relative limiting variances of both F(t) and C(t) are increased.

7. DELAY IN FIXING CONTRIBUTIONS

We now introduce a new parameter into the formula for fixing the contribution rate, C(t). We consider only individual funding methods, with the spread method choice for ADJ(t), and aggregate funding methods. We allow for a time delay in the pension scheme's funding process and use the fund level at time t-p in order to calculate C(t). So we would use

$$C(t) = NC + k(AL - F(t\text{-}p)) \tag{32}$$

$$C(t) = (PVB - F(t\text{-}p)).S/PVS \tag{33}$$

which replace equations (7) and (7a) respectively where p is a non negative integer. The delay p may arise because of the time taken to prepare the financial accounts or to assemble the valuation data and to complete the actuarial valuation exercise. Alternatively, we can think of the parameter p (like the spread period M) as being a control variable at the disposal of the actuary and which can be used to control the behaviour over time of C(t) or F(t).

7.1 Independent and Identically Distributed i(t)

In the case of independent and identically distributed i(t), Haberman (1992c) considers the case p=1, and obtains explicit formulae for E F(t) and E C(t) for finite t. In particular it can be shown that

$$E\ F(t+1) = u\ EF(t) - uk\ EF(t-1)+r \tag{34}$$

with u=1+i. The solution is then

$$E\ F(t) = a_0 + a_1 x_1^t + a_2 x_2^t \tag{35}$$

where x_1 and x_2 are roots of the polynomial

$$x^2 - \mathrm{ux} + \mathrm{uk} = 0$$

and a_i are determined by the initial conditions. He demonstrates that, if $M \geq 2$ and

$a_{m1} > 1$,

$$\lim_{t\to\infty} E\ F(t) = AL \quad \text{and} \quad \lim_{t\to\infty} E\ C(t) = NC\ .$$

Using the method of generating functions, expressions for $E(F(t)^2)$ and $E(C(t)^2)$ and for the covariances are also obtained and the following limiting values are obtained,

$$\lim_{t\to\infty} Var\ F(t) = \frac{\sigma^2 AL^2(1+uk)}{u^2(1+uk-(\sigma^2+u^2)(1-uk+k^2+uk^3))}$$

and $$\lim_{t\to\infty} VarC(t) = k^2 \lim_{t\to\infty} VarC(t)$$

providing that, for convergence, u and k satisfy certain complicated conditions (see Haberman (1992c) for details).

The limiting variance of F(t) can be rewritten as

$$\frac{\sigma^2 AL^2}{u^2(1-(\sigma^2+u^2)(1+k^2-2ug_1))} \tag{37}$$

where $g_1 = \frac{u}{1+uk} < 1$ if $M > 1$. As noted by Haberman (1993b), this means that a comparison of the limiting variances for the cases p=0 (equation (12)) and p=1 (equation (37)) shows that the presence of a one year delay results in an increase in the limiting variances of both F(t) and C(t). Haberman (1993b) also demonstrates that an optimal spread period exists when p=1, providing that σ^2 is not too *"large"* (i.e. under approximately 300%), and that the optimal spread period M_2^* is approximately equal to M^*, the value when p=0.

This argument has been successfully extended by Zimbidis and Haberman (1993) who have considered non-negative values of the delay p in general as well as the specific values p=2,3,... and so on. They construct a general framework involving sets of generating functions to obtain explicit formulae for E F(t) and EC(t) with t finite. Thus, it can be shown that

$$EF(t+1) = u\ EF(t) - uk\ EF(t-p) + r.$$

Then, the solution is

$$E\ F(t) = a_0 + \sum_{j=1}^{p+1} a_j x_j^t \tag{38}$$

where $x_1, x_2, \ldots, x_{p+1}$ are the roots of the polynomial equation.

$$x^{p+1} - ux^p + uk = 0$$

and the coefficients a_i are determined by the initial conditions. Zimbidis and Haberman (1993) also demonstrate that, under certain conditions,

$$\lim_{t\to\infty} EF(t) = AL \quad \text{and} \quad \lim_{t\to\infty} EC(t) = NC\ .$$

Similarly, they address higher moments and shows for example that, under certain conditions $\lim_{t\to\infty} Var\ F(t)$ is one of the form of equation (37) with the function g being different for different values of p. So far we have seen that g takes the following values:

$$g_0 = 1 \text{ for } p = 0$$

$$g_1 = \frac{u}{1+uk} \quad \text{for } p=1.$$

As an illustration, Zimbidis and Haberman (1993) show that,

$$g_2 = \frac{u^2(1-k)}{1+u^2k(1-k)} \quad \text{for p=2}$$

and $$g_3 = \frac{u^2(u-k-uk^2)}{1+uk+u^2k(u-k-uk^2)} \quad \text{for } p=3.$$

It is then possible to demonstrate that, if $M>1$, $g_0>g_1>g_2>g_3$ so that the limiting variance of F(t) increases as p increases. These results are intuitively reasonable, given our understanding of the entropy of systems. When we introduce a time delay, which means that we have lost (or do not have available) some information for the fund between times t-p and t, we should expect the variance (or, in other words, the entropy) of the fund level and contributions to be greater. These results confirm the findings of Balzer and Benjamin (1980) who report that the longer are the delays in information in a system, the longer are the resulting oscillations in that system. Zimbidis and Haberman (1993) also report on the conditions for oscillations to exist in the first two moments of F(t) as C(t) as p varies.

7.2 Autoregressive Rates of Return

Haberman (1993c) has investigated the effect of a one year delay (p=1) in the presence of a first order autoregressive representation of i(t), as in equation (23) reproduced below

$$i(t) = i+\varphi\ (i(t-1) - i)+e(t) \tag{23}$$

Haberman (1993c) demonstrates how, through the use of generating functions, equations (23), (32) and (33) can be combined (for the case p=1) to consider the behaviour of E F(t) and E C(t) for finite t and, in the limit as $t \to \infty$, and also the corresponding behaviour of $E\ (F(t)^2)$ and $E(C(t)^2)$, and hence of the variances.

8. CONTRIBUTION RATE RISK

In this section, we take a different viewpoint and consider the different risks which confront a defined benefit pension scheme.

Firstly, there is the *"contribution rate risk"*. Here the sponsor of the scheme, the employer, will be concerned that future investment performance is not such as to expose the pension fund to the risk of significant, unanticipated rises in contribution rate. Traditionally, this risk has been controlled by concentrating on real assets (e.g equities, property, indeed linked bonds). However, the concern remains about the variability of the levels of the contribution rate. Stability will also be a feature attractive to the finance manager and the shareholders of the employing/sponsoring company.

Secondly, the trustees, sponsor, members and advising actuary will be concerned that the pension fund can meet its liabilities. This is the *"solvency risk"*.

Thirdly, the investment manager will be concerned about how his performance is measured and hence about his own commercial viability.

Here, we consider only the first type of risk: the *"contribution rate risk"*. We hope to return to the *"solvency risk"* in later work.

With the assistance of the mathematical model described in section 3, we will consider the methods for controlling the variability in the present value of future contributions. We shall consider only the case of independent and identically distributed investment returns.

We begin with individual funding methods and the spread method for fixing ADJ(t) and no delays in the contribution rate fixing process (i.e $p=0$). Valuations will be taken as annual.

We shall define the present value of future contributions at time 0 to be

$$G(0) = \sum_{s=0}^{\infty} v^s C(s) \tag{39}$$

where present values have been taken using the real valuation rate of interest i_v, so $v=(1+i_v)^{-1}$. Although investment returns are stochastic, we shall calculate present values in a deterministic manner which would be the approach in a conventional actuarial valuation. Note also from assumption 6 (of section 3) that $i_v=i=Ei(t)$, so that the real valuation rate of interest is correct *"on average"*. We shall now assume that $i>0$ so that $v<1$ (for convergence).

Clearly we could similarly define $G(t)$ at any t.

To measure the *"contribution rate risk"*, we shall investigate the properties of the first two moments of G and we shall consider the choice of parameters, in particular k and M, that would lead to minimum values of risk.

We shall take $M>1$.

Then, $E\,G(0) = \sum_{s=0}^{\infty} v^s\, EC(s) = \sum_{s=0}^{\infty} v^s\, (NC+kAL-kF(s))$ from equation (7)

$$= \sum_{s=0}^{\infty} v^s \left[NC+k.AL-kF_0 q^s - \frac{kr}{1-q}(1-q^s) \right] \quad \text{from equation (9)}$$

$$= \frac{NC+kAL}{d} - \frac{k\,F_0}{1-vq} - \frac{kr}{d(1-q)} + \frac{kr}{(1-q)(1-vq)}$$

$$= \frac{NC}{d} + AL - F_0 \qquad (40)$$

since $r=AL(1-q)$ and $k=1-vq$. (Note that $0<q<1$ and $v<1$).

This result is intuitive: the expected present value of contributions is equal to the present value of the normal level of contributions plus the difference between F_0 and $AL=\lim F(t)$.

Indeed, if $F_0 = AL$, then $EG(0) = \dfrac{NC}{d}$.

If $F_0 = 0$, then $EG(0) = \dfrac{NC}{d} + AL = \dfrac{B}{d}$ from equation (3).

Then, the variance can be written as

$$VarG(0) \quad = \quad Var\left[\sum_{s=0}^{\infty} v^s \; C(s)\right]$$

$$= \quad Var\left[\sum_{s=1}^{\infty} v^s \; C(s)\right] \quad \text{as } Var\ C(0) = Var\ F(0) = 0.$$

$$= \quad \sum_{s=1}^{\infty} v^{2s} \; Var\ C(s) \; + \sum_{\substack{s,t \geq 1 \\ s \neq t}} v^s v^t \; Cov\ (C(s), C(t))$$

$$= \quad \sum_{s=1}^{\infty} v^{2s} \; Var\ C(s) \; + \; 2 \sum_{s=2}^{\infty} \sum_{t=1}^{s-1} v^s v^t q^{s-t} \; Var\ C(t) \tag{41}$$

since Dufresne (1986) has shown that Cov(C(t+u), C(t)) = q^u Var C(t).

As in the previous sections, we shall assume that $a < 1$.

From equations (7) and (10) we have that

$$Var\ C(t) \; = \; bk^2 a^t \sum_{j=1}^{t} a^{-j} \left\{ q^j F_0 + \frac{r(1-q^j)}{1-q} \right\}^2 \; . \tag{42}$$

Haberman (1993d) has explored in detail two special cases for the choice of F_0 viz

a) $F_0 = AL$ and b) $F_0 = 0$.

When $F_0 = AL$, Haberman demonstrates that

$$Var\ G(0) = \frac{bAL^2 v^2 (1 - v^2 q^2)}{(1 - v^2)(1 - q^2 v^2 (1 + b))} \tag{43}$$

which is an increasing function of q. Thus, we minimise Var G(0) by choosing $q = 0$ i.e. $k = 1$ i.e. $M = 1$. We note that for convergence we require $a < 1$ or $q < (1+b)^{-1/2} = q_{max}$, say.

Thus, if the initial size of the fund is equal to the stable value of the actuarial liability ($= E\ F(t)$), the optimum choice of M is <u>M=1</u>, i.e. pay off the unfunded liability at each valuation date without spreading payments into the future. Optimality is here determined by the criterion: minimise the variance of the present value of future contributions.

When the initial level of the fund is zero, $F_0 = 0$, Haberman shows that

$$VarG(0) = \frac{bAL^2 v^2 (1 + qv^2)(1 - q)^2}{(1 - v^2)(1 - qv^2)(1 - v^2 q^2 (1 + b))} \tag{44}$$

If we seek to minimise Var G(0) $= \gamma(q)$ say then we would be interested in solving $\gamma'(q) = 0$ which leads to a cubic equation in q (not given here, but discussed in Haberman (1993d)).

Numerical experiments indicate that there is one real solution, q^*, to this cubic equation. If $q^* < q_{max}$, then *Var G(0)* is minimised for $q = q^*$, while if $q^* > q_{max}$, then *Var G(0)* is minimised for $q = q_{max}$. Table 12 provides some illustrative numerical results. Here, it should be noted that i is a real rate of investment return.

These theoretical results indicate that for low values of σ, the optimal choice of M (hence q) is to make M as large as possible. As σ increases, the optimal set of values of M becomes finite and decreases as σ increases.

These conclusions differ somewhat from those of Dufresne (1986, 1988) as described in section 5.1, where we have defined an optimal region for M to be $(1, M^*)$ where M^* is the value of M that minimizes

$$\beta(M) = \frac{\lim_{t \to \infty} Var\ C(t)}{\left(\lim_{t \to \infty} E\ C(t)\right)^2} .$$

(The corresponding values are shown in Table 4). The two approaches are based on different criteria viz

i/ minimizing the variance of the present value of future contributions

ii/ minimizing the ultimate (stable) level of the variance of the rate of future contributions.

In i/ we consider all future contributions weighted by a discounting factor dependent on the valuation rate of interest. In ii/ we would consider only the ultimate level of contributions rather than the intermediate pathway and here the value of F_0 is immaterial.

The conflict between the results is illustrated by Tables 4 and 12, and is particularly apparent at *"low"* values of σ for which criterion i/ would suggest choosing much higher values of M than contained in the range $(1,M^*)$ advocated by criterion ii/. In practice, however, it is unlikely that values of M greater than the average remaining membership period would be used: for a group of male members with normal retirement age of 65 this would correspond to a maximum practicable value for M of 20-30 for a mature scheme and of 35-40 for a young scheme. So we should regard the values of M in Table 12 in terms of these practical upper bounds. Thus, the differences between the two sets of results (theoretically and practically) may be more apparent than real.

Haberman (1993d) has taken thus further by also considering aggregate funding methods as well as individual and aggregate funding methods in the presence of a delay ($p \neq 0$).

10. COMMENTS AND FURTHER DEVELOPMENTS

Varying levels of inflation and fluctuations in investment returns are problems with which the actuary must contend on an almost daily basis. Unlike mortality and other decrements or movements, for which deterministic and stochastic models are readily available, the movements of these economic factors are more difficult to model. Representation by identically distributed random variables or by simple stationary autoregressive models appear to be very appropriate for this purpose. An objective of this paper has been to show that explicit formulae are available for studying mathematically the variability of contributions and fund levels for a pension scheme. Practical implications for the choice of funding method are then considered as a consequence, and the effect of the choice of control parameter including spread period, valuation frequency and delay in fixing the contribution rate are discussed.

A number of interesting potentially useful directions for future research that come from the foregoing review are the following:

- use of moving average processes to represent i(t), the rate of investment return
- consideration of other descriptions for ADJ(t) used in practice
- consideration of the viewpoint of the scheme's sponsoring employer
- analysing further the effect of varying the control variables identified and the interactions between them
- consideration of the introduction of dynamic (rather than fixed) valuation assumptions: here it may be necessary to use simulation although control theory may be a promising line of attack - as advocated by Benjamin (1989).
- continuing the approach of section 9 to recognise explicitly the ***"contribution rate"*** and ***"solvency"*** risks. Thus, building on the earlier work of O'Brien (1986, 1987) and Vanderbroek (1990), Haberman and Sung (1994) introduce an objective function that allows simultaneous minimization of these two risks and leads to an optimal funding strategy (and hence choice of the contribution rate) subject to given constraints.

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TABLE 1
Maximum Length of Spread Period, M_0, for $a < 1$

	i		
σ	**.01**	**.03**	**.05**
.05	223	111	78
.10	112	68	51
.15	66	46	37
.20	42	33	28
.25	30	25	21
.30	22	19	17

TABLE 2
Maximum Length of Spread period, M_1, for Qc<1

	φ				
	0.1	0.3	0.5	0.7	0.9
i=1%					
υ=.05	362	234	161	100	37
.10	231	121	69	36	11
.15	161	71	37	17	5
.20	118	46	22	10	3
.25	89	32	15	7	2
.30	69	23	11	5	1
.35	55	17	8	3	1
i=3%					
υ=.05	158	113	86	61	28
.10	112	70	47	28	10
.15	86	47	28	15	5
.20	69	34	19	9	3
.25	57	25	13	6	2
.30	47	19	10	4	1
.35	39	15	7	3	1
i=5%					
υ=.05	106	79	62	46	24
.10	78	52	36	23	9
.15	62	37	24	13	4
.20	51	28	16	9	3
.25	43	21	12	6	2
.30	36	17	9	4	1
.35	31	13	7	3	1

TABLE 3
Maximum Length of Spread period, M_2, for $Q^2cw < 1$

	φ						
	-0.3	-0.1	0.1	0.3	0.5	0.7	0.9
i=1%							
υ=.05	467	262	188	135	96	57	20
.10	331	142	87	55	34	18	5
.15	254	88	48	28	16	8	2
.20	202	58	30	17	10	5	1
.25	165	41	20	11	6	3	1
.30	136	30	14	8	4	2	1
.35	114	23	11	6	3	2	1
i=3%							
υ=.05	194	123	97	77	59	40	17
.10	147	79	55	39	26	15	5
.15	120	55	35	23	14	8	2
.20	102	41	24	15	9	5	1
.25	88	31	17	10	6	3	1
.30	76	24	13	7	4	2	1
.35	67	19	10	6	3	2	1
i=5%							
υ=.05	128	85	68	56	45	32	15
.10	99	57	42	31	22	14	5
.15	83	42	28	20	13	7	2
.20	71	32	20	13	8	4	1
.25	63	25	15	9	6	3	1
.30	56	20	11	7	4	2	1
.35	50	17	9	5	3	2	1

TABLE 4
M^* as a Function of i and σ

i / σ	-.01	0	.01	.03	.05
.05	-	401	60	23	14
.10	-	101	42	20	13
.15	158	45	28	16	11
.20	41	26	19	13	10
.25	22	17	14	10	8

TABLE 5
RELATIVE STANDARD DEVIATIONS OF F(t) and C(t) AS $t \to \infty$
ASSUMPTIONS AS IN NUMERICAL EXAMPLE OF SECTION 5.2

[$i=0.01$, $\upsilon=0.05$]

Spread Period	$\frac{(\mathrm{Var}\ F(\infty))^{1/2}}{\lvert E\ F(\infty)\rvert}$					$\frac{(\mathrm{Var}\ C(\infty))^{1/2}}{\lvert E\ C(\infty)\rvert}$				
	$\varphi=-0.9$	-0.7	-0.5	-0.3	-0.1	$\varphi=-0.9$	-0.7	-0.5	-0.3	-0.1
M=1	5.0%	5.0%	5.0%	5.0%	5.0%	170%	170%	170.%	170%	170%
5	3.3	4.4	5.5	6.6	7.8	21.6	29.2	36.7	44.6	53.4
10	3.7	5.5	7.2	8.9	10.7	12.5	18.7	24.6	30.7	37.8
20	4.6	7.4	9.9	12.5	15.4	7.9	13.0	17.6	22.5	28.3
30	5.4	9.1	12.3	15.6	19.3	6.4	10.9	15.0	19.4	24.7
40	6.1	10.5	14.4	18.4	23.0	5.6	9.8	13.6	17.8	23.0
50	6.8	11.9	16.4	21.1	26.5	5.1	9.1	12.8	16.9	22.1
60	7.5	13.3	18.4	23.7	30.0	4.8	8.7	12.3	16.4	21.7
80	8.8	15.9	22.2	28.9	37.2	4.4	8.1	11.7	15.9	21.8

TABLE 6
RELATIVE STANDARD DEVIATIONS OF F(t) and C(t) AS $t \to \infty$
ASSUMPTIONS AS IN NUMERICAL EXAMPLE OF SECTION 5.2

[$i=0.01$, $\upsilon=0.05$]

Spread Period	$\frac{(\mathrm{Var}\ F(\infty))^{1/2}}{\lvert E\ F(\infty)\rvert}$				$\frac{(\mathrm{Var}\ C(\infty))^{1/2}}{\lvert E\ C(\infty)\rvert}$			
	$\varphi=+0.1$	+0.3	+0.5	+0.7	$\varphi=+0.1$	+0.3	+0.5	+0.7
M=1	5.0%	5.0%	5.0%	5.0%	170%	170%	170.%	170%
5	9.1	10.8	12.7	14.2	63.9	77.0	93.8	108
10	12.9	15.7	19.6	25.9	46.5	58.4	77.0	114
20	18.8	23.4	30.3	44.3	35.7	46.5	65.9	121
30	23.9	30.2	40.3	64.6	31.8	42.8	64.7	148
40	28.7	36.7	50.8	94.3	30.2	42.0	68.3	213
50	33.4	43.7	63.1	160	29.6	42.8	76.3	421
60	38.3	51.1	78.5	*	29.8	45.0	90.2	*
80	48.8	69.2	100	*	31.6	53.5	114	*

* not applicable as $Q^2cw > 1$

TABLE 7
CATEGORY OF α-β PROFILE AND WHERE APPROPRIATE OPTIMAL REGION FOR M, SPREAD PERIOD, φ=-0.3

i / υ	-.01	.005	.01	.03	.05
.05	B	B	B	A(1,30)	A(1,20)
.10	B	A(1,400)	A(1,250)	B	B
.15	B	A(1,250)	A(1,200)	B	B
.20	B	A(1,200)	A(1,140)	B	B
.25	B	A(1,150)	A(1,120)	B	B

TABLE 8
CATEGORY OF α-β PROFILE AND WHERE APPROPRIATE OPTIMAL REGION FOR M, SPREAD PERIOD, φ=-0.1

i / v	-.01	.005	.01	.03	.05
.05	B	A(1,130)	A(1,70)	A(1,25)	A(1,20)
.10	B	A(1,90)	A(1,60)	A(1,25)	A(1,20)
.15	B	A(1,55)	A(1,40)	A(1,25)	A(1,20)
.20	A(1,110)	A(1,35)	A(1,30)	A(1,20)	A(1,15)
.25	A(1,45)	A(1,25)	A(1,20)	A(1,15)	A(1,15)

TABLE 9
CATEGORY OF α-β PROFILE AND WHERE APPROPRIATE OPTIMAL REGION FOR M, SPREAD PERIOD, φ=0.1

i / v	-.01	.005	.01	.03	.05
.05	B	A(1,90)	A(1,50)	A(1,20)	A(1,15)
.10	B	A(1,40)	A(1,30)	A(1,15)	A(1,10)
.15	A(1,55)	A(1,25)	A(1,20)	A(1,10)	A(1,8)
.20	A(1,20)	A(1,15)	A(1,10)	A(1,8)	A(1,5)
.25	A(1,15)	A(1,9)	A(1,8)	A(1,6)	A(1,3)

TABLE 10
CATEGORY OF α-β PROFILE AND WHERE APPROPRIATE OPTIMAL REGION FOR M, SPREAD PERIOD, φ=0.3

v \ i	-.01	.005	.01	.03	.05
.05	B	A(1,60)	A(1,40)	A(1,15)	A(1,10)
.10	A(1,80)	A(1,20)	A(1,20)	A(1,9)	A(1,6)
.15	A(1,20)	A(1,9)	A(1,9)	A(1,5)	C
.20	A(1,7)	A(1,4)	A(1,3)	C	C
.25	C	C	C	C	C

TABLE 11
CATEGORY OF α-β PROFILE AND WHERE APPROPRIATE OPTIMAL REGION FOR M, SPREAD PERIOD, φ=0.5

v \ i	-.01	.005	.01	.03	.05
.05	B	A(1,40)	A(1,30)	A(1,15)	A(1,8)
.10	A(1,20)	A(1,10)	A(1,9)	A(1,4)	C
.15	C	C	C	C	C
.20	C	C	C	C	C
.25	C	C	C	C	C

TABLE 12:
VALUES OF M FOR WHICH Var G(0) IS A MINIMUM

σ	i=1%	i=3%	i=5%
.01	535	217	142
.05	223	110	77
.10	112	67	50
.15	66	45	35
.20	26*	32	26
.25	3*	24	20
.30	2*	18	16
.35	1*	10*	13

* denotes that $q^* < q_{max}$; otherwise q^* is chosen to be q_{max}.

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