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GUARANTEED EQUITY PRODUCTS

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1. INTRODUCTION

1.1 Guaranteed Equity Products (GEPs) provide exposure to equity performance, for example by linking benefits to an equity index, while offering either a minimum monetary guarantee or some other explicit limitation of the downside risk normally associated with equity investment.

1.2 GEPs are not new. Several insurance companies offered equity linked contracts with maturity guarantees in the late 1970s, until it was confirmed by the Maturity Guarantees Working Party(1) that the reserves that companies should be holding to meet those guarantees were rather higher than they would have liked. It was also thought that although policyholders wanted guarantees in their contracts, they would not be prepared to meet the cost of financing the necessary reserves.

1.3 The new generation of GEPs emerged in 1989. One of the main differences between the new and old generations of these products is that the new ones make use of derivative instruments to back the guarantees, thereby circumventing the need to establish contingency reserves against the guarantees. The cost of the guarantees on the new GEPs is therefore determined by the market in these derivative instruments. For the old generation of GEPs the cost was determined by a combination of experience and reserving requirements based on a particular stock market model and assumptions chosen by the actuarial profession.

1.4 The volume of GEPs currently sold, both by life assurance companies and other providers in the UK, is still fairly small when compared with the main classes of business written by those institutions. GEPs constitute a rapidly growing area of business and one which is likely to assume increasing importance in years to come. Reasons for the success of GEPs are not hard to find. The stockmarket falls of 1987 and 1990 have lingered in people's memories and deterred private investors from subsequently reinvesting in equities. GEPs provide a means of obtaining equity exposure while taking away some of the risk. A second reason is that GEPs are easy to understand. Benefits are capable of being clearly defined in terms of market values and monetary guarantees, and consequently disclosure of charges is largely irrelevant. Thirdly, policyholders like guarantees - a point which the life assurance industry all too often forgets. That is perhaps understandable in the light of the performance of equities over the period 1975-1987, but will that ever be repeated?

1.5 In general, GEPs sacrifice some potential gains from investing in equities in order to provide guarantees. They seek to reduce risk. The principles involved in their design can equally be used for risk enhancement through increasing the exposure to equities that can be obtained from a fixed amount of money, or for providing contracts whose performance reverses that of the stockmarket. There is great scope for ingenuity and creativity in designing GEPs, but a word of caution is in order. The most successful GEPs so far have been those with the simplest designs.

1.6 GEPs have, to date, been mainly life assurance rather than pension contracts. This is surprising since the nature of the guarantees offered seems particularly appropriate for use in the design of pension policies, especially for those policyholders approaching retirement. It could be argued that staggered vesting or switching to fixed interest funds in order to lock into annuity rates before retirement are better alternatives, but the former provides no guarantees and the latter removes equity exposure entirely.

1.7 The authors are grateful for the helpful comments received from Pat O'Keeffe, John O'Neill, Malcolm Kemp and Andrew Smith while preparing this paper. We would also like to thank Lynne Gillespie and Amanda Callanan for typing the various drafts. Responsibility for the contents of the paper rests solely with the authors.

2. PRODUCT DESIGN

2.1 Guaranteed Equity Bonds

2.1.1 All GEPs currently available are effectively single premium contracts and most insurance companies' products are classified as single premium bonds. The simplest form of such a bond has a fixed term, typically five years, and at maturity pays out the higher of:-

(i) The value of the equity link.
(ii) A guaranteed amount.

2.1.2 The guaranteed amount is often equal to the initial investment or single premium and the equity link is usually the FT-SE 100 Index. The value of the equity link at maturity would be the initial investment, or a defined percentage of the initial investment, increased in line with the growth in the index. The death benefit under an insurance bond would normally be guaranteed to be not lower than the single premium. Surrender values are, in general, not guaranteed.

2.1.3 The policyholder is in effect being offered an equity investment and a put option with an exercise price equal to the guaranteed amount. Alternatively, the contract can be viewed as a combination of a zero coupon bond, to provide the guaranteed amount, and a call option on the equity investment. For those readers unfamiliar with options, a brief description is provided in Appendix I.

2.1.4 The product provider could back its liabilities by purchasing a combination of suitable assets and tailored "over the counter" (OTC) options from an investment bank. In practice this might be done through a reinsurance company. Exchange traded options cover only a limited number of expiry dates and are not currently available at the longer durations required to back GEPs. In practice, the investment bank would often provide a specially constructed asset, which would include an option, to support the benefits offered by the GEP.

2.1.5 The use of an index for the equity link is not surprising. An investor would probably find a contract linked to the performance of one particular equity share unattractive. Option writers would usually be reluctant to provide options on insurance companies' internal linked funds or on unit trusts unless the investment strategy was tightly defined. Some products have been linked to internal funds, but those funds, by the nature of their investment strategies, are likely to perform similarly to index tracker funds.

2.1.6 If the equity link is, say, the FT-SE 100 Index, the maturity proceeds of the bond may typically be the single premium increased in line with any increase in the Index (but subject to a minimum amount equal to the single premium and a deduction for taxation where appropriate). Since the FT-SE 100 Index does not allow for dividends to be reinvested, the dividend income forgone is effectively being used to meet the cost of the guarantee, expenses and a profit margin.

2.1.7 The assets backing this contract could be either a zero coupon bond and a call option or shares and a put option. The price of the call option would make allowance for the absence of dividends in the FT-SE 100 Index. The shares and put option approach is more difficult to manage. There are dividend yield and tracking error risks, but the former can be mitigated and the latter eliminated by the alternative approach of holding cash, an index future or forward contract, and a put option.

2.2 Types of Guarantee

2.2.1 One of the attractions of the guaranteed equity bond is that it offers the customer an explicit mix of equity exposure and a guarantee, a mix that can be varied ad infinitum. At one extreme is the equity linked single premium bond with no guarantee (all shares, no put option), while at the other is the non profit contract (all zero coupon bond, no call option). In between is a whole range of possible combinations of equity gearing and level of guarantee.

2.2.2 The guarantee could include interest at a fixed rate on the initial investment combined with a reduced level of equity exposure. Such a contract could be regarded as a variety of fixed term deposit, and Building Societies tend to favour this style of contract. For the more adventurous investor the guarantee could be reduced to a return of say 80-90% of the initial investment, and the contract could offer increased equity exposure on this lower amount. For example, a contract could offer 150% of the rise in the FT-SE 100 Index based on, say, 90% of the initial investment in an attempt to offset the loss of dividend income, combined with a guarantee of 90% of the initial investment. Alternatively, a contract might offer, say, 150% of the rise in the FT-SE 100 Index on the full amount invested, but place a cap on the maximum return provided by the contract. Equity exposure beyond the cap could be sold by writing a call option with an exercise price equal to the cap. Many providers are now offering choices in the levels of equity exposure and guarantee.

2.2.3 The guarantee itself could be expressed in terms of the performance of an index. The use of the retail price index (RPI) is popular and the availability of index linked gilts provides a market price for the guarantee, but the amount of index linked stock is limited and indexation is lagged. A guarantee based on the RPI may be quite appropriate for pension contracts.

2.2.4 The maturity value could be expressed as an average of the index over, say, the six month period prior to maturity, rather than the index value on the maturity date. This provides an element of smoothing at no additional cost (in fact there is a cost saving, since the variability of returns is reduced by averaging).

2.3 Lock-In Guarantees

2.3.1 Many contracts lock in gains in the FT-SE 100 Index at predetermined levels, so that if, for example, the index rises to 25% above its initial value at any time during the contract, that higher value becomes the guaranteed amount. The initial guaranteed amount would normally equal the initial investment. Figure 2.1 shows an example of a five year bond with lock-ins at 25%, 50% and 75% increases in the index.

2.3.2 At the end of year 2 the index has risen by 25% so that rise gets locked in. During year 3 the index rises to 150% of its initial value so that level is locked in. In year 4 the index rises to 175% of its initial value, so the maturity payout is fixed at the 175% level, despite the subsequent fall in the index and the fact that at the maturity date the index has increased by only 60% from its initial value.

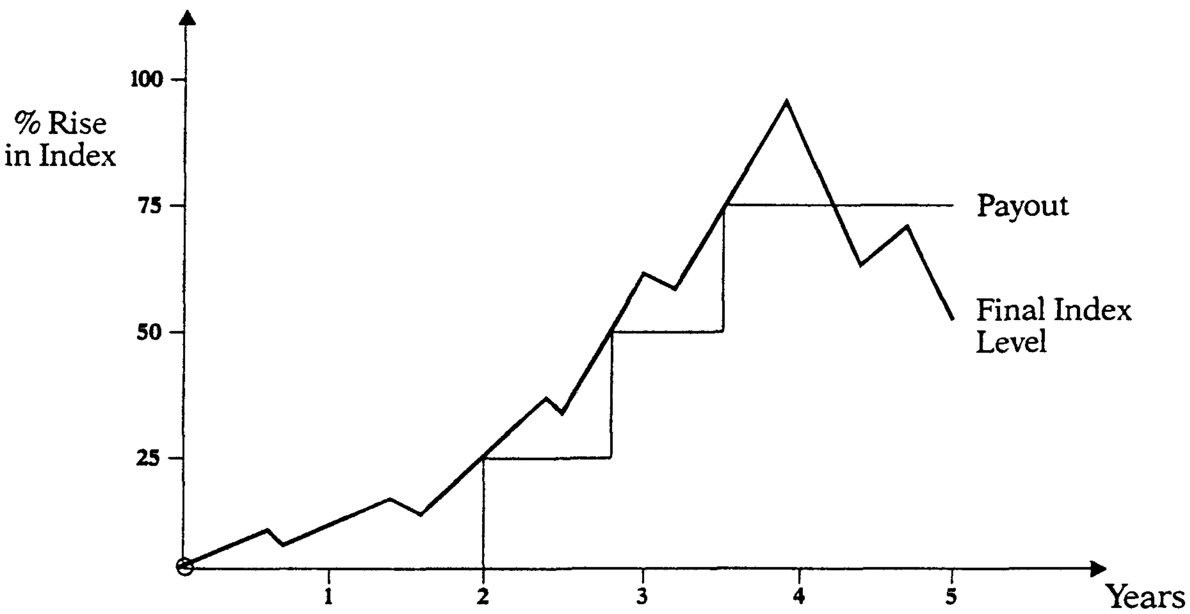

Figure 2.1

2.3.3 The cost of providing lock-in guarantees is, of course, higher than the cost of the simple money back guarantee, resulting in reduced equity exposure. In practice, contracts with lock-in guarantees normally provide equity exposure on an amount less than the single premium in order to meet the cost of the lock-in guarantees.

2.4 Rolling Guarantees

2.4.1 Some contracts are open-ended, having no fixed term. They provide a rolling guarantee, typically at annual or quarterly intervals. Under this type of contract, the value of the investment at the end of a specified period would be guaranteed to be not lower than the value at the start of the period (possibly adjusted for regular charges). The exposure to the equity index offered would then depend on investment conditions prevailing at the commencement of each guarantee period.

2.4.2 There are contracts which use derivatives such as options and futures to smooth out the peaks and troughs of equity market performance. Investment returns are thereby stabilised, but these contracts do not offer explicit guarantees in all circumstances and we have not therefore regarded them as GEPs.

2.5 Lookback Guarantees

2.5.1 In theory, at least, it is possible to design a contract whose maturity value is related to the highest index value or unit price achieved during the term of the contract. Such a contract provides a "lookback" guarantee, and can be regarded as the continuous form of a contract with lock-in guarantees with no upper limit. This type of guarantee is naturally quite expensive and at the time of writing (and to our knowledge) no such contract linked to an index has been offered. Managed funds with a guarantee that their value (or unit price) will not fall are equivalent to funds with a continuous lookback guarantee. While options may be used in these funds, the investment manager's discretion in the choice of assets held by the fund provides an aid to managing the guarantee not available on contracts linked to an index.

2.6 Guaranteed Surrender Values

2.6.1 The contract described in Section 2.1 provided no guarantee on surrender. The surrender value would be related to the value of the backing assets at any time. The put option provided to the policyholder was a European style put option, exercisable only on maturity. An American Style option permits exercise at any time up to maturity. American put options could therefore be used to provide surrender value guarantees. Few contracts currently offer guaranteed surrender terms.

2.6.2 In order to match guaranteed surrender values at any time, it must be possible for the American put option to be partially exercisable on any date. Although the guaranteed surrender value could be set equal to the single premium after an initial period, it will almost certainly be lower than the single premium initially. A specially constructed American style put option would therefore be needed to back a contract with surrender guarantees.

2.7 Marketing Process

2.7.1 The terms of a single premium GEP can be fixed shortly before receipt of the premiums. The usual process is that the product provider will fix a marketing period, after setting the contract terms, during which it receives subscriptions for the contract. The provider may reserve the right to close the offer early if it becomes "oversubscribed". The contract will then commence shortly after the end of the marketing period.

2.7.2 If the provider is purchasing a backing option (or a specially constructed asset containing an option) from an investment bank, it may agree to purchase a fixed amount of the option at a specified price on the commencement date of the contract. The provider bears a risk that the total subscriptions received differ from that fixed amount and the price of the option moves adversely. The provider can make a profit if the price moves the other way.

2.7.3 Alternatively, the provider could agree to buy a fixed amount (which could be zero) and take an option on a further amount at a specified price which would include an option premium (i.e. take an option on an option). The provider is then protected if the subscriptions received fall into that particular range, but pays a price for that protection.

2.8 Regular Premiums

2.8.1 The investment bank, in providing suitable backing assets, will determine the prices of any options contained in those assets, based on investment conditions at the time. Extending this process to future premiums, and in particular to regular premium contracts, produces additional uncertainties. The open-ended contracts described in Section 2.4 - which permit payment of future premiums - are only half-way solutions to this problem. They may provide a rolling capital guarantee indefinitely (which is equivalent to guaranteeing that interest rates will never be negative), but the levels of future equity exposure are not guaranteed in advance.

2.8.2 There is a difference between recurrent single premium contracts where a monetary guarantee applies separately to each single premium and regular fixed premium contracts with an aggregate monetary guarantee. The latter should be the cheaper, assuming all guarantees are fixed in advance, and there is unlikely to be demand for the former anyway.

2.8.3 The difficulties encountered in providing monetary guarantees for regular premium contracts are largely related to the long term interest rate guarantee underlying a maturity guarantee. If guarantees were offered on, say, unit-linked mortgage endowment contracts their cost would appear high, but this simply reflects the fact that premium rates for non-profit endowments would also look high in relation to those charged for the current designs of mortgage endowments. Premium rates for unit-linked endowments are typically set assuming unit price growth of 7.5% p.a. after tax. How many companies would price a non-profit endowment in current conditions using an assumed investment return of say 6% p.a. gross?

2.9 Exotic Contracts

2.9.1 It is possible to design more esoteric contracts, for example "negative funds" which mirror the performance of a stock index - if the index rises the fund falls and vice-versa. The required returns can be obtained by selling index futures (uncovered by stock). The fund can be protected by purchasing out of the money call options with an exercise price well in excess of the current market level. At least one Building Society has offered a similar contract related to Base Rates of interest.

2.10 High Income Products

2.10.1 In a time of low interest rates, there will be a demand for products which pay significantly higher income than that obtainable from deposit accounts or fixed interest investments. High Income Products seek to convert capital gains, or the prospect of capital gains, into income while providing some level of guarantee. The guarantee may relate to the combined return of income and capital.

2.10.2 There are two basic designs for high income products. The first approach, typically adopted by unit trusts, is to invest in equities, or a combination of equities and deposits, and sell call options on the equities. The premiums from the call options can then provide income at the expense of potential capital gains. The exercise price on the call options relative to the market value of equities determines the balance between income and potential capital gains. Protection against large falls in equity values could be obtained by purchasing out of the money put options.

2.10.3 An alternative approach, favoured by insurance companies, is to purchase a temporary annuity to provide a fixed income, invest in a zero coupon bond to provide a minimum capital guarantee and obtain exposure to equity price movements through the use of call options. At the money call options (exercise price equal to current market level) would be purchased to capture upward movement in equity prices. The overall cost could be reduced by writing out of the money call options (exercise price in excess of the current market level), thus setting a maximum limit on the equity gains achievable by the product.

2.10.4 Historically the latter approach would usually have produced more favourable returns than the former approach. This is because, over most periods of time, equities have outperformed fixed interest investments by a significant margin. In those conditions it is advantageous to gain exposure to upward movements in equity prices through purchasing call options, as opposed to selling that exposure by writing call options.

3. STRUCTURE OF INSURANCE COMPANY PRODUCTS

3.1 Characteristics of the Insurance GEP Market

3.1.1 At the end of 1993 the insurance GEP market had the following characteristics:-

(i) Approximately twenty five companies either were marketing, or had previously written, GEPs.

(ii) Most of the contracts were basic guaranteed equity bonds as described in Section 2.1, the most popular additional option being that of locking-in gains (see Section 2.3 for an example).

(iii) Very few contracts provided guaranteed surrender values.

(iv) Approximately half of the companies used reassurance facilities.

(v) The most usual classification of liabilities was "linked".

(vi) The most popular type of asset used to back the liabilities was a special bank deposit (the value of which increases in line with the FT-SE 100 Index subject to a minimum increase of zero), and the next most popular method combined a zero coupon bond and a call option.

3.2 The Insurance Regulatory Scene

3.2.1 The current regulations governing UK Life Insurance Companies are recognised as not being wholly appropriate in circumstances where an insurance company invests in derivative instruments to back contractual liabilities. The implementation of the Third Life Directive in the UK will require some of these regulations to be changed, and the proposed changes are set out in the DTI consultative document published in December 1993(9). In particular, the proposed regulations permit the use of derivatives "in connection with assets covering technical provisions" but subject to certain conditions. The proposed changes are described below and the potential impact on the future of GEPs is considered in Section 7.

3.2.2 The inappropriateness of the current regulations for GEPs can be overcome by seeking modification to these regulatory requirements as allowed by Sections 68 and 78 of the Insurance Companies Act 1982. Section 68 Orders can be sought to overcome the admissibility limits which apply to non-linked assets. The current admissibility limits are restrictive when investing in derivative instruments. Section 78 Orders can be sought to extend or modify the type of assets to which benefits can be linked, as set out in Schedule 13 (Permitted Links) of the Insurance Companies Regulations 1981. Options are currently not a permitted link.

3.2.3 The proposed new regulations allow for the use of derivative instruments to cover technical provisions for both linked and non-linked contracts, subject to certain conditions being fulfilled. The conditions that will need to be satisfied by derivative contracts in order that they may be either admissible or permitted links are as follows:-

(i) They are used either to "reduce investment risks" or for the purposes of "efficient portfolio management". The DTI proposes to issue guidance on the interpretation of these terms.

(ii) They must not be free-standing, but be associated with other assets which the insurer holds or plans to hold, subject to certain exceptions (for example, the use of cash and index futures to replicate an index).

(iii) They must either be traded on a regulated exchange or the counterparty must be approved.

(iv) The insurer should be able to close out the contract on a basis consistent with current rules for the valuation of the contract. That is, it should be readily realisable on reasonable terms.

(v) The underlying assets, to which the derivative contracts are related, must themselves be either admissible or permitted links.

(vi) They must be covered, that is the insurer must hold suitable assets in order to meet any obligations under the contracts. Uncovered derivative contracts will not be permitted in linked funds under any circumstances. If held outside the linked funds as inadmissible assets, a contingent liability reflecting the maximum likely loss that could arise under the uncovered contracts will need to be established.

3.3 Classification of Contracts

3.3.1 GEPs by their very nature provide benefits linked to equities or an index. Accordingly, these contracts should be classified as linked long term business, as defined in Schedule 1 to the Insurance Companies Act 1982. This includes contracts under which "the benefits are wholly or partly to be determined by reference to the value of, or the income from, property of any description (whether or not specified in the contracts) or by reference to fluctuations in, or in an index of, the value of property of any description (whether or not so specified)". In practice, however, some GEPs appear to have been classified as non-linked.

3.4 Classification of Assets

3.4.1 There are a number of different combinations of assets that can be used to match, partially or wholly and with differing degrees of accuracy, the liabilities of GEPs. Below is a list of possible combinations of assets that can be used to back the liabilities of GEPs linked to an index:-

(i) A special bank deposit, the value of which increases in line with the agreed index subject to a minimum increase in value of zero.

(ii) Equities (or securities underlying the index) to provide for the index exposure together with a put option on the index.

(iii) A zero coupon bond to provide for the underlying guarantee together with a call option on the index.

(iv) A futures contract to obtain exposure to the index, cash and a put option on the index.

(v) Bonds and equities (or futures and cash) held in varying proportions in an attempt to replicate the performance of the above combinations. This technique is known as dynamic hedging and is discussed in Section 6.

3.4.2 Assets are classified as linked assets if they are identified in the records of the company as being assets by reference to which linked benefits are determined. The above combinations of assets, except (v), can be classified as linked assets, but whether they should be depends on the terms of the contract. The assets held in following a dynamic hedging strategy do not determine the benefits payable and can therefore only be classified as non-linked assets. For a contract where the benefits are linked to the FT-SE 100 Index but the matching assets are one of (i) to (iv) above, the assets would be identified in Form 48 as "assets which are matching liabilities in respect of property linked benefits other than holdings in authorised unit trusts or internal linked funds" rather than in Form 49 as internal linked funds. If contracts are classified as non-linked, the backing assets should logically be accounted for in Form 45 of the DTI returns.

3.4.3 The new regulations, in return for providing greater freedom to use derivative instruments, will require greater disclosure. Additional information will be required in Forms 13, 45 and 49 of the DTI returns and the Appointed Actuary's valuation abstract will need to give details of investment guidelines and how derivatives have been allowed for. The details of these disclosure requirements are yet to be published.

3.5 Valuation of Liabilities

3.5.1 Valuation Principles

3.5.1.1 For the purposes of the statutory valuation, liabilities would need to be valued in accordance with the relevant regulations. The generally accepted accounting principles as they apply to life insurance contracts would need to be borne in mind (Section 52 of the Insurance Companies Regulations 1981). The current UK regulations for the valuation of liabilities do not distinguish between the valuation of non-linked and linked businesses, although the accounting for these businesses needs to be separated. There are, however, principles developed by the actuarial profession for the valuation of unit-linked business and in particular for the computation of the non-unit reserves[10][11]. The proposed new regulations lay down reserving principles for the unit liability of linked contracts.

3.5.1.2 The valuation regulations require that the nature and term of the assets be taken into account in determining the value of the liabilities. In considering the extent to which assets and liabilities are matched for GEPs the following types of liability need to be considered:-

(i) Guaranteed benefits on death, surrender and maturity.
(ii) Non-guaranteed benefits on death, surrender and maturity.
(iii) Expenses.
(iv) Counterparty risk.

3.5.2 Guaranteed Benefits

3.5.2.1 Maturity benefits are likely to be matched by appropriate assets, but guaranteed death benefits or surrender values may not necessarily be matched. To the extent that these guaranteed benefits are not matched, the valuation of liabilities would need to utilise actuarial methods to establish reserves. A discounted cash flow method could be used as follows:-

(i) Discount the minimum guaranteed benefit at maturity (which would be matched with a suitable asset), for example a return of the single premium, to various future points in time. Assume these amounts to be prudent realisable values at these future points in time.

(ii) Project the guaranteed benefits to future periods by inflating the current benefit, for example:

max [value of units; return of premium] at future points in time.

(iii) Calculate the sum at risk at future points in time as the difference between (ii) and (i) above.

(iv) Compute a reserve by applying mortality and surrender rates to the sums at risk, and discounting.

The assumptions used to discount and project the various cash flows, including the mortality and surrender rates, would need to be prudent. In setting the surrender rate assumptions a worst case scenario will need to be considered.

3.5.2.2 The proposed new regulations permit the use of derivative instruments to back technical provisions subject to certain conditions. The draft asset valuation regulations lay down the conditions for the valuation of unlisted derivatives; essentially a valuation basis should be in place and the assets should be realisable on that basis. The valuation of the non-matched guaranteed benefits could then be based on the expected realisable value of the underlying assets.

3.5.3 Non Guaranteed Benefits

3.5.3.1 The matching of assets and liabilities for non-guaranteed benefits is not relevant unless the asset value is negative (which should not happen for single premium contracts). Additional reserves for non-guaranteed benefits are unlikely to be required.

3.5.4 Expense Reserves

3.5.4.1 A reserve for future expenses may need to be established, at least for single premium contracts. In setting up the expense reserve the important assumptions are the level of future expenses and the rate at which to discount these (net of expense inflation). The unit growth rate is not relevant for GEPs which do not include fund charges in their design. To take account of the resilience reserve requirements (see Section 3.5.6 below) the rate of discount used to value the future expenses would need to be conservative.

3.5.5 Counterparty Risk Reserve

3.5.5.1 An insurance company could back its liabilities under GEPs by purchasing a special bank deposit or suitable assets and options from an investment bank. The company is then subject to counterparty risk or credit risk - the risk that the bank defaults on its obligations. If the contract between the policyholder and the insurance company does not explicitly pass the risk of such default onto the policyholder (that is the policyholder liability is not fully aligned with the asset) then consideration needs to be given to establishing a specific reserve. There are two particular aspects that are important to consider in setting up a reserve for such risks:-

(i) The fact that the risk of default may have effectively been ceded by the counterparty to a third party, through trading.

(ii) Exposures relating to all types of investments, not just those backing GEPs, need to be taken into account. Default risk is not unique to assets backing GEPs. For example, it would be appropriate to consider the risk of default on all deposits with a particular bank in assessing the concentration of risk. The new proposed regulations deal with this aspect by framing admissibility limits in terms of the aggregate exposure to any single company.

In assessing the need for such a reserve the Appointed Actuary ought to obtain a fuller picture of these and other aspects of the risk, if necessary by talking to the investment bank. In doing so an assessment of the bank's current and future investment policy should be made. An investment bank may be able to provide general comfort through some form of credit rating.

3.5.5.2 There are no satisfactory models available, which are both simple to understand and apply, to assist in evaluating counterparty risk and therefore a pragmatic approach is necessary.

3.5.5.3 Counterparty risk can be reduced by marking to market (a process in which there is a periodic cash settlement of any changes in the value of a derivative or other asset) or through a deposit back arrangement (an arrangement in which the underlying assets reside, or are deposited with, the insurer, with accounting processes set up to reflect such an arrangement). It would also be possible to insure against such risks (but one is then subject to the counterparty risk of the insurer).

3.5.6 Reserves for Maturity Guarantees and Resilience

3.5.6.1 Reserves for maturity guarantees on unit-linked contracts were considered by the Maturity Guarantees Working Party, whose recommendations are contained in their Report(1). Those recommendations were based on the premise that an office did not own, or could not purchase, assets which matched the guarantees provided under its contracts.

3.5.6.2 Insurance companies currently selling GEPs will usually purchase appropriate matching assets from an investment bank, sometimes through a reinsurer. It can, therefore, be argued that there is no need to hold maturity guarantee reserves for GEPs in these cases. This situation may change in the future depending on how the market develops. For example, for regular premium products (or long term contracts) there may not always be suitable assets available to match fully the guarantees provided.

3.5.6.3 The non-unit reserves need to be tested for resilience. The amount of resilience reserves required will depend on the extent of any mismatch between assets and liabilities. In the majority of cases the most significant non-unit reserve is likely to be the expense reserve. Non-unit reserves for future expenses are likely to be backed by suitable assets, such as gilts and resilience reserves are therefore likely to be small.

3.5.6.4 Assuming a resilience reserve is required then consideration must be given to the principles and methodology of establishing such reserves. One approach for establishing a resilience reserve is to use the discounted cash flow technique as mentioned in Section 3.5.2.1. but with appropriate changes to the assumptions and the value of non-matched assets.

3.5.6.5 An alternative approach for establishing a resilience reserve is to assess the sterling amount needed to purchase an appropriate matching asset. The Black-Scholes Model of option pricing (see Section 5) could be used to obtain a price for assessing the amount of sterling reserves required. The parameters for such pricing would be chosen to provide a prudent reserve rather than a realistic price.

3.5.7 **Minimum Solvency Margins**

3.5.7.1 The minimum solvency margin required for most GEPs is 4% of the mathematical reserves and 0.3% of the sum at risk. Where the asset risk, including any counterparty risk, has been passed on to the policyholders it can be argued that no investment guarantee is given resulting in a reduction in the required minimum solvency margin.

3.6 **Valuation of Assets**

3.6.1 The asset valuation regulations as set out in Part V of the Insurance Companies Regulations 1981, including the admissibility limits, apply only to assets backing non-linked liabilities. Linked assets are exempt from these regulations. There is, however, an overriding requirement to value linked assets at the amount that could be obtained for immediate assignment.

3.6.2 The current regulations are unnecessarily restrictive regarding the use of derivative instruments as assets to back liabilities on GEPs. For example, OTC options have no admissible value and the admissibility limits for traded options are very small.

3.6.3 The proposed new regulations cover the use of derivative instruments much more comprehensively. Subject to certain conditions being fulfilled (see Section 3.2.3 for a list of these), derivative instruments can be used to back technical provisions or mathematical reserves. The admissibility limits as set out in the 1981 Regulations are to be retained, and holdings of assets in excess of these limits will need to be left out of account when demonstrating solvency. These holdings will make allowance for the underlying economic exposure obtained through derivative contracts. The DTI is proposing to issue detailed guidance in this area.

3.7 Taxation

3.7.1 Taxation of life assurance business, and of options in particular, is complex and some companies obtain concessions or rulings on particular issues. In practice it is therefore very difficult to provide a comprehensive summary of the tax issues, and we have not sought to do so in this paper. In general, options are taxed under chargeable gains rules.

3.7.2 Without a suitable tax management policy the competitiveness of non-pension GEPs would suffer. There are a number of different ways of mitigating the tax burden including the following:-

(i) Reassurance - this is covered in Section 3.8 below.

(ii) Cross subsidy in a life assurance fund which is in an Excess E position.

(iii) Careful selection of assets chosen to back GEPs. For example, purchasing a deep discount or a zero coupon bond minimises income during the term of the policy. Capital gains tax can be avoided by structuring the bond so that it is a "Qualifying Corporate Bond" (as defined by Section 117 of the Taxation of Chargeable Gains Act 1992).

3.7.3 The tax regime for life assurance companies is currently under review, as announced by the Chancellor of the Exchequer in September 1993. This review is expected to be completed early in 1994.

3.7.4 Harmonization of taxation with the rest of Europe would lead to the abandonment of the current I-E tax regime. Whether this would result in a gross roll-up of policyholder funds would depend on the degree to which the company could reclaim tax credits or tax deducted at source from investment income. The taxation of policyholders may also be affected since the bases of taxation of life assurance funds and policyholders are linked in so far as the proceeds from a "qualifying" life assurance policy are exempt from all income and gains taxes in the hands of the policyholder. Assuming that the Exchequer will want no overall loss of revenue, any resulting changes may not necessarily benefit the competitiveness of insurance products in general and GEPs in particular. Under such a change to the tax regime the use of reassurance might decline.

3.7.5 A particular concern that the Inland Revenue has raised regarding the current I-E tax regime is that of the possible loss of revenue to the Exchequer through increased use of cross-border reassurance arrangements. One way of tackling this, while retaining the I-E regime is:-

(i) When accepting cross-border reassurance the tax basis would change from amalgamating this reassurance with the rest of the I-E system to taxing it separately on a profits basis. Thus the purchase of E would not reduce the I-E for the rest of the business.

(ii) When ceding cross-border reassurance the tax basis would impute investment returns which would be taxable in the hands of the cedent, so that it will no longer be beneficial to cede I.

3.7.6 The suggested changes in Section 3.7.5 would affect the competitiveness of the non-pension GEPs in that one of the main methods of tax mitigation would not be available to insurance companies.

3.8 **Reassurance**

3.8.1 Reassurance can be used for tax management, for financing (including the mitigation of EC Solvency Margins) and for managing mortality risk.

3.8.2 For GEPs written in the life assurance fund the main purpose of reassurance is likely to be tax management. The insurance company wishing to reassure GEPs will need to decide:-

(i) Whether to reassure with a UK company or an overseas company.

(ii) Whether to transfer the assets to the reassurer or not, and if not the nature of the deposit back arrangement. Under a deposit back arrangement, the reassurer takes an accounting interest in, rather than physical possession of, the underlying assets. The structure of a deposit back arrangement would usually involve first a reassurance of both the assets and liabilities and secondly a deposit back of the assets by the reassurer with the cedent together with the creation of a debt in the cedent's accounts. To avoid the physical transfer of assets these two parts of the reassurance are conducted simultaneously.

3.8.3 Under deposit back arrangements the cedent does not in general get exposed to credit risk, though this will depend on the nature of any offset clauses in the treaty. To ensure that a reassurance treaty with a deposit back arrangement is not considered as an artificial means merely of exploiting anomalies in the current tax regime, the details of the treaty will need careful consideration.

3.8.4 The UK reassurer route would normally involve the sale of Excess I to the reassurer, though a similar result could be achieved by the purchase of Excess E by the writer of a GEP. The price of the treaty will in general reflect the market for Excess E.

3.8.5 The overseas reassurer route will need to take account of any double taxation treaties and the tax basis in the overseas territory. In general, if the overseas reassurer is taxed on a profits basis and there are no adverse features of the double taxation treaties then, by fully reassuring GEPs to this territory, the tax basis effectively changes from an I-E basis to a profits basis.

3.8.6 Considerations affecting the reassurances of GEPs for financing reasons, for example to reduce the requirement for EC Solvency Margins, or for the reassurance of mortality risk, are similar to those for other types of savings contracts.

3.8.7 Reassurance may directly affect counterparty risk (see Section 3.5.5) since, depending upon the terms of the reassurance treaty, the ceding company's counterparty may effectively be the reassurer rather than the investment bank providing the backing assets.

4. OTHER PROVIDERS

4.1 The Wider Market

4.1.1 The market for GEPs is not restricted to the insurance industry and it may expand further in the future if certain providers, who currently face restrictions, are able to market such contracts.

4.1.2 The providers other than insurance companies who market, or who have marketed, GEPs are:

- Banks
- Building Societies
- Unit Trusts (by providing guarantees outside the unit trust)
- Business Expansion Schemes

There is also competition from offshore funds which use derivative instruments.

4.1.3 Examples of providers who face restrictions in offering GEPs are:-

- Unit Trusts
- Investment Trusts

4.1.4 Some institutions will have a choice over the provider of their GEPs. A couple of examples are:-

(i) Bancassurance groups can market either insurance GEPs or bank GEPs.

(ii) An insurance group which also operates an authorised unit trust could market either an insurance GEP or an equivalent unit trust product.

4.1.5 Product providers can utilise external fund managers and a number of GEPs have been marketed on this basis.

4.2 Factors for Consideration

4.2.1 The main factors that need consideration, both from the viewpoint of the consumer and that of the provider, in choosing between the different product providers are:-

(i) Regulatory aspects.
(ii) Taxation of both the provider and the investor.
(iii) Charging structure and competition.
(iv) Marketing.

Each of these is discussed in turn.

4.3 Regulatory Aspects

4.3.1 The regulatory aspects affecting the structure of insurance products were discussed in Section 3. In the main, the investment policy of insurance funds is not regulated but there are various other regulations in place; for example, the valuation of assets and liabilities when demonstrating solvency.

4.3.2 In contrast to insurance funds the main feature of regulation for unit trusts is the restrictive investment policy that they need to follow. In general unit trusts can invest in derivatives subject to satisfying the "efficient portfolio management" criteria. In 1991 the Securities and Investment Board (SIB) published rules which allowed unit trusts to set up futures and options funds (FOFs) and geared futures and options funds (GFOFs). These rules were amended in 1993. FOFs can use derivative instruments without satisfying the efficient portfolio management criteria but they need to comply with certain cover requirements laid down in the rules. GFOFs can offer a more speculative investment since their cover requirements are less stringent. Although these funds by their very nature can invest in derivative instruments they cannot directly provide the type of guarantees offered by an insurance company. The main difficulty is that different guarantees have to be supplied to different cohorts since these trusts are open to investment and disinvestment at any time. One way that unit trusts can provide guarantees is to obtain a guarantee from a third party such as a bank or an insurance company. There is currently a unit trust product on the market which utilises a guarantee provided by a general insurance contract to compensate investors if the proceeds from the unit trust on a particular date are less than the initial investment.

4.3.3 Investment trusts have faced a number of restrictions in being able to offer GEPs. The most obvious way of providing the guarantee is via an insurance policy as has been done by unit trusts. The restrictions which investment trusts have faced have been:

(i) They are obliged to distribute 85% of their income to shareholders. Thus any investment which earns an income to provide for the guarantee would not be available for the capital guarantee since a significant part would need to be distributed.

(ii) A significant part of the income of an investment trust must be derived from shares or securities. In practice this needs to be 70% or more. This places restrictions on the type of investment policy it can follow.

(iii) The current tax regime applicable to investment trusts relies on these companies seen to be trading rather than investing.

4.3.4 Banks and Building Societies have recently faced greater competition for retail funds. The standard insurance or unit trust single premium products generally compete for these deposits on the basis of comparative investment returns although the risks are quite different. Banks and Building Societies have responded by offering similar products either through their bancassurance life insurance operations or by way of a deposit account where the interest would be in line with say the FT-SE 100 Index with a minimum guaranteed rate over a fixed term of say five years.

4.3.5 Business Expansion Schemes (BESs) can no longer be marketed. The reason that they were given tax concessions was the relatively risky business that the schemes were investing in. Some BESs managed to reduce risk by utilising derivative instruments and therefore became primarily a vehicle for tax management.

4.3.6 Overseas or off-shore product providers are not subject to the same degree of regulation as, say, UK authorised unit trusts. The off-shore market providers have developed a market for funds utilising derivative instruments and such investments have been available from providers based in locations such as Bermuda, Channel Islands and Luxembourg for some years. These providers face problems of marketing these products in the on-shore market.

4.4 Tax Issues

4.4.1 The taxation of both the provider and the investor can be complex. Table 4.1 summarises the main taxation rules for each product provider.

4.4.2 From the investor's point of view the tax implications need to be understood. Different circumstances will dictate different types of products. The Inland Revenue have recently been considering various options regarding changes to the taxation regime for life assurance companies. Some of these options are discussed in Section 3.7.3 to 3.7.6. As a consequence, the balance between the different tax regimes may change in the future.

4.5 Charging Structure and Competition

4.5.1 Charging structures for GEPs are irrelevant information for the policyholder since the benefits provided are fixed as for non-profit contracts. The only factor that needs to be considered when analyzing the competitiveness of these products is the level of benefits that are being provided. Benefits that may need to be compared are the levels of equity exposure and the nature of the guarantee.

4.5.2 The charging structure is important for the provider since it will determine the competitiveness and profitability of the product.

4.5.3 GEP providers compete by providing attractive guarantees together with appropriate equity exposures, by timing their marketing periods and through product innovation.

Table 4.1
Summary of Taxation Rules

Product Provider	Taxation of Funds	Taxation of Policy Proceeds
Insurance Company	● Life assurance funds are taxed on the I-E basis. Since I is generally significantly larger than E for single premium savings contracts these funds can be considered "net" funds. ● Currently the I-E tax assessment can effectively be converted to a profits basis by the use of suitable reassurance contracts. ● Pension business is taxed on a profits basis and in general terms can be considered to be gross.	● Life assurance GEPs are generally "non-qualifying" policies and the proceeds are subject to the chargeable gains tax regulations. The policy gains are deemed to have borne basic rate income tax. A basic rate taxpayer would have no tax liability on the proceeds but a higher rate taxpayer would pay tax at the marginal rate on the assessed chargeable gains. ● For a non taxpayer there is no facility to reclaim tax paid by the insurance company.
Bank or Building Society	● The funds are taxed on profits and in general terms they would be considered to be gross funds.	● The proceeds at maturity or earlier surrender would be partially a return of capital and partially accrued interest. The interest component is generally deemed payable net of basic rate income tax during the tax year of maturity or earlier surrender. As an alternative interest can be received gross by a non taxpayer. The higher rate taxpayer would be subject to tax at the marginal rate on the grossed up amount.

Product Provider	Taxation of Funds	Taxation of Policy Proceeds
Unit Trusts	● Unit trusts are not liable to capital gains taxation. ● Franked income received by the unit trust will have a tax credit attached to it. Unfranked income less expenses are subject to corporation tax, levied at the basic income tax rate.	● Distributions from unit trusts are deemed to have been taxed at source. A basic rate taxpayer would have no further liability whilst a non taxpayer can reclaim the tax levied. ● Upon the sale of the units the unit holder is liable to capital gains tax on the chargeable gains. In assessing chargeable gains any personal allowances not utilised elsewhere can be used to offset liability. ● A "Bed and Breakfast" facility can be utilised to fully use the personal allowances in each year. ● GEPs offered by unit trusts could be put into a PEP to make them more tax efficient.
Offshore funds	● These can, in general, be considered to be gross funds.	● The investor would be liable to chargeable gains tax.

4.6 **Marketing**

4.6.1 The main thrust of marketing GEPs has been the provision of "best of both worlds" benefits. In doing so emphasis has been placed on:-

(i) The fact that equity investments have outperformed most other types of investments over long periods.

(ii) The risk of a stock market crash when proceeds are taken from such investments.

4.6.2 In effect, the marketing of GEPs relies to some extent on trying to match consumer expectations (that is the price consumers are willing to pay) and the market's expectation (that is the price that the market will charge).

4.6.3 The marketing literature published and used by GEP providers has come under scrutiny by the regulators who have shown concern about some aspects of the marketing of GEPs. The various regulatory bodies, such as LAUTRO and IMRO, have issued guidelines for the marketing of such products.

4.6.4 LAUTRO has set out six points that their members need to take account of in their marketing material used to support sales of GEPs. These are:-

(i) That the cost of guarantees should be made explicit.
(ii) It should be made clear that the growth in the relevant index does not include dividend reinvestment.
(iii) The amount allocated to tracking the index is made explicit.
(iv) Claims that returns are made gross of tax may be misleading, so illustrations should be shown net of tax where appropriate.
(v) It should be made clear to investors whether gains are subject to income or capital gains tax.
(vi) Early encashment terms are explained along with risk warnings for any penalties involved.

4.6.5 While guidelines (ii), (iv), (v) and (vi) in Section 4.6.4 are essential information for potential investors, it could be argued that guidelines (i) and (iii) are questionable since they are concerned with technical matters which do not directly affect benefits payable and which could conceivably be misinterpreted.

4.6.6 LAUTRO has recently published guidance, in their Enforcement Bulletin 27, regarding the marketing of high income products. Three particular points are mentioned:-

(i) That income derived in part from sources other than dividends or interest must be emphasised.

(ii) That if capital is not guaranteed it is not sufficient just to warn the investor of this. They should not be led into thinking that the product "is very likely to" return the investor's capital.

(iii) Any reference to lower future returns from competing products such as Building Society deposits should also require a reference that the high income product is not entirely free from the effects of changes in economic conditions.

4.6.7 FIMBRA has shown concern that their Guidance Note 6, which includes the phrase "if some benefits are guaranteed and some are not, members should say so and should give equal prominence to the description of benefits which are guaranteed and of benefits which are not", is too frequently overlooked.

4.6.8 Building Societies were forced to offer investors a cooling-off period for the first time in early 1993 under new guidelines covering guaranteed equity bonds.

4.6.9 In October 1993 IMRO issued guidance to its members after concerns that "some of the marketing material used in the promotion of high income investment products in recent months has given undue prominence to the high rate of income offered without fully or fairly describing the nature of the product and the risks involved".

5. OPTION PRICING MODELS

5.1 The pricing of GEPs can be based on well known actuarial techniques with one important exception, that of pricing for the guarantees to be provided. As explained in Section 2.1 the guarantees are akin to providing options, and a knowledge of option pricing theory is necessary in order to price the guarantees.

5.2 In 1973 Fischer Black and Myron Scholes published a paper[2] in which they derived a theoretical valuation formula for pricing European options on stocks under certain ideal conditions. The ideas behind the Black-Scholes Model are still widely used, though generally with modifications, and they have formed the basis for subsequent research into option pricing.

5.3 While it is possible to derive the Black-Scholes formula from the Capital Asset Pricing Model (CAPM), thus providing a link between the two models, it is not necessary to do so. For those who doubt the validity of CAPM, a preferable approach to deriving the Black-Scholes formula lies in the construction of what is called a hedged portfolio. One particular hedged portfolio consists of a combination of stock and written call options on that stock. At any stock price it is possible to find a particular combination of stock and written calls which is immunized against small movements in the stock price. The required combination changes as the stock price moves. That hedged portfolio is risk free and consequently should earn a risk free rate of return. The characteristics of the hedged portfolio together with a model for stock price movements are then used to derive a differential equation which can be solved to give the Black-Scholes formula for a call option. A derivation of the Black-Scholes formulae for both call and put options is given in Appendix II. The Black-Scholes formulae can also be derived by constructing a portfolio comprising stock and a risk free asset which replicates the characteristics of an option.

5.4 An alternative model, which uses a simpler approach not requiring stochastic calculus, is the Binomial Model developed by Cox, Ross & Rubinstein[3]. The Binomial Model is also based on constructing a hedged portfolio, but this time it is only necessary to consider two possible movements in the stock price (up or down by fixed amounts) over any one time period. This process is then applied over several time periods to derive the Binomial formula. In the limit, as the time periods become smaller and with suitably chosen up and down movements, the Binomial formula converges to the Black-Scholes formula. The Binomial Model is described in Appendix III.

5.5 The model for stock price changes assumed by Black and Scholes was a continuous random walk with a variance proportional to the square of the stock price. This can be represented by a generalised Gauss-Wiener process with constant volatility applied to the logarithm of the stock price. Such a process implies that the distribution of stock prices at any future date is log-normal. Some investigations have shown that this model may be a good approximation to reality over short time periods, whereas others have concluded that its use is questionable for longer periods. General reasoning might suggest the reverse is true, since sudden changes in stock prices would have more impact in the short term than in the long term. The Black-Scholes formula can, however, be derived using much weaker assumptions about the behaviour of stock prices. For example, it is not necessary to assume that stock prices follow a random walk or that returns are distributed log-normally.

5.6 The Black-Scholes Model (and the Binomial Model) assumes that the risk free rate of interest and the volatility of the stock price are constant over time. More sophisticated approaches assume that these variables are time dependent and may be either deterministic or stochastic. Deterministic approaches (and some stochastic approaches) will still result in the same form of Black-Scholes formula. Allowance can also be made for sudden discontinuous movements in the stock price. More complicated models do not generally produce a closed form solution (i.e. a simple formula for the value of an option), but require the application of numerical techniques.

5.7 The Black-Scholes and the Binomial Models apply only to European options. However, it is straightforward to show that, theoretically, it never pays to exercise an American call option before expiry, provided no dividends are due during the period of the option. Consequently the value of an American call option should equal that of a European call option if no dividends are payable - strictly if the stock does not go ex-dividend before expiry of the option. If a dividend is payable, the value of a call option can be determined by considering separately exercise at expiry of the option and exercise immediately prior to the ex-dividend date. The lattice structure of the Binomial Model makes it more easily adaptable, than is the Black-Scholes Model, for pricing American options.

5.8 The value of a European put option can be derived directly from the value of a European call option using Stoll's put-call parity theorem[4]. This states that the value of a unit of stock and a European put option is the same as the value of a European call option and cash equivalent to the exercise price discounted at the risk free rate of return. This relationship does not work for American options as it may be worthwhile exercising an American put option early. A closed form solution has not yet been found for an American put option, and may not exist.

5.9 Adjustments can be made to the standard Black-Scholes and Binomial formulae to allow for dividend payments. If the options are written on a stock index it may be appropriate to assume that dividends are payable continuously at a constant rate. The derivation of the Black-Scholes formulae given in Appendix II makes this assumption.

5.10 All the variables in the Black-Scholes formulae are either directly observable or capable of estimation from market prices and movements in those prices. The stock price, exercise price and time to expiry of the option are obviously known. The risk free rate of interest can be estimated from the yields available on fixed interest securities of suitable type and duration. The dividend yield on, for example, the FT-SE 100 Index can be determined from the prices of futures or forward contracts on the index. The volatility can either be determined from historical data or can be derived by substituting known option prices into the Black-Scholes formula to give market expectations of future volatilities.

5.11 The simplest form of guaranteed equity bond is, as explained in Section 2.1, effectively a combination of a zero coupon bond and a standard call option. Its pricing is therefore straightforward using, for example, the Black-Scholes formula with allowance for dividend payments on the FT-SE 100 Index. Guaranteed death benefits can also be allowed for fairly easily. The Black-Scholes Model is a general one and can be used to value more complex guarantees such as lock-ins and look backs. These adaptations are shown in Appendix IV.

5.12 For single premium contracts, the effect of the guarantee will depend solely on the equity price at commencement and at the date of a claim (maturity, death or surrender). The pricing of regular premium contracts is slightly more complicated because the cost of the guarantee depends on the equity prices at all premium payment dates. An option pricing model can be used to generate a formula which requires evaluation using numerical techniques. Examples of a possible approach are given by Bacinello and Ortu(5), who illustrate a method of deriving premiums for contracts where the guarantees are functions of the premiums paid.

6. **DYNAMIC HEDGING**

6.1 Providers of GEPs will generally insure the guarantees by purchasing suitable investments and options. They need not do so, but could instead try to manage the guarantees without purchasing options. Option writers may try to match their liabilities by purchasing offsetting options but they will not always be able to do so. They will, at least to some extent, need to manage their resulting net exposure. Somewhere there is an option writer of last resort.

6.2 When insurance companies provide contracts with guarantees they are usually passive in the management of those guarantees and establish appropriate reserves. By contrast, banks manage their investment risks actively by setting up and maintaining hedges.

6.3 Option pricing models can be used to quantify how options and their associated guarantees can be managed. Indeed the derivation of option pricing formulae through the construction of hedged portfolios shows us how to do this. As an example, consider a simple guaranteed equity bond offering equity exposure on the single premium with a money back guarantee. We could back this by purchasing a zero coupon bond to meet the guarantee and an at the money call option to provide the equity exposure. Using the Black-Scholes formula and the notation of Appendix II, the value of the bond and the call option, if the guaranteed amount is G (G = E, the exercise price of the option), is $V = Se^{-Dt} N(d_1) + Ge^{-rt} N(-d_2)$.

6.4 For dynamic hedging purposes, we want to construct a portfolio which is a combination of zero coupon bonds and the equity index (assuming the call option is based on an index) and which has similar characteristics to the "insured" portfolio consisting of bonds and a call option.

6.5 One of the most desirable characteristics that the portfolio should possess is that it should behave in a similar way with respect to movements in the equity index value. Suppose the portfolio consists of an amount A of the equity index and an amount Z of the zero coupon bond. Then $V = AS + Ze^{-rt}$, and we want the change in V as S changes to be the same as for the insured portfolio. So we require $\frac{\partial V}{\partial S} = A = \frac{\partial C}{\partial S} = e^{-Dt}N(d_1)$, where C is the value of the call option. The last expression is known as the option delta. (Note that $\frac{\partial V}{\partial S} = \frac{\partial C}{\partial S}$, since the zero coupon bond is independent of S, and that this is the hedging ratio in Appendix II.) Hence in our portfolio we should hold an amount e^{-Dt} $N(d_1)$ of the index and an amount Ge^{-rt} $N(-d_2)$ of zero coupon bonds.

6.6 The above example is an extremely simple one and, in practice, the process of managing the guarantees underlying options is complex. The example illustrates only one sensitivity (the portfolio delta), being that of movements in the equity index. It should be noted that the mix of the equity index and bonds is itself dependent on the value of the index. Consequently, in order to follow even this simplified approach to dynamic hedging, it is necessary to alter the mix of equities and bonds continuously. Frequent trading in equities and bonds is expensive, but the use of futures contracts allows transactions to take place at much lower costs. Nevertheless it is still necessary to make allowance for expenses when designing a dynamic hedging strategy. In any event, continuous trading is impractical.

6.7 As demonstrated in the option pricing formulae in the Appendices, the price of an option depends on factors other than the value of the underlying asset, and more importantly on factors which either do not affect the price of the underlying asset or which affect the price in a different way. These factors, such as sensitivities to changes in interest rates, volatility and time, create additional risks for option writers which need to be managed.

6.8 The various risks are outlined in the following paragraphs, by reference to a call option (with value C). The notation used is that of Appendix II.

6.8.1 Delta $\left(\frac{\partial C}{\partial S}\right)$

Delta measures the sensitivity of the option price to a change in the price of the underlying asset. As described in the simple example above, it can be hedged in the futures market. Figure 6.1 illustrates a typical Delta curve for a call option.

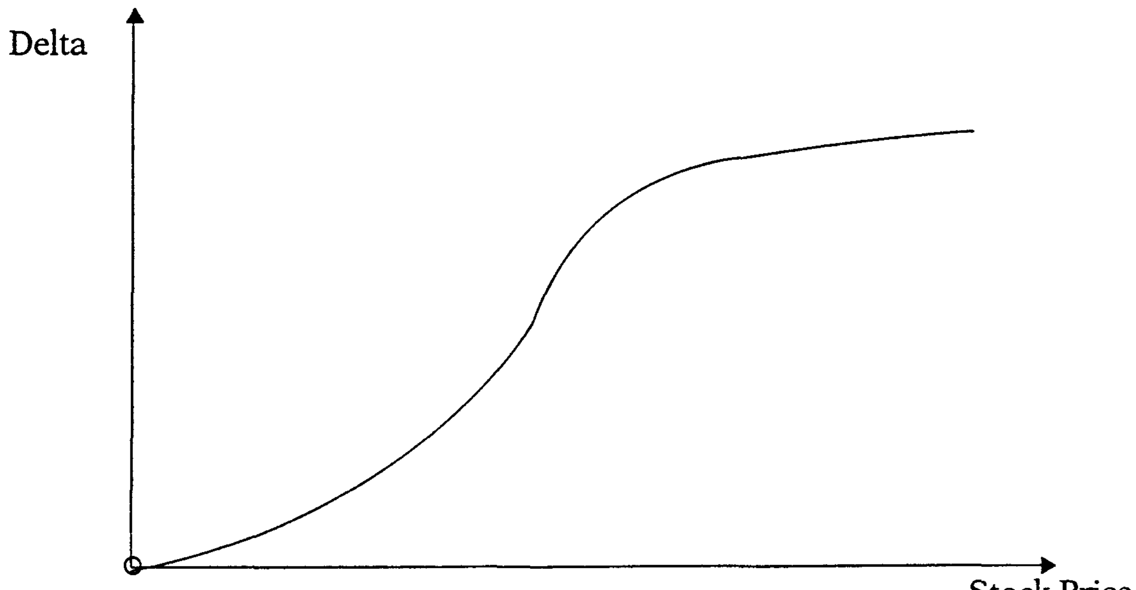

Figure 6.1 - Delta

6.8.2 Gamma $\left(\frac{\partial^2 C}{\partial S^2}\right)$

Gamma is the sensitivity of the option Delta to a change in the price of the underlying asset. It is highly dependent on the time to run until expiry of the option. Figure 6.2 illustrates a typical Gamma curve for a call option some months before expiry and Figure 6.3 shows how the Gamma curve develops as the option nears maturity.

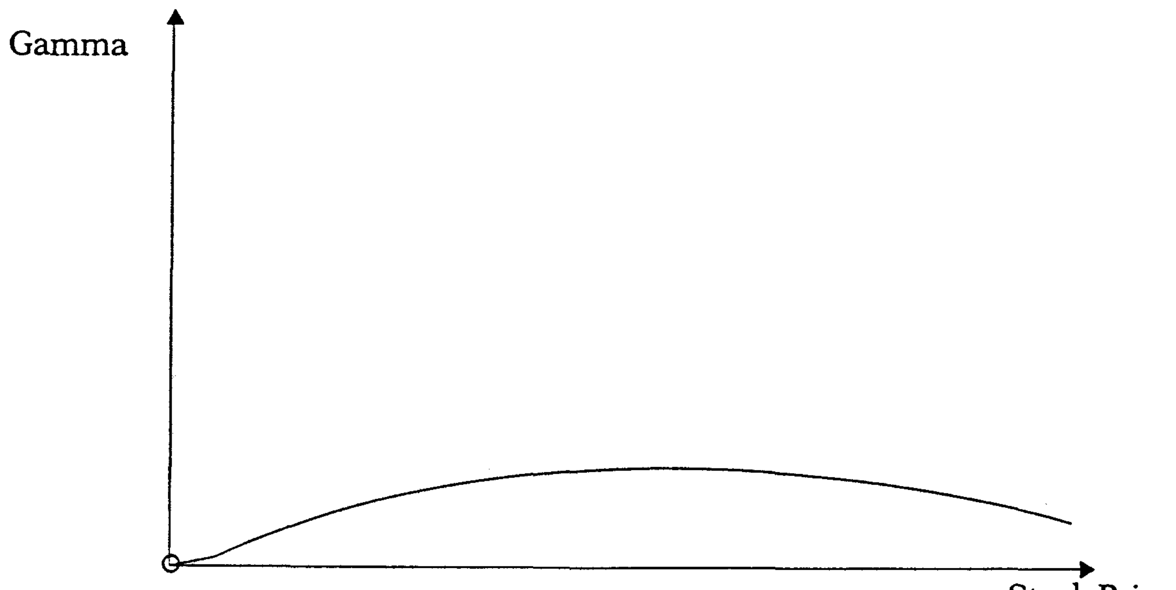

Figure 6.2 - Gamma several months before expiry

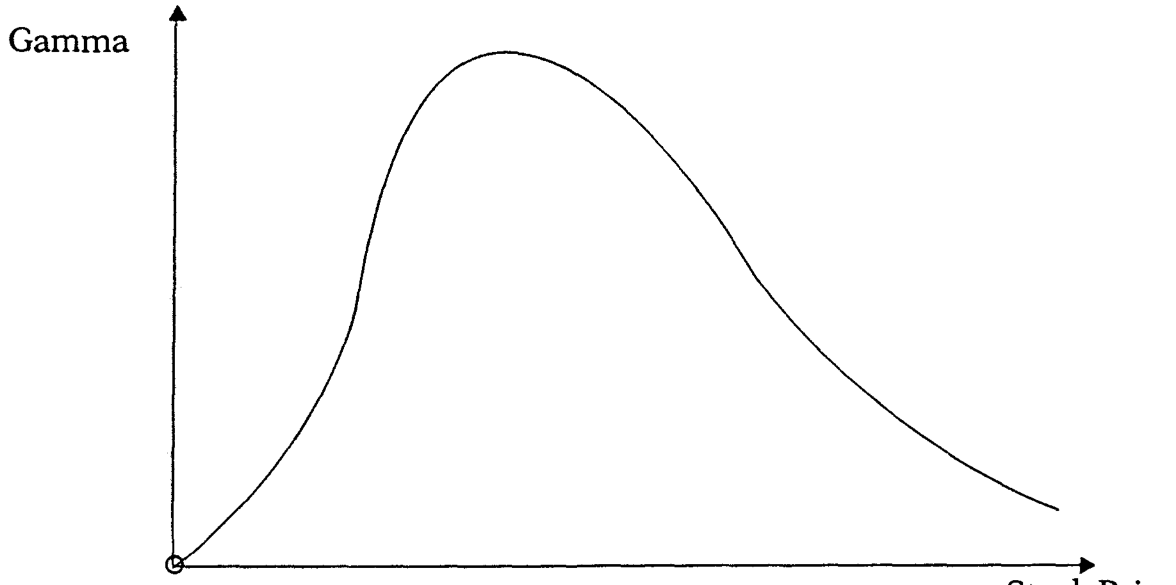

Figure 6.3 - Gamma close to expiry

The above figures show that the management of Gamma does not become critical until the option approaches expiry. At that stage it is of course possible to use the exchange traded option market to hedge the exposure.

6.8.3 Rho $\left(\frac{\partial C}{\partial r}\right)$

Rho is the sensitivity of the option price to a change in the risk free rate of interest. The value of a call option increases as the rate of interest increases, since holding a call option and cash instead of investing directly enables interest to be earned on the cash. The reverse is true for a put option. The risk can be managed to some extent by balancing puts and calls in a portfolio, by the use of interest rate futures, or by exchanging fixed interest payments for floating rate payments in the swaps market. Unfortunately Rho depends on the price of the underlying asset so the swap exposure needs to be changed as the price varies.

6.8.4 Lambda $\left(\frac{\partial C}{\partial D}\right)$

Lambda is the sensivity of the option price to dividend yield. It might be possible to hedge this risk by swapping actual dividends with an assumed constant dividend rate.

6.8.5 Theta $\left(\frac{\partial C}{\partial t}\right)$

Theta, the sensitivity of the option price to the time to run to expiry, is positive for call options and negative for put options. There is a linear relationship between Theta, Delta and Gamma, so the techniques described above can be used to hedge Theta. It is not possible to hedge Theta outside the options market.

6.8.6 Kappa $\left(\frac{\partial C}{\partial \sigma^2}\right)$

Kappa is the sensitivity of the option price to volatility. Like Theta, Kappa cannot be hedged without the use of options.

6.9 In practice it is necessary to monitor and manage the above risks on a portfolio of investments and derivatives (bought and sold). Successful management of these risks relies on the design and maintenance of sophisticated computer systems. Some of the risks may either be eliminated or at least mitigated by careful portfolio selection and management.

6.10 A further source of risk is that the market does not conform to the pricing model used. No model, however sophisticated, will replicate exactly the vagaries of the stock and option markets. The assumption for stock price behaviour underlying Black and Scholes' original work is a form of random walk. The Report of the Maturity Guarantees Working Party(1) suggests that a random walk is not a good model for stock market prices in the long term. Other studies have concluded that the random walk model is not unreasonable or at least cannot be disproved by history, in that any non random patterns observed in the past are not helpful to predicting the future. The Black-Scholes Model is widely used as a basis for pricing options, but as mentioned in Section 5.5, it does not necessarily depend on the assumptions originally made by Black and Scholes.

6.11 The Maturity Guarantees Working Party investigated dynamic hedging, referred to as an immunization strategy in their Report, and highlighted a number of the difficulties referred to above. The Working Party concluded that maturity guarantee reserves could not be reduced simply because a company adopted a dynamic hedging strategy. Although the difficulties remain, some may not be as extreme as in 1980 and advances have been made since then. It may therefore be opportune to review the ways in which insurance companies can provide for guarantees and the level of reserves required.

6.12 Options are usually available only on particular securities or indices. They are not generally written for example on insurance companies' internal linked funds. If a company wanted to provide a guarantee associated with investment in an internal linked fund it may not be possible to purchase suitable options. A dynamic hedging strategy may then be the only way to tackle the management of the guarantee.

7. THE FUTURE

7.1 GEPs

7.1.1 The following three aspects of the expected changes to the insurance regulations in the UK are likely to affect the GEP market:-

(i) Formalisation of some of the working rules in granting Section 68 and Section 78 orders.
(ii) The ability to use derivative instruments much more generally than is currently used to back GEPs.
(iii) Conditions that need to be fulfilled in using derivative instruments.

7.1.2 The first two factors in 7.1.1 should aid the growth of the current market for GEPs. The constraints on the use of derivatives envisaged by the proposed new regulations are unlikely to limit the development of GEPs and should instil confidence in the use of derivative instruments. Competition in the GEP market may result in further innovations as providers seek product differentiation. The real test for GEPs is whether regular premium contracts can successfully be introduced. If they can we may see a radical change in the savings and life assurance market.

7.1.3 The review of the taxation regime as discussed in Sections 3.7.3 to 3.7.6 is likely to affect the GEP market, for example in the use of reassurance facilities and the combination of assets used to back the liabilities.

7.1.4 Market opinion seems to be divided over whether GEPs have a future. They should continue to be attractive to cautious investors and to those who require contracts with clearly defined benefits. Their opponents may point to the fact that reducing interest rates makes GEPs appear to be less competitive, since the combined cost of a zero coupon bond and a call option increases as interest rates decline. This simply reflects the market view that returns from all types of investment will be lower in the future than in the recent past. It doesn't follow that equity prices will be less volatile. But all that is subjective. The only objective view is that provided by the investment market itself.

7.2 **Derivatives**

7.2.1 The use of options and other derivatives is now well established in both investment management and the provision of GEPs. These instruments, and the pricing theory underlying them, have many other potential applications in life assurance. A few possible applications are briefly discussed below.

7.2.2 Resilience Test

The resilience test requires an Appointed Actuary to make allowance for a 25% fall in equity prices when performing a statutory valuation. Protection against a less severe fall could be obtained by purchasing a put option, thereby reducing the size of any resilience reserve. The cost of buying the put option could be met (in part) by writing call options. Since options are generally only available on indices, it is of course necessary to take into account the deviation of the equity porfolio from the index. Care is needed in following a strategy of buying puts and selling calls. Although the puts may give the desired protection against a fall in equity prices, the sale of calls may cause difficulties in meeting the upward side of the resilience test. More complicated option structures are needed in practice.

7.2.3 With Profit Business

Traditional with profit contracts are generally backed by an investment portfolio comprising a high proportion of equities, although in recent years that proportion may have declined. The regular addition of reversionary bonuses effectively provides the policyholder with an increasing guarantee at maturity, or on earlier death, though not on early termination. A with profit policy can therefore be thought of as an investment in a mixed portfolio of assets, likely to have a high equity content, with a put option which has an increasing exercise price. In addition the contract provides an element of smoothing of investment returns. Asset shares with deductions for the cost of the option and the cost of smoothing could form the basis for assessing fair payouts to policyholders. Professor Wilkie illustrated the use of option pricing theory in with profit business in his 1986 Paper[(6)]. An alternative approach might be to design an investment policy, appropriate to the guarantees provided, based on a dynamic hedging strategy.

7.2.4 Immunization

All actuaries are familiar with Redington's theory of immunization for non-profit business[7]. In practice, the theory cannot always be applied when interest rates are high (will we ever have that problem again?), because the mean term, even of irredeemable stock, may be too low to match that of the liabilities. The purchase of appropriate bond futures can extend the mean term of a bond portfolio, enabling an immunization strategy to be maintained.

7.3 Education

7.3.1 If it is accepted that options and other derivatives are useful, perhaps indispensable, tools in life assurance (and equally in pensions and other areas), then more actuaries need to become familiar with them and learn and understand the fundamentals of option pricing theory. The principles involved - risk, arbitrage, hedging should not be new to actuaries. Most of the expertise currently resides in the banking sector, but the area is a natural one for actuaries to develop. There is both a role and an opportunity here for the actuarial profession.

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APPENDIX I

Options

A call option gives the holder the right to purchase a specified amount of a commodity on (or before) a particular date at a fixed price. The purchaser of an option pays a premium to the seller or writer of the option. A put option gives the holder the right to sell a specified amount of a commodity on (or before) a particular date at a fixed price. The commodity underlying the option could be an equity stock or an equity index.

Call options therefore give their holders exposure to increases in the value of the underlying commodity without actually owning the commodity. Holders of put options are protected against falls in the value of the commodity.

Options may be traded on recognised exchanges, or they may be tailored to meet a purchaser's particular requirements. The latter are known as "over the counter" options.

An option that can be exercised only on one particular date is known as a European option. An American option permits the holder to exercise the option at any time up to a particular date (the expiry date). The fixed price at which an option can be exercised is known as the strike price or the exercise price.

A call option is said to be "in the money" if the price of the underlying commodity exceeds the exercise price. If the price of the commodity is below the exercise price, the call option is "out of the money". The reverse holds for put options. Options where the exercise price is equal or close to the current market price of the underlying commodity are said to be "at the money".

The maximum amount an option holder can lose is the premium paid to purchase the option. A writer of a call option can stand to lose an indefinite amount, unless he has taken the precaution of purchasing that amount of the underlying commodity necessary to cover (i.e. satisfy his potential obligation under) the call. The writer of a put option has limited liability (since the value of a commodity cannot usually fall below zero), but he is still at risk of losing a large amount.

Figure 1 shows the profit (value at exercise less premium) for a call option with exercise price E and premium P.

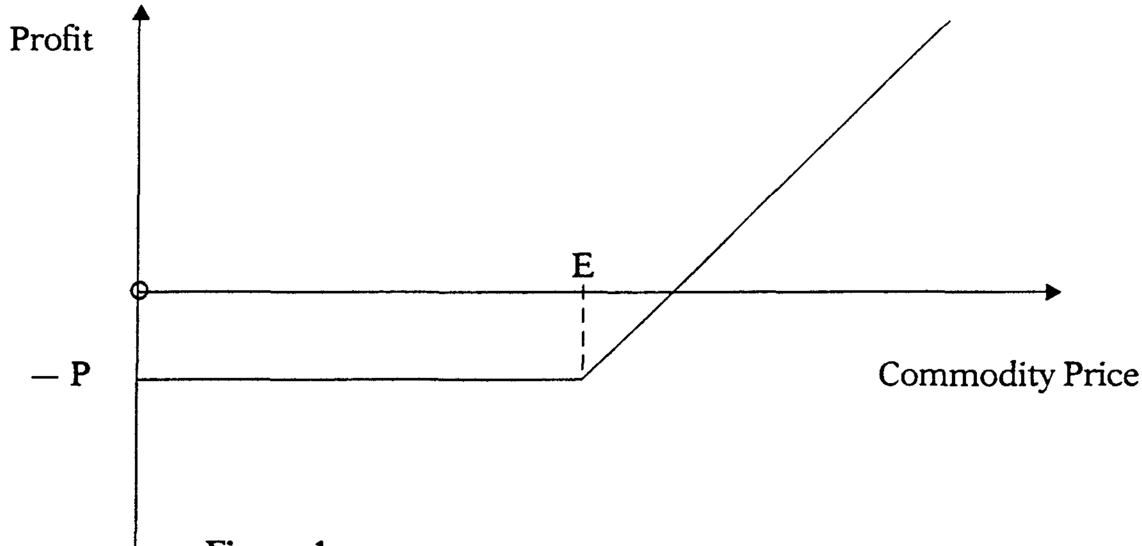


Figure 1.

The option is worth exercising if the commodity price on the exercise date exceeds E, and the profit is positive if the commodity price on exercise exceeds E+P. The profit to a writer of the same call option is the mirror image of Figure 1 in the commodity price axis.

Figure 2 shows the profit for a put option with exercise price E and Premium P.

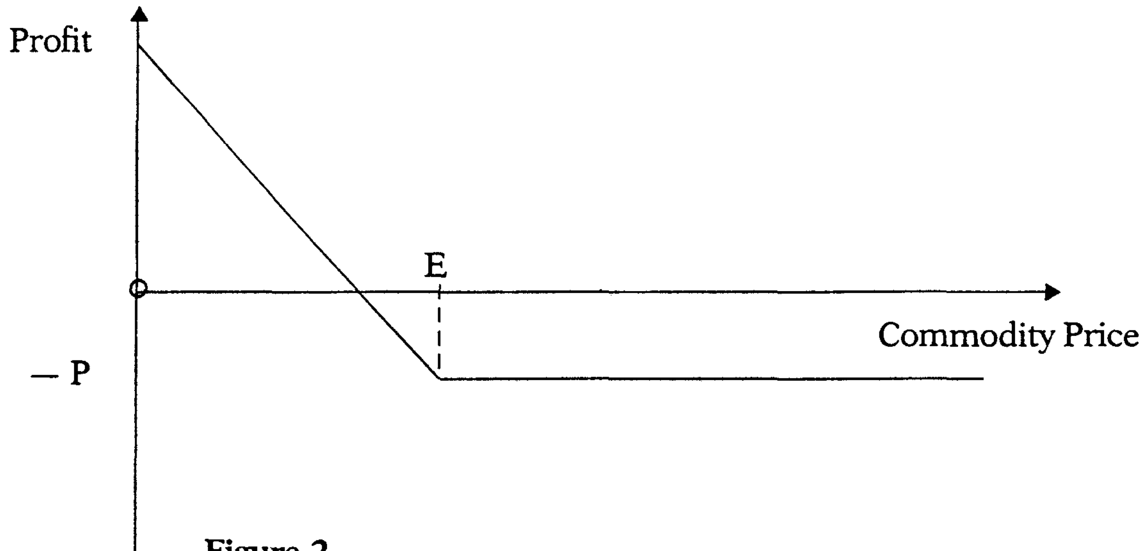


Figure 2.

The option is worth exercising if the commodity price on the exercise date is below E, and the profit is positive if the commodity price on exercise is below E–P. The profit to a writer of the same put option is again the mirror image of Figure 2 in the commodity price axis.

A put option might be held together with the underlying commodity in order to provide protection against a fall in value of the commodity. The profit from a combination of a unit of the commodity and a put option on that unit, with an exercise price E equal to the purchase price of the commodity unit and premium P, is shown in Figure 3.

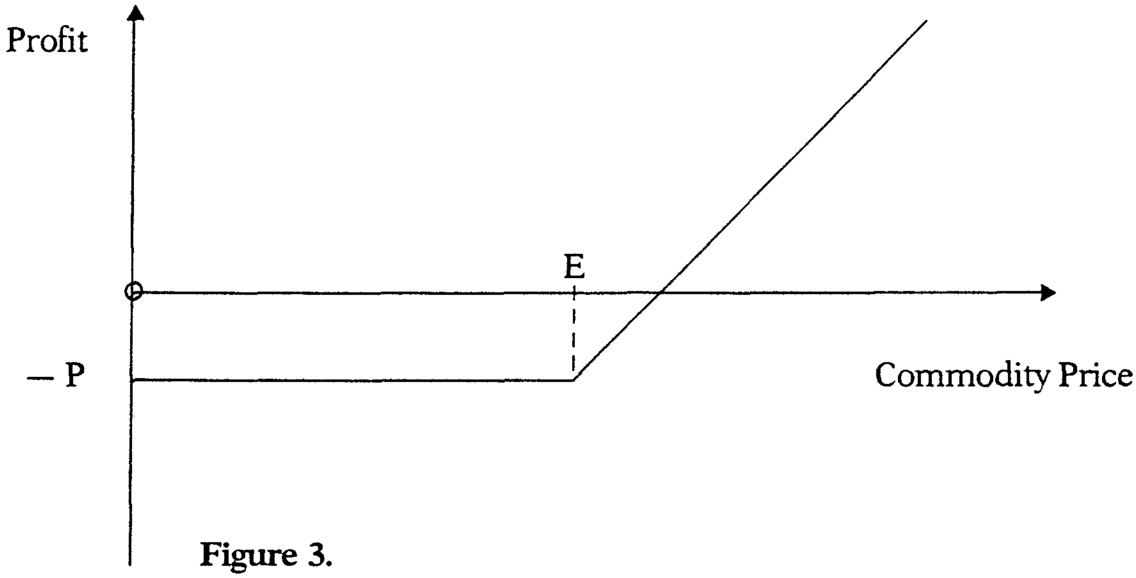


Figure 3.

Figure 3 is identical to Figure 1, illustrating the well-known result that the underlying commodity and a put option can be combined in such a way to produce the same pay-off as a call option. In general, cash plus call is equivalent to commodity plus put.

Futures and Forward Contracts

Forward contracts are agreements to buy or sell a specified amount of a commodity at a fixed price on a particular future date. They are widely used in the foreign exchange market. Futures are similar to forwards, except that profits and losses are usually settled on a daily basis. This process is known as marking to market.

Both parties to a futures or forward contract are obliged to proceed with the transaction, though in practice the contract may be settled by a cash payment rather than by physical delivery of the commodity. A futures or forward contract may be unwound by an equal and opposite transaction at a later date. By contrast, the holder of an option has the right, but not an obligation, to exercise it.

APPENDIX II

Black-Scholes Model

The standard Black-Scholes Model for options on stocks makes the following assumptions:

(i) The options are European.
(ii) There is a constant risk-free rate of interest (e.g. yield on government debt of appropriate maturity).
(iii) There are no transaction costs or taxes.
(iv) The logarithm of the stock price follows a Gauss-Wiener process, with constant volatility.
(v) The stock does not pay dividends.

Since it is straightforward to derive the Black-Scholes formula for an option on a stock which is assumed to have a constant dividend yield, and the resulting formula is more useful in practice than the standard formula, we dispense with the last assumption above and instead assume that the underlying stock has a constant dividend yield.

To derive the value of a call option on a unit of stock, let

S = market price of the stock
D = dividend yield
C = value of a call option on the stock
A = amount of stock held
B = amount of call option held
r = risk free force of interest
t = time.

S is a function of t, and C is a function of S and of t. A and B are chosen so that the value of the portfolio

$$V = AS+BC$$

is immunised against small changes in S, except to the extent that dividends are received.

The logarithm of S follows a Gauss-Wiener process, so

$$dS = \mu S dt + \sigma S dz \tag{1}$$

where μ = mean (which can be dependent on both S and t)
σ^2 = constant volatility
dz is a normal random variable with zero mean and variance dt.

From Ito's lemma of stochastic calculus,

$$dC = \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial t} dt + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} dt \tag{2}$$

This is similar to the Taylor series for a deterministic function, but since S is stochastic there is an additional term in the expansion.

Using (1) and (2) and rearranging terms,

$$dV = \left[A\sigma S + B\sigma S\frac{\partial C}{\partial S}\right] dz + A\mu S\, dt + B\left[\mu S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right] dt.$$

If V is immunised, the stochastic term in dz must be zero.

Hence $A + B\frac{\partial C}{\partial S} = 0$, which defines the hedged portfolio.

If at a particular point in time $A = \frac{\partial C}{\partial S}$, $B = -1$, so a hedged portfolio could comprise a written call on a unit of stock and an amount $\frac{\partial C}{\partial S}$ of the stock. $\frac{\partial C}{\partial S}$ is the hedging ratio.

Substituting $A = -B\frac{\partial C}{\partial S}$ in the above expression for dV gives

$$dV = B\frac{\partial C}{\partial t} dt + \tfrac{1}{2}\sigma^2 S^2 B\frac{\partial^2 C}{\partial S^2} dt$$

But if V is a hedged portfolio it is risk free and so should earn the risk free rate of return r.

Hence $dV + DASdt = rVdt$, and so

$$B\frac{\partial C}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 B\frac{\partial^2 C}{\partial S^2} = r\left[-BS\frac{\partial C}{\partial S} + BC\right] + DBS\frac{\partial C}{\partial S}$$

$$\text{or } \frac{\partial C}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC + (r-D)S\frac{\partial C}{\sigma S} = 0. \tag{3}$$

At the time of exercise of the call option, we know that

$$C = S - E \text{ if } S > E$$
$$\quad 0 \text{ if } S \leqslant E,$$

where E is the exercise price and this gives the boundary condition for the above differential equation.

The boundary condition determines a unique solution to the differential equation and the solution is:-

$$C = Se^{-Dt} N(d_1) - Ee^{-rt} N(d_2)$$

$$\text{where } d_1 = \frac{\ln S/E + (r - D + \sigma^2/2)t}{\sigma\sqrt{t}},$$

$$d_2 = d_1 - \sigma\sqrt{t},$$

and t has been redefined to be the time to run to expiry.

$N(d_1)$ and $N(d_2)$ are cumulative probalities for the standard Normal distribution, so that

$$N(d_i) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{d_i} e^{-x^2/2}\, dx.$$

The value, P, of a European put option with exercise price E can be found from the relationship

$$S + P = Ee^{-rt} + C + S(1 - e^{-Dt})$$

that is stock + put = cash + call + dividends,

$$\text{so} \quad P = Se^{-Dt} [N(d_1) - 1] + Ee^{-rt} [1 - N(d_2)]$$
$$= Ee^{-rt} N(-d_2) - Se^{-Dt} N(-d_1).$$

Alternatively, it can be shown that P satisfies the differential equation (3) above by constructing a similar hedged portfolio. The boundary condition at expiry of the option is

$$P = E - S \text{ if } E > S$$
$$\quad 0 \text{ if } E \leqslant S,$$

and again this is sufficient to determine a unique solution.

A more general approach, which can be used to derive the differential equation (3) for any type of option and indeed for any financial instrument, involves the construction of a portfolio comprising solely stock and a risk-free asset which replicates the characteristics of the option or other financial instrument.

APPENDIX III

Binomial Model

The Binomial Model starts by assuming that the price of a stock can be either higher or lower by fixed amounts at the end of a discrete period of time. As in Appendix II, we assume that there are no transaction costs or taxes and that there is a risk free rate of interest. We want to price a European call option in this simplified environment.

Let S be the market value of the stock and C the value of a call option at the beginning of the time period, and let r be the nominal risk free rate of interest over that period.

The standard Binomial Model (like the standard Black-Scholes Model) makes no allowance for dividends. It is easily modified to allow for a constant dividend yield of D on the stock.

Suppose that the stock price at the end of the period can be either uS or $dS(u > 1 + r - D, d < 1 + r - D)$. Let the value of the call option at the end of the period be C(u) or C(d) depending on whether the stock price is uS or dS respectively.

Consider a portfolio which consists of an amount A of stock and a written call option on a unit of stock (that is the owner of the stock writes the call option).

The value of the portfolio at the beginning of the time period is AS−C.

At the end of the time period the value of the portfolio is either AuS−C(u) or AdS−C(d), and dividends of DAS are assumed to be received. [Alternative dividend assumptions could be made.]

We now choose A so that the value of the portfolio at the end of the period is the same whether the stock rises or falls in value. The portfolio is therefore hedged and earns the risk free rate of return.

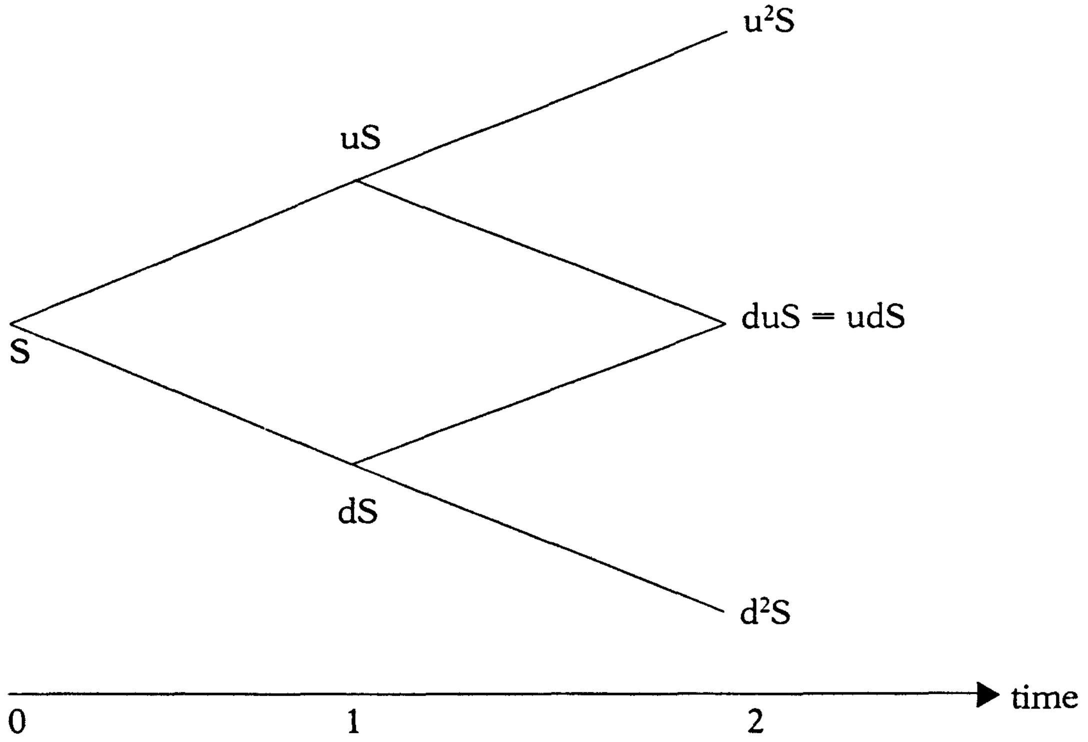
Hence $(AS-C)(1+r) = AuS-C(u) + DAS = AdS-C(d) + DAS$.

Eliminating AS and writing

$$P = \frac{1+r-D-d}{u-d}, \ (0 < P < 1 \text{ from the above constraints}),$$

$$\text{gives } C = \frac{C(u)P + C(d)(1-P)}{1+r}. \tag{1}$$

Suppose we now have two time periods with identical possible up and down movements of the stock price in each. The following diagram shows the possible stock values.



Let the possible values of the call option at the end of the second time period (time = 2) be $C(u^2)$, $C(ud)$ which is equal to $C(du)$, and $C(d^2)$.

Applying formula (1) to the second time period gives

$$C(u) = \frac{C(u^2)P + C(ud)(1-P)}{1+r}$$

$$\text{and } C(d) = \frac{C(ud)P + C(d^2)(1-P)}{1+r}$$

Substituting these values back into (1) yields

$$C = \frac{P^2C(u^2) + 2P(1-P)C(ud) + (1-P)^2C(d^2)}{(1+r)^2}$$

This process can be repeated and by induction it can be shown that over n time periods

$$C = (1+r)^{-n}\left[\sum_{t=0}^{n} {}^{n}C_{t}P^{t}(1-P)^{n-t}C(u^{t}d^{n-t})\right]$$

Now let us assume that the n time periods cover the whole period to expiry of the option. If the exercise price of the option is E, then

$C(u^t d^{n-t}) = \text{Max}\ [Su^t d^{n-t} - E, O]$, the value of the option at expiry.

We now set α such that

$$u^{\alpha-1} d^{n-\alpha+1} S < E \leqslant u^{\alpha} d^{n-\alpha} S \qquad (2)$$

so α is the minimum number of upward movements in the stock price necessary to make exercise of the option worthwhile.

For $\quad t \geqslant \alpha,\ C(u^t d^{n-t}) = Su^t d^{n-t} - E$

and for $t < \alpha,\ C(u^t d^{n-t}) = 0.$

$$\text{Hence}\quad C = (1+r)^{-n}\left[\sum_{t=\alpha}^{n} {}^nC_t P^t (1-P)^t (Su^t d^{n-t} - E)\right]$$

$$= (1+r)^{-n} S \sum_{t=\alpha}^{n} {}^nC_t (uP)^t [d(1-P)]^{n-t} - (1+r)^{-n} E \sum_{t=\alpha}^{n} {}^nC_t P^t (1-P)^{n-t}$$

Writing $Q = \dfrac{uP}{1+r-D}$, so $\dfrac{d(1-P)}{1+r-D} = 1-Q,$

we have $C = S\left(\dfrac{1+r-D}{1+r}\right)^n B[\alpha,n,Q] - E(1+r)^{-n} B[\alpha, n, P],$

where $B[\alpha,n,Q]$ and $B[\alpha,n,P]$ are binomial probabilities of at least α successes in n trials with the probability of success Q and P respectively.

The value of a put option using the Binomial Model can be derived in a similar way.

In order to value an option using the Binomial Model we need to choose appropriate values for u and d. If, for example, we choose

$$u = \exp\left[\frac{\mu t}{n} + \sigma\sqrt{t/n}\right] \text{ and } d = \exp\left[\frac{\mu t}{n} - \sigma\sqrt{t/n}\right]$$

where t = duration to expiry of the option,
n = number of time periods in the model,
then μ = mean return of the logarithm of stock prices,
and σ = variance of the logarithm of stock prices, as in the Gauss-Wiener process.

Note that we do not need to specify probabilities for u and d. The result is independent of any assumed trend in stock prices in exactly the same way that the Black-Scholes formula does not contain μ, the mean in the Gauss-Wiener process.

If we substitute for u and d in (2), we have

$$\mu t + \ln S/E = 2\sigma\sqrt{t/n}\,(n/2 - \alpha).$$

For large n

$$B[\alpha,n,Q] \approx N\left(\frac{nQ-\alpha}{\sqrt{nQ(1-Q)}}\right) \tag{3A}$$

$$B[\alpha,n,P] \approx N\left(\frac{nP-\alpha}{\sqrt{nP(1-P)}}\right) \tag{3B}$$

where N(x) is the probability that a standard Normal random variable takes a value below x.

Substituting for u and d in the expressions for P and Q, letting $n \to \infty$, and using the approximation $e^x \approx 1+x+\frac{1}{2}x^2$ for small x, we find that

$$Q \approx \frac{1}{2} + \frac{(r-D-\mu)}{2\sigma}\sqrt{\frac{t}{n}} + \frac{1}{4}\sigma\sqrt{\frac{t}{n}},$$

$$\text{and} \quad P \approx \frac{1}{2} + \frac{(r-D-\mu)}{2\sigma}\sqrt{\frac{t}{n}} - \frac{1}{4}\sigma\sqrt{\frac{t}{n}},$$

$$\text{and} \quad nQ(1-Q) \approx nP(1-P) \approx \frac{1}{4}n.$$

Hence

$$\frac{nQ-\alpha}{\sqrt{nQ(1-Q)}} \to \frac{2}{\sqrt{n}}\left[\left(\frac{1}{2}n-\alpha\right) + \sqrt{tn}\left(\frac{(r-D-\mu)}{2\sigma} + \frac{\sigma}{4}\right)\right]$$

$$= \frac{2}{\sqrt{n}}\left[\frac{(\mu t + \ln S/E)}{2\sigma\sqrt{t/n}} + \sqrt{tn}\left(\frac{r-D-\mu}{2\sigma} + \frac{\sigma}{4}\right)\right]$$

$$= \frac{\ln S/E + (r-D+\sigma^2/2)t}{\sigma\sqrt{t}},$$

and similarly

$$\frac{nP-\alpha}{\sqrt{nP(1-P)}} \to \frac{\ln S/E + (r-D-\sigma^2/2)t}{\sigma\sqrt{t}}, \text{ as } n \to \infty.$$

Also, as $n \to \infty$, $\left(\frac{1+r-D}{1+r}\right)^n \to e^{-Dt}$.

Substituting these values into (3A) and (3B) and then into the Binomial formula yields the Black-Scholes formula, demonstrating that in the limit $n \to \infty$, the Binomial formula converges to the Black-Scholes formula.

APPENDIX IV

Knock-In and Lookback Options

A knock-in option comes into effect when the price of the underlying commodity reaches a certain level, known as the barrier. A lookback option relates the exercise price to the history of prices of the underlying commodity over the period until expiry of the option.

Knock-In Options

To illustrate the use of knock-in options, consider a guaranteed equity bond which provides growth in line with the FT-SE 100 Index on 90% of the single premium, a money back guarantee and a lock-in if the index rises by 50% during the 5 year term of the bond. The contract could be backed by the following three assets:

(i) A zero coupon bond to provide the money back guarantee.

(ii) An out of the money European call option with an exercise price 11% above the value of the index at the start of the contract, to provide the required exposure to increases in the index.

(iii) A knock-in European put option with a barrier and an exercise price equal to 150% of the index level at commencement of the bond.

The value of the knock-in European put option can be derived by modifying the Black Scholes formula for an ordinary put option. Knock-in options satisfy the same differential equation (equation (3) in Appendix II) as ordinary options, but the boundary conditions are different.

The boundary conditions are

$P = 0$ if $S \geqslant E$ at expiry,

$= 0$ if $S < E$ at expiry and the price S has never been above E,

$= E\text{-}S$ if $S < E$ at expiry and the price S has been above E at some point before expiry,

where in this example E is both the exercise price and the barrier of the knock-in put option. [In general, the exercise price may be different to the barrier.]

The solution at a time before S hits the barrier E is

$$P = Ee^{-rt}(^E/_S)^{\lambda-1}N(-d_2) - Se^{-Dt}(^E/_S)^{\lambda+1}N(-d_1),$$

where $\lambda = \dfrac{2(r - D)}{\sigma^2}$,

$$d_1 = \frac{\ln {}^E/_S + (r - D + \sigma^2/_2)t}{\sigma\sqrt{t}}$$

and $d_2 = d_1 - \sigma\sqrt{t}$.

When S hits the barrier, the formula for the value of the knock-in put option becomes that for an ordinary put option.

The value of the knock-in put option needed for the contract described above can be derived by putting E = 1.5S in the above formula and setting S = 90% of the single premium.

The asset backing for this contract described in (i), (ii) and (iii) above is more than necessary to cover the liabilities. This can be seen by considering the pay off from the assets if the index rises above 150% of its starting level but is below 111% of its starting level at maturity of the bond. If, for example, the final level of the index is the same as the starting level the pay off from the assets would be:

(i) 100% of the single premium from the zero coupon bond.

(ii) Nothing from the call option.

(iii) 45% (50% of 90%) of the single premium from the knock-in put option.

However, it is only necessary to pay out 135% (150% of 90%) of the single premium on maturity. The difference between the asset and liability pay offs can be removed by writing a knock-in put option with an exercise price of 111% of the starting level of the index with the barrier set at 150% of the starting level of the index, based on an amount equal to 90% of the single premium. This provides income in the form of the premium received for writing the option, enabling improved terms to be offered. In practice the various components of the asset structure would be packaged by the wholesale provider and the price would reflect the precise pay offs required.

The value of this last knock-in put option can be derived as

$$P = 1.11\,Se^{-rt}(1.5)^{\lambda-1}N(-d_2) - Se^{-Dt}(1.5)^{\lambda+1}N(-d_1),$$

where $\lambda = \dfrac{2(r-D)}{\sigma^2}$,

$$d_1 = \frac{\ln 2.03^{(1)} + (r - D + \sigma^2/_2)t}{\sigma\sqrt{t}}$$

and $d_2 = d_1 - \sigma\sqrt{t}$.

[(1) $2.03 = \frac{(1.5)^2}{1.11}$, and in general is $\frac{B^2}{SE}$, where B = barrier level, S = commodity price and E = exercise price.]

The above asset structure can be extended to cater for a series of lock-in levels (e.g. 25%, 50%, 75% and 100% increases in the index) by developing a "ladder" of knock-in options — one bought and one written for each lock-in level — which are added to the zero coupon bond and the call option.

Knock-out options are the converse of knock-in options. They become worthless when the price of the underlying commodity reaches a specified barrier.

Lookback Options

The simplest form of lookback option is a put option with an exercise price equal to the highest price obtained by the underlying commodity between inception and expiry of the option. As mentioned in Section 2.5, the lookback guarantee can be viewed as the continuous form of a series of lock-in guarantees with no upper limit. The value of the lookback put option can be derived by considering a series of knock-in put options and moving to the continuous limit. The value of the lookback put at inception, using the notation and Black Scholes Model of Appendix II, is:

$$P = \frac{\sigma^2 Se^{-Dt}}{2(r-D)}N(d_1) - \frac{\sigma^2 Se^{-rt}}{2(r-D)}N(d_2),$$

where $d_1 = \dfrac{(r - D + \sigma^2/_2)t}{\sigma\sqrt{t}}$

and $d_2 = d_1 - \sigma\sqrt{t}$,

and S is the price of the underlying commodity at inception of the option.

More complicated lookback options where the exercise price is a function of the price history of the underlying commodity can be evaluated in a similar manner, but not all functions will lead to a closed form solution.

The generalised formulae for knock-out options can be found in Merton[8], and the formulae for knock-in options derived from those. The formulae for lookback options are contained in Goldman, Sosin and Gatto[12].