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ACTUARIAL NOTE

FURTHER REMARKS ON THE BASIC MORTALITY FUNCTIONS

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§1. *Introduction*

After a seminar at the University of Michigan in December 1971, it was suggested by the second author that some of the results of reference 3 might be presented in a more general form, and that to do so it might be helpful to express them in terms of life table functions rather than survival functions. Such expression will be given and the generalizations obtained. We show that two well-known formulas of life contingencies may be deduced as limiting cases of these more general results. Some further study is made of the case where the central death rate functions $m_x^{(1)}$, $m_x^{(2)}$ are such that $m_x^{(1)} = m_x^{(2)}$ for all $x \leqq x_0$ but the corresponding annual rates $q_x^{(1)}$ and $q_x^{(2)}$ differ. Finally, two main relations of reference 3 are extended to an emerging population subject to a fixed set of mortality and fertility rates and to the associated *stable population*.

It should be noted that throughout it is assumed or follows that q_x, m_x, μ_x, l_x and L_x are continuous, positive functions of a real variable x, $0 \leqq x < \infty$, and also that l_x and L_x are differentiable. It is mathematically simpler to proceed by assuming a smooth attenuation of the survivorship function l_x rather than to assume a fixed limiting age beyond which $l_x = 0$. If a fixed limiting age ω is assumed, problems of indeterminacy arise concerning μ_x and m_x in the neighbourhood of ω. The attenuated values of l_x may be assumed to be sufficiently small that their practical effect is negligible.

§2. *Some earlier results in terms of life table functions*

A main relation, corresponding to formula (2.10) of reference 3, is

$$L_x = L_0 . \exp\left\{-\int_0^x m_y dy\right\} \qquad (2.1)$$

where m_y is the central death rate for year of age y to $y+1$. This formula is easily established from the relation

$$m_y = \frac{d_y}{L_y} = -\frac{1}{L_y} \cdot \frac{d}{dy}(L_y). \tag{2.2}$$

We now examine relations between the life table survivorship function l_x and the central death rate function m_x. Firstly we note that a life table survivorship function l_x is assumed to be such that:

$$l_x \text{ is a positive function of } x,\ 0 \leqq x < \infty; \tag{2.3}$$

$$l_x \text{ is strictly monotonically decreasing}; \tag{2.4}$$

$$\lim_{x \to \infty} l_x = 0. \tag{2.5}$$

Given l_x, a corresponding central rate m_x is determined by

$$m_x = \frac{l_x - l_{x+1}}{\int_x^{x+1} l_y\, dy},\ 0 \leqq x < \infty. \tag{2.6}$$

The correspondence between m_x and l_x is elaborated in the following statements:

A. If m_x is the central death rate function corresponding to a given life table survivorship function l_x, then

$$m_x \text{ is a positive function},\ 0 \leqq x < \infty; \tag{2.7}$$

$$\int_0^\infty m_x dx = \infty; \tag{2.8}$$

the series $\sum_{n=0}^{\infty} m_{x+n} \cdot \exp\left\{-\int_0^{x+n} m_y dy\right\}$ converges for all values of x; (2.9)

the function defined by the above series is strictly monotonically decreasing. (2.10)

In fact, the series is

$$\frac{\sum_{n=0}^{\infty} m_{x+n} . L_{x+n}}{L_0} = \frac{\sum_{n=0}^{\infty} d_{x+n}}{L_0} = \frac{l_x}{L_0}. \tag{2.11}$$

B. Conversely, if m_x is a given positive function that satisfies conditions (2.8), (2.9) and (2.10), it may be shown that the definitions

$$L_x = L_0 . \exp\left\{-\int_0^x m_y dy\right\},\ L_0 \text{ arbitrary}, \tag{2.12}$$

and

$$d_x = L_x . m_x \tag{2.13}$$

lead to a unique life table survivorship function

$$l_x = \sum_{n=0}^{\infty} d_{x+n} \tag{2.14}$$

for which $L_x = \int_x^{x+1} l_y dy$ and m_x is the corresponding central rate function.

The proof of statement A follows easily from the properties (2.3), (2.4) and (2.5) of l_x, the relationships of l_x with L_x and m_x, and formula (2.1). The proof of the converse statement B can be given along lines similar to those in reference 3, and will not be repeated here.

From the foregoing we obtain the relation

$$l_x = L_0 \, . \sum_{n=0}^{\infty} m_{x+n} \, . \exp\left\{ -\int_0^{x+n} m_y dy \right\}, \tag{2.15}$$

giving l_x in terms of m_x.

From formula (2.15) with $x = 0$ it follows that

$$\sum_{n=0}^{\infty} m_n . \exp\left\{ -\int_0^n m_y dy \right\} = \frac{l_0}{L_0} = \frac{l_0}{\int_0^1 l_y dy}. \tag{2.16}$$

Thus, if two life tables, of which the second has functions denoted by primed symbols, are such that

$$\mathring{e}_{0:\overline{1}|} = \mathring{e}'_{0:\overline{1}|}, \tag{2.17}$$

then

$$\sum_{n=0}^{\infty} m_n . \exp\left\{ -\int_0^n m_y dy \right\} = \sum_{n=0}^{\infty} m'_n . \exp\left\{ -\int_0^n m'_y dy \right\}. \tag{2.18}$$

§3. *Generalizations and applications*

Generalizations of formulas (2.1) and (2.15) may be obtained by introducing

$${}_hd_y = l_y - l_{y+h} = \int_y^{y+h} l_z \mu_z dz, \tag{3.1}$$

$${}_hL_y = \int_y^{y+h} l_z dz, \tag{3.2}$$

and

$${}_hm_y = \frac{{}_hd_y}{{}_hL_y} = -\frac{1}{{}_hL_y} \cdot \frac{d}{dy}({}_hL_y). \tag{3.3}$$

Then, corresponding to formula (2.1), we have

$$ {}_h\mathrm{L}_x = {}_h\mathrm{L}_0 . \exp\left\{ -\int_0^x {}_hm_y dy \right\}, \tag{3.4} $$

and to formula (2.15)

$$ l_x = {}_h\mathrm{L}_0 \cdot \sum_{n=0}^{\infty} {}_hm_{x+nh} \cdot \exp\left\{ -\int_0^{x+nh} {}_hm_y dy \right\} \tag{3.5} $$

since $l_x = \sum_{n=0}^{\infty} {}_hd_{x+nh} = \sum_{n=0}^{\infty} {}_hm_{x+nh} \cdot {}_h\mathrm{L}_{x+nh}$.

From formulas (3.1), (3.2) and (3.3) we have

$$ {}_hm_y = \frac{\int_y^{y+h} l_z\mu_z dz}{\int_y^{y+h} l_z dz} . \tag{3.6} $$

Since μ has been assumed to be a continuous function, then (cf. reference 1, p. 230, ex. 6.8)

$$ \int_y^{y+h} l_z\mu_z dz = \mu_{y+\theta h} \cdot \int_y^{y+h} l_z dz \tag{3.7} $$

for some θ such that $0 \leqq \theta \leqq 1$. Combining equations (3.6) and (3.7), we see that

$$ {}_hm_y = \mu_{y+\theta h} \tag{3.8} $$

and

$$ \lim_{h\to 0} {}_hm_y = \mu_y. \tag{3.9} $$

We have also as a direct consequence of definition (3.3) that

$$ \lim_{h\to\infty} {}_hm_y = \frac{l_y}{\mathrm{T}_y} = \frac{1}{\mathring{e}_y}. \tag{3.10} $$

which is the aggregate death rate for persons aged y or more in the stationary population represented by the mortality table. The force of mortality μ_y and the aggregate death rate l_y/T_y may therefore be regarded as limiting values of ${}_hm_y$. This suggests letting h tend to 0 or infinity in the results of §2 above.

From equation (3.8) we see that there exists a value of θ ($0 \leqq \theta \leqq 1$) such that

$$ | {}_hm_y - \mu_y | = | \mu_{y+\theta h} - \mu_y |. \tag{3.11} $$

Since a continuous function is uniformly continuous on a closed bounded interval (cf. reference 1, p. 96) this last equation implies that, if μ is continuous, then the convergence of ${}_hm_y$ to μ_y as h tends to 0 is uniform on the interval $0 \leqq y \leqq x$ for each positive real number x. In this case therefore (cf. reference 1, p. 283)

$$\lim_{h\to+0}\int_0^x {}_hm_y dy = \int_0^x \lim_{h\to+0} {}_hm_y dy = \int_0^x \mu_y dy$$

by formula (3.9). Hence

$$\lim_{h\to+0} \exp\left\{-\int_0^x {}_hm_y dy\right\} = \exp\left\{-\int_0^x \mu_y dy\right\}. \tag{3.12}$$

We note also from equation (3.4) that

$$\exp\left\{-\int_0^x {}_hm_y dy\right\} = \frac{{}_h\mathrm{L}_x}{{}_h\mathrm{L}_0}. \tag{3.13}$$

Now

$$\lim_{h\to+0}\frac{{}_h\mathrm{L}_x}{{}_h\mathrm{L}_0} = \lim_{h\to+0}\frac{\frac{d}{dh}\,{}_h\mathrm{L}_x}{\frac{d}{dh}\,{}_h\mathrm{L}_0} = \lim_{h\to+0}\frac{l_{x+h}}{l_h} = \frac{l_x}{l_0}. \tag{3.14}$$

Combining equations (3.12)-(3.14), we obtain

$$l_x = l_0.\exp\left\{-\int_0^x \mu_y dy\right\}, \tag{3.15}$$

which classical formula is now seen as a limiting case of formula (3.4).

By similar but slightly simpler arguments one may show that letting $h\to\infty$ in formula (3.4) leads to

$$\mathrm{T}_x = \mathrm{T}_0.\exp\left\{-\int_0^x \lim_{h\to\infty} {}_hm_y dy\right\} = \mathrm{T}_0.\exp\left\{-\int_0^x \frac{1}{\mathring{e}_y}\, dy\right\}, \tag{3.16}$$

which is another familiar formula of life contingencies.

By expressing the right member of (3.5) as $\sum_{n=0}^{\infty} {}_hm_{x+nh}\cdot{}_h\mathrm{L}_{x+nh}$ and letting $h\to 0$, one obtains

$$l_x = \int_0^\infty l_{x+t}\,\mu_{x+t}dt. \tag{3.17}$$

However, letting $h\to\infty$ in the right member of (3.5) leads to a trivial relation.

§4. *The functions* $\bar{p}_x$, $\bar{q}_x$

In reference 3 an illustration was given of two life tables for which the central death rate functions $m_x^{(1)}$ and $m_x^{(2)}$ are such that $m_x^{(1)} = m_x^{(2)}$ for all $x \leqq x_0$ yet the corresponding annual rates $q_x^{(1)}$ and $q_x^{(2)}$ have $q_n^{(1)} > q_n^{(2)}$ for *every* integer $n \geqq 0$. To gain more insight into this situation we observe from formulas (2.1) and (3.15) that L_x may be considered as a life table survivorship function (i.e. as an l_x) for which the force of mortality is m_y. Carrying the analogy further, we can define

$$\bar{p}_x = \exp\left\{-\int_x^{x+1} m_y dy\right\} = \frac{L_{x+1}}{L_x}. \tag{4.1}$$

and

$$\bar{q}_x = 1-\bar{p}_x = \frac{(L_x - L_{x+1})}{L_x}. \tag{4.2}$$

Here

$$\bar{q}_x = \frac{\int_x^{x+1} (l_y - l_{y+1}) dy}{\int_x^{x+1} l_y dy}$$

$$= \frac{\int_x^{x+1} l_y q_y dy}{\int_x^{x+1} l_y dy} \tag{4.3}$$

is a weighted mean of q_y, $x \leqq y \leqq x+1$. One can also express $\bar{q}_x$ as

$$\bar{q}_x = \frac{\int_x^{x+1} d_y dy}{L_x},$$

that is

$$\bar{q}_x = \frac{\int_x^{x+1} L_y m_y dy}{L_x}$$

$$= \int_x^{x+1} {}_{y-x}\bar{p}_x m_y dy \tag{4.4}$$

$\left(\text{where } {}_{y-x}\bar{p}_x = \frac{L_y}{L_x}\right)$, which is the analogue of

$$q_x = \int_x^{x+1} {}_{y-x}p_x \mu_y dy. \tag{4.5}$$

If now $m_x^{(1)} = m_x^{(2)}$ for $x \leqq x_0$, it is clear from formulas (4.1) and (4.2) that $\bar{p}_x^{(1)} = \bar{p}_x^{(2)}$ and $\bar{q}_x^{(1)} = \bar{q}_x^{(2)}$ for $x \leqq x_0-1$. Thus the weighted means $\bar{q}_x^{(1)}$ and $\bar{q}_x^{(2)}$ are equal for all $x \leqq x_0-1$ even though as shown in reference 3 it is possible that $q_x^{(1)} \neq q_x^{(2)}$. It is also clear that $L_x^{(1)} = L_x^{(2)}$ for all $x \leqq x_0$, provided $L_0^{(1)}$ and $L_0^{(2)}$ are taken equal, so that below age x_0 the two life tables cannot differ very substantially.

§5. *Extension to a stable population*

We consider here a female population for which survivorship is according to a given life table with survivorship function l_x and for which the annual rate of female birth at age x is according to a given function $f(x)$. In this population neither the number of births nor the number of deaths per year need be stationary; however, if the fixed survival and birth rates apply for a sufficiently long time, one expects intuitively that some form of stability would emerge in the population. A full account of such population models is given in reference 2 (see for instance §§5.1, 5.2 and 7.2).

Let $B(t)$ denote the density of female births at time t (i.e. the annual rate at which births are coming into the population at time t) and let α and β be the lowest and highest ages at which child-bearing occurs. If one is interested in only the general form of the eventual population, one may consider that for $t > \beta$ the density function satisfies the integral equation

$$B(t) = \int_\alpha^\beta B(t-x) \,.\, \frac{l_x}{l_0} \,.\, f(x)dx. \tag{5.1}$$

This integral equation has solutions of the form Ce^{rt} where r satisfies the equation

$$1 = \int_\alpha^\beta e^{-rx} \,.\, \frac{l_x}{l_0} \,.\, f(x)dx. \tag{5.2}$$

For $r = 0$, the right member of formula (5.2) becomes

$$\int_\alpha^\beta \frac{l_x}{l_0} \,.\, f(x)\, dx \tag{5.3}$$

which is the net reproduction rate for females, that is, the expected number of girl children to be born (in the future) in respect to a newborn girl. The equation (5.2) has a unique real root (say r_1) which is positive if the expression (5.3) is greater than 1. In demographic examples other roots appear in pairs of conjugate complex numbers with negative real parts, but in the general solution

$$C_1e^{r_1t} + C_2e^{r_2t} + C_3e^{r_3t} + \ldots \tag{5.4}$$

of equation (5.1), for t sufficiently large, only the first term remains significant. By suitable choice of origin and l_0, and by setting

$r_1 = r$, $C_1 = B(O) = l_0$,

we have

$$B(t) = B(O).e^{rt} = l_0.e^{rt}. \tag{5.5}$$

The density $l_{x(t)}$ of women aged x at time t is then

$$l_{x(t)} = B(t-x)\frac{l_x}{l_0} = e^{rt}.e^{-rx}.l_x \tag{5.6}$$

and the total population at time t is

$$T_{(t)} = \int_0^\infty l_{x(t)}dt = e^{rt}\int_0^\infty e^{-rx}l_x dx$$

$$= e^{rt}.T \tag{5.7}$$

where $T = \int_0^\infty e^{-rx}l_x dx$ (cf. equations (7.2.4) and (7.2.5) in reference 2).

Thus the population at time t may be regarded as the amplification by the growth factor e^{rt} of a special population with population density function $e^{-rx}l_x$ and total number T. This special population is called the *stable population* (corresponding to the root r of equation (5.2)). Since the population at time t is proportional by the factor e^{rt} to the stable population, then formulas involving ratios of two functions or rates for the stable population are immediately translateable into formulas for the population at time t. For instance, central death rates are identical for the stable population and the growing population. In the following we shall limit our attention mainly to the stable population, realizing that corresponding formulas may be obtained for the population at time t. We note that the stable population is simply the population at time 0 (where the origin has been chosen at a time when a stable age distribution has been attained).

To aid in the examination of the stable population, we write $e^{-rx}l_x = D_x$ and utilize familiar properties of the commutation function D_x. We have first that the total number in the stable population is

$$T = \int_0^\infty D_x dx = \overline{N}_0. \tag{5.8}$$

The density of births is seen from equation (5.2) to be

$$\int_\alpha^\beta D_x f(x)dx = l_0, \tag{5.9}$$

so that the aggregate birth rate is

$$\frac{l_0}{\mathrm{T}} = \mathrm{b}, \text{ say.} \tag{5.10}$$

By our remark above, b is also the aggregate birth rate in the population at t, and is independent of t. The density of deaths is

$$\int_0^\infty \mathrm{D}_x\mu_x dx = \int_0^\infty \mathrm{D}_x(\overline{\mu_x+r}-r)dx$$
$$= l_0 - r\mathrm{T}, \tag{5.11}$$

so that the aggregate death rate is

$$\frac{l_0 - r\mathrm{T}}{\mathrm{T}} = \mathrm{b} - r = \mathrm{d}, \tag{5.12}$$

and then

$$r = \mathrm{b} - \mathrm{d}. \tag{5.13}$$

Thus the actual population is growing continuously at the rate r equal to the excess of the aggregate birth rate over the aggregate death rate. If $b = d$, then $r = 0$, and we are back to the familiar stationary population model. For the population growing at rate r the relative age distribution is the same as in the stable population, that is,

$$\frac{l_{x(t)}}{\mathrm{T}_{(t)}} = \frac{\mathrm{D}_x}{\mathrm{T}} \tag{5.14}$$

which is independent of t.

We denote by ${}_h m'_y$ the central death rate for the stable population (and the population at t), where the prime is to distinguish this rate from the central rate ${}_h m_y$ of the given life table. Working in terms of the stable population and using ${}_h\overline{\mathrm{D}}_y$ to denote $\int_y^{y+h} \mathrm{D}_z dz$, we have

$${}_h m'_y = \frac{\int_y^{y+h} \mathrm{D}_z\mu_z dz}{{}_h\overline{\mathrm{D}}_y} \tag{5.15}$$

$$= \frac{\int_y^{y+h} \mathrm{D}_z(\overline{\mu_z+r}-r)dz}{{}_h\overline{\mathrm{D}}_y}$$

$$= \frac{\mathrm{D}_y - \mathrm{D}_{y+h}}{{}_h\overline{\mathrm{D}}_y} - r,$$

so that

$$ {}_hm'_y + r = -\frac{d}{dy}(\log {}_h\overline{D}_y) \tag{5.16}$$

and

$$ {}_h\overline{D}_x = {}_h\overline{D}_0 . \exp\left\{-\int_0^x ({}_hm'_y + r)dy\right\}. \tag{5.17}$$

Multiplying this last equation by e^{rt}, we get for the population at time t

$$ {}_hL_{x(t)} = {}_hL_{0(t)} . \exp\left\{-\int_0^x ({}_hm'_y + r)dy\right\} \tag{5.18}$$

where

$$ {}_hL_{x(t)} = \int_x^{x+h} l_{y(t)}dy = \int_x^{x+h} e^{rt}D_y dy = e^{rt}{}_h\overline{D}_x. \tag{5.19}$$

There is an interesting and useful alternative way for arriving at formula (5.17). Instead of regarding D_x as the population density function for the stable population, one notes that D_x has properties (2.3), (2.4) and (2.5) and can be considered as a life table survivorship function for which the corresponding central death rate for the interval of age y to $y+h$ is

$$\frac{D_y - D_{y+h}}{{}_h\overline{D}_y} = {}_hm'_y + r$$

(cf. formula (5.15)). Now, an application of formula (3.4), with l_x replaced by D_x and ${}_hL_x$ replaced by ${}_h\overline{D}_x$, yields formula (5.17).

If, further, we set

$$\phi_x = \sum_{n=0}^{\infty} ({}_hm'_{x+nh} + r) . \exp\left\{-\int_0^{x+nh} ({}_hm'_y + r)dy\right\} \tag{5.20}$$

and apply formula (3.5) to D_x considered as a survivorship function, we obtain

$$D_x = {}_h\overline{D}_0 . \phi_x. \tag{5.21}$$

On multiplying through by e^{rt} in formula (5.21), we have

$$l_{x(t)} = {}_hL_{0(t)} . \phi_x, \tag{5.22}$$

the analogue for the growing population of formula (3.5).

We note finally that

$$\frac{l_{x(t)}}{l_{0(t)}} = \frac{D_x}{D_0} = \frac{\phi_x}{\phi_0}, \tag{5.23}$$

which yields

$$\frac{l_x}{l_0} = e^{rx}.\frac{\phi_x}{\phi_0}. \tag{5.24}$$

Thus the survival function $s(x) = \frac{l_x}{l_0}$ of the given table may be expressed explicitly in terms of ${}_h m'_y$ and r, where ${}_h m'_y$ is the central death rate in the growing population. We have not observed any direct relation between ${}_h m'_y$ and the central rate ${}_h m_y$ of the given life table. However, if ${}_h m'_y$ and r are known, formula (5.24) could be used to obtain $s(x)$ from which ${}_h m_y$ may be determined.

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